

RELATIVE TO ANY NONRECURSIVE SET

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ABSTRACT. There is a countable first order structure \mathfrak{M} such that for any set of integers X , X is not recursive if and only if there is a presentation of \mathfrak{M} which is recursive in X .

1. INTRODUCTION

It is a well-known truth that if a set $R \subseteq \omega$ is recursive relative to every nonrecursive subset of ω , then R is recursive (see [Kleene and Post, 1954]). Similarly, if R is recursive in every element of a co-meager set, then R is recursive; if R is recursive in every element of a set of positive measure, then R is recursive; and if R is recursive in every element of a nonempty Π_1^0 set P , then R is recursive in the real appearing in the definition of P . In fact, one can consider a wide variety of notions of forcing \mathbb{P} , and conclude that if R is recursive in every generic set (extending the condition p), then R is recursive in \mathbb{P} (and p). Thus, in wide generality, there is no $R \subseteq \omega$ which constitutes information common to all nonrecursive reals or to all generic reals.

Even so, one need not abandon the possibility that there is a mathematical object which instantiates the property of being nonrecursive. The following question of S. Lempp suggests one such possibility.

Question 1.1 (Lempp). *Does there exist a countable first order structure \mathfrak{M} such that, for every X , X is nonrecursive if and only if there is a presentation of \mathfrak{M} which is recursive in X ?*

In Section 2, we answer Question 1.1 by showing that there is such an \mathfrak{M} . In Theorem 2.1, we construct an \mathfrak{M} which has no recursive presentation and is uniformly-recursively presented relative to every nonrecursive real.

Independently and shortly before our obtaining Theorem 2.1, Wehner [1996] answered a question of J. Knight, and obtained an example of a similar type. In the following, say that a family of sets \mathcal{F} is uniformly-recursively enumerable if there is a recursively enumerable set of pairs W such that

$$(\forall F) [F \in \mathcal{F} \iff (\exists i)[F = \{n : (i, n) \in W\}]].$$

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Theorem 1.2 (Wehner). *There is a family \mathcal{F} of subsets of ω which is not uniformly-recursively enumerable, such that for all X , X is nonrecursive if and only if there is a uniformly-recursive-in- X enumeration of \mathcal{F} .*

Shortly after our obtaining Theorem 2.1, A. McAllister showed that Theorem 2.1 can be derived from Wehner's Theorem. The Σ_1 -types of the elements of McAllister's model correspond to the sets in Wehner's family. Conversely, it is not difficult to deduce Wehner's theorem by analyzing the Σ_1 -types realized in our model.

2. THE MODEL

Here, we present the central result of this paper.

Theorem 2.1. *There is a countable model \mathfrak{M} and there is a recursive function $M : X \mapsto M(X)$, which maps subsets of ω to presentations of countable first order structures with the following properties.*

1. *For all X , if X is not recursive, then $M(X)$ is a presentation of \mathfrak{M} .*
2. *There is no recursive presentation of \mathfrak{M} .*

In this section, we will specify the language of \mathfrak{M} and describe the general features of \mathfrak{M} , construct \mathfrak{M} , and then prove that \mathfrak{M} has the required properties.

2.1. Describing \mathfrak{M} . The language of \mathfrak{M} consists of a single constant symbol 0 , a unary function symbol s , and three binary relation symbols R , $<_T$ and $<_L$. We will write $R(x, y)$, $x <_T y$, and $x <_L y$. Let \mathcal{L} denote this language.

In \mathfrak{M} , we use 0 and s to represent the natural numbers with successor. That is, we let $s^i(0)^{\mathfrak{M}}$ denote the i th iterate of $s^{\mathfrak{M}}$ applied to $0^{\mathfrak{M}}$, and set $s^{\mathfrak{M}}$ to be the identity on all points not in $\{s^i(0) : i \in \omega\}^{\mathfrak{M}}$. We use $<_T^{\mathfrak{M}}$ to represent a countable family of nonempty binary trees, and informally let $T_i^{\mathfrak{M}}$ denote the i th tree $\{x : x >_T s^i(0)\}^{\mathfrak{M}}$ with the ordering given by $<_T^{\mathfrak{M}}$. We let $<_L^{\mathfrak{M}}$ uniformly order the levels of the $T_i^{\mathfrak{M}}$'s.

Finally, we use $R^{\mathfrak{M}}$ to pick out paths within $T_i^{\mathfrak{M}}$, as follows. Let $R^{-1}(T_i)^{\mathfrak{M}}$ denote $\{p : R(p, s^i(0))\}^{\mathfrak{M}}$. For each p in $R^{-1}(T_i)^{\mathfrak{M}}$, we let $\zeta(p)^{\mathfrak{M}}$ denote the set $\{x : R(p, x)\}^{\mathfrak{M}}$. We will ensure that $\zeta(p)^{\mathfrak{M}}$ is a maximal finite path in $T_i^{\mathfrak{M}}$. Further, we will ensure that there are maximal finite paths in $T_i^{\mathfrak{M}}$, and ensure that for each maximal finite path ζ in $T_i^{\mathfrak{M}}$ there are infinitely many p such that $\zeta = \zeta(p)^{\mathfrak{M}}$ and so $R^{-1}(T_i)^{\mathfrak{M}}$ is infinite.

In Figure 1, we show a finite subset of the diagram of \mathfrak{M} . In this picture, $\zeta(p)^{\mathfrak{M}}$ is equal to the collection of points below x in the sense of $<_T^{\mathfrak{M}}$.

Let $\langle \mathfrak{A}_i, i \in \omega \rangle$ be a recursive enumeration of the recursively-enumerably presented \mathcal{L} -structures.

We will ensure that either $R^{-1}(T_i)^{\mathfrak{A}_i} = \emptyset$, or $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{A}_i}$ is not isomorphic to $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{M}}$, or there is a p in $R^{-1}(T_i)^{\mathfrak{A}_i}$ such that $\zeta(p)^{\mathfrak{A}_i}$ is not maximal, or there is a p in $R^{-1}(T_i)^{\mathfrak{A}_i}$ such that $\zeta(p)^{\mathfrak{A}_i}$ is infinite. Since none

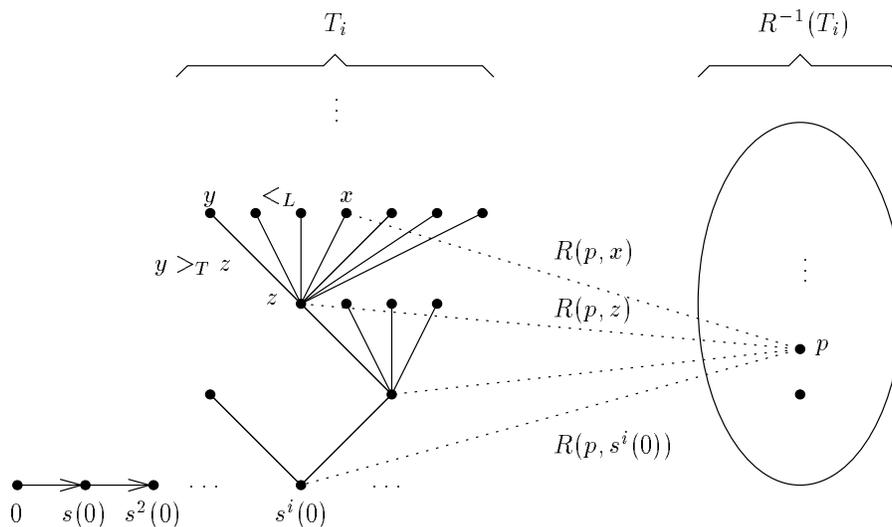


FIGURE 1. Part of the diagram of \mathfrak{M}

of these disjuncts apply to \mathfrak{M} , we will thus ensure that \mathfrak{M} has no recursive presentation.

Although we have been discussing \mathfrak{M} , we will only define \mathfrak{M} at the end of our proof. During the proof, we will define a Σ_1^0 function M , mapping subsets X of ω to relatively recursively enumerable presentations of models $M(X)$, interpreting the language \mathcal{L} . Since a recursively enumerable presentation of an infinite structure can be uniformly converted to a recursive presentation of the same structure, we can conclude that there is a uniformly recursive function mapping X to a recursive presentation of $M(X)$.

Our intension is to ensure that if X is not recursive, then for all i and all p in $R^{-1}(s^i(0))^{M(X)}$, $\zeta(p)^{M(X)}$ is a maximal finite path in $T_i^{M(X)}$. To diagonalize against all of the recursive isomorphism types, we may make $T_i^{M(X)}$ infinite, and so $T_i^{M(X)}$ may have an infinite path as well. For arbitrary X , we cannot avoid the possibility that $\zeta(p)^{M(X)}$ may be infinite, so we ensure that if $\zeta(p)^{M(X)}$ is infinite, then X is recursive. To this end, we ensure that $T_i^{\mathfrak{M}}$ is recursively isomorphic to a recursive tree with at most one infinite path, which would necessarily be recursive. Then, we make the function M sufficiently injective so that for each $p \in \omega$, there can be at most one X such that $\zeta(p)^{M(X)}$ is an infinite path in $T_i^{M(X)}$. Then, this X is recursive.

2.2. Constructing \mathfrak{M} . We divide our construction of $M(X)$ into two parts, the part that is recursively enumerable without reference to X and and the part that is recursively enumerable with essential references to X .

2.2.1. The part of $M(X)$ which does not refer to X . We begin by recursively enumerating the diagram of a model M , which will interpret 0 , s , $<_T$, and $<_L$. Only the interpretation of R within $M(X)$ will depend on X .

We let 0 and s be a recursive presentation of the natural numbers with 0 and the successor function such that the domain of s is a recursive coinfinite set.

We enumerate $<_T$ above $s^i(0)$ while monitoring the i th recursive model \mathfrak{R}_i of the same signature as \mathfrak{M} . In the following, we use the suffix “[s]” to denote a finite approximation given by what has been enumerated by stage s . For example, $\mathfrak{R}_i[s]$ is the finite subset of the diagram of \mathfrak{R}_i enumerated by stage s , and $T_i^M[s]$ is the finite subtree of T_i^M which we have enumerated by stage s . Now, we enumerate the remaining part of M as follows.

1. (a) While there is a no x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, we do not enumerate any successors of $s^i(0)$.
 - (b) If there is an element x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, then let p_0 be the first such in the enumeration of \mathfrak{R}_i .
2. (a) While either the reduced structure $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{R}_i}[s]$ is not isomorphic to $\langle T_i, <_L \upharpoonright T_i \rangle^M[s]$ or the set $\zeta(p_0)^{\mathfrak{R}_i}[s]$ is not a maximal chain in $T_i^{\mathfrak{R}_i}[s]$, we do not enumerate any further elements into T_i^M .
 - (b) Otherwise, we let π denote the unique isomorphism mapping $\langle T_i, <_L \upharpoonright T_i \rangle^M[s]$ to $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{R}_i}[s]$. We let $z_i^M[s]$ be the maximal element of $T_i^M[s]$ such that $\pi(z_i^M[s])$ is the maximal element of $\zeta(p_0)^{\mathfrak{R}_i}[s]$. We enumerate 2^s many immediate $<_T$ -successors of $z_i^M[s]$ into T_i and extend $<_L^M$ so that it linearly orders these new elements of T_i^M . At the next stage, we return to 2(a).

The effect of our enumeration is that either the structures $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{R}_i}$ and $\langle T_i, <_L \upharpoonright T_i \rangle^M$ are not isomorphic, or they are isomorphic by a recursive isomorphism $\pi : \langle T_i, <_L \upharpoonright T_i \rangle^M \rightarrow \langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{R}_i}$, T_i^M is an infinite finitely-branching tree, and $\pi^{-1}(\zeta(p_0)^{\mathfrak{R}_i})$ is isomorphic to its unique branch.

Note, we may safely assume that our enumeration of M proceeds so that the universe of M is a co-infinite recursive subset of ω .

2.2.2. The part of $M(X)$ which refers to X . We begin by fixing some notation. Let \preceq be the lexicographic ordering of $2^{<\omega}$, and for σ in 2^s let $|\sigma|_{\preceq}$ be the number of elements of 2^s which are less than or equal to σ under \preceq .

We define $R^{M(X)}$ by recursively enumerating a collection of pairs $\langle \sigma, \langle p, z \rangle \rangle$ into W_R , where σ is a finite binary sequence, meant to be an initial segment of X , and $\langle p, z \rangle$ is a pair, meant to belong to $R^{M(X)}$. Thus, $\langle p, z \rangle$ is an element of $R^{M(X)}$ if and only if there is a finite initial segment σ of X such that $\langle \sigma, \langle p, z \rangle \rangle$ is an element of W_R . During stage s , we proceed as follows.

1. (a) While there is a no x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, we execute only Step 3, below.
 - (b) If there is an element x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, then let p_0 be the first such in the enumeration of \mathfrak{R}_i , and proceed to Step 2.

2. (a) While either the reduced structure $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{R}_i}[s]$ is not isomorphic to $\langle T_i, <_L \upharpoonright T_i \rangle^M[s]$ or $\zeta(p_0)^{\mathfrak{R}_i}[s]$ is not a maximal chain in $T_i^{\mathfrak{R}_i}[s]$, we execute only Step 3.
- (b) Otherwise, we let π denote the unique isomorphism mapping $\langle T_i, <_L \upharpoonright T_i \rangle^M[s]$ to $\langle T_i, <_L \upharpoonright T_i \rangle^{\mathfrak{R}_i}[s]$. We let $z_i^M[s]$ be the maximal element of $T_i^M[s]$ such that $\pi(z_i^M[s])$ is the maximal element of $\zeta(p_0)^{\mathfrak{R}_i}[s]$. For each σ in 2^s , let k_σ be the $|\sigma|_{\leq}$ th immediate successor of $z_i^M[s]$ (added during stage s , as above); for each p such that $R^\sigma(p, z_i^M[s])$ holds during stage s , enumerate the pair $\langle \sigma, \langle p, k_\sigma \rangle \rangle$ into W_R .
3. For each maximal chain ζ in $T_i^M[s]$ and for each σ in 2^s , we choose a number p^* greater than any number which has previously been mentioned in the construction, and for each point z in ζ enumerate $\langle \sigma, \langle p, z \rangle \rangle$ into W_R .

Definition 2.2. *A countable extension M^* of M to an \mathcal{L} -structure is appropriate if the following conditions hold.*

1. $0, s, <_T$ and $<_L$ have identical interpretations in M and M^* .
2. For each p and z in M^* , if $M^* \models R(p, z)$, then $p \in M^* \setminus M$.
3. For each p in $M^* \setminus M$, $\{z : M^* \models R(p, z)\}$ is a maximal path in one of the T_i^M .

By virtue of our construction, for every X , $M(X)$ is an appropriate extension of \mathfrak{M} .

2.3. Verifying the theorem.

Lemma 2.3. *Suppose that \mathfrak{M}_1 and \mathfrak{M}_2 are two appropriate extensions of M such that for k equal to 1 or 2,*

1. *for every i and every maximal finite path ζ in T_i , there are infinitely many p in \mathfrak{M}_k such that $\zeta(p)^{\mathfrak{M}_k} = \zeta$;*
2. *and there is no p in \mathfrak{M}_k such that $\zeta(p)^{\mathfrak{M}_k}$ is infinite.*

Then, \mathfrak{M}_1 and \mathfrak{M}_2 are isomorphic.

Proof. Each of \mathfrak{M}_1 and \mathfrak{M}_2 consists of M and infinitely many additional elements which it relates to the elements in M by its interpretation of R . Any bijection between the new elements of \mathfrak{M}_1 and the new elements of \mathfrak{M}_2 which preserves R will then be an isomorphism.

Each new element p of \mathfrak{M}_1 is uniquely associated with a maximal finite path $\zeta = \{z \in T_i : R(p, z)\}$ contained in some T_i . Further, for each such ζ there are infinitely many p such that $\zeta = \{z \in T_i : R^{\mathfrak{M}_1}(p, z)\}$. The same conditions hold for \mathfrak{M}_2 .

Let π be any bijection between $\mathfrak{M}_1 \setminus M$ and $\mathfrak{M}_2 \setminus M$ so that for each p in $\mathfrak{M}_1 \setminus M$,

$$\zeta(p)^{\mathfrak{M}_1} = \zeta(\pi(p))^{\mathfrak{M}_2}.$$

Since the only pairs in R are those which associate new elements outside of M to paths in the T_i , for all x and y , $R^{\mathfrak{M}_1}(x, y)$ if and only if $R^{\mathfrak{M}_2}(\pi(x), \pi(y))$. Thus, \mathfrak{M}_1 and \mathfrak{M}_2 are isomorphic. \square

Definition 2.4. *Let \mathfrak{M} be the isomorphism type of the extension of M to a model of the type mentioned in Lemma 2.3*

Corollary 2.5. *If for each p in $M(X) \setminus M$, $\zeta(p)$ is finite, then $M(X)$ is isomorphic to \mathfrak{M} .*

Proof. Apply Lemma 2.3. \square

Lemma 2.6. *There is no recursive presentation of \mathfrak{M} .*

Proof. Suppose that \mathfrak{M} has a recursive presentation, and fix i so that \mathfrak{R}_i is isomorphic to \mathfrak{M} . Since $R^{-1}(s^i(0))^{\mathfrak{M}}$ is not empty, we may let p_0 be the first element in the enumeration of $R^{-1}(s^i(0))^{\mathfrak{R}_i}$.

In Section 2.2.1, we defined T_i in the following way. We waited for an element to appear in $R^{-1}(\{s^i(0)\})^{\mathfrak{R}_i}$. Given that p_0 is the first such element, we then ensured that either T_i^M is finite and $\zeta(p_0)^{\mathfrak{R}_i}$ is not a maximal path in $T_i^{\mathfrak{R}_i}$, or $\zeta(p_0)^{\mathfrak{R}_i}$ is the unique infinite path in $T_i^{\mathfrak{R}_i}$.

For every element p of $R^{-1}(T_i)^{\mathfrak{M}}$, $\zeta(p)^{\mathfrak{M}}$ is a finite maximal chain in $T_i^{\mathfrak{M}}$. But then, p cannot be isomorphic to p_0 , since $\zeta(p_0)^{\mathfrak{R}_i}$ is either not maximal or is infinite. \square

Lemma 2.7. *For each X and $i \in \omega$, one of the following conditions holds.*

1. *For every $p \in R^{-1}(T_i)^{M(X)}$, $\zeta(p)^{M(X)}$ is a finite maximal path in T_i^M .*
2. *X is recursive.*

Proof. As mentioned above, we defined T_i^M in the following way. We waited for an element to appear in $R^{-1}(\{s^i(0)\})^{\mathfrak{R}_i}$. For the first such element, p_0 , to appear in the enumeration of \mathfrak{R}_i , we ensured that either T_i^M is finite and either $T_i^{\mathfrak{R}_i}$ is not isomorphic to T_i^M or $\zeta(p_0)^{\mathfrak{R}_i}$ is not a maximal path in $T_i^{\mathfrak{R}_i}$, or T_i^M is infinite and $\zeta(p_0)^{\mathfrak{R}_i}$ is the unique infinite path in $T_i^{\mathfrak{R}_i}$.

Item (1) is a clear consequence of the first case. So we may assume that the second case occurs.

In Section 2.2.2, we defined R as follows. If during stage s , $\zeta(p_0)^{\mathfrak{R}_i}[s]$ increases in length to become an maximal chain in $T_i^{\mathfrak{R}_i}[s]$ and π is the isomorphism from $T_i^M[s]$ to $T_i^{\mathfrak{R}_i}[s]$, then for each σ in 2^s and each p such that $\pi(\zeta(p)^{M(\sigma)})[s]$ is equal to $\zeta(p_0)^{\mathfrak{R}_i}[s]$, we ensure that for each X extending σ , only one of the immediate predecessors of the maximum of $\zeta(p)^{M(\sigma)}[s]$ is related to p by $R^{M(X)}$, namely k_σ . Note that we choose k_σ as an injective function of σ .

Now, suppose that X and p are given so that $\pi(\zeta(p)^{M(X)}) = \zeta(p_0)^{\mathfrak{R}_i}$. We can compute X as follows. Given a number x , wait for a stage s in the construction such that $s > x$ and $\zeta(p_0)^{\mathfrak{R}_i}[s]$ is larger than ever before. During stage s , we enumerated R so that for each extension ζ of $\pi^{-1}(\zeta(p_0)^{\mathfrak{R}_i})[s]$ in T_i^M , there is at most one binary sequence σ of length s such that $R(\sigma, \langle p, k \rangle)$

holds for all of the elements k appearing in ζ . But then, there is only one such ζ such that $\pi(\zeta)$ is an initial segment of $\zeta(p_0)^{\mathfrak{M}_i}[s]$, and only one such σ for which $R(\sigma, \langle p, k \rangle)$ holds for all of the elements k appearing in that ζ . Since $\zeta(p)^{M(X)}$ is infinite, this unique σ must be an initial segment of X , and $X(x)$ is equal to $\sigma(x)$. \square

We can now prove Theorem 2.1. If X is not recursive, then Lemma 2.7 states that there is no p such that $\zeta(p)^{M(X)}$ is infinite. But then Corollary 2.5 states that $M(X)$ is isomorphic to \mathfrak{M} . Thus, we can conclude the first claim of Theorem 2.1, that we have a uniformly recursive method to present \mathfrak{M} relative to any nonrecursive set X . By Lemma 2.6, there is no recursive presentation of \mathfrak{M} , and we can conclude the remaining claim of Theorem 2.1.

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