

# Consistency— What’s Logic Got to Do with It?

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April 27, 1996

Most modern philosophers who study logic as part of their training, know something about the development of modern logic and something about the fact that the issue of consistency had something to do with this development.

In this paper, I want to explore the origin of the modern conception of the idea of consistency in logic in the work of German mathematician David Hilbert. My interest in the development of the modern idea of consistency arises from my belief that an overriding concern with a strict requirement of consistency, borrowed primarily from the rigors of modern developments in logic, has prevented latter day twentieth century philosophers from producing philosophical systems of the type produced in earlier times.

To give an idea of what I mean, let me attempt to discuss the following question: How, if in any way, does an overriding concern with consistency affect philosophy? Well, it seems that some notion of consistency is operative when philosophers reason about anything. On the other hand, philosophers do not always expect that the objects of their study satisfy some consistency criteria. With only rare exceptions, do we hear a call for consistency in love, which has been an object of philosophical musing at least since Plato wrote his *Symposium* [Ham89] and probably long before. In fact, when asked “Consistency—What’s Love Got to Do with It?”, we would probably say “not much” unless we insist on interpreting consistency in love as love’s consistency with scientific theories about anthropology and human physiology and genetics. But that sort of perspective will take us philosophers perhaps too far from the original ideas found in the *Symposium* and I’m not sure how far philosophers want to get from Plato. However, if we interpret the consistency in love as love’s fitness in a philosophical system, it seems we will be back in a philosophical sort of business. Leaving the detailed discussion of the connection between modern philosophy and the notion of consistency to another occasion, let me return to the main topic of

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\*I would like to thank professor Paolo Mancosu for having motivated the writing of this paper.

this paper after merely noting that the criterion of consistency has always been a measure of good philosophy but the way this criterion has been defined and applied has changed a great deal, with the developments in mathematical logic having a lot to do with the change in this century.

## 1 Hilbert's Way

David Hilbert came from a family of artisans turned lawyers and received his early training in the classics, being groomed for a legal career by his father. But he seems to have had a unique passion for mathematics even as he was studying the classics as a boy—a passion that determined his advance, in a decade, from a young mathematics student in Königsberg to a world famous mathematician with broad acclaim for

1. the original approach in his study of the invariant properties of algebraic forms in his doctoral dissertation under Ferdinand Lindemann—the man who proved transcendency of  $\pi$ ,
2. the solution of Gordan's problem, which had puzzled the very best mathematicians of the time, and
3. the development of an axiomatic approach exemplified by his *Grundlagen der Geometrie* [Hil72].

All of these achievements demonstrated the basic Hilbertian style of mathematical reasoning, a style which later comes to be called the “axiomatic method.”<sup>1</sup>

The place that the *Grundlagen* occupies in the development of mathematical logic can hardly be overlooked. It is a work that was compared with Euclid in its lucidity and simplicity of development. However, while Euclid never questions the evidence of his axioms and concerns himself with the way the axioms help define the physical space, for Hilbert the physical truth of the axioms and explicit definitions of objects are irrelevant to the basic task of developing consistent logical consequences of an axiom system upon the deletion or addition of particular axioms, while the axiom system itself provides the implicit definitions of any mathematical objects it mentions<sup>2</sup>. This Hilbertian style of work has dominated, if not much of mathematics, much of mathematical logic, culminating in the proof of the independence of the continuum hypothesis from the conventional axioms of set theory<sup>3</sup>.

In a letter to Hilbert, written soon after the publication of the *Grundlagen*, Frege demanded that a clear place be given to definitions in any mathematical theory. Furthermore, Frege seems not to have appreciated why Hilbert and

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<sup>1</sup>For an accessible discussion of Hilbert's life and contributions, see Constance Reid's excellent biography [Rei70].

<sup>2</sup>For a similar analysis, see Weyl[Wey44]

<sup>3</sup>For an accessible presentation of the proof, see [Kun80].

others were interested in demonstrating that any contradiction in Euclidean geometry must reduce to one in the axioms of arithmetic. Concerning axioms of Euclidean Geometry, Frege says:

I call axioms propositions that are true but that are not proved because our understanding of them derives from that nonlogical basis which may be called intuition of space. *From the fact that axioms are true, it follows of itself that they do not contradict one another*<sup>4</sup>.

Hilbert summarized his views succinctly in his response to Frege's criticism:

... each axiom contributes something to the definition, and therefore each new axiom alters the concept. "Point" is always something different in Euclidean, non-Euclidean, Archimedean, and non-Archimedean geometry respectively ... I think of my points as some system or other of things, e.g. the system of love, of law, or of chimney sweeps ... and then conceive of all my axioms as relations between these things, then my theorems, e.g. the Pythagorean one, will hold of these things as well. In other words, each and every theory can always be applied to infinitely many systems of basic elements<sup>5</sup>.

Frege, of course, seized on one controversial point which Hilbert raised in his letter. He noted that contrary to Hilbert's view, even if we may be able to construct non-contradictory propositions about an object, it does not follow that the said object exists. Unfortunately, Hilbert never responded to Frege's critique of his view of existence, neither did he make his concept of existence any more clear, nor did the future development of his thought shed light on this. As far as I know, Hilbert never moved far from his original position on these topics.

In Hilbert's style of mathematics, the main focus of activity is placed on

1. how one can probe an axiom system in order to arrive at interesting theorems consistent with the axioms and meaningful of the mathematical structure being investigated,

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<sup>4</sup>The emphasis is mine. For Hilbert/Frege correspondence on the role of definitions and axioms in geometry see their correspondence [Fre71].

<sup>5</sup>Again, the quote is from [Fre71]. I should also note that this view of the definitions and axioms of geometry was apparently shared by many mathematicians, even those who spoused opposed to Hilbert's on other matters. See for example [Poi52b], where Poincaré says:

In other words, *the axioms of geometry* (I do not speak of those of arithmetic) *are only definitions in disguise*. What are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true, and if the old weights and measures are false; if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another; it can only be more convenient.

2. how a change in the axioms is reflected in a change in the mathematical structure,
3. whether an axiom is independent of others in a system of axioms, and
4. how the question of consistency of a particular axiom system can be reduced to the question of consistency of some other axiom system.

Having transferred the question of consistency from one mathematical structure to another obviously does not settle the consistency question for any axiomatic system. This problem, along with the problems associated with all the antinomies that logicians were discovering at the time, could only produce a deep interest in how one would need to prove the consistency of a certain set of axioms, an associated question being how mathematics could be used to talk about consistency in mathematics.

“On the Foundations of Logic and Arithmetic” [Hil04], which Hilbert delivered to the 1904 Congress of Mathematicians at Heidelberg, inaugurated his proof theory and became an inspirational starting point for many mathematicians in their study of Hilbert’s famous “Second Problem,” which had to do with the consistency of arithmetic. This problem was among the many problems which Hilbert had already described in his speech delivered before the Second International Congress of Mathematicians in Paris in 1900 <sup>6</sup>. These were supposed to be problems that were mathematically significant and that would keep mathematicians busy in the new century, as indeed they did.

Hilbert’s 1904 paper is certainly important as a watershed due to the response it elicited [Poi52a] and for the mathematical work it inspired [Bar35]<sup>7</sup>. This paper is also significant because it is Hilbert’s first serious search for a solution to his “second problem.” Although a sketchy description of his vision of what a proof theory could look like, and how one could go about dealing with questions of consistency in mathematics, the paper was instrumental in setting in full motion Hilbert’s program of axiomatization of various scientific fields and of provision of consistency proofs for the ensuing axiomatic systems [Bar35].

Having already demonstrated for geometry that it was provably consistent if arithmetic could be shown to be consistent, Hilbert now noted that “recourse to another fundamental discipline does not seem to be allowed when the foundations of arithmetic are at issue.”

The whole idea of giving a logical foundation to arithmetic had been part of a mounting response to the Kantian views on mathematics [Kan65]. Kant had shown, to the satisfaction of many, that mathematics consisted of *a priori synthetic* judgments<sup>8</sup>. Kant had thrown the ball of mathematics in the Humean

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<sup>6</sup>For an English translation of the speech see [Hil00]. Also reprinted in [Bro76].

<sup>7</sup>Barnays [Bar35] while giving account of the influence and historical place of Hilbert’s work, comments only on the general problems of the 1904 paper, such as lack of clear notions. However, the paper also suffers from some mathematical mistakes

<sup>8</sup>For modern interpretations of Kant’s views on mathematics, see [Pos92].

court. It was obvious that if the followers of Hume were to adhere certainty to mathematics, they should also take care not to deny others the certainty they attached to other *a priori synthetic* judgements unless the Humeans could come up with a clear demonstration of why some *a priori synthetic* judgments could be true and others false. For many mathematicians and mathematical philosophers, this seemed like a heavy bill for mathematicians to pay. Bolzano was among the first and Dedekind followed him [Rus80, Bol10, Ded63] in the mathematical investigations that ensued in order to find a non-synthetic foundation for mathematics. It was Frege, who conceived the inherently fanciful nature of the task the mathematicians had set for themselves as he himself went about trying to demolish the Kantian perspective which placed the origins of arithmetic in our intuition of time:

... moreover, we see how pure thought irrespective of any content given by the senses or even by an intuition *a priori*, can, solely from the content that results from its own constitution, bring forth judgments that at first sight appear to be possible only on the basis of some intuition. *This can be compared with condensation, through which it is possible to transform the air that to the child's consciousness appears as nothing into a visible fluid that forms drops*<sup>9</sup>.

But even thin air must have some water molecules before pressure can produce condensation. It was in constructing the seeds of this condensation that the program ran into serious problems. The paradoxical set theory into which Frege and Dedekind had run into, now forced Hilbert to seek another solution:

It is my opinion that all the difficulties touched upon can be overcome and that we can provide a rigorous and completely satisfying foundation for the notion of number, and in fact by a method that I would call axiomatic and whose fundamental idea I wish to develop briefly in what follows[Hil04].

What was necessary, Hilbert argued, was “a partly simultaneous development of the laws of logic and of arithmetic” to avoid paradoxes and “turning in circles” which arises due to the dependence of laws of logic on “certain fundamental arithmetic notions,” such as number and whole. The idea was that if one could axiomatize arithmetic in such a way which would allow a logical demonstration of its consistency with no recourse to arithmetic notions such as number and induction, then arithmetic would find an *a priori* foundation. But so far this program had failed even in its less ambitious formulations. Hence, the call for the “simultaneous development of the laws of logic and of arithmetic.”

At this stage and also afterwards, in the works of Hilbert and his followers, we have a distinction—at times subtle and subconscious, and at times consciously

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<sup>9</sup>The emphasis is mine. See [Fre79].

construed—between formal axiomatic structures under study and a proof theory devoid even of a simple formal language, not to mention a formal structural description. This is due to the fact that the distinction between mathematics and metamathematics has not yet been demarcated in Hilbert’s 1904 paper. Hilbert’s original attempt is based on an algebraic treatment of symbols, regardless of their meaning or what they may stand for. However, this technique is rather dangerous if symbols are made to stand for different things and basic notions are used for different purposes at different levels and it can lead to the type of error Hilbert makes in his 1904 paper. The necessity of drawing a sharp distinction and its relationship to the formalist/intuitionist arguments of the time had become significant by 1927, when von Neumann published his paper entitled “On Hilbertian Proof Theory”. He wrote:

Here, one must always draw a sharp distinction between two different kinds of “proofs”: the formal (“mathematical”) proofs within the formal system and the contentual (“metamathematical”) proofs about the system. While the first is a kind of random logical game (which must broadly remain analogous to the classical mathematics), the latter is a concatenation of immediate, evident, contentual judgements. These latter “contentual proofs” need to be carried out in accordance with intuitionistic logic of Brouwer and Weyl: Proof theory, therefore, needs to give a foundation to classical mathematics on an intuitionistic basis and then lead strict Intuitionism to absurdity ...<sup>10</sup>

## 2 The Outcome

Despite the first errors and the stumbling moves, Hilbert blazed a path which was followed to some degree by all early mathematical logicians. The emphasis that Hilbert’s program put on the question of consistency certainly affected the simultaneous growth of logic for which he had called. The modern developments in mathematical logic were motivated in a large part by the Hilbertian type of questions about the independence of axioms and consistency of axiom systems, the major culmination being Paul Cohen’s result on the independence of the continuum hypothesis from the conventional axioms of set theory[Coh66].

The logical language and techniques developed by Hilbert and those who followed him, were primarily algebraic in their flavor and sought to give consistent accounts of what they spoke of. In fact, if one was required to give a completely consistent account of a scientific subject matter, one found it necessary to use or create a formal logic as the language of inquiry, of clarification, and even of discovery<sup>11</sup>. The spread of the formalist methods to other fields was swift.

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<sup>10</sup>The translation is mine. For the German original see [Neu27].

<sup>11</sup>For example, see the various axiomatizations of quantum mechanics[Neu55, Lud85].

Today, most university students in non-art departments are encouraged to give self-consistent accounts of the topics on which they are asked to write. Rhetorical skill often takes precedence over appreciation and understanding of a subject. What does it mean to give a self-consistent philosophical account of emotions for example? Why should self-consistency override reality and become a criterion of truth? I believe the answers to these questions lie in the great success of science which it owes, to a large degree, to the main currents and ideas of modern logic. Physical scientists need to give consistent definitions for objects of their study in order to be able to communicate with each other <sup>12</sup>. But if discovery of new scientific theory inherently requires that one should contradict the older theory in some domain, does not the criterion of self-consistency play a merely functional, pragmatic role in scientific inquiry?

Philosophy has always been defined—by philosophers if not by others— as pursuit and love of wisdom not as a pedantic concern with consistency and contradictoriness <sup>13</sup>.

To what extent should philosophers attach themselves to this modern view of what logic is and what logic can do? In many ways, it is easier to study modern logic than to pore over classical scholastic logic of Aristotelians and neo-Platonists. Modern logic—in addition to its serene beauty—does indeed convey the main, technically useful notions of classical logic in a succinct manner. It does provide an immensely useful tool to the scientist and even the engineer who seeks to model “artificial reasoning” and to arrive at minimal “logic circuits” which make computer chips more efficient. But the breadth of philosophy often overflows the boundaries that modern logic has set for its investigations. Today the philosopher is left to conclude that logic is something smaller in breadth and deeper in its reflections, something that says things about consistency and is very useful in the philosophical investigations having to do with language. Logic was known as a tool that philosophers used in the whole breadth of their philosophical musing. A small portion of it is still a tool for the philosopher but not all that now springs from the vastness of modern logic can be employed by one who seeks to explore philosophical questions dealing with ethics and emotions or with the sort of mysteries we encounter in the Platonic dialogues such as the *Symposium*.

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<sup>12</sup>If I have one understanding of what force is and you have another, we will eventually meet in discord on some standing problem.

<sup>13</sup>This is the translation of the German word *Widerspruchfreiheit*, which Hilbert and his followers used in referring to consistency of mathematical systems.

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