Graphs, groupoids and Cuntz-Krieger algebras

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Abstract

We associate to each locally finite directed graph G two locally compact groupoids $\mathcal{G}$ and $\mathcal{G}(\star)$. The unit space of $\mathcal{G}$ is the space of one–sided infinite paths in G, and $\mathcal{G}(\star)$ is the reduction of $\mathcal{G}$ to the space of paths emanating from a distinguished vertex $\star$. We show that under certain conditions their C^*–algebras are Morita equivalent; the groupoid C^*–algebra $C^*(\mathcal{G})$ is the Cuntz–Krieger algebra of an infinite $\{0,1\}$ matrix defined by G, and that the algebras $C^*(\mathcal{G}(\star))$ contain the C^*–algebras used by Doplicher and Roberts in their duality theory for compact groups. We then analyse the ideal structure of these groupoid C^*–algebras using the general theory of Renault, and calculate their K-theory.

1 Introduction

Over the past fifteen years many C^*-algebras and classes of C^*-algebras have been given groupoid models. Here we consider locally finite directed graphs, which may have infinitely many vertices, but only finitely many edges in and out of each vertex. We associate to each such graph G a locally compact groupoid $\mathcal{G}$, and show that its groupoid C^*-algebra $C^*(\mathcal{G})$ is the universal C^*-algebra generated by (possibly infinite) families of partial isometries satisfying Cuntz-Krieger relations determined by G. We then use Renault's structure theory for groupoid C^*-algebras [16, 17] to analyse the ideal structure of these Cuntz-Krieger algebras, thereby extending important results of Cuntz and Cuntz-Krieger [2, 3] to the case of infinite, locally finite $\{0,1\}$-matrices.

Our original motivation was to understand the C^*-algebras arising in the duality theory for compact groups of Doplicher and Roberts [4, 5]. Renault's philosophy suggests that the algebra $\mathcal{O}_\rho$ associated by Doplicher-Roberts to a special unitary representation $\rho : K \to SU(n)$ should be realisable as a groupoid C^*-algebra, and the basic structural properties of $\mathcal{O}_\rho$, such as simplicity, should follow from the general theory of groupoid algebras. This program was begun in [10]. The algebra $\mathcal{O}_\rho$ is constructed from the intertwiners of tensor powers ρ^n of ρ, which depend on the decompositions of ρ^n into irreducibles. The construction of the groupoid $\mathcal{G}_\rho$ in [10] is based on a directed graph G_ρ with vertex set $\widehat{K}$, and edges describing the decompositions of $\{\rho \otimes \pi : \pi \in \widehat{K}\}$ as direct sums of representations in $\widehat{K}$; the unit space of $\mathcal{G}_\rho$ is the infinite path space of the graph, and the groupoid itself is given, loosely speaking, by tail equivalence with lag. The algebra $\mathcal{O}_\rho$ is shown in [10] to be the enveloping algebra of a subalgebra ${}^0\mathcal{O}_\rho$ of $C_c(\mathcal{G}_\rho)$, but unfortunately it is not clear how to prove that the enveloping norms on ${}^0\mathcal{O}_\rho$ and $C_c(\mathcal{G}_\rho)$ agree — even though we know they must, because both enveloping algebras are known to be simple. The situation described in [10], therefore, is unsatisfactory: one has a groupoid model, but needs the established theory to complete the identification. Although the ideas of [10] were subsequently adapted to give a new algebraic approach to the Doplicher-Roberts theory [14], this approach avoids the technical difficulty rather than solves it.

The present paper brings the program of [10] to a more satisfactory conclusion. We associate a groupoid $\mathcal{G}(\star)$ to every pointed directed graph G, using tail equivalence with lag on the infinite path space, so that applying our construction to G_ρ gives $\mathcal{G}_\rho$. We show that this groupoid is r-discrete and amenable, and give conditions on G which ensure that $\mathcal{G}(\star)$ is essentially principal, so that the structure theory of [16, §II.4] and [17] applies. We then prove directly that *-representations of ${}^0\mathcal{O}_\rho$ extend to *-representations of $C_c(\mathcal{G}_\rho)$, and hence that the completions $\mathcal{O}_\rho$ and $C^*(\mathcal{G}_\rho)$ coincide. We can then deduce from [17] that $\mathcal{O}_\rho$ is simple. But having done all this work, we now believe the most interesting aspect of our results to be their applications to the Cuntz-Krieger algebras of infinite $\{0,1\}$-matrices.

Cuntz-Krieger algebras $\mathcal{O}_A$ arise naturally when one tries to compute the K-theory of the algebras $\mathcal{O}_\rho$. For finite groups, $\mathcal{O}_\rho$ is a corner in an appropriate $\mathcal{O}_A$, and one can use the known computations of $K_*(\mathcal{O}_A)$ to find $K_*(\mathcal{O}_\rho)$ [9]. For compact groups, the corresponding matrix A is an infinite, locally finite $\{0,1\}$-matrix, but not much is known about the corresponding $\mathcal{O}_A$. So when we computed $K_*(\mathcal{O}_\rho)$ in [13], we first had to extend the basic theorems of Cuntz and Krieger [3] to an appropriate class of infinite A, which we did by approximating $\mathcal{O}_A$ by the algebras $\mathcal{O}_B$ of suitable finite B [13, §2]. In our present approach, groupoid models for $\mathcal{O}_A$ appear naturally alongside those of the corresponding $\mathcal{O}_\rho$; indeed, because we are now working with arbitrary directed graphs, we have models of $\mathcal{O}_A$ for every locally finite A. Under an appropriate Condition (K) on A, which generalises the Condition (II) of [2], the theory of [17] applies, and the main theorem of [2] carries over. It is interesting that Renault's theory is deep enough to give the full strength of [2, Theorem 2.5] in the case of finite A, though not the full uniqueness theorem of [3]. We are not aware, incidentally, that this analysis has previously been carried out even for finite A.

We begin by constructing the groupoid $\mathcal{G}$ of a directed graph $G = (V, E)$. We need to assume the graph is row-finite (that each vertex emits only finitely many edges) to ensure that the one-sided path space $P(G)$ is locally compact. The groupoid $\mathcal{G}$ is then r-discrete, locally compact and Hausdorff, with unit space $P(G)$. We also introduce a pointed version $\mathcal{G}(\star)$, by fixing a distinguished vertex $\star$ and restricting attention to paths starting at $\star$. In §2, we show that the groupoids $\mathcal{G}$ and $\mathcal{G}(\star)$ are equivalent in the sense of [11], so that their C^*-algebras are Morita equivalent: indeed, $C^*(\mathcal{G}(\star))$ embeds as a corner in $C^*(\mathcal{G})$. Since $\mathcal{G}$ will model Cuntz-Krieger algebras, and suitable $\mathcal{G}(\star) = \mathcal{G}_\rho$ will model Doplicher-Roberts algebras, this will later embed $\mathcal{O}_\rho$ as a corner in a Cuntz-Krieger algebra.

In §3, we identify the groupoid algebra $C^*(\mathcal{G})$ as the universal C^*-algebra generated by families of partial isometries $\{S_e : e \in E\}$ parametrised by the edge set E of G which satisfy the Cuntz-Krieger relations

$$S_e^* S_e = \sum_{\{f \in E : s(f) = r(e)\}} S_f S_f^* =: \sum_{f \in E} A(e,f) S_f S_f^*.$$

It is quite easy to write down a Cuntz-Krieger family which generates $C^*(\mathcal{G})$ (Proposition 4.1), but we have to work to show that every such family determines a *-representation of $C_c(\mathcal{G})$. This difficulty is similar to that encountered in extending representations of ${}^0\mathcal{O}_\rho$ to $\mathcal{G}_\rho$, and we give a version of our construction for pointed graphs which will be used in §6 to settle this question.

A main technical hypothesis in Renault's theory is amenability, and in §4 we check this for our groupoids. As suggested in [10], this can be done by realising $\mathcal{G}$ and $\mathcal{G}(\star)$ as reductions of a semidirect product $\mathcal{R} \times \mathbf{Z}$, in which $\mathcal{R}$ is an equivalence relation on a two-sided product space. Nevertheless, we had to make substantial changes to the procedure outlined in [10], and had to assume that the graph is locally finite to know that $\mathcal{R}$ is AF, and hence amenable. In retrospect, our construction is similar to that carried out in [3, p. 259].

Our main results are in §5, where we apply Renault's theory to compute the ideals of $C^*(\mathcal{G})$. We need to impose a structural condition on our graphs G to ensure that the corresponding groupoids $\mathcal{G}$ are essentially principal. Our Condition (K) is an analogue of the Condition (II) imposed by Cuntz in [2]. We discuss the relationship of (K) and (II) at the start of §5, and give some examples of new phenomena which arise for infinite graphs, and which had to be accommodated by (K). Our main Theorem 6.6 is a direct generalisation of [2, Theorem 2.5]. We also give a version of the Cuntz-Krieger Uniqueness Theorem which is not quite so satisfactory: we leave open the question of how best to extend Condition (I) of [3] to the infinite case, and merely observe that the present approach will not suffice, since the groupoids need not be essentially principal.

In our final section, we show how the Doplicher-Roberts algebras $\mathcal{O}_\rho$ are naturally isomorphic to the groupoid algebra $C^*(\mathcal{G}(\star))$ of the graph G_ρ, pointed at the trivial representation ι. As we mentioned earlier, the constructions in §3 effectively solve the main technical problem in the program of [10]. Since we have already shown that $C^*(\mathcal{G}(\star))$ is a corner in $C^*(\mathcal{G})$, the results of the previous section give the simplicity and the K-theory of $\mathcal{O}_\rho$, by methods which are independent of the previous results of Doplicher-Roberts and Cuntz.

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2 Directed graphs and their groupoids.

A *directed graph* $G = (V, E, r, s)$ consists of a countable set V of vertices, a set E of edges, and maps $r, s : E \to V$ describing the range and source of edges. To avoid pathological cases, we assume that the map $s : E \to V$ is onto, so that every vertex emits at least one edge. A directed graph $G = (V, E, r, s)$ is *row finite* if $s^{-1}(v) \subseteq E$ is a finite set for all $v \in V$, and *locally finite* if both $s^{-1}(v)$ and $r^{-1}(v)$ are finite for all $v \in V$. It is *pointed* if there is a distinguished vertex $\star \in V$.

A *finite path* in a directed graph G is a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of edges in G with $s(\alpha_{j+1}) = r(\alpha_j)$ for $1 \le j \le k-1$; we write $s(\alpha) = s(\alpha_1)$ and $r(\alpha) = r(\alpha_k)$, and $|\alpha| := k$ for the *length* of α. We write $F(G)$ for the set of all finite paths in G (we denote by v the path of length 0 with $s(v) = r(v) = v$), and $F(G, v)$ for the set of finite paths $\alpha \in F(G)$ such that $s(\alpha)$ is a fixed vertex v. We let $P(G)$ and $P(G, v)$ denote the corresponding sets of infinite paths $\alpha = (\alpha_1, \alpha_2, \cdots)$ in G.

For $\alpha, \mu \in F(G)$ satisfying $r(\alpha) = s(\mu)$, we define a path $\alpha\mu \in F(G)$ of length $|\alpha| + |\mu|$ by $\alpha\mu = (\alpha_1, \dots, \alpha_{|\alpha|}, \mu_1, \dots, \mu_{|\mu|})$. We can similarly define $\alpha x \in P(G)$ for $\alpha \in F(G)$ and $x \in P(G)$ satisfying $s(x) = r(\alpha)$.

The infinite path space $P(G)$, being a subset of the product space $\prod_1^\infty E$, has a natural product topology, for which the *cylinder sets*

$$Z(\alpha) = \{x \in P(G) : x_1 = \alpha_1, \dots, x_{|\alpha|} = \alpha_{|\alpha|}\}$$

parametrised by $\alpha \in F(G)$ form a basis of open sets. This is not immediately obvious: one has to prove that finite intersections of $Z(\alpha)$'s contain enough $Z(\beta)$'s to separate points. The following straightforward Lemma guarantees this:

2.1 Lemma: *For $\alpha, \beta \in F(G)$, we have*

$$Z(\alpha) \cap Z(\beta) = \begin{cases} Z(\alpha) & \text{if } \alpha = \beta\alpha' \text{ for some } \alpha' \in F(G), \\ Z(\beta) & \text{if } \beta = \alpha\beta' \text{ for some } \beta' \in F(G), \\ \emptyset & \text{otherwise.} \end{cases}$$

When the graph G is row finite, only a finite set $E_k(v)$ of edges can be reached by paths of length k starting at a given vertex v. Thus each cylinder set $Z(\alpha)$ is (homeomorphic to) a subset of the compact product space $\prod_{k=1}^\infty E_k(r(\alpha))$, and is itself compact. Hence:

2.2 Corollary: *If G is a row-finite directed graph, the cylinder sets $\{Z(\alpha) : \alpha \in F(G)\}$ form a basis of compact open sets for a locally compact, σ-compact, totally disconnected, Hausdorff topology on $P(G)$, which coincides with the product topology obtained by viewing $P(G)$ as a subset of $\prod E$.*

We aim to define a groupoid with unit space $P(G)$, associated to an equivalence relation on $P(G)$: two paths $x, y \in P(G)$ are *shift equivalent with lag* $k \in \mathbf{Z}$ (written $x \sim_k y$) if there exists $N \in \mathbf{N}$ such that $x_i = y_{i+k}$ for all $i \ge N$. It is easy to check that shift equivalence is an equivalence relation: $x \sim_k y \Rightarrow y \sim_{-k} x$, and $x \sim_k y,\ y \sim_l z \Rightarrow x \sim_{k+l} z$.

2.3 Definition: Let $\mathcal{G} = \{(x, k, y) \in P(G) \times \mathbf{Z} \times P(G) : x \sim_k y\}$. For pairs in

$$\mathcal{G}^2 := \{(\,(x, k, y), (y, l, z)\,) : (x, k, y), (y, l, z) \in \mathcal{G}\},$$

we define

$$(x, k, y) \cdot (y, l, z) := (x, k + l, z), \tag{1}$$

and for arbitrary (x, k, y), we define

$$(x, k, y)^{-1} := (y, -k, x). \tag{2}$$

2.4 Lemma: *With the operations (1) and (2), and range, source maps $r, s : \mathcal{G} \to P(G)$ given by $r(x, k, y) = x$, $s(x, k, y) = y$, $\mathcal{G}$ is a groupoid with unit space $P(G)$.*

We want to make $\mathcal{G}$ into a locally compact groupoid with (topological) unit space $P(G)$. The idea is to use the sets

$$Z(\alpha, \beta) := \{(x, k, y) : x \in Z(\alpha), y \in Z(\beta), k = |\beta| - |\alpha|, x_i = y_{i+k} \text{ for } i > |\alpha|\},$$

where α and β are paths in $F(G)$ with $r(\alpha) = r(\beta)$, as a neighbourhood base. We allow α or $\beta = \emptyset$. To see that this is possible requires some work. The first Lemma is straightforward.

2.5 Lemma: *For $\alpha, \beta, \gamma, \delta \in F(G)$ with $r(\alpha) = r(\beta), r(\gamma) = r(\delta)$, we have*

$$Z(\alpha,\beta) \cap Z(\gamma,\delta) = \begin{cases} Z(\alpha,\beta) & \text{if there exists } \epsilon \in F(G) \text{ such that } \alpha = \gamma\epsilon, \beta = \delta\epsilon, \\ Z(\gamma,\delta) & \text{if there exists } \epsilon \in F(G) \text{ such that } \gamma = \alpha\epsilon, \delta = \beta\epsilon, \\ \emptyset & \text{otherwise.} \end{cases}$$

2.6 Proposition: *Let G be a row-finite directed graph. The sets*

$$\{Z(\alpha,\beta) : \alpha, \beta \in F(G),\ r(\alpha) = r(\beta)\}$$

form a basis for a locally compact Hausdorff topology on $\mathcal{G}$. With this topology, $\mathcal{G}$ is a second countable, r-discrete locally compact groupoid in which each $Z(\alpha,\beta)$ is a compact open $\mathcal{G}$-set. The product topology on the unit space $P(G)$ agrees with the topology it inherits by viewing it as the subset $\mathcal{G}^0 = \{(x,0,x) : x \in P(G)\}$ of $\mathcal{G}$. The counting measures form a left Haar system for $\mathcal{G}$.

Proof: Lemma 2.5 implies that each finite intersection of $Z(\alpha,\beta)$'s is another $Z(\alpha,\beta)$, and hence the sets $Z(\alpha,\beta)$ form a (countable) basis for a topology on $\mathcal{G}$. Since two distinct points differ in lag or in some initial segment of range or source, this is a Hausdorff topology. For each fixed $\alpha, \beta \in F(G)$, the map $h_{\alpha,\beta} : x \mapsto (\alpha x, |\beta| - |\alpha|, \beta x)$ is a bijection of $P(G, r(\alpha))$ onto $Z(\alpha,\beta)$, which is continuous because the basic open sets intersecting $Z(\alpha,\beta)$ have the form $Z(\alpha\epsilon, \beta\epsilon)$, and $h^{-1}_{\alpha,\beta}(Z(\alpha\epsilon,\beta\epsilon)) = Z(\epsilon)$ is open in $P(G)$. Because G is row finite, $P(G, r(\alpha))$ is compact, and the continuous bijection $h_{\alpha,\beta}$ is automatically a homeomorphism. Thus the sets $Z(\alpha,\beta)$ are compact as well as open, and the topology is locally compact.

Inversion is continuous, because it maps $Z(\alpha,\beta)$ onto $Z(\beta,\alpha)$. To see that the product is continuous, suppose $(x,k,y) \cdot (y,l,z) = (x, k+l, z)$ is in $Z(\alpha,\beta)$. There exists $N \in \mathbf{N}$ such that $x_i = y_{i+k}$ for $i \geq N$ and $y_j = z_{j+l}$ for $j \geq N + k$; we may as well suppose $N \geq |\alpha|$ and $N \geq |\beta| - l - k$. Let

$$\begin{aligned} \alpha' &= (x_1, \cdots, x_N) = (\alpha_1, \cdots, \alpha_{|\alpha|}, x_{|\alpha|+1}, \cdots, x_N), \\ \gamma &= (y_1, \cdots, y_{N+k}), \quad \text{and} \\ \beta' &= (z_1, \cdots, z_{N+k+l}) = (\beta_1, \cdots, \beta_{|\beta|}, z_{|\beta|+1}, \cdots, z_{N+k+l}). \end{aligned}$$

Then $(x,k,y) \in Z(\alpha',\gamma)$, $(y,l,z) \in Z(\gamma,\beta')$, and the product maps the open set $\mathcal{G}^2 \cap (Z(\alpha',\gamma) \times Z(\gamma,\beta'))$ into $Z(\alpha,\beta)$.

Next note that r is a homeomorphism of $Z(\alpha,\beta)$ onto $Z(\alpha)$ (it is the composition of $h^{-1}_{\alpha,\beta}$ with the homeomorphism $x \mapsto \alpha x$ of $P(G, r(\alpha))$ onto $Z(\alpha)$), which shows both that $\mathcal{G}$ is r-discrete with the counting measures as a Haar system [16, p.18], and that $Z(\alpha,\beta)$ is a $\mathcal{G}$-set. For the statement about the topology on $P(G)$, observe that $Z(\alpha,\beta)$ meets $\mathcal{G}^0$ only if $\alpha = \beta$, and thus the map $x \mapsto (x,0,x)$ is a homeomorphism of $P(G)$ onto $\mathcal{G}^0$. □

Remark. When the graph G is pointed, there is another locally compact groupoid $\mathcal{G}(\star)$ naturally associated to G, which is based on the space $P(G,\star)$ of infinite paths starting at the distinguished vertex $\star$ rather than $P(G)$. To avoid repeating the construction above (we shall be forced to do that quite enough as it is), we just define $\mathcal{G}(\star)$ to be the reduction of the groupoid $\mathcal{G}$ to the compact subset $P(G,\star)$ of its unit space.

3 Equivalence of the full and pointed groupoids.

We say that a vertex v in a directed graph G is *cofinal* if, for every infinite path $x \in P(G)$, there is a finite path $\alpha \in F(G)$ such that $s(\alpha) = v$ and $r(\alpha) = r(x_n)$ for some n.

3.1 Theorem: *Let $(G,\star)$ be a pointed directed graph with associated groupoid $\mathcal{G}$; suppose that G is row-finite and that the distinguished vertex $\star$ is cofinal. Let $N = P(G,\star)$ be the (compact-open) subset of $P(G)$ consisting of the paths starting at the distinguished vertex $\star$, so that $\mathcal{G}(\star)$ is by definition the reduction $\mathcal{G}^N_N$. Then the characteristic function $1_N \in C_c(\mathcal{G})$ is a full projection in $C^*(\mathcal{G})$, and the inclusion of $C_c(\mathcal{G}^N_N)$ in $C_c(\mathcal{G})$ induces an isomorphism of $C^*(\mathcal{G}(\star))$ onto the full corner $1_N C^*(\mathcal{G}) 1_N$.*

3.2 Remarks: (1) Since $r, s : \mathcal{G} \to P(G)$ are continuous, and N is open and closed in $P(G)$, $\mathcal{G}(\star) = \mathcal{G}_N^N = r^{-1}(N) \cap s^{-1}(N)$ is both open and closed in $\mathcal{G}$. Thus we can view functions in $C_c(\mathcal{G}_N^N)$ as continuous functions of compact support on $\mathcal{G}$. Similarly, if $\mathcal{G}_N = s^{-1}(N)$, we can view $C_c(\mathcal{G}_N)$ as a subset of $C_c(\mathcal{G})$. We shall do this without comment, but we shall only write (e.g.) $f *_{C_c(\mathcal{G})} g$ if we are viewing $f, g \in C_c(\mathcal{G}_N^N)$ as functions in $C_c(\mathcal{G})$.

(2) Since the product topology on $P(G)$ agrees with the topology it inherits as an (open and closed) subset of $\mathcal{G}$, the characteristic function 1_N can indeed be viewed as a continuous function of compact support on $\mathcal{G}_N^N$, $\mathcal{G}_N$ and $\mathcal{G}$.

3.3 Lemma: *The characteristic function* 1_N *satisfies* $1_N *_{C_c(\mathcal{G})} 1_N = 1_N = 1_N^*$*, and the inclusion of* $C_c(\mathcal{G}_N^N)$ *in* $C_c(\mathcal{G})$ *is a* **-homomorphism of* $C_c(\mathcal{G}_N^N)$ *onto* $1_N * C_c(\mathcal{G}) * 1_N$*.*

Proof: If $g \in C_c(\mathcal{G})$ and $(x, k, y) \in \mathcal{G}$, then

$$g *_{C_c(\mathcal{G})} 1_N(x, k, y) = \sum_{\{(\ell,v):(x,\ell,v)\in\mathcal{G}\}} g(x, \ell, v) 1_N(v, k - \ell, y).$$

The only non-zero summand occurs when $(v, k - \ell, y) = (y, 0, y)$ and $y \in N = \mathcal{P}(G, \star)$, so $g *_{C_c(\mathcal{G})} 1_N$ is the restriction of g to the subset $\mathcal{G}_N = s^{-1}(N)$. Similarly, $1_N *_{C_c(\mathcal{G})} g$ is the restriction of g to $\mathcal{G}^N := r^{-1}(N)$; in particular, $1_N *_{C_c(\mathcal{G})} 1_N = 1_N$. Since we trivially have $1_N^* = 1_N$, it follows immediately that inclusion maps $C_c(\mathcal{G}_N^N)$ onto $1_N * C_c(\mathcal{G}) * 1_N$. It is easy to verify that $f \in C_c(\mathcal{G}_N^N)$ has the same adjoint in $C_c(\mathcal{G}_N^N)$ and $C_c(\mathcal{G})$, and by looking at the formula for $*_{C_c(\mathcal{G})}$ we can see that $f *_{C_c(\mathcal{G})} g = f *_{C_c(\mathcal{G}_N^N)} g$ whenever f, g, and hence $f *_{C_c(\mathcal{G}_N^N)} g$, have support in $\mathcal{G}_N^N$. Thus the inclusion is actually a *-homomorphism of $C_c(\mathcal{G}_N^N)$ onto $1_N * C_c(\mathcal{G}) * 1_N$. □

By cofinality of $\star$, the set N is an abstract transversal for $\mathcal{G}$ in the sense of [11]: indeed, if $x \in P(G)$, there is a finite path α from $\star$ to some $s(x_{n+1}) = r(x_n)$, and if we write x' for the path $x_{n+1}x_{n+2}\cdots$, then $(\alpha x', n - |\alpha|, x) \in \mathcal{G}$ has range $\alpha x'$ in N and source x, so N meets the orbit of x. (That r and s are open on $\mathcal{G}_N$ is automatic because N is open and closed, and r, s are local homeomorphisms.) It therefore follows from [11, Example 2.7] that $\mathcal{G}_N = s^{-1}(N)$ is a $(\mathcal{G}, \mathcal{G}_N^N)$-equivalence. We can now deduce from [11, Theorem 2.8] that $C_c(\mathcal{G}_N)$ can be naturally made into a $C_c(\mathcal{G})$–$C_c(\mathcal{G}_N^N)$ imprimitivity bimodule. (The formulas for the module actions are given on [11, p. 11], and those for the inner products at the top of [11, p. 12]. We shall write ${}_{C_c(\mathcal{G})}\langle \cdot, \cdot \rangle$ for the $C_c(\mathcal{G})$-valued inner product to stress that it is $C_c(\mathcal{G})$ which acts on the left.)

3.4 Lemma: *The inner products on* $C_c(\mathcal{G}_N)$ *in* [11, p. 12] *are given by*

$$\langle f, g \rangle_{C_c(\mathcal{G}_N^N)} = f^* *_{C_c(\mathcal{G})} g \quad \text{and} \quad {}_{C_c(\mathcal{G})}\langle f, g \rangle = f *_{C_c(\mathcal{G})} g^*.$$

(We are asserting, inter alia*, that* $f^* *_{C_c(\mathcal{G})} g$ *has support in* $\mathcal{G}_N^N$*.)*

Proof: The formula for $\langle f, g \rangle_{C_c(\mathcal{G}_N^N)}(x, k, y)$ in [11, p. 12] requires a choice of $z \in \mathcal{G}_N$ with $s(z) = r(x, k, y) = x$ (but is then independent of the choice): we choose $z = (x, 0, x)$. Then

$$\langle f, g \rangle_{C_c(\mathcal{G}_N^N)}(x, k, y) = \sum_{\{(\ell,v):(x,\ell,v)\in\mathcal{G}\}} \overline{f(v, -\ell, x)} g(v, k - \ell, y) = f^* *_{C_c(\mathcal{G})} g(x, k, y).$$

Since f, g have support in $\mathcal{G}_N$, $f^* *_{C_c(\mathcal{G})} g(x, k, y)$ is zero unless $x, y \in N$, which forces $f^* *_{C_c(\mathcal{G})} g \in C_c(\mathcal{G}_N^N)$.

In the formula for ${}_{C_c(\mathcal{G})}\langle f, g \rangle(x, k, y)$, we need to choose $z \in \mathcal{G}_N$ with $r(z) = s(x, k, y) = y$: this time we use cofinality to choose $\alpha \in F(G)$ such that $s(\alpha) = \star$, $r(\alpha)$ is the source of some end segment $y' := y_{n+1}y_{n+2}\cdots$ of y, and take $z := (y, |\alpha| - n, \alpha y')$. Then

$${}_{C_c(\mathcal{G})}\langle f, g \rangle(x, k, y) = \sum_{\{(\ell,v):(\alpha y',\ell,v)\in\mathcal{G}_N^N\}} f(x, k + |\alpha| - n + \ell, v) \overline{g(y, \ell + |\alpha| - n, v)}.$$

Since $\operatorname{supp} g \subset \mathcal{G}_N$, $g(y, \ell + |\alpha| - n, v) = 0$ unless $v \in N$, and this sum equals

$$\sum_{\{(\ell,v):(\alpha y',\ell,v)\in\mathcal{G}\}} f(x, k + |\alpha| - n + \ell, v) \overline{g(y, \ell + |\alpha| - n, v)}.$$

Since $(\alpha y', \ell, v)$ belongs to $\mathcal{G}$ iff $(y, \ell + |\alpha| - n, v)$ belongs to $\mathcal{G}$, this sum reduces to the formula for $f *_{C_c(\mathcal{G})} g^*(x, k, y)$.
□

3.5 Lemma: *For $f \in C_c(\mathcal{G}_N^N)$, we have $\|f\|_{C^*(\mathcal{G}_N^N)} = \|f\|_{C^*(\mathcal{G})}$.*

Proof: We have

$$\begin{aligned}
\|f\|^2_{C^*(\mathcal{G}_N^N)} &= \|f^* *_{C_c(\mathcal{G}_N^N)} f\|_{C^*(\mathcal{G}_N^N)} && \text{by definition} \\
&= \|f^* *_{C_c(\mathcal{G})} f\|_{C^*(\mathcal{G}_N^N)} && \text{because } *_{C_c(\mathcal{G}_N^N)} \text{ and } *_{C_c(\mathcal{G})} \text{ agree on } C_c(\mathcal{G}_N^N) \\
&= \|\langle f, f\rangle_{C_c(\mathcal{G}_N^N)}\|_{C^*(\mathcal{G}_N^N)} && \text{by Lemma 3.4} \\
&= \|{}_{C_c(\mathcal{G})}\langle f, f\rangle\|_{C^*(\mathcal{G})} && \text{by [18, Proposition 3.1]} \\
&= \|f *_{C_c(\mathcal{G})} f^*\|_{C^*(\mathcal{G})} && \text{by Lemma 3.4} \\
&= \|f\|^2_{C^*(\mathcal{G})},
\end{aligned}$$

as required. □

Lemmas 3.3 and 3.5 imply that the inclusion is a *-homomorphism of $C_c(\mathcal{G}_N^N)$ onto $1_N * C_c(\mathcal{G}) * 1_N$ which is isometric for the enveloping C^*-norms, and hence extends to an isomorphism of $C^*(\mathcal{G}_N^N)$ onto the closure $1_N C^*(\mathcal{G}) 1_N$ of $1_N * C_c(\mathcal{G}) * 1_N$ in $C^*(\mathcal{G})$. Finally, we recall from the proof of Lemma 3.3 that $1_N *_{C_c(\mathcal{G})} g^* = g^*$ when $g \in C_c(\mathcal{G}_N)$, and hence for any $f, g \in C_c(\mathcal{G}_N)$, the function

$${}_{C_c(\mathcal{G})}\langle f, g\rangle = f *_{C_c(\mathcal{G})} g^* = f *_{C_c(\mathcal{G})} 1_N *_{C_c(\mathcal{G})} g^*$$

belongs to the ideal generated by 1_N. Since we know from [11] that the inner products span a dense ideal of $C^*(\mathcal{G})$, it follows that 1_N is full. Thus we have completed the proof of Theorem 3.1.

3.6 Remark: We can replace the distinguished vertex $\star$ by a finite subset F of V, and $P(G, \star)$ by the space $P(G, F)$ of paths starting in F. Provided the set F is cofinal, the above proof carries over to show that $C^*(\mathcal{G}(F))$ embeds as the corner $1_{P(G,F)} C^*(\mathcal{G}) 1_{P(G,F)}$. Similar arguments probably work even for infinite cofinal subsets S of V, except that the characteristic function $1_{P(G,S)}$ would no longer have compact support, and hence would not belong to $C^*(\mathcal{G})$. So one would first have to show that the pair (p', p'') of maps $p' : g \mapsto g|_{\mathcal{G}^{P(G,S)}}$, $p'' : g \mapsto g|_{\mathcal{G}_{P(G,S)}}$ on $C_c(\mathcal{G})$ extends to a multiplier of $C^*(\mathcal{G})$. We shall not follow this up since we do not need it here, but we remark that the techniques of [13, §3] will very likely suffice.

4 Cuntz-Krieger algebras.

Associated to a directed graph $G = (V, E, r, s)$ are two integer matrices: the *edge matrix* $A = A_G$ is the $E \times E$ matrix defined by

$$A_G(e, f) := \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise,} \end{cases}$$

and the *vertex matrix* B_G is the $V \times V$ matrix defined by

$$B_G(v, w) := \#\{e \in E : s(e) = v, r(e) = w\}.$$

Every $n \times n$ matrix B with entries $B(i, j)$ in $\mathbf{N} \cup \{\infty\}$ is the vertex matrix of a graph G: just take n points $v_1, v_2, \cdots, v_n$ as vertices, and draw $B(i, j)$ edges from v_i to v_j to obtain a graph G_B. One trivially checks that $B = B_{G_B}$ and $G_{B_G} = G$. On the other hand, not all $\{0, 1\}$ matrices arise as A_G: for example, one can have $A_G = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, but there is no graph G with $A_G = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

4.1 Proposition: *Let G be a row-finite directed graph with edge matrix A and vertex matrix B. Then $C^*(\mathcal{G})$ is generated by a Cuntz-Krieger A-family — that is, by a family $\{S_e : e \in E\}$ of partial isometries with orthogonal ranges satisfying*

$$S_e^* S_e = \sum_{f \in E} A(e, f) S_f S_f^*;$$

indeed, we can take S_e to be the characteristic function $1_{Z(e, r(e))}$ of the basic open set

$$Z(e, r(e)) = \{(x, -1, y) : x_0 = e, x_i = y_{i-1} \text{ for } i \geq 1\}.$$

If B is a $\{0, 1\}$ matrix, $C^*(\mathcal{G})$ is also generated by a Cuntz-Krieger B-family.

Remark In the finite case, we insist also that $\sum_{e\in E} S_e S_e^* = 1$. In the infinite case, this does not make sense in the C^*–algebra $C^*(\mathcal{G})$, and the assumed orthogonality of the range projections is a substitute for this condition (it is implied by $\sum_{e\in E} S_e S_e^* = 1$).

Proof: Because the Haar system is counting measure, we have

$$S_e^* S_f(x,k,y) = \sum_{\{\ell,z:(x,\ell,z)\in\mathcal{G}\}} \overline{S_e(z,-\ell,x)} S_f(z,k-\ell,y);$$

and since $(z,-\ell,x) \in Z(e,r(e))$, $(z,k-\ell,y) \in Z(f,r(f))$ force $e=f$, $l=1$, $k=0$, $y=x$ and $z=ex$, the sum collapses to at most one term. We deduce that $S_e^*S_f = 0$ unless $e=f$, and then is the characteristic function 1_{D_e} of the subset $D_e := \{(x,0,x) : s(x) = r(e)\}$ of the unit space $\mathcal{G}^0$ (viewed as a subset of $\mathcal{G}$). Since $1_{D_e} = (1_{D_e})^2 = 1_{D_e}^*$, this proves in particular that S_e is a partial isometry. Similarly, one can verify that $S_eS_e^* = 1_{Z(e,e)}$, and hence the S_e are orthogonal. Since

$$D_e = \{(x,0,x) : s(x) = r(e)\} = \bigcup\{Z(f,f) : f \in E \text{ satisfies } s(f) = r(e)\},$$

it follows that

$$S_e^*S_e = 1_{D_e} = \sum_{\{f:s(f)=r(e)\}} 1_{Z(f,f)} = \sum_f A(e,f) S_f S_f^*,$$

so that $\{S_e : e \in E\}$ is a Cuntz-Krieger A-family.

More calculations like the one above show that the product

$$S_\alpha S_\beta^* := S_{\alpha_1} \dots S_{\alpha_{|\alpha|}} S_{\beta_{|\beta|}}^* \dots S_{\beta_1}^*$$

is zero unless $\alpha, \beta \in F(G)$ satisfy $r(\alpha) = r(\beta)$ (which happens iff there exists $f \in E$ such that $A(\alpha_{|\alpha|}, f) = 1 = A(\beta_{|\beta|}, f)$), and then is the characteristic function $1_{Z(\alpha,\beta)}$. It is a standard observation in Cuntz-Krieger theory that the C^*-algebra generated by a Cuntz-Krieger family $\{S_e\}$ is the closed span of the elements $S_\alpha S_\beta^*$, so to prove $\{S_e\}$ generates $C^*(\mathcal{G})$ we just need to prove that $\operatorname{sp}\{1_{Z(\alpha,\beta)} : \alpha,\beta \in F(G)\}$ is dense in $C^*(\mathcal{G})$. Since $C_c(\mathcal{G})$ is dense in $C^*(\mathcal{G})$, and the C^*-norm is dominated by Renault's I-norm, it is enough to show that $\operatorname{sp}\{1_{Z(\alpha,\beta)}\}$ is $\|\cdot\|_I$-dense in $C_c(\mathcal{G})$. The support of any fixed $f \in C_c(\mathcal{G})$ is a disjoint union of compact subsets of the form $Z(\gamma,\delta)$, and by Lemma 2.5 we may assume these compact open sets are disjoint. Thus f is a finite sum of functions with support in some $Z(\gamma,\delta)$, and it is enough to approximate $f \in C(Z(\gamma,\delta))$ by something in $\operatorname{sp}\{1_{Z(\alpha,\beta)}\}$. Because each $Z(\gamma,\delta)$ is a $\mathcal{G}$-set (i.e. r and s are one-one on $Z(\gamma,\delta)$), the uniform norm on $C(Z(\gamma,\delta))$ dominates the $\|\cdot\|_I$-norm, and it is enough to approximate f in the uniform norm. But now we can use the Stone-Weierstrass Theorem to see that the *-subalgebra

$$\operatorname{sp}\{1_{Z(\alpha,\beta)}\} \bigcap C(Z(\gamma,\delta)) = \operatorname{sp}\{1_{Z(\gamma\mu,\delta\mu)}\}$$

is uniformly dense in $C(Z(\gamma,\delta))$, which is a C^*-algebra with pointwise operations.

We have now proved that $C^*(\mathcal{G})$ is generated by the Cuntz-Krieger A_G-family $\{S_e\}$. If the vertex matrix B has entries in $\{0,1\}$, then the argument of [9, Proposition 4.1(1)] shows that

$$T_v := \sum_{\{e:s(e)=v\}} S_e$$

is a Cuntz-Krieger B-family generating $C^*(\mathcal{G})$: indeed, we can recover S_e as $T_{s(e)}T_{r(e)}T_{r(e)}^*$ (this is not immediately obvious, and requires that the entries of B are in $\{0,1\}$, so that $s(e) = s(f)$ and $r(e) = r(f)$ imply $e = f$). This completes the proof of the Proposition. □

4.2 Theorem: *Let G be a row-finite directed graph with edge matrix A and groupoid $\mathcal{G}$, and suppose that $\{S_e : e \in E\}$ is a Cuntz-Krieger A-family of partial isometries on a Hilbert space H. Then there is a representation π of $C^*(\mathcal{G})$ on H such that $\pi(1_{Z(e,r(e))}) = S_e$ for every $e \in E$.*

To prove this Theorem, we shall construct a representation of $C_c(\mathcal{G})$, and then use the general theory of [16] to extend this to $C^*(\mathcal{G})$. Since the support of any $f \in C_c(\mathcal{G})$ is the disjoint union of basic open sets $Z(\alpha,\beta)$, and $f = \sum f|_{Z(\alpha,\beta)}$, we shall always be able to reduce to the case where $\operatorname{supp} f \subset Z(\alpha,\beta)$. Thus we start by constructing representations of $C(Z(\alpha,\beta))$. Since $Z(\alpha,\beta)$ is a compact space, $C(Z(\alpha,\beta))$ is a commutative C^*-algebra under pointwise operations, and we shall be exploiting this; however, it is important to remember that these pointwise operations are *not* those obtained by viewing $C(Z(\alpha,\beta))$ as a subspace of the convolution algebra $C_c(\mathcal{G})$ unless $\alpha = \beta$, in which case $C(Z(\alpha,\beta)) \subset C_c(\mathcal{G}^0)$.

Recall that, for fixed $\alpha,\beta \in F(G)$ with $r(\alpha) = r(\beta)$, the map $h_{\alpha,\beta} : x \mapsto (\alpha x, |\beta| - |\alpha|, \beta x)$ is a homeomorphism of $P(G, r(\alpha))$ onto $Z(\alpha,\beta)$. This homeomorphism induces an isomorphism $\phi_{\alpha,\beta}$ of $C(P(G,r(\alpha)))$ onto $C(Z(\alpha,\beta))$.

4.3 Lemma: *For each vertex v of G, there is a representation π_v of $C(P(G,v))$ on H such that $\pi_v(1_{Z(\gamma)}) = S_\gamma S_\gamma^*$ for each $\gamma \in F(G,v)$.*

Proof: For each $k \in \mathbf{N}$, we let

$$C_k := \operatorname{sp}\{1_{Z(\gamma)} : \gamma \in F(G,v) \text{ has length } |\gamma| = k\},$$

which because G is row-finite is a finite-dimensional C^*-subalgebra of $C(P(G,v))$ spanned by the mutually orthogonal projections $1_{Z(\gamma)}$. The cylinder set $Z(\gamma)$ is the disjoint union of the sets

$$\{Z(\gamma e, \gamma e) : e \in E, s(e) = r(\gamma)\},$$

which implies that $1_{Z(\gamma)} = \sum_{s(e)=r(\gamma)} 1_{Z(\gamma e,\gamma e)}$, and $C_k \subset C_{k+1}$. Since the sets $Z(\gamma)$ are a basis for the topology on $P(G,v)$, the Stone-Weierstrass Theorem implies that $C(P(G,v)) = \overline{\cup C_k}$. For a fixed k, the projections $\{S_\gamma S_\gamma^* : |\gamma| = k\}$ are mutually orthogonal, and hence there is a *-homomorphism $\phi_k : C_k \to B(H)$ such that $\phi_k(1_{Z(\gamma)}) = S_\gamma S_\gamma^*$. Because

$$S_\gamma S_\gamma^* = S_\gamma \Big(\sum_{e\in E} S_e S_e^*\Big) S_\gamma^* = \sum_{\{e:s(e)=r(\gamma)\}} S_\gamma S_e S_e^* S_\gamma^* = \sum_{\{e:s(e)=r(\gamma)\}} S_{\gamma e} S_{\gamma e}^*,$$

we have $\phi_{k+1}|_{C_k} = \phi_k$, and together the ϕ_k give a *-homomorphism ϕ of $\cup C_k$ into $B(H)$. Since the C_k's are C^*-subalgebras of $C(P(G,v))$, the homomorphisms ϕ_k are automatically norm-decreasing, and hence so is ϕ. Thus ϕ extends to a representation π_v of the closure $C(P(G,v))$ of $\cup C_k$ with the required property. $\square$

The representations π_v immediately give representations $\pi_{r(\alpha)} \circ \phi_{\alpha,\beta}^{-1}$ of each $C(Z(\alpha,\beta))$. However, there is more than one way of writing a compact set as the union of $Z(\alpha,\beta)$'s, and we have to check that the representations $\pi_{r(\alpha)} \circ \phi_{\alpha,\beta}^{-1}$ are consistent.

4.4 Lemma: *Suppose $\alpha, \beta \in F(G)$ satisfy $r(\alpha) = r(\beta)$, and $f \in C(Z(\alpha,\beta))$. Then for any $k \geq 1$ we have*

$$\pi_{r(\alpha)}\left(\phi_{\alpha,\beta}^{-1}(f)\right) = \sum_{\{\gamma\in F(G,r(\alpha)):|\gamma|=k\}} S_\gamma \pi_{r(\gamma)}\big(\phi_{\alpha\gamma,\beta\gamma}^{-1}(f|_{Z(\alpha\gamma,\beta\gamma)})\big) S_\gamma^* \tag{3}$$

Proof: Both sides of (3) are continuous and linear in $f \in C(Z(\alpha,\beta))$. Hence we may suppose that $f = \phi_{\alpha,\beta}(1_{Z(\chi)}) = 1_{Z(\alpha\chi,\beta\chi)}$ for some $\chi \in F(G, r(\alpha))$. The left-hand side of (3) is then

$$\pi_{r(\alpha)}\left(\phi_{\alpha,\beta}^{-1}(1_{Z(\alpha\chi,\beta\chi)})\right) = \pi_{r(\alpha)}(1_{Z(\chi)}) = S_\chi S_\chi^*.$$

If $k < |\chi|$, then the only non-zero summand on the right of (3) occurs when γ is the initial segment of χ; then, with $\chi = \gamma\chi'$, the right-hand side becomes

$$S_\gamma \pi_{r(\gamma)}\left(\phi_{\alpha\gamma,\beta\gamma}^{-1}(1_{Z(\alpha\gamma\chi',\beta\gamma\chi')})\right) S_\gamma^* = S_\gamma \pi_{r(\gamma)}(1_{Z(\chi')}) S_\gamma^* = S_\gamma S_{\chi'} S_{\chi'}^* S_\gamma^* = S_\chi S_\chi^*.$$

If $k \geq |\chi|$, the non-zero summands occur when γ has the form $\chi\gamma'$ for some $\gamma' \in F(G, r(\chi))$, and the right-hand side of (3) is

$$\sum_{\{\gamma'\in F(G,r(\chi)):|\gamma'|=k-|\chi|\}} S_{\chi\gamma'} \pi_{r(\gamma')}\left(\phi_{\alpha\chi\gamma',\beta\chi\gamma'}^{-1}(1_{Z(\alpha\chi\gamma',\beta\chi\gamma')})\right) S_{\chi\gamma'}^*,$$

which reduces to

$$\sum_{\gamma'} S_{\chi\gamma'} \pi_{r(\gamma')}(1_{P(G,r(\gamma'))}) S_{\chi\gamma'}^* = \sum_{\gamma'} S_{\chi\gamma'} S_{\chi\gamma'}^* = S_\chi S_\chi^*,$$

as required. $\square$

Now we aim to define $\pi : C_c(\mathcal{G}) \to B(H)$ as follows. For $f \in C_c(\mathcal{G})$, write $\operatorname{supp} f$ as the disjoint union $\cup_i Z(\alpha^i, \beta^i)$ of basic open sets, and take

$$\pi(f) = \sum_i S_{\alpha^i}\left(\pi_{r(\alpha^i)}\big(\phi_{\alpha^i,\beta^i}^{-1}(f|_{Z(\alpha^i,\beta^i)})\big)\right) S_{\beta^i}^*. \tag{4}$$

It is true, but not immediately obvious, that this process is independent of the description of $\operatorname{supp} f$:

4.5 Lemma: *There is a well-defined linear map $\pi : C_c(\mathcal{G}) \to B(H)$, continuous in the inductive limit topology, such that* (4) *holds whenever* $\operatorname{supp} f \subset \cup Z(\alpha^i, \beta^i)$ *and* $Z(\alpha^i, \beta^i)$ *are disjoint.*

Proof: Since the isomorphisms $\phi_{\alpha,\beta}$ are homeomorphisms for the uniform topology on $C(P(G, r(\alpha)))$ and the inductive limit topology, and the $\pi_{\alpha,\beta}$ are uniformly continuous, the only issue is whether $\pi(f)$ is well-defined by (4). Suppose we had an alternative description of $\operatorname{supp} f$ as $\cup Z(\gamma, \delta)$. Since $Z(\alpha, \beta) \cap Z(\gamma, \delta) \neq \emptyset$ only when one set is contained in the other, each $Z(\alpha^i, \beta^i)$ is contained in a disjoint union of $Z(\gamma, \delta)$'s, or vice-versa. Suppose without loss of generality that

$$Z(\alpha^i, \beta^i) = \bigcup_{j=1}^{n} Z(\gamma^j, \delta^j).$$

(We now drop the superscript i.) Then each (γ^j, δ^j) has the form $(\alpha\mu^j, \beta\mu^j)$. If $k := \max |\mu^j|$, then the paths γ arising in the decomposition

$$Z(\alpha, \beta) = \bigcup_{\{\gamma \in F(G, r(\alpha)) : |\gamma| = k\}} Z(\alpha\gamma, \beta\gamma) \tag{5}$$

must group together in subsets $F_j := \{\mu^j \nu : |\nu| = k - |\mu^j|\}$ to form decompositions of $Z(\gamma^j, \delta^j)$, and (5) can be rewritten

$$Z(\alpha, \beta) = \bigcup_j \Big(\bigcup_{\mu^j\nu \in F_j} Z(\alpha\mu^j\nu, \beta\mu^j\nu) \Big) = \bigcup_j \Big(\bigcup_{\mu^j\nu \in F_j} Z(\gamma^j\nu, \delta^j\nu) \Big).$$

Several applications of the previous lemma give

$$\begin{aligned} S_\alpha \left(\pi_{r(\alpha)}(\phi_{\alpha,\beta}^{-1}(f|_{Z(\alpha,\beta)})) \right) S_\beta^* &= \sum_{\{\gamma \in F(G, r(\alpha)) : |\gamma| = k\}} S_{\alpha\gamma} \left(\pi_{r(\gamma)}(\phi_{\alpha\gamma,\beta\gamma}^{-1}(f|_{Z(\alpha\gamma,\beta\gamma)})) \right) S_{\beta\gamma}^* \\ &= \sum_j \sum_{\mu^j\nu \in F_j} S_{\gamma^j\nu} \left(\pi_{r(\nu)}(\phi_{\gamma^j\nu,\delta^j\nu}^{-1}(f|_{Z(\gamma^j\nu,\delta^j\nu)})) \right) S_{\delta^j\nu}^* \\ &= \sum_j S_{\gamma^j} \left(\pi_{r(\gamma^j)}(\phi_{\gamma^j,\delta^j}^{-1}(f|_{Z(\gamma^j,\delta^j)})) \right) S_{\delta^j}^*. \end{aligned}$$

Repeated applications of this process, to decompositions of $Z(\alpha^i, \beta^i)$ in terms of $Z(\gamma, \delta)$'s or vice-versa, show that (4) is independent of the choice of cover of $\operatorname{supp} f$. □

We now want to prove that the linear map π is a *-homomorphism on $C_c(\mathcal{G})$. That it is adjoint-preserving is straightforward: if $\operatorname{supp} f \subset Z(\alpha, \beta)$, then $f^* = \phi_{\beta,\alpha} \circ \overline{\phi_{\alpha,\beta}^{-1}(f)}$, and hence

$$\begin{aligned} \pi(f)^* &= \left(S_\alpha \pi_{r(\alpha)}(\phi_{\alpha,\beta}^{-1}(f)) S_\beta^*\right)^* = S_\beta \pi_{r(\alpha)}(\overline{\phi_{\alpha,\beta}^{-1}(f)}) S_\alpha^* \\ &= S_\beta \pi_{r(\alpha)}(\phi_{\beta,\alpha}^{-1}(f^*)) S_\alpha^* = \pi(f^*). \end{aligned}$$

This extends to arbitrary $f \in C_c(\mathcal{G})$ by linearity.

Next, let $f, g \in C_c(\mathcal{G})$. Again, linearity allows us to reduce to the case where $\operatorname{supp} f \subset Z(\alpha, \beta)$ and $\operatorname{supp} g \subset Z(\gamma, \delta)$. If $Z(\beta) \cap Z(\gamma) = \emptyset$, then $f * g = 0$ and $S_\beta^* S_\gamma = 0$, which forces $\pi(f)\pi(g) = 0$. So we may suppose $Z(\beta) \cap Z(\gamma) \neq \emptyset$. Then either $\beta = \gamma\beta'$ or $\gamma = \beta\gamma'$; suppose for the sake of argument that $\beta = \gamma\beta'$. We can use the alternative decomposition

$$g = \sum_{\{\nu \in F(G, r(\gamma)) : |\nu| = |\beta| - |\gamma|\}} g|_{Z(\gamma\nu, \delta\nu)},$$

and by our earlier reasoning discard all but the term $g|_{Z(\gamma\beta', \delta\beta')}$. So we may as well assume that $\operatorname{supp} f \subset Z(\alpha, \beta)$ and $\operatorname{supp} g \subset Z(\beta, \delta)$. But then $\operatorname{supp} f * g \subset Z(\alpha, \delta)$, so

$$\pi(f)\pi(g) = S_\alpha \pi_{r(\beta)}\big(\phi_{\alpha,\beta}^{-1}(f)\big) S_\beta^* S_\beta \pi_{r(\beta)}\big(\phi_{\beta,\delta}^{-1}(f)\big) S_\delta^*.$$

We can remove the term $S_\beta^* S_\beta = S_{r(\beta)}^* S_{r(\beta)}$, because $S_{r(\beta)}^* S_{r(\beta)} S_\gamma = S_\gamma$ whenever $s(\gamma) = r(\beta)$. Thus we only have to check that

$$\phi_{\alpha,\beta}^{-1}(f)\phi_{\beta,\delta}^{-1}(g) = \phi_{\alpha,\delta}^{-1}(f * g), \tag{6}$$

and by linearity and sup-norm continuity it is enough to do this when $f = 1_{Z(\alpha\chi,\beta\chi)}$ and $g = 1_{Z(\beta\mu,\delta\mu)}$. But then the left-hand side of (6) is $1_{Z(\chi)}1_{Z(\mu)}$, which is 0, $1_{Z(\chi)}$ or $1_{Z(\mu)}$ depending on whether $Z(\chi) \cap Z(\mu)$ is empty, $Z(\chi)$ or $Z(\mu)$, and

$$f * g = \begin{cases} 0 & \text{if } Z(\chi) \cap Z(\mu) = \emptyset \\ 1_{Z(\alpha\chi,\beta\chi)} & \text{if } Z(\chi) \subset Z(\mu) \\ 1_{Z(\alpha\mu,\beta\mu)} & \text{if } Z(\chi) \supset Z(\mu), \end{cases}$$

from which (6) follows.

We have now shown that there is a well-defined representation π of $C_c(\mathcal{G})$ on H characterised by Equation (4). Since $\mathcal{G}$ is r-discrete, this representation is automatically $\|\cdot\|_I$-bounded by [16, Corollary II.1.22], and hence extends to the C^*-enveloping algebra $C^*(\mathcal{G})$.

Since (4) implies that $\pi(1_{Z(e,r(e))}) = S_e$, this completes the proof of Theorem 4.2.

4.6 Theorem: *Let G be a row-finite pointed directed graph, and suppose we have a *-representation π of the *-subalgebra*

$$C := \mathrm{sp}\{1_{Z(\alpha,\beta)} : s(\alpha) = s(\beta) = \star \text{ and } r(\alpha) = r(\beta)\}$$

of $C_c(\mathcal{G}(\star))$ on a Hilbert space H. Then π extends to a representation of $C_c(\mathcal{G}(\star))$ which is continuous for the inductive limit topology, and hence also to a representation of $C^*(\mathcal{G}(\star))$ on H.

This pointed analogue of Theorem 4.2 will require slightly different arguments, since the generating partial isometries S_e do not belong to $C_c(\mathcal{G}(\star))$ unless $s(e) = \star$. In fact, there may not even be such partial isometries acting on H: for if $C_c(\mathcal{G})$ acts on K, the subalgebra $C_c(\mathcal{G}(\star))$ acts on $H := 1_{P(G,\star)}(K)$, and the partial isometries $S_e = 1_{Z(e,r(e))}$ do not. (In [9], we got round this in the case of finite graphs by enlarging the Hilbert space of the representation.) The problem starts when we try to construct the representations π_v: we have no analogue of S_γ unless $s(\gamma) = \star$.

However, for each $\alpha \in F(G,\star)$, the procedure of Lemma 4.4 still gives a representation π_α of $C(P(G, r(\alpha)))$ such that $\pi_\alpha(1_{Z(\gamma)}) = \pi(1_{Z(\alpha\gamma,\alpha\gamma)})$. If $r(\alpha) = r(\beta)$, the corresponding representations of $C(P(G, r(\alpha)))$ are related by

$$\pi_\alpha(f) = \pi(1_{Z(\alpha,\beta)})\pi_\beta(f)\pi(1_{Z(\beta,\alpha)}); \tag{7}$$

this can be checked by a routine calculation on f of the form $1_{Z(\gamma)}$, and extended to arbitrary $f \in C(P(G, r(\alpha)))$ by linearity and continuity. We now aim to define π on $C(Z(\alpha,\beta))$ by

$$\pi(f) := \pi(1_{Z(\alpha,\beta)})\pi_\beta(\phi_{\alpha,\beta}^{-1}(f)) = \pi_\alpha(\phi_{\alpha,\beta}^{-1}(f))\pi(1_{Z(\alpha,\beta)}),$$

but, as before, we have to check that the result does not change if we subdivide $Z(\alpha,\beta)$.

4.7 Lemma: *For $\alpha, \beta \in F(G,\star)$ with $r(\alpha) = r(\beta)$, and $f \in C(Z(\alpha,\beta))$, we have*

$$\pi(1_{Z(\alpha,\beta)})\pi_\beta(\phi_{\alpha,\beta}^{-1}(f)) = \sum_{\{\gamma : |\gamma| = k, s(\gamma) = r(\alpha)\}} \pi(1_{Z(\alpha\gamma,\beta\gamma)})\pi_{\beta\gamma}(\phi_{\alpha\gamma,\beta\gamma}^{-1}(f|_{Z(\alpha\gamma,\beta\gamma)})).$$

Proof: Both sides are linear and uniformly continuous in $f \in C(Z(\alpha,\beta))$, so we may assume $f = 1_{Z(\alpha\chi,\beta\chi)}$ for some $\chi \in F(G)$ with $s(\chi) = r(\alpha)$. There are two cases for which the right-hand side is non-zero: $\chi = \gamma\chi'$ for just one γ with $|\gamma| = k$, or $|\chi| < k$, in which case

$$f = \sum_{\{\gamma' : |\gamma'| = k - |\chi|, s(\gamma') = r(\chi)\}} 1_{Z(\alpha\chi\gamma',\beta\chi\gamma')}.$$

In either case, because π is a representation on C, the left-hand side is

$$\pi(1_{Z(\alpha,\beta)})\pi_\beta(1_{Z(\chi)}) = \pi(1_{Z(\alpha,\beta)}1_{Z(\beta\chi,\beta\chi)}) = \pi(1_{Z(\alpha\chi,\beta\chi)}).$$

If $\chi = \gamma\chi'$, the right-hand side is

$$\pi\big(1_{Z(\alpha\gamma,\beta\gamma)}1_{Z(\beta\gamma\chi',\beta\gamma\chi')}\big) = \pi\big(1_{Z(\alpha\gamma\chi',\beta\gamma\chi')}\big) = \pi(1_{Z(\alpha\chi,\beta\chi)}).$$

If $|\chi| < k$, the right-hand side is

$$\sum_{\gamma'} \pi\big(1_{Z(\alpha\chi\gamma',\beta\chi\gamma')}1_{Z(\beta\chi\gamma',\beta\chi\gamma')}\big) = \sum_{\gamma'} \pi\big(1_{Z(\alpha\chi\gamma',\beta\chi\gamma')}\big) = \pi(1_{Z(\alpha\chi,\beta\chi)}),$$

and the result follows. □

We now define $\pi(f)$ for $f \in C_c(\mathcal{G}(\star))$ by writing $\operatorname{supp} f$ as the disjoint union of basic sets $Z(\alpha,\beta)$, and setting

$$\pi(f) := \sum_{\alpha,\beta} \pi(1_{Z(\alpha,\beta)})\pi_\beta\left(\phi_{\alpha,\beta}^{-1}(f|_{Z(\alpha,\beta)})\right).$$

The argument of Lemma 4.5, using Lemma 4.7 in place of Lemma 4.4, shows that the operator $\pi(f)$ does not depend on the decomposition of $\operatorname{supp} f$, and hence we have a well-defined linear map $\pi : C_c(\mathcal{G}(\star)) \to B(H)$ which is continuous for the inductive limit topology.

That $\pi(f^*) = \pi(f)^*$ follows from (7). As for the previous Theorem, it is enough to check multiplicativity on $f \in C(Z(\alpha,\beta))$ and $g \in C(Z(\beta,\delta))$, for which $f * g \in C(Z(\alpha,\delta))$. But then we can use (6) and (7) to see that

$$\begin{aligned}
\pi(f)\pi(g) &= \left(\pi(1_{Z(\alpha,\beta)})\pi_\beta(\phi_{\alpha,\beta}^{-1}(f))\right)\left(\pi_\beta(\phi_{\beta,\delta}^{-1}(g))\pi(1_{Z(\beta,\delta)})\right)\\
&= \pi(1_{Z(\alpha,\beta)})\pi_\beta\left(\phi_{\alpha,\beta}^{-1}(f)\phi_{\beta,\delta}^{-1}(f)\right)\pi(1_{Z(\beta,\delta)})\\
&= \pi(1_{Z(\alpha,\beta)})\pi_\beta(\phi_{\alpha,\delta}^{-1}(f*g))\pi(1_{Z(\beta,\delta)})\\
&= \pi_\alpha(\phi_{\alpha,\delta}^{-1}(f*g))\pi(1_{Z(\alpha,\beta)}1_{Z(\beta,\delta)})\\
&= \pi_\alpha(\phi_{\alpha,\delta}^{-1}(f*g))\pi(1_{Z(\alpha,\delta)})\\
&= \pi(f*g),
\end{aligned}$$

as required.

This completes the proof of Theorem 4.6.

4.8 Corollary: *Let G be a row-finite directed graph with associated groupoid $\mathcal{G}$. There is a continuous gauge action α of $\mathbf{T}$ on $C^*(\mathcal{G})$ such that $\alpha_z(1_{Z(\alpha,\beta)}) = z^{|\alpha|-|\beta|}1_{Z(\alpha,\beta)}$. If G is pointed, there is a similar gauge action on $C^*(\mathcal{G}(\star))$.*

Proof: For $z \in \mathbf{T}$, the functions $S_e := z1_{Z(e,r(e))}$ form a Cuntz-Krieger A-family, which also generates $C^*(\mathcal{G})$. Thus by representing $C^*(\mathcal{G})$ faithfully on H, and applying Theorem 4.2, we obtain a homomorphism α_z of $C^*(\mathcal{G})$ onto $C^*(\mathcal{G})$ such that $\alpha_z(1_{Z(e,r(e))}) = z1_{Z(e,r(e))}$. It is an isomorphism because $\alpha_{z^{-1}}$ is an inverse, and it is easy to check that $\alpha_z(1_{Z(\alpha,\beta)})$ is as described. For the last part it is quickest to recall from Theorem 3.1 that $C^*(\mathcal{G}(\star))$ is isomorphic to the corner $1_N C^*(\mathcal{G})1_N$. Since

$$1_N = \sum_{\{e:s(e)=\star\}} 1_{Z(e,e)},$$

we trivially have $\alpha_z(1_N) = 1_N$, so α induces an action on the corner, and this has the required property. □

5 Amenability.

We need to know that the groupoids $\mathcal{G}$ and $\mathcal{G}(\star)$ of a locally finite graph are amenable in the sense of [16, p.92]. Following the treatment of the Cuntz groupoids in [16, §III.2], we aim to realise them as reductions of the semidirect product of an obviously amenable groupoid by a shift automorphism. Straightforward variations of the approach in [16, §III.2] do not appear to work: without making assumptions on the underlying graph G, it is not clear how to embed $P(G)$ in the two-sided infinite path space $P_{-\infty}^{\infty}(G)$ so that one-sided tail equivalence on $P(G)$ is compatible with the two-sided tail equivalence for which $P_{-\infty}^{\infty}(G)$ is an AF-groupoid. To get round this problem, we use a much smaller space Y of two-sided paths, which is just large enough to contain $P(G)$ and admit a shift automorphism, but small enough to give an amenable groupoid with respect to one-sided tail equivalence. We assume throughout that the graph G is row-finite. To begin with, we assume also that $r : E \to V$ is onto, so that every vertex receives an edge, and remove this assumption at the end.

In this section, we shall have to keep careful track of indices. We therefore write, for $m \geq n \in \mathbf{Z}$,

$$F_n^m(G) := \left\{ \alpha \in \prod_{i=n}^{m} E : r(\alpha_i) = s(\alpha_{i+1}) \text{ for } n \leq i \leq m-1 \right\},$$

and, for $n \in \mathbf{Z} \cup \{-\infty\}, m \in \mathbf{Z} \cup \{\infty\}$ satisfying $m \geq n$,

$$P_n^m(G) := \left\{ x \in \prod_{i=n}^{m} E : r(x_i) = s(x_{i+1}) \text{ for } n-1 < i < m \right\}.$$

(If $m = \infty$, we sometimes leave it out of the notation.) If $n \neq -\infty$, the cylinder sets

$$Z(\alpha) = \{x \in P_n(G) : x_i = \alpha_i \text{ for } n \leq i \leq m\},$$

parametrised by $\alpha \in \cup_{m \geq n} F_n^m(G)$, are a basis of compact open sets for the product topology on $P_n(G)$. If $n = -\infty$, the sets

$$W(\alpha) := \{x \in P_{-\infty}^{\infty}(G) : x_i = \alpha_i \text{ for } -n \leq i \leq n\},$$

parametrised by $\alpha \in \cup_{n \geq 0} F_{-n}^n(G)$, are a basis of open sets for the product topology, but are not compact unless the graph is locally finite.

For each $v \in V$, we choose once and for all an edge $e(v)$ with range $r(e(v)) = v$. Then for each $v \in V$, putting these edges together gives a path $\gamma(v) \in P_{-\infty}^0(G)$ ending at v: formally, we define $\gamma(v)$ inductively by

$$\gamma(v)_0 = e(v), \quad \gamma(v)_{-n} = e\left(s(\gamma(v)_{-n+1})\right) \text{ for } n \geq 1.$$

Premultiplying by the appropriate $\gamma(v)$ gives embeddings k_n of $P_{-n}(G)$ in $P_{-\infty}(G)$: $k_n(x) := \gamma(s(x))x$ (where, as in the previous section, we fix the starting point of the right-hand path x, and move the left-hand path to fit, so that, e.g., $(\gamma(s(x))x)_{-n-1} = \gamma(s(x))_0 = e(s(x)))$. Since the $e(v)$'s give a unique left-infinite path ending at each vertex (formally: $\gamma\big(s((\gamma(v))_{-n})\big)_{-m} = \gamma(v)_{-(n+m)}$), these embeddings satisfy $k_n(P_{-n}(G)) \subset k_{n+1}(P_{-n-1}(G))$.

We write Y_n for the image $k_n(P_{-n}(G))$, so that the previous comment says $Y_n \subset Y_{n+1}$. The image of each basic open set $Z(\alpha) \subset P_{-n}(G)$ in Y_n is either a set of the form $W(\beta) \cap Y_n$ or a union of such sets; conversely, each $k_n^{-1}(W(\beta) \cap Y_n)$ is a union of $Z(\alpha)$'s. Thus k_n is a homeomorphism of $P_{-n}(G)$ onto Y_n. Since specifying the initial segment of a path gives open sets in any of the path spaces, each Y_n is open in Y_{n+1}. We let $Y := \cup_n Y_n$, so that Y is by definition a subset of $P_{-\infty}(G)$, but give Y the inductive limit topology in which a subset V is open iff $V \cap Y_n$ is open in Y_n for all n. In particular, each Y_n is open in Y; since Y_n is homeomorphic to the totally disconnected, locally compact Hausdorff path space $P_{-n}(G)$, Y is itself a totally disconnected, locally compact Hausdorff space.

Remark: This inductive limit topology is *not* necessarily the subspace topology on Y inherited from the product topology on $P_{-\infty}^{\infty}(G)$. To see this, fix a vertex v and consider

$$Z(\gamma(v)) := \{\gamma(v)x : x \in P_0(G), s(x) = v\}.$$

Then for any $n > 0$, $Z(\gamma(v)) \cap Y_n = \{k_n(y) : y \in Z(\alpha)$, where $\alpha \in F_{-n}^{-1}(G)$ is the restriction of $\gamma(v)$: $\alpha_i = \gamma(v)_i$ for $-n \leq i < 0$. Thus $Z(\gamma(v))$ is open in the inductive limit topology. But it is not necessarily open in the product topology. Indeed, if all the vertices along $\gamma(v)$ receive two distinct left-infinite paths, then any $W(\beta)$ which meets $Z(\gamma(v))$ will contain points of Y which do not agree with $\gamma(v)$ somewhere out towards $-\infty$, and hence no basic open set $W(\beta) \cap Y$ is contained in $Z(\gamma(v))$.

Our next goal is to turn the one-sided tail equivalence relation $\mathcal{R}$ on Y into a locally compact amenable groupoid. For each $n \geq 0$, we define

$$\mathcal{R}_n = \{(x, y) \in Y_n \times Y_n : x_i = y_i \text{ for } i > n\}.$$

Then $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$, and $\mathcal{R} = \cup_{n \geq 0} \mathcal{R}_n$. For each $n \geq 0$ and each pair $\alpha, \beta \in F_{-n}^n(G)$ with $r(\alpha) = r(\beta)$, we define

$$Y(\alpha, \beta) := \{(u, v) \in Y_n \times Y_n : u \in W(\alpha), v \in W(\beta) \text{ and } u_i = v_i \text{ for } i > n\}.$$

5.1 Lemma: *If G is a row-finite directed graph,*

$$\{Y(\alpha, \beta) : \alpha, \beta \in F_{-n}^n(G) \text{ satisfy } r(\alpha) = r(\beta)\}$$

is a basis of compact open sets for a locally compact topology on $\mathcal{R}$, in which each $\mathcal{R}_n$ is an open set. With $\mathcal{R}^2 := \big\{\big((x, y), (y, z)\big)\big\}$, $(x, y).(y, z) := (x, z)$, $(x, y)^{-1} := (y, x)$, $r(x, y) := x$, and $s(x, y) := y$, the space $\mathcal{R}$ is a locally compact r-discrete groupoid with unit space Y, such that the counting measures form a left Haar system.

Proof: As in Lemma 2.5, $Y(\alpha,\beta)\cap Y(\gamma,\delta)$ is either $Y(\alpha,\beta)$ or $Y(\gamma,\delta)$, so the family $\{Y(\alpha,\beta)\}$ does form a basis of open sets for a topology on $\mathcal{R}$, and the map $x\mapsto\big(\gamma(s(\alpha))\alpha x,\gamma(s(\beta))\beta x\big)$ is a homeomorphism of the compact path space $P_n(G,r(\alpha))$ onto $Y(\alpha,\beta)$. One verifies almost exactly as in §1 that $\mathcal{R}$ is an r-discrete groupoid with left Haar system as claimed. □

5.2 Lemma: *For $n>0$, let $\mathcal{S}_n$ denote the reduction of $\mathcal{R}$ to the (locally compact) subset Y_n of its unit space Y. If the graph G is locally finite, then each $\mathcal{S}_n$ is an AF-groupoid in the sense of* [16, p.123].

Proof: For $n>0$, $k>0$ and $x\in P_{k+1}(G)$, let

$$\mathcal{S}_n^k(x):=\{(u,v)\in\mathcal{S}_n : r(u_k)=r(v_k)=s(x),\ \text{and } u_i=v_i=x_i \text{ for } i>k\}.$$

Thus $\mathcal{S}_n^k(x)$ is the reduction of $\mathcal{S}_n$ with unit space

$$\{\gamma(s(\alpha))\alpha x : \alpha\in F_{-n}^k(G) \text{ and } r(\alpha)=s(x)\},$$

which is finite because G is locally finite. If $E_{-n}^k(v)$ denotes the transitive and principal groupoid on the finite set $\{\alpha\in F_{-n}^k(G) : r(\alpha)=v\}$, then $(\alpha,\beta)\mapsto\big(\gamma(s(\alpha))\alpha x,\gamma(s(\beta))\beta x\big)$ is an isomorphism of $E_{-n}^k(s(x))$ onto $\mathcal{S}_n^k(x)$. Thus for any fixed vertex v,

$$\mathcal{S}_n^k(v):=\bigcup\{\mathcal{S}_n^k(x) : x\in P_{k+1}(G) \text{ satisfies } s(x)=v\}$$

is isomorphic to the product $E_{-n}^k(v)\times P_{k+1}(G,v)$ of $E_{-n}^k(v)$ and the trivial equivalence relation on the totally disconnected path space $P_{k+1}(G,v)$. In other words, $\mathcal{S}_n^k(v)$ is an elementary groupoid of type $\#(\{\alpha\in F_{-n}^k(G) : r(\alpha)=v\})$, as in [16, p.123].

It now follows that $\mathcal{S}_n^k:=\cup\{\mathcal{S}_n^k(v) : v\in V\}$ is an elementary groupoid, as in [16, p.123]. The original reduction $\mathcal{S}_n:=\mathcal{R}|_{Y_n}$ is the increasing union of the family $\{\mathcal{S}_n^k : k\in\mathbf{N}\}$, and each $\mathcal{S}_n^k$ is open in $\mathcal{S}_n$, being a union of basic open sets $Y(\alpha,\beta)$. Thus $\mathcal{S}_n$ is the inductive limit of the sequence $\mathcal{S}_n^k$ as in [16, p.122], and hence is approximately elementary. Finally, since its unit space Y_n is totally disconnected, we deduce that $\mathcal{S}_n$ is an AF-groupoid. □

5.3 Corollary: *If the graph G is locally finite, the groupoid $\mathcal{R}$ is amenable.*

Proof: Since every AF-groupoid is amenable, $\mathcal{R}=\cup\{\mathcal{S}_n : n\in\mathbf{N}\}$ is the increasing union of amenable groupoids $\mathcal{S}_n$. Although $\mathcal{R}$ is not the inductive limit of $\{\mathcal{S}_n : n\in\mathbf{N}\}$ in the strict sense of [16, p.122] (the unit spaces vary), the argument at the top of [16, p.123] carries over, and allows us to deduce that $\mathcal{R}$ is amenable. □

Any two of our left-hand tails $\gamma(v)$ or their translates which pass through the same vertex agree to the left of that vertex. Thus the shift h on $P_{-\infty}(G)$ defined by $h(x)_i:=x_{i+1}$ maps the subset Y onto itself: since it maps Y_n homeomorphically onto Y_{n+1} for every $n\geq 0$, h restricts to a homeomorphism of the inductive limit Y. Since h preserves right-tail equivalence, it induces a groupoid homomorphism $\sigma:(u,v)\mapsto(h(u),h(v))$ of $\mathcal{R}$ onto itself. This bijection σ is a homeomorphism for the topology with basis $\{Y(\alpha,\beta)\}$, and hence is an automorphism of the locally compact r-discrete groupoid $\mathcal{R}$. It leaves the left Haar system of counting measures invariant. Thus the semi-direct product $\mathcal{R}\times_\sigma\mathbf{Z}$ is a locally compact amenable r-discrete groupoid [16, p.96].

Recall for convenience that $\mathcal{R}\times_\sigma\mathbf{Z}$ has unit space Y, range and source maps given by $r_\sigma((u,v),k)=u$, $s_\sigma((u,v),k)=h^{-k}(v)$, inverse given by $((u,v),k)^{-1}=\big(\sigma^{-k}(v,u),-k\big)=\big((h^{-k}(v),h^{-k}(u)),-k\big)$, and multiplication defined by

$$((u,v),k)((p,q),\ell)=((u,h^k(q)),k+\ell)\ \text{ provided } v=h^k(p).$$

5.4 Proposition: *Let G be a row-finite directed graph such that every vertex receives an edge. Then the map $\psi:\mathcal{G}\to\mathcal{R}\times_\sigma\mathbf{Z}$ defined by*

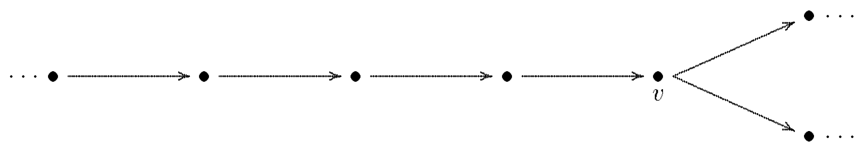
$$\psi\big((x,k,y)\big):=\big(((k_0(x),h^k(k_0(y))),k\big)$$

is an isomorphism of $\mathcal{G}$ onto the reduction of $\mathcal{R}\times_\sigma\mathbf{Z}$ to the closed subset $Y_0\cong P(G)$ of its unit space Y. Further, ψ restricts to an isomorphism of $\mathcal{G}(\star)$ onto the reduction of $\mathcal{R}\times_\sigma\mathbf{Z}$ to the closed subset $Y_0(\star):=k_0\big(P(G,\star)\big)$ of its unit space.

Proof: That ψ is an algebraic isomorphism of groupoids follows from elementary calculations using the above formulas for $\mathcal{R} \times_\sigma \mathbf{Z}$ and the ones in §1 for $\mathcal{G}$. The subset Y_0 is closed because it is the disjoint union of the compact open subsets $k_0(Z(e))$ parametrised by the edges $e \in E$. To see that ψ is a homeomorphism, one just has to verify that it carries basic open sets into unions of basic open sets, and this is routine (but messy). The last observation is straightforward. □

5.5 Corollary: *If G is a locally finite, directed and pointed graph, both $\mathcal{G}$ and $\mathcal{G}(\star)$ are amenable.*

Proof: First suppose that $r^{-1}(v) \neq \emptyset$ for all $v \in V$. Then we have already observed that the semi-direct product $\mathcal{R} \times_\sigma \mathbf{Z}$ is amenable, and hence so are its reductions to Y_0 and $Y_0(*)$ [16, p.92]. For the general case, construct a larger graph H with $r^{-1}(v) \neq \emptyset$ for all v by adding an infinite tail to each $v \in V(G)$ with $r^{-1}(v) = \emptyset$:



Then the path spaces $P(G)$ and $P(G, \star)$ are closed subsets of the path space $P(H)$, and the groupoids $\mathcal{G}$ and $\mathcal{G}(\star)$ are reductions of the groupoid for H. Hence they are also amenable. □

6 Ideal theory.

Let G be a directed graph. We denote by V_2 the set of vertices v for which there are two distinct finite loops $\alpha, \beta \in F(G)$ based at v; that is, $r(\alpha) = r(\beta) = s(\alpha) = s(\beta) = v$, $r(\alpha_i) \neq v$ for $1 \leq i < |\alpha|$, and $r(\beta_j) \neq v$ for $1 \leq j < |\beta|$. We similarly denote by V_1 the set of vertices v for which there is precisely one loop based at v, and by V_0 the set of vertices v for which there is no finite path starting and finishing at v. We say that G *satisfies condition* (K) if $V_1 = \emptyset$, i.e, if $V = V_2 \cup V_0$.

Condition (K) will be our analogue of Cuntz's Condition (II), which is not appropriate for infinite graphs. (In the notation of [2], the set Γ_A of "equivalence classes" could easily be empty.) We shall show below that (K) is equivalent to (II) for finite graphs (Lemma 6.1), and that the weaker condition (I) of [3] is not enough to make the theory of [16, 17] apply, even in the finite case (see Remark 6.4). Condition (K) is logically independent of the analogue (J) of (I) used in [13].

In comparing our results with those of [3] and [2], we shall use the notation $v \geq w$ of [2] to mean that the vertex w can be reached from v.

6.1 Lemma: *A finite directed graph G satisfies* (K) *if and only if the associated edge matrix A satisfies* (II).

Proof: The key observation is that when V is finite, each infinite path must pass through at least one vertex more than once, so there are always loops in G. Thus there is always at least one v such that the equivalence class

$$[v] := \{w \in V : v \geq w \geq v\} \tag{8}$$

is nonempty. Then by definition we have $[v] \subset V_1 \cup V_2$, and to prove the result we just have to observe that $[v] \subset V_2$ if and only if the block $A_{[v]}$ is not a permutation matrix. □

6.2 Lemma: *Suppose G is a directed graph which is irreducible in the sense that there is a finite path joining any two given vertices. Then G satisfies* (K) *if and only if some vertex emits more than one edge.*

Proof: Because G is irreducible, $V_0 = \emptyset$, and G satisfies (K) iff $V = V_2$. In fact $V_2 \neq \emptyset$ is enough to imply that some vertex emits two edges, since distinct loops based at the same point must diverge somewhere. On the other hand, if $s(e) = v = s(f)$, the irreducibility of G gives paths from $r(e)$ to v and $r(f)$ to v, and hence distinct loops based at v. The irreducibility also allows us to transport these distinct loops to any other vertex, and hence V_2 is all of V. □

6.3 Proposition: *If a row-finite directed graph G satisfies* (K)*, then the corresponding groupoids $\mathcal{G}$ and $\mathcal{G}(\star)$ are essentially principal.*

Proof: Let F be a nonempty closed invariant subset of $P(G) = \mathcal{G}^0$: we have to show that the set of points with trivial isotropy is dense in F. So fix $x \in F$, and a basic open neighbourhood $Z(\alpha) \cap F$ of x; note that x must have the form αz. Because G satisfies (K), if z never passes through V_2, it lies entirely within V_0, and hence passes through each vertex exactly once. Thus $(\alpha z, k, \alpha z)$ can belong to $\mathcal{G}$ only if $k = 0$, and x itself has trivial isotropy. So we may suppose that z passes through some $v \in V_2$: say $z = \beta w$ with $r(\beta) = v$. Let μ, ν be distinct loops based at v, and define paths $y_n \in P(G)$ by

$$y_n := \alpha\beta\mu\nu\mu\mu\nu\nu\cdots\overbrace{\mu\cdots\mu}^{n}\overbrace{\nu\cdots\nu}^{n}w.$$

Each y_n is equivalent to $x = \alpha\beta w$ (with a substantial lag), and hence lies in the invariant set F. The sequence y_n converges in the product topology to the aperiodic path

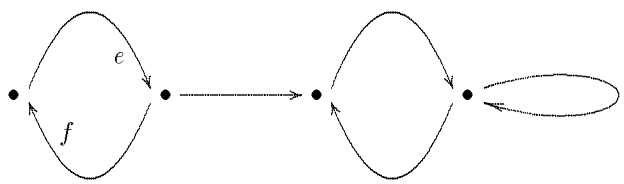
$$y := \alpha\beta\mu\nu\mu\mu\nu\nu\cdots\overbrace{\mu\cdots\mu}^{n}\overbrace{\nu\cdots\nu}^{n}\cdots,$$

which has trivial isotropy (i.e., there is no point of the form (y, k, y) with $k \neq 0$ in $\mathcal{G}$). Since F is closed, $y \in Z(\alpha) \cap F$, and we have approximated x by a point of trivial isotropy. Thus $\mathcal{G}$ is essentially principal, and so is its reduction $\mathcal{G}(\star)$.
□

6.4 Remark: It is important here that we are using an analogue of Condition (II): even if G is finite, Condition (I) is not enough to ensure that the groupoid is essentially principal. For example, consider the graph G with *vertex* matrix

$$B := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

so that

$G :=$ 

Then the infinite path $x := efefef\cdots$ has isotropy group $\mathcal{G}_x^x = \{(x, 2k, x)\}$ isomorphic to $2\mathbf{Z}$, and together with the path $fefefe\cdots$ forms a closed invariant subset of the unit space $P(G)$. Thus the groupoid $\mathcal{G}$ is not essentially principal. The $\{0,1\}$-edge matrix $A = A_G$ satisfies (I) but not (II): $\{e, f\}$ is an equivalence class such that $A_{\{e,f\}}$ is a permutation matrix.

That Renault's theory does not apply in this case is not surprising, since we know that the ideal theory of Cuntz-Krieger algebras $\mathcal{O}_A$ is more complicated when A satisfies (I) but not (II): permutation blocks on the diagonal of A give copies of $\mathbf{T}$ in the primitive ideal space [1]. From our point of view, the copy of $\mathbf{T}$ comes from the representation theory of the isotropy group $\mathcal{G}_x^x \cong \mathbf{Z}$.

For each open invariant subspace U of the unit space $P(G)$, the space

$$C_c(\mathcal{G}_U^U) := \{f \in C_c(\mathcal{G}) : \operatorname{supp} f \subset \mathcal{G}_U^U\}$$

is a self-adjoint two-sided ideal in $C_c(\mathcal{G})$, and hence its closure is an ideal $I(U)$ in $C^*(\mathcal{G})$. When G satisfies (K), Proposition 6.3 allows us to apply [17, Corollary 4.9] to see that $U \mapsto I(U)$ is an isomorphism of the lattice of open invariant subsets of $P(G)$ onto the lattice of (closed two-sided) ideals of $C_r^*(\mathcal{G})$. If G is in addition locally finite, the groupoid $\mathcal{G}$ is amenable (see §4), and all representations of $\mathcal{G}$ are obtained by integration [16, II.1.21], so $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$, and we have a description of the ideals of $C^*(\mathcal{G})$. To make this description useful, we need to describe the open invariant subsets of $P(G)$ in terms of the graph G. Naturally our description is similar to that of Cuntz in the finite case, though we have to make some changes because the set V_2 could be empty in the infinite case (see Remark 6.7 below).

We call a subset H of V *hereditary* if $v \in H$ and $v \geq w$ imply $w \in H$, and *saturated* if

$$\big[r(e) \in H \text{ for all } e \in E \text{ with } s(e) = v\big] \Longrightarrow v \in H.$$

For each nonempty open subset U of $P(G)$ let

$$H(U) := \{v \in V : \exists\, \alpha \in F(G) \text{ such that } r(\alpha) = v \text{ and } Z(\alpha) \subset U\}$$

(note that U open implies $H(U)$ nonempty), and for each subset H of V let

$$U(H) := \{x \in P(G) : r(x_n) \in H \text{ for some } n\}.$$

6.5 Lemma: *For any directed graph, $H \mapsto U(H)$ is an isomorphism between the lattices of saturated hereditary subsets of V and open invariant subsets of $P(G)$, with inverse given by $U \mapsto H(U)$.*

Proof: We begin by verifying that each $H(U)$ is hereditary and saturated. The first follows easily from the observation that $Z(\alpha) \supset Z(\alpha\beta)$ for any $\beta \in F(G)$ with $s(\beta) = r(\alpha)$. To see that $H(U)$ is saturated, suppose all the edges $\{e^i\}$ starting at v end in $H(U)$, so that there are $\alpha^i \in F(G)$ such that $r(\alpha^i) = r(e^i)$ and $Z(\alpha^i) \subset U$. Because U is invariant, and $e^i x$ is lag equivalent to $\alpha^i x$, we have $Z(e^i) \subset U$ for all i. Thus every path starting at v is in U, so that $Z(\beta) \subset U$ whenever $r(\beta) = v$, and $v \in H(U)$.

Since

$$x \in U(H) \Longrightarrow r(x_n) \in H \Longrightarrow Z(x_1 x_2 \cdots x_n) \subset U(H),$$

$U(H)$ is certainly open, and it is easy to verify that $U(H)$ is invariant whenever H is hereditary. So it remains to prove that $H(U(H)) = H$ whenever H is saturated and hereditary, and $U(H(U)) = U$ whenever U is open and invariant.

We trivially have $H \subset H(U(H))$, so we suppose $v \in H(U(H))$ and aim to prove $v \in H$. We know there exists $\alpha \in F(G)$ such that $r(\alpha) = v$ and $Z(\alpha) \subset U(H)$; thus for every $\alpha y \in Z(\alpha)$, there is some n such that $r((\alpha y)_n) \in H$. If $v \notin H$, then because H is saturated there would be an infinite path $y \in P(G)$ with $s(y) = v$ but $r(y_k) \notin H$ for all k. But then αy would be an element of $Z(\alpha)$ which did not pass through H, and we have a contradiction. Thus we must have $v \in H$, as required.

We also trivially have $U \subset U(H(U))$, so we suppose $x \in U(H(U))$, say $r(x_n) \in H(U)$. Then there exits $\alpha \in F(G)$ with $r(\alpha) = r(x_n)$ and $Z(\alpha) \subset U$. But then $x = x_1 \cdots x_n x'$ is lag equivalent to $\alpha x' \in U$, and the invariance of U implies $x \in U$.

It remains to show that $H \mapsto U(H)$ preserves the lattice operations, which are given by intersection and union of open sets, and intersection and union-followed-by-saturation of hereditary subsets. Of these, the only tricky one is $U(H_1 \vee H_2) \subset U(H_1) \cup U(H_2)$, and this follows from the observation that if $r(x_k) \notin H_1 \cup H_2$ for all $k \geq n$, then $r(x_n)$ is not in the saturation $H_1 \vee H_2$ of $H_1 \cup H_2$ (for then $V \setminus \{r(x_k) : k \geq n\}$ would be a saturated set containing $H_1 \cup H_2$, and hence also $H_1 \vee H_2$). □

6.6 Theorem: *Let G be a locally finite directed graph with associated locally compact groupoid $\mathcal{G}$, and suppose that G satisfies Condition* (K). *For $H \subset V$, let*

$$I(H) := \overline{\text{sp}}\{1_{Z(\alpha,\beta)} : \alpha, \beta \in F(G) \textit{ satisfy } r(\alpha) = r(\beta) \in H\}.$$

Then $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of V onto the lattice of ideals in $C^*(\mathcal{G})$. The quotient $C^*(\mathcal{G})/I(H)$ is naturally isomorphic to the groupoid algebra $C^*(\mathcal{F})$ of the directed graph $F := (V \setminus H, \{e : r(e) \notin H\})$. The ideal $I(H)$ is Morita equivalent to the groupoid algebra $C^*(\mathcal{E})$ of the directed graph $E := (H, \{e : s(e) \in H\})$.

Proof: From Lemma 6.5, the amenability of $\mathcal{G}$, and [17, Corollary 4.9], we deduce that $H \mapsto I(U(H))$ is a lattice isomorphism, so we only have to identify $I(U(H))$ with $I(H)$. Since $\operatorname{supp} 1_{Z(\alpha,\beta)} = Z(\alpha, \beta)$, and

$$Z(\alpha, \beta) \subset \mathcal{G}_U^U \iff Z(\alpha) \subset U \text{ and } Z(\beta) \subset U, \tag{9}$$

it follows immediately from the definition of $U(H)$ that $I(H) \subset I(U(H))$. So it is enough to show that an arbitrary $f \in C_c(\mathcal{G}_{U(H)}^{U(H)})$ can be approximated by elements of $I(H)$. Since $\operatorname{supp} f$ is a disjoint union of (compact open) sets of the form $Z(\alpha,\beta)$, and $\operatorname{supp} f \subset \mathcal{G}_{U(H)}^{U(H)}$, it follows from (9) that each of these $Z(\alpha,\beta)$'s satisfies $r(\alpha) = r(\beta) \in H(U(H)) = H$. Since f is the (finite) sum of the functions $f1_{Z(\alpha,\beta)} \in C(Z(\alpha,\beta))$, it is enough to consider $f \in C(Z(\alpha,\beta))$ where $r(\alpha) = r(\beta) \in H$. But as in the proof of Proposition 4.1, we can approximate such an f uniformly, and hence in $\|\cdot\|_I$, by a linear combination of functions of the form $1_{Z(\alpha\gamma,\beta\gamma)}$. The hereditary property of H implies that $r(\gamma) \in H$ whenever $Z(\alpha\gamma,\beta\gamma) \neq \emptyset$, so that each $1_{Z(\alpha\gamma,\beta\gamma)} \in I(H)$. Thus $I(H)$ is dense in the closure $I(U(H))$ of $C_c(\mathcal{G}_{U(H)}^{U(H)})$, and we have proved that $I(H) = I(U(H))$.

To identify the quotient, we apply [16, Proposition II.4.5], which says that $I(U(H))$ and $C^*(\mathcal{G})/I(U(H))$ are isomorphic to the groupoid algebras $C^*(\mathcal{G}_{U(H)}^{U(H)})$ and $C^*(\mathcal{G}_{P(G)\setminus U(H)}^{P(G)\setminus U(H)})$, respectively. Since H is saturated and hereditary, $P(G) \setminus U(H) = P(F)$, and the groupoid $\mathcal{F}$ can be naturally identified with the reduction of $\mathcal{G}$ to $P(G) \setminus U(H)$. The space $U(H)$ is generally larger than the space $P(E)$ of paths in E: paths in $U(H)$ can have initial segments lying outside H. But the closed subset $P(E)$ is an abstract transversal for $U(H)$, and hence the groupoid $\mathcal{G}_{U(H)}^{U(H)}$ is equivalent to $\mathcal{G}_{P(E)}^{P(E)} = \mathcal{E}$ [11, Example 2.7]. It therefore follows from [11, Theorem 2.8] that $I(U(H)) = C^*(\mathcal{G}_{U(H)}^{U(H)})$ is Morita equivalent to $C^*(\mathcal{E})$. □

6.7 Remark: When the graph G is finite, Condition (K) coincides with Condition (II) of [2], and our Theorem reduces to that of [2]. For if G is finite, every saturated hereditary subset of V contains equivalence classes $[v]$ as in (8). The relation $\geq$ on vertices induces a partial order on $\Gamma_G := \{[v] : v \in V, [v] \neq \emptyset\}$, and saturated hereditary subsets of V are determined by their intersections with Γ_G, which are hereditary with respect to this partial order. Thus for finite G, our Theorem gives a bijection between the hereditary subsets of Γ_G and the ideals in $C^*(\mathcal{G}(G))$, as in [2, Theorem 2.5]. This description does not carry over to the infinite case, because Γ_G could be empty: there need not be any loops in G. For example, consider

$$G := \quad \cdots \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$$

6.8 Corollary: *Suppose G is a locally finite directed graph which satisfies* (K). *Then $C^*(\mathcal{G}(G))$ is simple if and only if G is cofinal, in the sense that for every $v \in V$ and $x \in P(G)$, there exist $\alpha \in F(G)$ and $n \in \mathbf{N}$ such that $s(\alpha) = v$ and $r(\alpha) = r(x_n)$.*

Proof: Suppose first that G is cofinal. Then it is enough by Theorem 6.6 to show that there are no non-trivial saturated hereditary subsets of V. Suppose H is such a subset, and $H \neq \emptyset$. Fix $v \in H$, and let w be an arbitrary vertex of G. If $w \notin H$, the saturation of H implies that there is an edge starting at w which does not end in H, and by induction there is an infinite path x such that $s(x) = w$ and $r(x_n) \notin H$ for all n. But cofinality implies that there is a path $\beta \in F(G)$ with $s(\beta) = v$ and $r(\beta) = r(x_n)$ for some n. Because H is hereditary, $v \in H$ implies $r(\beta) \in H$, which forces $r(x_n) \in H$. This is a contradiction, and hence w must be in H. Thus H is all of V, there is no proper saturated hereditary subset of V, and $C^*(\mathcal{G})$ is simple.

If $C^*(\mathcal{G})$ is not simple, there is a proper saturated hereditary subset H of V. Suppose $v \notin H$. Then because H is saturated, there has to be an infinite path x such that $s(x) = v$ and $r(x_n) \notin H$ for all n. Because H is hereditary, no vertex in H can connect to x, and G is not cofinal. □

6.9 Remark: When G is finite, this Corollary reduces to the simplicity theorem of Cuntz and Krieger: if G satisfies (K) and is cofinal, then the edge matrix A_G is irreducible and is not a permutation matrix. To prove this we have to prove that the graph is transitive, i.e., that there is a path joining any two vertices. Then Condition (K) will rule out the possibility of a permutation matrix.

First note that every infinite path ends up in V_2: it must pass through at least one vertex v twice, so there are loops in G, and since $V_1 = \emptyset$, v must be in V_2. On the other hand, we claim that every vertex v can be reached from a vertex in V_2. To see this, consider an infinite path going backwards out of v. It too must loop at some vertex w, and again $V_1 = \emptyset$ implies $w \in V_2$. Since we are considering a backwards path, we trivially have $w \geq v$, justifying the claim. It is therefore enough for us to prove that $v \geq w$ for any two vertices $v, w \in V_2$. But if α is a loop based at w, then the cofinality implies that v is connected to the path $\alpha\alpha\alpha\cdots$, and hence to w.

6.10 Corollary: *Suppose B is a locally finite $\{0,1\}$-matrix such that the directed graph G with vertex matrix B satisfies* (K). *If $\{S_i\}$ and $\{T_i\}$ are families of non-zero partial isometries satisfying*

$$S_i^* S_i = \sum_j B(i,j) S_j S_j^*, \quad T_i^* T_i = \sum_j B(i,j) T_j T_j^*,$$

then there is an isomorphism of $C^*(S_i)$ onto $C^*(T_i)$ which takes S_i to T_i.

Proof: It is enough to prove that the representation π of $C^*(\mathcal{G}(G))$ corresponding to such a Cuntz-Krieger family $\{S_i\}$ is faithful. Every ideal of $C^*(\mathcal{G})$ has the form $I(H)$ for some saturated hereditary subset H of V, so any nontrivial ideal must contain some $1_{Z(e,\emptyset)}$. Since each $\pi(1_{Z(e,\emptyset)}) = S_{s(e)} S_{r(e)} S_{r(e)}^*$ (see the end of the proof of Proposition 4.1) is non-zero, $\ker \pi$ must be $\{0\}$, and π is faithful. □

6.11 Remark: This Corollary allows us to *define* the Cuntz-Krieger algebra $\mathcal{O}_B$ for any locally finite $0,1$-matrix (with graph G_B) satisfying (K), as the unique C^*-algebra generated by a Cuntz-Krieger B-family of non-zero partial isometries. If H is a saturated hereditary subset of $V(G_B)$, then both subgraphs $F := (V \setminus H, \{e : r(e) \notin H\})$ and $E := (H, \{e : s(e) \in H\})$ also satisfy (K). Thus our main theorem asserts that every ideal in $\mathcal{O}_B$ is Morita equivalent to some $\mathcal{O}_{B'}$, and that every quotient is isomorphic to some $\mathcal{O}_{B''}$. Since for finite matrices (K) is equivalent to (II), this is a direct generalisation of [2, Theorem 2.5].

6.12 Corollary: *let G be a row-finite directed graph with associated groupoid $\mathcal{G}$ and edge-matrix A. Then $K_0(C^*(\mathcal{G})) \cong \mathbf{Z}^{|E|}/(1 - A^t)\mathbf{Z}^{|E|}$ and $K_1(C^*(\mathcal{G})) \cong \ker\{1 - A^t : \mathbf{Z}^{|E|} \to \mathbf{Z}^{|E|}\}$. If G is pointed, we can replace $\mathcal{G}$ by $\mathcal{G}(\star)$.*

Proof: This will follow by applying the construction of [13, §4] to the gauge action α of $\mathbf{T}$ on $C^*(\mathcal{G})$, provided we verify that α has large spectral subspaces. But if $k \in \mathbf{Z}$, then

$$C^*(\mathcal{G})^\alpha(k) \supset \mathrm{sp}\{1_{Z(\alpha,\beta)} : |\beta| - |\alpha| = k\}.$$

If $|\delta| - |\gamma| = |\beta| - |\alpha|$, then the usual arguments involving the cases $\gamma = \alpha\gamma'$ and $\alpha = \gamma\alpha'$ show that $1_{Z(\gamma,\delta)}^* 1_{Z(\alpha,\beta)}$ has the form $1_{Z(\mu,\nu)}$ for some μ, ν with $|\mu| = |\nu|$. Thus

$$\begin{aligned} C^*(\mathcal{G})^\alpha(k)^* C^*(\mathcal{G})^\alpha(k) &\supset \mathrm{sp}\{1_{Z(\mu,\nu)} : |\mu| = |\nu|, |\mu| \geq \max(0,k)\} \\ &= \mathrm{sp}\{1_{Z(\mu,\nu)} : |\mu| = |\nu|\}. \end{aligned}$$

On the other hand, because $C^*(\mathcal{G}) = \overline{\mathrm{sp}}\{1_{Z(\mu,\nu)}\}$, and the expectation $f \mapsto \int \alpha_z(f)\, dz$ is bounded, the elements $1_{Z(\mu,\nu)}$ with $|\mu| = |\nu|$ span a dense subspace of $C^*(\mathcal{G})^\alpha$. Thus α does indeed have large spectral subspaces, as required. The last statement follows from Theorem 3.1. □

6.13 Remark: In view of the realisation of the Doplicher-Roberts algebras $\mathcal{O}_\rho$ as $C^*(\mathcal{G}(\star))$ (to be proved in the next section), this Corollary generalises [13, Corollary 5.1.1] as well as [2, Proposition 3.1].

7 Doplicher-Roberts algebras.

Suppose K is a compact group and ρ is a fixed finite-dimensional unitary representation of K on a Hilbert space H_ρ. We define a pointed directed graph G_ρ by taking V to be the set $\widehat{K}$ of equivalence classes of irreducible unitary representations of K, taking the number of edges from v to w to be the multiplicity of w in $v \otimes \rho$, and taking for $\star$ the trivial one-dimensional representation ι. (We confuse an irreducible representation v of K with its equivalence class $v \in \widehat{K}$.) As in [9], let ρ^n denote the nth tensor power of ρ, acting in H_ρ^n, and let (ρ^m, ρ^n) denote the space of intertwining operators $T : H_\rho^n \to H_\rho^m$. If we identify $T \in (\rho^m, \rho^n)$ with $T \otimes 1 \in (\rho^{m+k}, \rho^{n+k})$, then we can compose arbitrary pairs of elements of ${}^0\mathcal{O}_\rho := \cup_{m,n}(\rho^m, \rho^n)$; with the natural involution $T \mapsto T^*$, ${}^0\mathcal{O}_\rho$ becomes a *-algebra, and the *Doplicher-Roberts algebra* $\mathcal{O}_\rho$ is its C^*-enveloping algebra.

It is shown in [9, Theorem 2.1] that, if A_ρ is the edge matrix of G_ρ, and $\{S_e : e \in E\}$ is a Cuntz-Krieger A_ρ-family, then there is a *-homomorphism ϕ of ${}^0\mathcal{O}_\rho$ onto a corner in the *-algebra generated by $\{S_e\}$. (The idea is, loosely speaking, that pairs of paths (α, β) in $F(G_\rho, \iota)$ with $r(\alpha) = r(\beta)$ and $|\alpha| = m, |\beta| = n$ determine a basis $\{T_{\alpha,\beta}\}$ for (ρ^m, ρ^n), and the *-homomorphism ϕ takes $T_{\alpha,\beta}$ to $S_\alpha S_\beta^*$.) This result applies in particular to the partial isometries $S_e = 1_{Z(e,\emptyset)}$ in $C_c(\mathcal{G}(G_\rho))$, and we obtain a *-homomorphism ϕ of ${}^0\mathcal{O}_\rho$ onto the *-subalgebra

$$C := \mathrm{sp}\{1_{Z(\alpha,\beta)} : \alpha, \beta \in F(G_\rho, \iota) \text{ and } r(\alpha) = r(\beta)\}$$

of $C_c(\mathcal{G}(G_\rho))$, which is the corner in $\mathrm{sp}\{1_{Z(\alpha,\beta)} : r(\alpha) = r(\beta)\}$ corresponding to the projection $P := \sum\{1_{Z(e,e)} : s(e) = \iota\}$.

We claim that ϕ is an isomorphism of ${}^0\mathcal{O}_\rho$ onto C. To see this, we define an inverse ψ for ϕ. Since the lag splits the pointed groupoid $\mathcal{G}(\iota)$ into disjoint open and closed subsets $\{\mathcal{G}_k(\iota) : k \in \mathbf{Z}\}$, and $f \in C_c(\mathcal{G}(\iota))$ can be uniquely written as a sum $f = \sum f_k$ with $f_k \in C_c(\mathcal{G}_k(\iota))$, it is enough to define $\psi_k : C_c(\mathcal{G}_k(\iota)) \to {}^0\mathcal{O}_\rho$ such that $\psi_k \circ \phi(f) = f$ when $\mathrm{supp}\, f \subset \mathcal{G}_k(\iota)$. But now we can use the path length of (say) α to write $C_k := C \cap C_c(\mathcal{G}_k(\iota))$ as the union of

$$C_{n,k} := \mathrm{sp}\{1_{Z(\alpha,\beta)} : \alpha, \beta \in F(G_\rho, \iota), r(\alpha) = r(\beta), |\alpha| = n, |\beta| - |\alpha| = k\} :$$

the embeddings $C_{n,k} \to C_{n+1,k}$ are given by

$$1_{Z(\alpha,\beta)} \mapsto \sum_{\{e \in E : s(e) = r(\alpha)\}} 1_{Z(\alpha e, \beta e)}.$$

Since these tally exactly with the behaviour of the bases $\{T_{\alpha,\beta}\}$ under the embeddings of (ρ^n, ρ^{n+k}) in $(\rho^{n+1}, \rho^{n+1+k})$ (see the calculation at the top of p.230 of [9]), we can define $\psi_{n,k}(1_{Z(\alpha,\beta)}) := T_{\alpha,\beta}$, and the family $\{\psi_{n,k}\}$ extends to a linear map ψ_k of $C_k = \cup C_{n,k}$ into $\varinjlim(\rho^n, \rho^{n+k})$ with the required property. We conclude that ϕ is an isomorphism of ${}^0\mathcal{O}_\rho$ onto C, as claimed.

Any representation π of ${}^0\mathcal{O}_\rho$ on Hilbert space gives a representation $\pi \circ \phi^{-1}$ of the corner C, and by Theorem 4.6 this extends to representations of $C_c(\mathcal{G}(\iota))$ and $C^*(\mathcal{G}(\iota))$. Thus:

7.1 Theorem: *Let $\rho : K \to U(n)$ be a representation of a compact group K, and let $\mathcal{G}(\iota)$ be the associated pointed groupoid, as described above. Then the isomorphism ϕ of ${}^0\mathcal{O}_\rho$ into $C_c(\mathcal{G}(\iota))$ extends to an isomorphism of the Doplicher-Roberts algebra $\mathcal{O}_\rho$ onto $C^*(\mathcal{G}(\iota))$.*

7.2 Lemma: *If ρ is a faithful representation of a compact group K and $\rho(K) \subset SU(H_\rho)$, then the graph G_ρ is irreducible and locally finite.*

Proof: We first establish that every $\pi \in V := \widehat{K}$ can be reached from ι, by showing that

$$R := \{\pi \in \widehat{K} : \pi \text{ is equivalent to a summand of some power } \rho^n\}$$

is all of $\widehat{K}$. Because ρ is faithful, this will follow from [7, 27.39] if we can prove that R is closed under conjugation (closure under the other operation $\times$ in [7, §27] is automatic for our R). For any matrix $T \in M_n$, the nth tensor power acts on the space $\mathrm{sp}\{e_1 \wedge \cdots \wedge e_n\}$ of antisymmetric tensors by multiplication by $\det T$, and since $\det \rho_s = 1$ for all $s \in G$, this implies that the nth tensor power ρ^n contains the trivial representation ι. From the orthogonality relations [7, 27.30] for the corresponding characters, we deduce that $(\chi_{\rho^n}, \chi_\iota) \geq 1$. But this implies

$$(\chi_{\rho^{n-1}}, \chi_{\overline{\rho}}) = (\chi_\rho^{n-1}, \overline{\chi_\rho}) = (\chi_\rho^n, 1) = (\chi_{\rho^n}, \chi_\iota) \geq 1,$$

and the conjugate $\overline{\rho}$ is contained in ρ^{n-1}. Thus $\overline{\rho} \in R$, all powers of $\overline{\rho}$ are in R, and $\pi \in R$ implies $\overline{\pi} \in R$. This establishes the claim, i.e. that every vertex π can be reached from ι.

Since we can reach ι from any vertex π ($\overline{\pi} \subset \rho^n$ for some n, and ι is a summand of $\pi \otimes \overline{\pi} \subset \pi \otimes \rho^n$), we can go from each vertex to any other one, and the graph G_ρ is irreducible. It is row-finite because each $\pi \otimes \rho$ has only finitely many irreducible summands. Reversing all the arrows in G_ρ gives the graph $G_{\overline{\rho}}$ of the conjugate, because, again by the orthogonality relations, π_2 is a summand of $\pi_1 \otimes \rho$ iff π_1 is a summand of $\pi_2 \otimes \overline{\rho}$. Since $G_{\overline{\rho}}$ is row-finite, G_ρ is column-finite. □

7.3 Corollary: ([4, Theorem 3.1]) *If K is a compact group and $\rho : K \to SU(n)$ is a faithful special unitary representation, then the Doplicher-Roberts algebra $\mathcal{O}_\rho$ is simple.*

Proof: Since $\mathcal{O}_\rho \cong C^*(\mathcal{G}(\iota))$ is a corner in $C^*(\mathcal{G})$ by Theorem 3.1, and the graph G_ρ is irreducible by the Lemma, the result follows from Corollary 6.8. □

7.4 Remark: That ρ takes values in $SU(n)$ was only used to ensure that every $\pi \in \widehat{K}$ is a summand of some power of ρ. Thus the Lemma and the Corollary apply whenever ρ has this property. This is automatic if ρ is a representation of a finite group [9, Lemma 3.1], or even if $\det\rho$ takes values in a finite subgroup of $\mathbf{T}$ (by adapting the argument of the Lemma); the resulting generalisation of Corollary 7.3 is due to Pinzari [15, Theorem 2.2].

In principle, Theorem 7.1 and Corollary 6.12 give us the K-theory of the Doplicher-Roberts algebras $\mathcal{O}_\rho$. As we pointed out in [13], one can interpret the resulting description in terms of the representation ring of the group K; indeed, this approach will show that $K_1(\mathcal{O}_\rho) = 0$, and allow us to identify generators of $K_0(\mathcal{O}_\rho)$. Before we give the details, we recall that the representation ring $R(K)$ is by definition the set of formal differences of classes of representations of K, under the operations $\oplus$ and $\otimes$; additively, $R(K)$ can be identified with the free abelian group on $\widehat{K}$.

Our first lemma is well-known; it can, for example, be deduced from [12, Proposition 1.2]. We give a short proof based on Exel's elegant description of K_0 [6] (which, unlike that in [12], does not need separability hypotheses).

7.5 Lemma: *If p is a full projection in a C^*-algebra A, then the inclusion map $i : pAp \to A$ induces an isomorphism of $K_0(pAp)$ onto $K_0(A)$.*

Proof: Write $B = pAp$, and suppose that $e \in M_m(B)$ and $f \in M_n(B)$ are projections. In Exel's picture, the element $[e]-[f]$ of $K_0(B)$ is represented as $\operatorname{Ind} 0$, where 0 denotes the trivial homomorphism of the projective (Hilbert) module eB^m into fB^n. The isomorphism X_* induced by the B-A imprimitivity bimodule $X := pA$ is given by $X_*(\operatorname{Ind} T) = \operatorname{Ind}(T \otimes I_X)$ [6, Theorem 5.1], and hence $X_*([e]-[f])$ is represented by the difference $[eB^m \otimes_B pA] - [fB^n \otimes_B pA]$ of projective modules. Since

$$eB^m \otimes_B pA = e\big((pAp)^m \otimes_{pAp} pA\big) \cong e(pA)^m = (ep)A^m = i(e)A^m,$$

the result follows. □

We are now ready to formulate our description of $K_0(\mathcal{O}_\rho)$. By Lemma 7.2, each $\pi \in \widehat{K}$ embeds as a summand of ρ^n for some n. If e_π is the projection of the tensor power $\mathcal{H}_\rho^n$ onto a summand where ρ^n acts as π, then e_π is a projection in $(\rho^n, \rho^n) \subset {}^o\mathcal{O}_\rho \subset \mathcal{O}_\rho$.

7.6 Proposition: *The map $\psi : [\pi] \to [e_\pi]$ is a well-defined map of $\widehat{K}$ into $K_0(\mathcal{O}_\rho)$, and $K_0(\mathcal{O}_\rho)$ is generated as an abelian group by the elements $\{[e_\pi] : \pi \in \widehat{K}\}$ subject to the relations*

$$[e_\pi] = \sum_{i=1}^{d} [e_{\pi_i}] \quad \textit{where } \pi \otimes \rho = \bigoplus_{i=1}^{d} \pi_i. \tag{10}$$

Proof: To see that $[\pi] \mapsto [e_\pi]$ is well–defined, suppose that π is also a summand of ρ^m, yielding a projection f_π in (ρ^m, ρ^m). If we choose the basis $\{T_x T_y^*\}$ for (ρ^m, ρ^n) described in [9, Proposition 1.1], then e_π is the projection $T_x T_x^*$, where x is the unique path of length n in the graph G_ρ starting at the trivial representation ι and finishing at the summand π; the partial isometry $T_x : \mathcal{H}_\rho^n \to \mathcal{H}_\pi$ by definition embeds π as a summand of ρ^n. Similarly, f_π has the form $T_y T_y^* \in (\rho^m, \rho^m)$ for some path y in $G_\rho(\iota)$ of length m and range π. But then $T_y T_x^* \in (\rho^m, \rho^n) \subset {}^o\mathcal{O}_\rho$ intertwines the two copies of π, and hence satisfies

$$(T_y T_x^*)(T_y T_x^*)^* = T_y T_y^* = f_\pi \quad \text{and} \quad (T_y T_x^*)^*(T_y T_x^*) = T_x T_x^* = e_\pi;$$

thus $[e_\pi] = [f_\pi]$ in $K_0(\mathcal{O}_\rho)$.

In the same notation, the irreducible summands π_i of $\pi \otimes \rho$ are embedded in (ρ^{n+1}, ρ^{n+1}) via the partial isometries $T_{xi} : \mathcal{H}_\rho^{n+1} \to \mathcal{H}_{\pi_i}$; thus we have

$$[e_\pi] = [e_\pi \otimes 1_{\mathcal{H}_\rho}] = \left[\bigoplus_{i=1}^{d} T_{xi} T_{xi}^*\right] = \sum_{i=1}^{d} [e_{\pi_i}],$$

so the classes $[e_\pi]$ satisfy the relations (10).

Because the matrix A_ρ is irreducible and not a permutation matrix, the isomorphism ϕ of Theorem 7.1 is an isomorphism of $\mathcal{O}_\rho$ onto a full corner in $\mathcal{O}_{A_\rho}$. Thus by Lemma 7.5, it will be enough to prove that the images $S_x S_x^* = \phi(e_\pi)$ generate $K_0(\mathcal{O}_{A_\rho})$ subject only to the relations (10). However, we know from [13, Corollary 4.2.5] that $\mathcal{O}_{A_\rho}$ is generated by the projections $P_x = S_x S_x^*$ subject only to the relations

$$[P_x] = \sum_{y \in E} A_\rho(x, y)\,[P_y]\,. \tag{11}$$

Since the value of $A_\rho(x, z)$ depends only on $r(x)$, the projections $S_x^* S_x$ depend only on $r(x)$; hence if $r(x) = r(y)$, we have

$$\begin{aligned} [P_x] &= [S_x S_x^*] = [S_x^* S_x] = \left[\sum_{z \in E} A_\rho(x, z) S_z S_z^*\right] \\ &= \left[\sum_{z \in E} A_\rho(y, z) S_z S_z^*\right] = \left[S_y^* S_y\right] = \left[S_y S_y^*\right] = [P_y]\,. \end{aligned}$$

We deduce that there is one generator $[q_\pi] \in K_0(\mathcal{O}_{A\rho})$ for each $\pi = r(x)$ in $\widehat{K}$. The relations (11) become

$$[q_\pi] = \sum_{\tau \in \widehat{K}} B_\rho(\pi, \tau)\,[q_\tau] = \sum_{i=1}^{d} [q_{\pi_i}]\,,$$

where $B_\rho(\pi, \tau)$ is the multiplicity of τ in $\pi \otimes \rho$. Since $[S_x S_x^*] = \phi([e_\pi])$, the result follows. □

7.7 Corollary: *Let $R(K)$ be the representation ring of an infinite compact Lie group K, and ρ a faithful representation of K in $SU(\mathcal{H}_\rho)$ with $1 < dim \mathcal{H}_\rho < \infty$. Let β_ρ be the endomorphism $[\pi] \mapsto [\pi \otimes \rho]$ of $R(K)$. Then $K_0(\mathcal{O}_\rho) \cong R(K)/\mathrm{Im}\,(\mathrm{id} - \beta_\rho)$ and $K_1(\mathcal{O}_\rho) \cong 0$.*

Proof: The map ψ extends to an additive homomorphism of $R(K)$ onto $K_0(\mathcal{O}_\rho)$. If $\pi \otimes \rho = \oplus_{i=1}^d \pi_i$, then from (10) we have

$$\psi\left(\beta_\rho\,[\pi]\right) = \psi\left([\pi \otimes \rho]\right) = [e_{\pi\otimes\rho}] = \left[\bigoplus_{i=1}^{d} e_{\pi_i}\right] = \sum_{i=1}^{d} [e_{\pi_i}] = [e_\pi] = \psi\left([\pi]\right)$$

in $K_0(\mathcal{O}_\rho)$, and hence the Theorem identifies the kernel of ψ with the image of $(\mathrm{id} - \beta_\rho)$. Thus $K_0(\mathcal{O}_\rho)$ is the cokernel of $(\mathrm{id} - \beta_\rho)$, as claimed.

Since K is a compact Lie group, the embedding of a maximal torus T in K induces am embedding of $R(K)$ in $R(T)$ (see [8, p.172]); thus $R(K)$ is a subring of a Laurent polynomial ring. Since multiplication by a Laurent polynomial p has no fixed elements unless $p = 1$, and since ρ does not restrict to the trivial character on T, we deduce that β_ρ has no fixed points, and hence Ker $(\mathrm{id} - \beta_\rho) = \{[\iota]\}$.

If we identify $R(K)$ with $\mathbf{Z}^{\widehat{K}}$, then β_ρ is given by the matrix B_ρ^t, and hence $\mathrm{Ker}(\mathrm{id} - B_\rho^t) = \{0\}$. Since we know from [9, Lemma 4.2] that $\mathrm{Ker}(\mathrm{id} - A_\rho^t) \cong \mathrm{Ker}(\mathrm{id} - B_\rho^t)$, we deduce that

$$K_1(\mathcal{O}_\rho) \cong K_1(\mathcal{O}_{A_\rho}) = \mathrm{Ker}(\mathrm{id} - A_\rho^t) = \mathrm{Ker}(\mathrm{id} - B_\rho^t) = \{0\},$$

which completes the proof. □

7.8 Remark: Example 4.4 of [9], in which K is the alternating group A_5, shows that this result does not hold for finite groups, even if ρ is irreducible.

7.9 Example:

Let $K = SU(\mathbf{C}^n)$. Then by [8, p.179], the representation ring $R(K)$ is a polynomial ring in $n-1$ generators $\lambda_1, \ldots, \lambda_{n-1}$, where λ_1 is the natural action of $SU(\mathbf{C}^n)$ on $\mathbf{C}^n$ and λ_i its ith exterior power. If ρ is the representation λ_1, then the cokernel of $(\mathrm{id} - \beta_\rho)$ is isomorphic to $\mathbf{Z}[\lambda_2, \ldots, \lambda_{n-1}]$. Thus for $n \geq 3$, $K_0(\mathcal{O}_\rho)$ is isomorphic to the free abelian group on infinitely many generators. This example illustrates the value of the representation ring approach: a direct calculation of $\mathrm{coker}(1 - B_\rho^t)$ would be a formidable undertaking.

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