World Applied Sciences Journal 7 (12): 1597-11604, 2009
ISSN 1818-4952

An Iterative Scheme for the Solution of Generalized System of Linear Fuzzy Differential Equations

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Abstract: In this paper, a numerical scheme based on variational iteration method is given. The proposed method is an approximate-analytical algorithm for solving a generalized non-homogeneous n-dimensional system of linear fuzzy differential equations $\dot{x}(t) = A\dot{x} + Bx(t) + f(t),\ x(0) = x_0$, where A and B are real $n \times n$ matrices and the initial condition x_0 is described by a vector made up of n fuzzy numbers. A theorem for the convergence of the method is presented. The efficiency of the proposed method is illustrated by a numerical example.

Key words: System of linear fuzzy differential equation . iterative method . approximation solution

INTRODUCTION

Consider the non-homogeneous n-dimensional system of linear fuzzy differential equations

$$\begin{cases} \dot{x}(t) = A\dot{x}(t) + Bx(t) + f(t) \\ \qquad x(0) = x_0, \end{cases} \tag{1}$$

where $x(t) = (x_1, x_2,\dots,x_n)^T$ in which x_i, $i = 1,\dots,n$ are unknown fuzzy functions of variable t, A, $B \in R^{n\times n}$, $f(t) = (f_1, f_2,\dots,f_n)^T$ in which f_i, $i = 1,\dots,n$ are known fuzzy functions of variable t and the initial condition x_0 is described by a vector made up of n fuzzy numbers. The independent variable t belongs to an interval [0, T] for some T>0.

It is well-known that the solution of (1), in the classical form (crisp case) is of the form

$$x(t) = e^{Mt}\left(x_0 + \int_0^t e^{-Ms} f(s)ds\right) \tag{2}$$

where, $e^{M(t-s)}$ is the state transition matrix and $M = (I-A)^{-1}B$.

Hence, for computing the solution x(t), e^{Mt} Z(t) should be computed for some vector function Z (t). We know well, the computation of e^{Mt} either leads to the eigenvalue problem for the matrix M or a formal series of , *i.e.*

$$e^{Mt} = I + tM + \frac{t^2}{2!}M^2 + \cdots$$

In the first case, we need to compute the Jordan canonical form of the matrix which has its own difficulties.

There are some numerical methods to approximate $e^{Mt}Z(t)$, for example, Pade approximation, method based on the Krylov subspace, restrictive Taylor method, *etc*. Also there are several methods for computing the analytical solution.

In the present paper, we apply an iterative scheme for computing an approximate-analytical solution to (1). The convergence of the method is also studied.

This paper is organized as follows. In the next section, some basic definitions and published results are presented. Then we transform the non-homogeneous n-dimensional system of linear fuzzy differential equations to a system of linear ODEs. Then a brief description of iterative method is introduced. The next section is devoted to the proposed method and its convergence. Finally, the last sections give the illustrative examples and conclude the paper.

PRELIMINARIES

In this section, we introduce the notations that will be used in this paper.

Basic notations

Definition 2.1: A fuzzy number u is a fuzzy subset of the real line with a normal, convex and upper semicontinuous membership function of bounded support. The family of fuzzy numbers will be denoted by E.

We represent an arbitrary fuzzy number by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right), 0 \le r \le 1$, which satisfies the following requirements [1]:

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1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0,1], with respect to any r,
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over [0,1], with respect to any r,
3. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$

For $0<r\le 1$, denote $[u]^r = \{x\in R;\ u(x)\ge r\}$ and $[u]^0 = \overline{\{x \in R; u(x) > 0\}}$. Then, it is well-known that for any $r\in[0,1]$, $[u]^r$ is a bounded closed interval.

For u, $v\in E$ and $\lambda\in R$, the sum $u+v$ and the product λu are defined by $[u+v]^r = [u]^r + [v]^r$, $[\lambda.u]^r = \lambda[u]^r, \forall r \in [0,1]$, where $[u]^r+[v]^r$ means the usual addition of two intervals of R and $\lambda[u]^r$ means the usual product between a scalar and a subset of R.

Also note that a crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, 0 \le r \le 1$.

Definition 2.2: A triangular fuzzy number is defined as a fuzzy set in E, which is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a\le b\le c$ such that $[u]^r = [u_-^r, u_+^r]$ are the endpoints of r-level sets for all $r\in[0,1]$, where $u_-^r = a + (b - a)r$ and $u_+^r = c - (c - b)r$. Here, $u_-^0 = a, u_+^0 = c, u_-^1 = u_+^1 = b$, which is denoted by u^1.

In the following, E is the set of triangular fuzzy number.

Definition 2.3: A mapping $f:T\to E$ for some interval $T\subseteq R$ is called a fuzzy process. Therefore, its r-level set can be written as follows,

$$[f(t)]^r = [f_-^r(t), f_+^r(t)],\ t \in T,\ r \in [0,1]$$

Definition 2.4: [1] The supremum metric d_∞ on E is defined by

$$d_\infty(u,v) = \sup\{d_H([u]^r,[v]^r): r \in [0,1]\}$$

where d_H is the Hausdorff metric for non-empty compact sets in R. With the supremum metric, the space (E, d_∞) is a complete metric space.

Definition 2.5: For u, $v\in E$, $w\in E$ is called the Hukuhara difference of u and v, if $u = v+w$ and it is denoted by $w = u \underline{H} v$.

Let us recall the definition of Hukuhara differentiability introduced in [2].

Definition 2.6: A function $f: T\to E$ is Hukuhara differentiable at $t_0\in T$, if there exists an element $f'(t_0)\in E$ such that for all $h>0$ sufficiently small,

$\exists f(t_0 + h) \underline{H} f(t_0), f(t_0) \underline{H} f(t_0 - h)$ and the limits (in the metric d_∞)

$$\lim_{h\searrow 0^+} \frac{f(t_0 + h) \underline{H} f(t_0)}{h} = \lim_{h\searrow 0^+} \frac{f(t_0) \underline{H} f(t_0 - h)}{h} = f'(t_0)$$

Recall that $u \underline{H} v = w\in E$ are defined on level sets, where $[u]^r \underline{H} [v]^r = [w]^r$ for all $0\le r\le 1$. Considering the definition of the metric d_∞, all the level set of mappings $[f(.)]^r$ are Hukuhara differentiable at t_0 with Hukuhara derivatives $[f'(t_0)]^r$ for each $0\le r\le 1$, when $f: T\to E$ is Hukuhara differentiable at t_0 with Hukuhara derivative $f'(t_0)$. Hence, we have the following [3],

Theorem 2.7: Let $f: T\to E$ be Hukuhara differentiable and denote $[f(t)]^r = [f_-^r(t), f_+^r(t)]$. Then, the boundary function $f_-^r(t)$ and $f_+^r(t)$ are differentiable (Seikkala differentiable [4]) and

$$[f'(t)]^r = [(f_-^r(t))', (f_+^r(t))'],\ r\in [0,1].$$

Definition 2.8: Let $f: T\to E$. The fuzzy integral over T, denoted by $\int_T f(t)dt$, is defined level-wise by the equation

$$\left[\int_T f(t)dt\right]^r = \left[\int_T f_-^r(t)dt, \int_T f_+^r(t)dt\right],$$

for all $0\le r\le 1$.

From [5], it is known that if $f: T\to E$ is continuous, it is a fuzzy integrable.

Definition 2.9: If $f: T\to E$ is Seikkala differentiable and its Seikkala derivative f' is integrable over T, then

$$f(t) = f(t_0) + \int_{t_0}^{t} f'(s)ds,$$

for all values of $t_0, t\in T$.

Fuzzy linear system

Definition 2.10: The $n\times n$ linear system of equations

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n, \end{aligned} \qquad (3)$$

where, the coefficients matrix $A = (a_{ij}), 1\le i, j\le n$ is a real (crisp) $n\times n$ matrix and the given $y_i \in E, 1\le i\le n$, with the unknown $x_j \in E, 1 \le j \le n$, is called a fuzzy linear system (FLS).

Definition 2.11: A fuzzy number vector $(x_1, x_2,\ldots,x_n)^T$ given by

$$x_j = (\underline{x}_j(r), \overline{x}_j(r)), 1 \le j \le n, 0 \le r \le 1,$$

is called a solution of (3) if

$$\underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i,$$
$$\overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i, \quad (4)$$

From (4), we have a 2n×2n crisp linear system as follows:

$$SX = Y \quad (5)$$

here,

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}, \; X = \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix}$$

and

$$Y = \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix}$$

where,

$$X = (\underline{X}, \overline{X})^T = (\underline{x}_1, \ldots, \underline{x}_n, \overline{x}_1, \ldots, \overline{x}_n)^T$$

$$Y = (\underline{Y}, \overline{Y})^T = (\underline{y}_1, \ldots, \underline{y}_n, \overline{y}_1, \ldots, \overline{y}_n)^T$$

and s_{ij} are determined as

$$a_{ij} \ge 0 \Rightarrow s_{ij} = s_{i+n,\, j+n} = a_{ij},$$

$$a_{ij} \le 0 \Rightarrow s_{i+n,\, j} = s_{i,\, j+n} = a_{ij},$$

and any s_{ij} which is not determined is zero such that,

$$A = S_1 + S_2.$$

Theorem 2.12: [6] The matrix S is a non-singular if and if only the matrices $A = S_1+S_2$ and S_1-S_2 are both non-singular.

Theorem 13: [6] If S^{-1} exists it must have the same structure as S, *i.e.*

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix} \quad (6)$$

where

$$D = \frac{1}{2}[(S_1 + S_2)^{-1} + (S_1 - S_2)^{-1}$$

$$D = \frac{1}{2}[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}]$$

METAMORPHOSIS

In this section, we are going to transform the system of linear FDE

$$\begin{cases} \dot{x}(t) = A\dot{x}(t) + Bx(t) + f(t) \\ \quad x(0) = x_0, \end{cases}$$

to a system of linear ODEs.

Recall that, $x(t) = (x_1, x_2, \ldots, x_n)^T$ in which x_i, i =1,…,n are unknown fuzzy functions of variable t, A, $B \in R^{n\times n}$, $f(t) = (f_1, f_2, \ldots, f_n)^T$ in which f_i, i = 1,…n are known fuzzy functions of variable t and the initial condition x_0 is described by a vector made up of n fuzzy numbers.

Before entering the main discussion, let us to remind that for all non-crisp fuzzy number $u \in E$, $u + (-u) \ne 0$. Therefore, the system of linear fuzzy differential equations (1) can not be equivalently replaced by the system of linear FDEs

$$\begin{cases} (I - A)\dot{x}(t) = Bx(t) + f(t) \\ \quad x(0) = x_0, \end{cases}$$

which had been investigated.

In order to transform the n×n coefficients $A = (a_{ij})$ and $B = (b_{ij})$ of the system of linear FDEs (1) into 2n×2n matrices as in the (5), we define the following matrices

$$P = (p_{ij}), Q = (q_{ij}); 1 \le i, j \le n,$$

where

$$\begin{aligned} a_{ij} \ge 0 &\Rightarrow p_{ij} = p_{i+n,\, j+n} = a_{ij}, \\ a_{ij} \le 0 &\Rightarrow p_{i+n,\, j} = p_{i,\, j+n} = a_{ij}, \\ b_{ij} \ge 0 &\Rightarrow q_{ij} = q_{i+n,\, j+n} = b_{ij}, \\ b_{ij} \le 0 &\Rightarrow q_{i+n,\, j} = q_{i,\, j+n} = b_{ij}, \end{aligned} \quad (7)$$

while all the remaining p_{ij}, q_{ij} are taken zero.
Using matrix notation, we get

$$\begin{cases} \dot{X}^r(t) = P\dot{X}^r(t) + QX^r(t) + F^r(t), \\ \\ X^r(0) = X_0^r, \end{cases} \quad (8)$$

where

$$r \in [0,1], P = (p_{ij}), Q = q_{ij}, 1 \le i, j \le 2n$$

and

$$\dot{X}^r(t) = \left(\underline{x}_1^r(t),\ldots,\underline{x}_n^r(t),\overline{x}_1^r(t),\ldots,\overline{x}_n^r(t)\right)^T,$$
$$X^r(t) = \left(\underline{x}_1^r(t),\ldots,\underline{x}_n^r(t),\overline{x}_1^r(t),\ldots,\overline{x}_n^r(t)\right)^T,$$
$$F^r(t) = \left(\underline{f}_1^r(t),\ldots,\underline{f}_n^r(t),\overline{f}_1^r(t),\ldots,\overline{f}_n^r(t)\right)^T,$$
$$X^r(0) = \left(\underline{x}_{01}^r,\ldots,\underline{x}_{0n}^r,\overline{x}_{01}^r,\ldots,\overline{x}_{0n}^r\right)^T.$$

Assuming that I-P is non-singular, we obtain

$$\begin{cases}\dot{X}^r(t) = MX^r(t) + NF^r(t),\\ X^r(0) = X_0^r,\end{cases} \tag{9}$$

here, $r\in[0,1]$, $M = (I-P)^{-1}Q$ and $N = (I-P)^{-1}$.

Now, the following immediate and important question arises, *under what conditions the vector $\dot{X}$ is an appropriate fuzzy number vector?*

The next theorem provides the sufficient conditions for $\dot{X}$ be a fuzzy number vector.

Theorem 3.1: For a given fuzzy number vector X, $\dot{X}$ in the system of linear FDES (1) is a fuzzy number vector if and only if $(I-P)^{-1}$ and $(I-P)^{-1}Q$ exist and all their entries are non-negative.

Proof: This is similar to the proof of Theorem 1 pp. 57 [7]. By (7), the system of linear FDEs (1) is equivalent to the system (8) and it follows that

$$(I-P)\dot{X}^r(t) = QX^r(t) + F^r(t) \tag{10}$$

If $(I-P)^{-1}$ exists, $\dot{X}^r(t)(0 \le r \le 1)$ is obtained uniquely by Eq.(10) and clearly if $(I-P)_{ij}^{-1} \ge 0$ and $\left((I-P)^{-1}Q\right)_{ij}$ for all i,j, then by virtue of Lemma 2 in [7], $\left(t\underline{x}_i^r(t),\overline{x}_i^r(t)\right)_{i=1}^n$ is a fuzzy number vector.

Before proceeding to the main subject, for a better clarification, the He's method with system (1) in the form of a system of ODEs is introduced.

ITARATIVE SCHEME

Consider the following differential equation (DE) in the crisp case

$$Lu(t) + Nu(t) = g(t) \tag{11}$$

where L is a linear operator, N a nonlinear operator and g(t) an inhomogeneous term.

In He's variational iteration method, a correctional functional such as

$$u_{m+1}(t) = u_m + \int_0^t \lambda(Lu_m(s) + N\tilde{u}_m(s) - g(s))ds,\ m = 0,1,2,\ldots,$$

is constructed, where λ is a general Lagrangian multiplier [8-12], which can be identified optimally via the variational theory. Obviously, the successive approximations, u_j, j = 0,1,..., can be computed by determining λ. Here, the function $\tilde{u}_m$ is a restricted variation, which means $\delta\tilde{u}_m = 0$.

Now, we consider the system of ordinary differential equations in the crisp case with constant coefficients

$$\begin{cases}\dot{X}(t) = MX(t) + NF(t), \quad 0 \le t \le 1,\\ X(0) = X_0,\end{cases} \tag{12}$$

where $X(t) = (x_1, x_2,..., x_n)^T$ in which x_i, i = 1,…,n are unknown real functions of variable t, M and $N\in R^{n\times n}$, $F(t) = (f_1, f_2,..., f_n)^T$ where f_i, i = 1,…,n are known real functions of t and X_0 is a given vector in R^n.

It is well-known that, the solution of system (12) is

$$X(t) = \Phi(t)X_0 + \int_0^t \Phi(t)\Phi(s)^{-1}NF(s)ds, \tag{13}$$

where $\Phi(t,s) = \Phi(t)\Phi(s)^{-1}$ is the state transition matrix and $\Phi(t)$ satisfies the matrix DE

$$\dot{\Phi}(t) = M\Phi(t),\ \Phi(0) = I. \tag{14}$$

Since, here, M is time-independent, Φ (t) appears in the matrix exponential, *i.e.*, $\Phi(t) = e^{tM}$ and therefore, $\Phi(t,s) = e^{(t-s)M}$.

For solving problem (12) using the variational iteration method, the matrix $M = (m_{ij})$ is decomposed into two matrices D and M_1, such that $M = D+M_1$, where $D = diag(m_{11}, m_{22},\ldots,m_{nn})$ and $M_1 = M-D$.

Then, we construct the following correction functional for Y

$$X_{l+1}(t) = X_l(t) + \int_0^t \Lambda(\dot{X}_l(s) - DX_l(s) - M_1\tilde{X}_l(s) - NF(s))ds \tag{15}$$

where $\Lambda = diag(\lambda_1, \lambda_2,\ldots,\lambda_n)$, in which λ_i, i = 1,2,…,n are the Lagrange multipliers and $\tilde{X}_l$ denotes the restrictive variation, *i.e.*,

$$\delta\tilde{X}_1 = (\delta\tilde{x}_1, \delta ex_2, \ldots, \delta\tilde{x}_n)^T = 0 .$$

Here, M_1X is considered as a nonlinear term, although it is clearly not.

Using integration by parts and also by constructing the correction functional,

$$\begin{aligned}\delta X_{l+1}(t) &= \delta X_l(t) + \delta\int_0^t \Lambda(\dot{X}_l(s) - DX_l(s) \\ &\quad - M_1\tilde{X}_l(s) - NF(s))ds \\ &= \delta X_l(t) + \Lambda(s)\delta X_l(s)|_{s=t} - \delta\int_0^t (\dot{\Lambda} + \Lambda D)(X_l(s) \\ &\quad + \Lambda(M_1\tilde{X}_l(s) + NF(s))ds,\end{aligned}$$

the stationary conditions would be as follows

$$I + \Lambda(s)|_{s=t} = 0,$$

$$\dot{\Lambda}(s) + \Lambda(s)D = 0.$$

For a fixed i, we consider two cases. If $m_{ii} = 0$, then it follows that $\lambda_i(s) = -1$ and if $m_{ii} \neq 0$, then $\lambda i = -e^{-m_{ii}(s-t)}$. Hence, we have $\Lambda = -e^{-(s-t)D}$

Therefore, from (15), the following iteration formula for computing $X_l(t)$ may be obtained,

$$X_{l+1}(t) = X_l(t) - \int_0^t e^{-(s-t)D}(\dot{X}_l(s) - MX_m(s) - NF(s))ds, \quad l = 0,1,\ldots. \quad (16)$$

CONVERGENCE

Having introduced the iterative scheme, here in this section, system (9) is solved by this method.

Using (16) for (9), we obtain the following variational iteration method for $l = 0,1,\ldots$

$$X_{l+1}^r(t) = X_l^r(t) - \int_0^t e^{-(s-t)D}(\dot{X}_l^r(s) - MX_l^r(s) - NF^r(s))ds. \quad (17)$$

Now, we show that the sequence $\{X_l^r(t)\}_{l=l}^{\infty}$ defined by (17), with $X^r(0) = X_0^r$, converges to the exact solutions of (9) for all $r \in [0,1]$.

Theorem 5.1: Let

$$X^r(t), X_l^r(t) \in \left(C^1[0,T]\right)^{2n}, l = 0,1,\ldots.$$

The sequence defined by (17) with $X^r(0) = (x_0)^r$, converges to $X^r(t)$, the exact solution of (9).

Proof: It is sufficient to show $\lim_{l\to\infty} X_l^r(t) = X^r(t)$, for a fixed $r \in [0,1]$. From (9) we obtain,

$$X^r(t) = X^r(t) - \int_0^t e^{-(s-t)D}(\dot{X}^r(s) - MX^r(s) - NF^r(s))ds. \quad (18)$$

Subtracting (18) from (17), we get

$$E_{l+1}^r(t) = E_l^r(t) - \int_0^t e^{-(s-t)D}(\dot{E}_l^r(s) - ME_l^r(s))ds, \quad (19)$$

where,

$$E_j^r(t) = X_j^r(t) - X^r(t), \quad j = 1,2,\ldots.$$

Noting that for $l = 0,1,\ldots$, we have $E_l^r(0) = 0$. Then integration by parts for (19) gives,

$$\begin{aligned}E_{l+1}^r(t) &= E_l^r(t) - \int_0^t e^{(s(M-D)+tD)}\frac{d}{ds}(e^{-sM}E_l^r(s))ds, \\ &= E_l^r(t) - \int_0^t e^{(sM_1+tD)}\frac{d}{ds}(e^{-sM}E_l^r(s))ds, \quad (20) \\ &= M_l^r\int_0^t e^{-(s-t)D}E_l^r(s)ds.\end{aligned}$$

Therefore,

$$\| E_{l+1}^r(t) \|_\infty \leq \| M_l^r \|_\infty \int_0^t \| e^{-(s-t)D} \|_\infty \| E_l^r(s) \|_\infty ds. \quad (21)$$

On the other hand, since s<t<T, we conclude that

$$\| e^{-(s-t)D} \|_\infty \leq e^{\|-(s-t)D\|_\infty} \leq e^{|s-t|\|D\|_\infty} \leq e^{2T\max_i|m_{ii}|}.$$

Let $B_1 = \| M_1 \|_\infty e^{2T\max_i|m_{ii}|}$. Thus, (21) can be written as

$$\| E_{l+1}^r(t) \|_\infty \leq B_1\int_0^t \| E_l^r(s) \|_\infty ds. \quad (22)$$

Proceeding as follows

$$\begin{aligned}&\| E_1^r(t) \|_\infty \leq B_1\int_0^t \| E_0^r(s) \|_\infty ds \leq \\ &B_1 \| E_0^r(s) \|_\infty \int_0^t ds \leq B_1 \| E_0^r(s) \|_\infty t,\end{aligned}$$

$$\begin{aligned}&\| E_2^r(t) \|_\infty \leq B_1\int_0^t \| E_1^r(s) \|_\infty ds \leq \\ &B_1 \| E_0^r(s) \|_\infty \int_0^t tds = \left(B_1^r\right)^2 \| E_0^r(s) \|_\infty \frac{t^2}{2!},\end{aligned}$$

$$\| E_3^r(t) \|_\infty \le B_1 \int_0^t \| E_2^r(s) \|_\infty \, ds \le$$

$$B_1 \| E_0^r(s) \|_\infty \int_0^t \frac{t^2}{2!} ds = (B_1)^3 \| E_0^r(s) \|_\infty \frac{t^3}{3!},$$

$$\vdots$$

$$\| E_l^r(t) \|_\infty \le B_1 \int_0^t \| E_{l-1}^r(s) \|_\infty \, ds \le B_1 \| E_0^r(s) \|_\infty \int_0^t \frac{t^{l-1}}{(l-1)!} ds$$

$$= (B_1)^l \| E_0^r(s) \|_\infty \frac{t^l}{l!} = \| E_0^r(s) \|_\infty \frac{(B_1 t)^l}{l!}$$

Therefore,

$$\| E_0^r(s) \|_\infty \frac{(B_1 t)^l}{l!} \le \| E_0^r(s) \|_\infty \frac{(B_1 T)^l}{l!} \longrightarrow 0,$$

as $l \to \infty$ and these complete the proof.

Remark 5.2: For improving the convergence rate of the method, as in the classical Gauss-Seidel method for solving linear system of equations, as soon as a component of X_{l+1}^r is computed then it is used in computing the next component of X_{l+1}^r.

NUMERICAL RESULT

To illustrate the efficiency of the method, an example is given. For this example we used only 4 iterations of the method. The numerical calculations are all undertaken by MATLAB software.

Example 1: Consider the problem (1), with

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ \frac{1}{3} & 0 \end{pmatrix}, x^r(0) = \begin{pmatrix} [1+r, 3-r] \\ [r, 2-r] \end{pmatrix}, r \in [0,1].$$

In this example $f(t) \equiv 0$. The exact solutions of the problem for r = 0 are

$$\underline{x}_1^0(t) = 1.1734e^{3.7554t} - 0.8034e^{-1.3629t} + 0.8266e^{-0.0888t}$$
$$-0.1965e^{-0.0815t},$$

$$\underline{x}_2^0(t) = 2.4509e^{3.7554t} - 1.4104e^{-1.3629t} - 1.4509e^{-0.0888t}$$
$$+0.4104e^{-0.0815t},$$

$$\overline{x}_1^0(t) = 1.1734e^{3.7554t} + 0.8034e^{-1.3629t} + 0.8266e^{-0.0888t}$$
$$+0.1965e^{-0.0815t},$$

$$\overline{x}_2^0(t) = 2.4509e^{3.7554t} + 1.4104e^{-1.3629t} - 1.4509e^{-0.0888t}$$
$$-0.4104e^{-0.0815t}.$$

By (7), we have

$$P = \begin{pmatrix} -1 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}.$$

Therefore, from (9) we get

$$\begin{pmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \\ \dot{\overline{x}}_1 \\ \dot{\overline{x}}_2 \end{pmatrix} = \begin{pmatrix} 0.4444 & 0.3333 & 1.2222 & 0.6667 \\ 1.222 & 0.6667 & 2.4444 & 1.3333 \\ 1.222 & 0.6667 & 0.4444 & 0.3333 \\ 2.4444 & 1.3333 & 1.2222 & 0.6667 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \overline{x}_1 \\ \overline{x}_2 \end{pmatrix}.$$

Using the proposed method, the computed approximate solutions are as follows

$$\underline{x}_1^0(t) = -\frac{1676655}{2048} + \frac{201035787464975}{2048} e^{\frac{4}{9}t}$$
$$-\frac{6282368305821}{64} e^{\frac{2}{3}t} + \frac{5012309285577}{512} te^{\frac{4}{9}t}$$
$$+\frac{384771581037}{32} te^{\frac{2}{3}t} + \frac{8843614313963}{20736} t^2 e^{\frac{4}{9}t}$$
$$-\frac{43184486889}{64} t^2 e^{\frac{2}{3}t} + \frac{481772070197}{46656} t^3 e^{\frac{4}{9}t}$$
$$+\frac{133594235}{6} t^3 e^{\frac{2}{3}t} + \frac{22884424717}{157464} t^4 e^{\frac{4}{9}t} - \frac{86449625}{192} t^4 e^{\frac{2}{3}t}$$
$$+\frac{1318846639}{1180980} t^5 e^{\frac{4}{9}t} + \frac{154375}{32} t^5 e^{\frac{2}{3}t} + \frac{86806489}{23914845} t^6 e^{\frac{4}{9}t},$$

$$\underline{x}_2^0(t) = \frac{2215575}{4096} - \frac{4326885398355751}{4096} e^{\frac{4}{9}t} + \frac{270430337258761}{256} e^{\frac{2}{3}t}$$
$$-\frac{291802581908299}{3072} te^{\frac{4}{9}t} - \frac{26834061458879}{192} te^{\frac{2}{3}t} -$$
$$\frac{153840071712019}{41472} t^2 e^{\frac{4}{9}t} + \frac{5002134318653}{576} t^2 e^{\frac{2}{3}t}$$
$$-\frac{7479499488439}{93312} t^3 e^{\frac{4}{9}t} - \frac{1715304873793}{5184} t^3 e^{\frac{2}{3}t}$$
$$-\frac{158189928985}{157464} t^4 e^{\frac{4}{9}t} + \frac{1208311825}{144} t^4 e^{\frac{2}{3}t}$$
$$-\frac{16243442809}{2361960} t^5 e^{\frac{4}{9}t} - \frac{81015625}{576} t^5 e^{\frac{2}{3}t} - \frac{954871379}{47829690} t^6 e^{\frac{4}{9}t}$$
$$+\frac{771875}{576} t^6 e^{\frac{2}{3}t},$$

$$
\begin{aligned}
\overline{x}_1^0(t) = & \frac{836331}{512} - \frac{3008274890965161}{512} e^{\frac{4}{9}t} \\
& + \frac{1504137445065183}{256} e^{\frac{2}{3}t} - \frac{333105682346833}{576} t e^{\frac{4}{9}t} \\
& - \frac{46551448862945}{64} t e^{\frac{2}{3}t} - \frac{43913525591947}{1728} t^2 e^{\frac{4}{9}t} \\
& + \frac{16118365405987}{384} t^2 e^{\frac{2}{3}t} - \frac{45061551488635}{69984} t^3 e^{\frac{4}{9}t} \\
& - \frac{1270516649069}{864} t^3 e^{\frac{2}{3}t} - \frac{707251147405}{69984} t^4 e^{\frac{4}{9}t} \\
& + \frac{12977825975}{384} t^4 e^{\frac{2}{3}t} - \frac{691795697857}{7085880} t^5 e^{\frac{4}{9}t} \\
& - \frac{48149375}{96} t^5 e^{\frac{2}{3}t} - \frac{1702470121}{3188646} t^6 e^{\frac{4}{9}t} \\
& + \frac{771875}{192} t^6 e^{\frac{2}{3}t} - \frac{272820394}{215233605} t^7 e^{\frac{4}{9}t},
\end{aligned}
$$

Table 1: Numerical results for $\underline{x}_2^0(t)$

t	Our method	Exact
0.0	0.0000	0.0000
0.1	1.3063	1.3063
0.2	3.0988	3.0988
0.3	5.6123	5.6125
0.4	9.1863	9.1877
0.5	14.1330	14.1385
0.6	21.6921	21.7228
0.7	31.3404	32.3404
0.8	47.6947	47.6947
0.9	69.7886	70.6075
1.0	101.4740	103.4758

Table 2: Numerical results for $\overline{x}_1^0(t)$

t	Our method	Exact
0.0	3.0000	3.0000
0.1	3.4235	3.4235
0.2	4.1040	4.1040
0.3	5.1507	5.1507
0.4	6.7239	6.7241
0.5	9.0568	9.0583
0.6	12.4885	12.4949
0.7	17.5085	17.5321
0.8	24.8213	24.8954
0.9	35.4333	35.6417
1.0	50.7748	51.3098

and finally

$$
\begin{aligned}
\overline{x}_2^0(t) = & -\frac{1100467}{1024} + \frac{64728667841354847}{1024} e^{\frac{4}{9}t} \\
& - \frac{16182166960063083}{256} e^{\frac{2}{3}t} + \frac{6563253193565453}{1152} t e^{\frac{4}{9}t} \\
& + \frac{2137536392434433}{256} t e^{\frac{2}{3}t} + \frac{32869299073259}{144} t^2 e^{\frac{4}{9}t} \\
& - \frac{200824077920761}{384} t^2 e^{\frac{2}{3}t} + \frac{735322959899165}{139968} t^3 e^{\frac{4}{9}t} \\
& + \frac{70539580581911}{3456} t^3 e^{\frac{2}{3}t} + \frac{10457159802485}{139968} t^4 e^{\frac{4}{9}t} \\
& - \frac{5661022116955}{10368} t^4 e^{\frac{2}{3}t} + \frac{9258643706543}{14171760} t^5 e^{\frac{4}{9}t} \\
& + \frac{35523656125}{3456} t^5 e^{\frac{2}{3}t} + \frac{6878971363}{2125764} t^6 e^{\frac{4}{9}t} \\
& - \frac{75720625}{576} t^6 e^{\frac{2}{3}t} + \frac{1500512167}{215233605} t^7 e^{\frac{4}{9}t} + \frac{3859375}{4032} t^7 e^{\frac{2}{3}t}.
\end{aligned}
$$

The exact and approximate solutions are depicted in Table 1 and 2. As we see, there is a good agreement between the approximate solutions obtained by the presented method and the exact solutions.

CONCLUSION

We have successfully used an iterative scheme for solving non-homogeneous n-dimensional system of first order linear FDEs with constant coefficients in the Seikkala derivative sense. A theorem for the convergence of the method has been given in details. The Results obtained by the method confirm the robustness and efficiency of the method. The method can also be implemented easily.

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