Subdivision on Arbitrary Meshes: Algorithms and Theory

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Subdivision surfaces have become a standard geometric modeling tool for a variety of applications. This survey is an introduction to subdivision algorithms for arbitrary meshes and related mathematical theory; we review the most important subdivision schemes the theory of smoothness of subidivision surfaces, and known facts about approximation properties of subdivision bases.

1 Introduction

This survey is based on a series of lectures presented at the IMS-IDR-CWAIP Joint Workshop on Data Representation at the National University of Singapore in August 2004.

Our primary goal is to present a brief introduction to the algorithms and theory related to subdivision surfaces from basic facts about subdivision to more recent research developments. This tutorial is intended for a broad audience of computer scientists and mathematicians. While not being comprehensive by any measure, it aims to provide an overview of what the author considers the most important aspects of subdivision algorithms and theory as well as provide references for further study.

A large variety of algorithms and a comprehensive theory exist for subdivision schemes on *regular grids*, which are only briefly mentioned in this survey. Subdivision on regular grids, being closely related to wavelet constructions, has an important applied role in many applications. However, ability to handle *arbitrary* control meshes was one of the primary reasons for the rapid increase in popularity of subdivision for computer graphics and geometric modeling applications during the last decade. This motivates our focus on schemes designed for such meshes.

We start with a brief survey of applications of subdivision in computer graphics and geometric modeling in Section 1. In Section 2, we introduce the basic concepts for both curve and surface subdivision. In the third section we review different types of subdivision rules focusing on the most commonly used in practice (Loop and Catmull-Clark subdivision).

In contrast to the regular case, fewer general theoretical results and tools are available for subdivision schemes on arbitrary meshes; in many aspects the theory is somewhat behind the practice. The most important theoretical results on smoothness and approximation properties of subdivision surfaces are reviewed in Sections 5 and 6.

Sections 2–5 are partially based on the notes for the SIGGRAPH course "Subdivision for Modeling and Animation" co-taught by the author in 1998-2000. There is a number of excellent books and review articles on subdivision which the author highly recommends for further reading: the monograph of Cavaretta et al. [12] on subdivision on regular grids, survey articles by Dyn and Levin [19,20], the book by Warren and Weiner [82], the articles by Sabin[69,68] and Schröder [72,73].

1.1 *Subdivision in Computer Graphics and Geometric Modeling*

The idea of constructing smooth surfaces from arbitrary meshes using recursive refinement was introduced in papers by Catmull and Clark [11] and Doo and Sabin [18] in 1978. These papers built on subdivision algorithms for regular control meshes, found in the spline literature, which can be traced back to late 40s when G. de Rham used "corner cutting" to describe smooth curves.

Wide adoption of subdivision techniques in computer graphics applications occurred in the mid-nineties: with an increase in complexity of the models, the need to extend traditional NURBS-based tools bacame apparent.

Constructing surfaces through subdivision elegantly addresses many issues with which computer graphics and computer-aided design practitioners are confronted. Most importantly, the need to handle control meshes of *arbitrary topology*, while maintaining surface smoothness and visual quality automatically. Subdivision surfaces easily admit multiresolution extensions, thus enabling efficient hierarchical representations of complex surfaces. At the same time, most popular subdivision schemes extend splines (and produce piecewise-polynomial surfaces for regular control meshes), thus maintaining continuity with previously used representations and inheriting some of the appealing qualities of splines. Another important advantage of subdivision surfaces is that simple local modifications of subdivision rules make it possible to introduce surface features of many different types [26,9]. Finally, subdivision surfaces can be extended to hierarchical representations either of wavelet [48], pyramid type [93], or related displaced subdivision surfaces [39].

Over the past few years, a number of crucial geometric algorithms were developed for subdivision surfaces and subdivision-based multiresolution representations. One of the important steps that enabled many practical applications was development of direct evaluation methods [76], that made it possible to evaluate, in constant time, recursively defined subdivision surfaces at arbitrary points. Algorithms were developed for trimming [44], performing boolean operations [8], filleting and blending [85,57], fitting [35], computing surface volumes [60], lofting [54,55,56,70] and other operations. Subdivision surfaces were demonstrated to be a useful too for complex interactive surface editing [37,93,10,31].

Subdivision surfaces became a mature technology, used in a variety of applications. Examples of applications include representing and registering complex range scan data [2], face modeling [75,45] and three dimensional extensions of subdivision used in large-scale visualization [42,4].

As subdivision algorithms can be used to define bases on arbitrary mesh domains, they are a natural candidate for higher-order finite element calculations for engineering applications, shell problems in particular. First steps in this direction were made in [13,14]. Natural refinement structure of subdivision surfaces leads to adaptive hierarchal finite element constructions [36]. Subdivision-based mesh generation for FEM is explored in [40,41].

2 Basics

In this section we introduce the basic concepts of subdivision needed to define various subdivision schemes considered in Section 3

2.1 *Subdivision curves*

The goal of this section is to introduce the basic concepts using subdivision curves as an example. The apparatus of subdivision matrices we introduce is not essential for curves, as the same formulas can be obtained by other means; however, it is indispensable for subdivision surfaces.

Subdivision algorithm. We can summarize the basic idea of subdivision as follows: subdivision defines a smooth curve or surface as the limit of of successive refinements of an initial sequence of control points.

In this section, to simplify exposition, we only consider curves defined by infinite sequences of control points indexed by integers and only one type of refinement: a new control point is added to the sequence between two old control points and the positions of old points are recomputed (Figure 1).

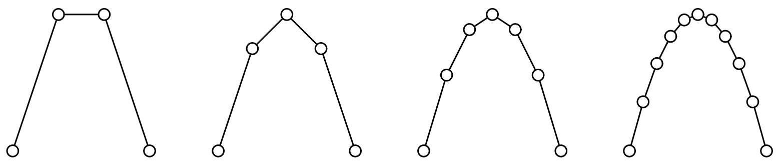

Fig. 1. Subdivision steps for a cubic spline.

The numbering for the refined sequence is chosen so that the point i in the original sequence has even number $2i$ in the new sequence. We use notation p^j for the sequence of control points after j subdivision steps.

The most general definition of a linear subdivision rule is that it is a collection of linear maps S^j, mapping p^j to p^{j+1}. In this survey we consider subdivision rules which satisfy two additional requirements: the rules are *stationary* and have finite support.

More formally, for the type of one-dimensional refinement described above, *stationary subdivision rules* can be specified by two sequences of coefficients $\{a_i^e, |i \in \mathbf{Z}\}$ and $\{a_i^o | i \in \mathbf{Z}\}$ which are usually referred to as even and odd masks. For a given sequence of control points $p = (p_i \in \mathbf{R}^n, i \in \mathbf{Z})$, a single subdivision step produces a new refined sequence p' of control points p'_i, defined by

$$\begin{aligned} p'_{2i} &= \sum_{j \in \mathbf{Z}} a^e_{i-j} p_j \\ p'_{2i+1} &= \sum_{j \in \mathbf{Z}} a^o_{i-j} p_j \end{aligned} \tag{2.1}$$

For our choice of numbering, the even-numbered points correspond to the repositioned original control points, and odd-numbered points are the newly added points. For stationary subdivision, the linear map from p^j to p^{j+1} does not depend on the level, i.e. there is a single linear operator S, such that $p^{j+1} = Sp^j$.

The rules have *finite support* if only a finite number of coefficients a_i^o and a_i^e are nonzero. The set of indices for which the mask coefficients are not zero is called *mask support*.

The most common subdivision scheme for uniform cubic B-splines has masks with nonzero entries $(1/8, 3/4, 1/8)$ with indices $(-1, 0, 1)$ and $(1/2, 1/2)$ with indices $(-1, 0)$, for even and odd control points respectively (Figure 1).

We can view the initial control points p^0 as values assigned to integer points in $\mathbf{R}$. It is natural to assign control points p^1 to half-integers, and in general control points p^j to points of the form $i/2^j$ in $\mathbf{R}$.

For each subdivision level j we then have a unique piecewise linear function $L[p^j]$, defined on $\mathbf{R}$ which interpolates the control points p^j: $L(i/2^j) = p_i^j$. We say that the subdivision scheme *converges* if for any initial control points p^0, the associated sequence of piecewise linear functions $L[p^{(j)}]$ converges pointwise.

In particular, for the cubic spline masks defined above, the limit curve is a cubic polynomial on each integer interval $[i, i+1]$. The reason for this is that this set of masks is derived from the well-known refinement relation for uniform cubic B-splines:

$$B(t) = \frac{1}{8}\left(B(2t-2) + 4B(2t-1) + 6B(2t) + 4B(2t+1) + B(2t+2)\right). \tag{2.2}$$

A cubic spline curve has the form $\sum_{i\in\mathbf{Z}} p_i B(t-i)$; applying the refinement relation (2.2) to $B(t-i)$ and collecting the terms, we obtain

$$\sum_{i\in\mathbf{Z}} p_i B(t-i) = \sum_{i\in\mathbf{Z}} p_i' B(2t-i)$$

with $p_{2i}' = (1/8)(p_{i-1} + 6p_i + p_{i+1})$ and $p_{2i+1}' = (1/2)(p_i + p_{i+1})$, i.e. with p_i' defined by the subdivision rules stated above. We conclude that sequences p_i and p_i' *define the same spline curve*. However, the refined control points p_i' correspond to scaled basis functions $B(2t)$ with smaller support and are spaced closer to each other. As we refine, we get control points for the same cubic curve $f(t)$ but split into shorter polynomial segments. One can show the piecewise linear functions, connecting the control points, converge to $f(t)$ pointwise.

While spline subdivision is a starting point for many subdivision constructions, deriving subdivision masks from spline refinement is not essential for obtaining convergent schemes or schemes producing smooth curves or surfaces. For example, one can replace the $(1/8, 3/4, 1/8)$ rule by three perturbed coefficients $1/8 - w, 3/4 + 2w, 1/8 - w)$, and still maintain convergence and tangent continuity of limit curves for sufficiently small w. However, the limit curves for the modified rules in general cannot be expressed in closed form.

Modified coefficients are usually chosen to meet a set of requirements necessary for desirable scheme behavior. The most basic requirement is

Affine invariance. *If the points of sequence q are obtained by applying an affine transformation T to points of p, then $[Sq]_i = T[Sp]_i$, $i \in \mathbf{Z}$.*

By considering translations by t, $q_i = p_i + t$, and substituting into the subdivision rules 2.1, we immediately obtain that the coefficients of masks should sum up to one:

$$\sum_{i\in\mathbf{Z}} a_i^e = 1, \quad \sum_{i\in\mathbf{Z}} a_i^o = 1.$$

In other words, the subdivision operator S should have a eigenvector with constant components $p_i = 1$, for all i, and eigenvalue 1. It can also be shown this is necessary (but not sufficient) condition for convergence.

Subdivision matrices. As we have seen above, a subdivision step can be represented by a linear operator acting on sequences. It is often useful to consider local subdivision matrices of finite dimension. Such matrices have an important role, both in practice and in theory, as they can be used for limit control point positions and tangent vectors and analysis of convergence and continuity. These local matrices are restrictions of the infinite subdivision matrices to *invariant neighborhoods* of points.

Fix an integer i; then the invariant neighborhood N_m of size m for i is the set of indices $\{i-m, \dots i+m\}$, such that the control points p_j^1, $j = 2i-m \dots 2i+m$, can be computed using only points p_i^0, for $i \in N_m$. The minimal size of the invariant neighborhood depends only on the support of the masks. For example, the minimal size m for the cubic B-spline subdivision rules is 1 because one can compute points p_{2i-1}^1, p_{2i}^1 and p_{2i+1}^1 given points p_{i-1}^0, p_i^0 and p_{i+1}^0.

We often need to consider invariant neighborhoods of larger size, such that the control points in the neighborhood define the curve completely on some interval containing the point of interest. For cubic splines, a curve segment, corresponding to an integer interval $[i, i+1]$, requires four control points. To obtain a part of the curve, containing i in the interior of its domain, we need to consider both $[i-1, i]$ and $[i, i+1]$ for a total of five points, which correspond to the neighborhood of size 2.

The subdivision rules for computing five control points, centered at i, on level $j+1$ from five control points, centered at i on level j can be written as

$$\begin{pmatrix} p_{2i-2}^{j+1} \\ p_{2i-1}^{j+1} \\ p_{2i}^{j+1} \\ p_{2i+1}^{j+1} \\ p_{2i+2}^{j+1} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} p_{i-2}^{j} \\ p_{i-1}^{j} \\ p_{i}^{j} \\ p_{i+1}^{j} \\ p_{i+2}^{j} \end{pmatrix}.$$

The 5 by 5 matrix in this expression is the *subdivision matrix*. If the same subdivision rules are used everywhere, this matrix does not depend on the choice of i.

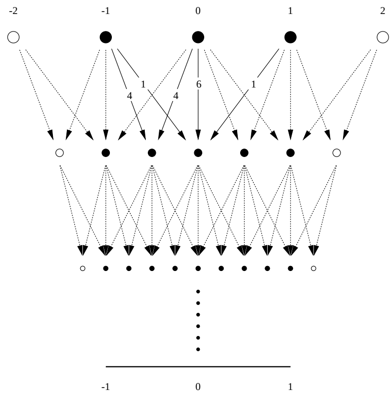

Fig. 2. In the case of cubic B-spline subdivision, the invariant neighborhood is of size 2. It takes 5 control points at the coarsest level to determine the behavior of the subdivision limit curve over the two segments adjacent to the origin. At each level, we need one more control point on the outside of the interval $t \in [-1, 1]$ in order to continue on to the next subdivision level. 3 initial control points for example would not be enough.

The eigenvalues and eigenvectors of the subdivision matrix allow one to analyze how the control points in the invariant neighborhoods change from level to level.

Suppose an $n \times n$ subdivision matrix is non-defective, i.e. has n independent eigenvectors x_i, $i = 0, \ldots n-1$. Then, any vector of initial control points p can be written as a linear combination of eigenvectors of the matrix: $p = \sum_{i=0}^{n} a_i x_i$. The coefficients a_i can be computed using eigenvectors as

$$a_i = (l_i \cdot p),$$

using the dual basis of left eigenvectors l_i, $i = 0 \ldots n-1$, satisfying $(x_i \cdot l_k) = \delta_{ik}$. In this form, the result of applying the subdivision matrix j times, i.e. the control points on j-th subdivision level in the invariant neighborhood, can be written as

$$S^j p = \sum_{i=0}^{n} \lambda^j a_i x_i \tag{2.3}$$

where λ_i, $i = 0 \ldots n-1$, are the eigenvalues.

Limit positions. One can immediately observe that for convergence it is necessary that all eigenvalues of the matrix have magnitudes no greater than one. Furthermore, one can easily show that if there is more than one eigenvalue of magnitude one, the scheme does not converge either. At the same time, $\lambda_0 = 1$ is an eigenvalue corresponding to eigenvector $[1, 1, \dots 1]$. The reason is that multiplying S by this eigenvector is equivalent to summing up the entries in each row, and by affine invariance, these entries sum up to one.

Next, we observe that for $i \geq 1$, $|\lambda_i| < 1$, all terms excluding the first on the right-hand side of (2.3) vanish, leaving only the term $a_0 x_0 = [a_0, a_0, \dots a_0]$. This means that in the limit, all points in the invariant neighborhood approach a_0, i.e. a_0 is the value of the limit subdivision curve at the center of the invariant neighborhood.

Tangent vectors. If we further assume that $|\lambda_1| > |\lambda_2|$ and λ_2 is real and positive, consideration of the first two dominant terms in (2.3) makes it possible to compute the tangent to the curve under some additional conditions on the subdivision scheme, which will be considered in Section 5 for surfaces. Consider the vector of differences $S^j p - a_0 x_0$ between all points in the invariant neighborhood at level j and the center of the invariant neighborhood. if we scale this vector by $1/\lambda_1$, it converges to $a_1 x_1 = [a_1 x_1^1, a_1 x_1^2, \dots a_1 x_1^n]$, i.e. all limit difference vectors are collinear and parallel to a_1. This suggestS (but does not guarantee without additional assumptions, which hold for most common schemes) that $a_1 = (l_1 \cdot p)$ is a tangent vector to the curve.

The observations above show the left eigenvectors, corresponding to the eigenvalue 1 and the second largest eigenvalue λ_1, play a special role, defining the limit positions and tangents for a subdivision curve.

Example. The eigenvalues and eignevectors of the subdivision matrix for cubic splines are

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

$$(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 0 & -\frac{1}{11} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{11} & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The left eigenvectors of eigenvalue 1 and subdominant eigenvalue $1/2$ are $[0, 1/6, 2/3, 1/6, 0]$ and $[0, -1, 0, 1, 0]$, which yield the formulas for the curve point and tangent

$$a_0 = \frac{1}{6}(p_{i-1} + 4p_i + p_{i+1}); \quad a_1 = p_{i+1} - p_{i-1},$$

which coincide with the formulas obtained by direct evaluation of cubic B-spline curves.

2.2 *Subdivision surfaces*

Most of the concepts we have introduced for subdivision curves can be extended to surfaces, but significant differences exist. While the control points for a curve have a natural ordering, this is no longer true for arbitrary meshes. Furthermore, for an arbitrary mesh, local mesh structure may vary: e.g. a vertex can share an edge with an arbitrary number of neighbors, rather than only one or two, as is the case for a curve and the polygonal faces of the mesh which may have different numbers of sides. A finer mesh can be obtained from a given coarser mesh in many different ways. Thus, in the case of subdivision for meshes, one needs to define *refinement rules*, which specify how the connectivity of the mesh is changed when it is refined, and *geometric rules*, which specify the way the control point positions are computed for the refined mesh.

Another important difference is that while the curves can always be considered to be functions on a domain in $\mathbf{R}$, there is no simple natural domain for surfaces. To be able to define subdivision surfaces as a limit of refinement, we need to construct a suitable domain out of the control mesh of the surface. We start with a specific example, the Loop subdivision scheme, to motivate the formal constructions we need to introduce.

Refinement of triangular manifold meshes. This scheme uses *triangular manifold control meshes*. Such control mesh consists of a *complex* K, which is a triple (V, E, F) of sets of vertices, edges and faces, and control points p^0, associated with each vertex in V. We use notation $p^j(v)$ for a control point at refinement level j associated with vertex v. The sets of vertices, edges and faces satisfy the following constraints:

- each edge is a pair of distinct vertices;
- each face is a set of three distinct vertices;
- each pair of vertices of a face is an edge;
- the intersection of two faces is either empty or an edge;
- each edge belongs to exactly two faces;
- the link of a vertex v (the set of edges of all faces containing v, excluding the edges that contain v themselves) can be ordered cyclically such that each two sequential edges share a vertex.

Two complexes are *isomorphic* if between their vertices there is a one-to-one map, which maps faces to faces and edges to edges.

Similarly to the curve case, we define neighborhoods on meshes. A 1-neighborhood $N_1(v, K)$ of a vertex v is a set of faces, consisting of all triangles, containing v. A 1-neighborhood $N_1(G, K)$ of a set of faces G consists of all triangles of 1-neighborhoods of the vertices of G. An m-neighborhood $N_m(v, K)$ is defined recursively as 1-neighborhood of $m - 1$ neighborhood.

The most common refinement rule for such meshes is *face quadrisection*. The new mesh is formed as follows: all old vertices are retained; a new vertex is added for each edge, splitting it into two; each edge is replaced by two new edges and each face by four new faces. One can easily see that all new vertices inserted using this refinement rule have valence 6, and only the vertices of the original mesh may have a different valence. The vertices of valence 6 are called *regular*, and the vertices of other valences are called *extraordinary*.

The Loop subdivision scheme. To define how the control points are computed, we need to specify rules for updating the positions of existing control points and for computing newly inserted control points.

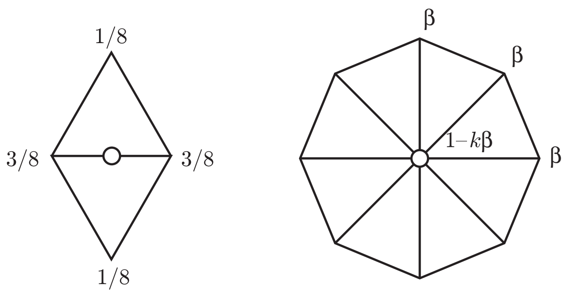

Fig. 3. Loop subdivision masks for new control points and updated positions of old control points. Vertices, for which control points are computed, are marked with circles.

These rules for the Loop subdivision scheme are shown in Figure 3. The rule for a vertex, inserted on an edge e, uses the control points for two triangles sharing e:

$$p^{j+1}(w) = \frac{3}{8}p^j(v_1) + \frac{3}{8}p^j(v_2) + \frac{1}{8}p^j(v_3) + \frac{1}{8}p^j(v_4),$$

where v_1, v_2 are edge endpoints, and v_3 and v_4 are the two remaining vertices of triangles sharing e.

The rule for updating positions of existing vertices is actually a parametric family of rules, with coefficients depending on the valence k of the vertex.

$$p^{j+1}(v) = (1 - k\beta) + \beta \sum_{v_i \in N_1(v)} p^j(v)$$

where β can be taken to be $3/8k$, for $k > 3$, and $\beta = 1/16$ for $k = 3$ (this is the simplest choice of β different choices of β are possible).

If the mesh is fully regular, i.e. all vertices have valence 6, these rules reduce to the subdivision rules for quartic box splines and can be derived from scaling relations similar to (2.2).

We note that these rules only depend on the local structure of the mesh, using only points within a fixed-size neighborhood of the point being computed: if we measure the neighborhood size in the refined mesh, both types of rules use level j control points, corresponding to vertices within the 2-neighborhood at level $j+1$; this is the analog of finite support in the curve case.

Furthermore, we observe that the rules depend only on the mesh structure of the 1-neighborhood of the vertex (specifically, the number of adjacent vertices), not on the subdivision level, or vertex numbering. This is the analog of being stationary in the curve case. We will give a more precise definition below.

To reason about convergence of this scheme, we also need to define the piecewise linear interpolants similar to $L[p^j]$, defined for curves. Unfortunately, there is no natural way to map the vertices of an arbitrary mesh to points in the plane or some other standard domain, so one cannot use a similar simple construction. For mesh subdivision to be able to define the limit surfaces rigorously, we need to construct special domains for each complex; subdivision surfaces are defined as functions on these domains.

Domains for subdivision surfaces. The simplest construction of the domain for the subdivision surface requires an additional assumption. For triangular meshes, the control points p^0 in $\mathbf{R}^n$ can be used to define an geometric realization of a complex. Each face of K (i.e. a triple of vertices (u, v, w)) corresponds to the triangle in $\mathbf{R}^n$, defined by three control points $(p^0(u), p^0(v), p^0(w))$. We additionally require that no two control points coincide, and for any two triangles in $\mathbf{R}^n$, corresponding to faces of K, their intersection is either a control point, a triangle edge, empty, or, informally, the initial control mesh has no self-intersections. With this additional assumption, one can use the initial mesh as the domain on which the linear interpolants of control points at different levels of refinement are defined. We denote this domain $|K^0|$.

The initial control points p^0 are already associated with the points in the domain (the control points themselves). It remains to associate the control points on finer levels with points on the initial mesh. This can be done recursively. Suppose a vertex w of the refined complex K^{j+1} is inserted on the edge connecting vertices u and v of K^j. Suppose these vertices are already associated with points $t(u)$ and $t(v)$ on $|K^0|$, contained in the same triangle T of $|K^0|$. Then we associate w with the midpoint $(1/2)(t(u) + t(v))$, which, by convexity, is also contained in the same triangle T. It is easy to show that no two vertices can be assigned to the same point in the domain: the points obtained after j refinement steps form a regular grid on each triangle of $|K^0|$.

Now we can define the piecewise linear interpolants, similar to the ones used for curves. Fix a refinement level j and a triangle T of $|K^0|$. The vertices of K^j form a regular grid on T, with triangles corresponding to faces of K^j. For points of $|K^0|$ inside each subtriangle (u, v, w) of T, we define $L[p^j]$ to be the linear interpolant between $p^j(u)$, $p^j(v)$ and $p^j(w)$.

In this way, we obtain a sequence of functions $L[p^j]$ defined on $|K^0|$; the limit subdivision surface is a the pointwise limit of this sequence, if it exists. Thus the subdivision surface is defined as a function on $|K^0|$ with values in $\mathbf{R}^n$.

Stationary subdivision in 2D. The Loop subdivision scheme is an example of a stationary subdivision scheme. More generally, for any complex K and its refinements K^j, $K^0 = K^j$, a linear subdivision scheme gives a sequence of linear operators $S^j(K)$, mapping control points for vertices V^j to control points for vertices V^{j+1}. This means that for a given vertex w of K^{j+1},

$$p^{j+1}(w) = \sum_{v \in V} a_{vw} p^j(v) \tag{2.4}$$

We say that a scheme is *finitely supported* if there is an M, such that for any w and $v \notin N_M(w, K^{j+1})$, $a_{vw} = 0$. The *support* $\operatorname{supp} w$ of the mask of the scheme at w is the minimal subcomplex containing all vertices v of K^j such that $a_{vw} \neq 0$. We say that the scheme is stationary or invariant, with respect to isomorphisms, if the coefficients a_{vw} coincide for vertices, for which supports are isomorphic. More precisely, if there is an isomorphism $\iota : \operatorname{supp} w_1 \to \operatorname{supp} w_2$, and $\iota(w_1) = w_2$ then $a_{\iota(v)w_2} = a_{vw_1}$. The invariance can be also defined with respect to a restricted set of isomorphisms, e.g. if the mesh is tagged.

Subdivision matrices in 2D. The definition of invariant neighborhoods and the construction of subdivision matrices for subdivision on meshes is completely analogous to the curve case. However, the size of the matrix is variable and depends on the number of points in the invariant neighborhood. Another difference is related to the fact that invariant neighborhoods may not exist for a finite number of initial subdivision levels, as the mesh structure changes with each refinement.

For a given neighborhood size m, however, after a sufficient number of subdivision steps, each extraordinary vertex v is surrounded by sufficiently many layers of regular vertices, and m-neighborhoods of v on different subdivision levels are similar.

For example, for the Loop scheme, the invariant neighborhood size is 2. For a vertex of valence k, it contains $3k+1$ vertices. The subdivision matrix has the following general form:

$$\left(\begin{array}{c|ccc} 1-k\beta & a_{01}^T & 0 & 0 \\ a_{10} & A_{11} & 0 & 0 \\ a_{20} & A_{21} & A_{22} & 0 \\ a_{30} & A_{32} & A_{32} & A_{33} \end{array}\right),$$

where all vectors a_{ij} are of length k and have constant elements ($a_{01} = \beta\mathbf{1}$, $a_{02} = (3/8)\mathbf{1}$, $a_{02} = (1/8)\mathbf{1}$, $a_{03} = (1/16)\mathbf{1}$. The blocks A_{ij} are cyclic $k \times k$, defined as follows,

$$\begin{aligned} &A_{11} = \frac{1}{8}\text{Cyclic}(3, 1, 0, \ldots 0, 1), \\ &A_{21} = \frac{1}{8}\text{Cyclic}(3, 3, 0 \ldots 0), \quad A_{22} = \frac{1}{8}\text{Cyclic}(1, 0, \ldots 0), \\ &A_{31} = \frac{1}{16}\text{Cyclic}(10, 1, 0, \ldots 0, 1), \quad A_{32} = \frac{1}{16}\text{Cyclic}(1, 0, \ldots 0, 1), \\ &A_{33} = \frac{1}{16}\text{Cyclic}(1, 0, \ldots 0). \end{aligned}$$

Limit positions and tangent vectors in 2D. The computation of the limit positions for mesh subdivision scheme is the same as for curves: one needs to compute the dot product of the left eigenvector of eigenvalue 1 with the vector of control points in the invariant neighborhood.

The computation of tangent vectors is slightly different. Instead of a unique tangent vector, a smooth subdivision surface has at least two nonuniquely defined independent tangent vectors spanning the tangent plane. In the case of surfaces, we further assume that the eigenvalues of the subdivision matrix satisfy $1 = |\lambda_0| > |\lambda_1| \geq |\lambda_2| > |\lambda_3|$ and $\lambda_{1,2}$ are real. This is not necessary for tangent plane continuity, but this assumption commonly holds and greatly simplifies the exposition. In this case, again under some additional assumptions to be discussed in Section 5, one can compute the tangent vectors to the surface using right eigenvectors l_1 and l_2, corresponding to the eigenvalues λ_1 and λ_2.

For the Loop scheme, the masks for limit positions and tangent vectors are quite simple: both have supports in the 1-neighborhood of a vertex. The coefficients of the mask for the limit position, i.e. the entries of the left eigenvector l_0, have the same form as the vertex rule, with β replaced with $\beta_{limit} = 8\beta/(3+8k\beta)$. The two tangent masks l_1 and l_2 can be chosen to be $\cos 2\pi j/k$ and $\sin 2\pi j/k$ for the vertices of 1-neighborhood distinct from the center indexed by j. The coefficient for the center itself is 0. This choice is not unique: for the Loop scheme and most other commonly used schemes, $\lambda_1 = \lambda_2$, and any linear combination $c_1 l_1 + c_2 l_2$ is also a left eigenvector.

3 Overview of Subdivision Schemes

In this section we review a number of stationary subdivision schemes generating C^1-continuous surfaces on arbitrary meshes. Our discussion is not exhaustive even for stationary schemes. We discuss two most common schemes (Loop and Catmull-Clark) and their variations in considerable detail, and briefly several examples of other types of schemes; more detailed information on other schemes can be found in provided references.

3.1 *Classification of subdivision schemes*

Refinement rules. The variety of stationary subdivision schemes for surfaces is primarily due to the many possible ways to define refinement of complexes. Several classifications of refinement rules (e.g. [28,1,25]) were proposed; our discussion mostly follows [28].

Almost all refinement rules are extensions of refinement rules for periodic tilings of the plane. The principal reason is there is an extensive theory for analysis of subdivision on regular planar grids which can be used to analyze the surface, constructed from an arbitrary mesh everywhere excluding a set of isolated points.

A single refinement step typically maps a tiling to a finer tiling, which is obtained by scaling and optionally rotating the original tiling; however, some schemes may alternate between different tiling types.

All known schemes with one exception are based on refinements of *regular monohedral tilings*, for which all tiles are regular polygons. The 4-8 scheme [81,80], originally formulated using a tiling with right triangles, can be reformulated using regular quad tilings, i.e. it also fits into this category. There are only three regular tilings: triangular, quadrilateral and hexagonal. Hexagonal tilings are rarely used, and stationary schemes for such tilings were considered in detail only recently [15,86,58].

Once the tiling is fixed, there are still many ways to define how it is refined, even if we require that the refined tiling is of the same type. Dodgson [16] lists a set of heuristics that are typically used to limit the variety of possible refinement rules. Here we briefly review these heuristics and their motivation.

1 Refinement of regular tilings is used. While other tiling types, such as periodic (e.g. Laves or Archimedes tilings) or aperiodic (Penrose tilings) can be considered, all schemes proposed so far meet this requirement.

2/3 A refinement rule either maps all vertices of the original tiling to the vertices of the refined tiling, or it maps them to the face centers of the refined tiling. Again, it is possible to consider other types of rules, but all known schemes are in one of these categories.

4 If a point is a center of rotational symmetry of order k in the tiling (i.e. the rotations by $2\pi j/k$ around this vertex map the tiling to itself), then in the refined tiling, it should be a center of rotational symmetry of at least the same order. If this requirement is not satisfied, one can show that the result of refinement depends on the way the vertices of a tiling are enumerated. Given the first 3 heuristics, this heuristic excludes refinement rules, mapping triangle vertices to centers, and hexagon centers to vertices.

5 For some number s, s times refined tiling is aligned with the original tiling, i.e. is obtained by uniform scaling. This is also justified by symmetry considerations, although, as pointed out in [16] is not strictly necessary. However, all schemes satisfying heuristic 7 in the stronger form that we use also satisfy this heuristic.

6 Triangle and quadrilateral schemes are generally useful but hexahedral schemes are more limited in their applications. One reason is that hexahedral tiling does not contain any multiple-edge straight lines, which can be used for meshes with boundaries and features.

7 Low *arity* (the ratio of the edge length of the refined tiling to the original tiling) is preferable. According to [16] arities higher than four are not likely to be useful. All practical and most known schemes, with exception of three recently proposed schemes, have arity two or less. As schemes of high arities result in very rapid decrease in the edge length, which is often undesirable, it is likely that only schemes of arity two or less will be used in applications.

These heuristics reduce the number of possible refinement rules to just six: four for quadrilateral tilings and two for triangle tilings (Figure 4).

We note that classifications, based on considering various possible transformations of tilings, do not yield an immediate recipe for refinement purely in terms of mesh connectivity; generalization to arbitrary connectivity meshes is not automatic either.

The remaining six refinement rules are uniquely identified by three parame-

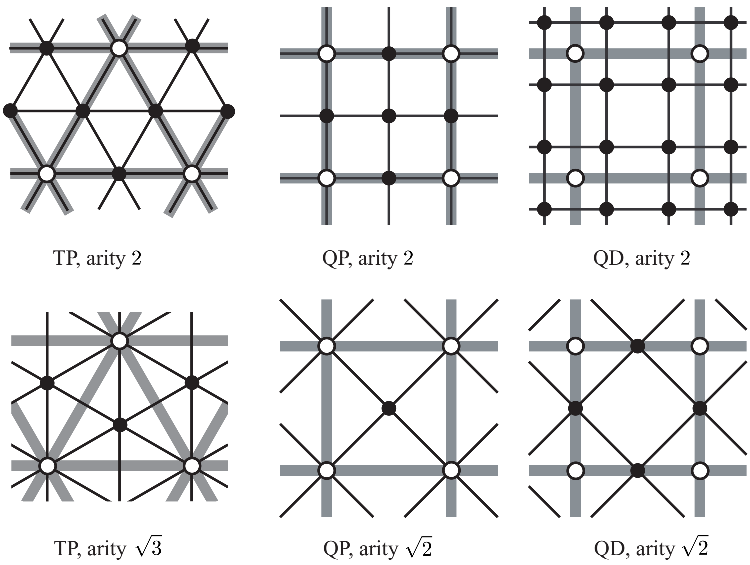

Fig. 4. Different refinement rules.

ters:

Tiling. The tile can be triangle or quadrilateral.
Vertex mapping. Vertices are mapped to vertices (*primal*) or vertices are mapped to faces (*dual*). Dual triangle refinement are excluded by heuristics 4.
Arity. For triangle tilings can be 2 or $\sqrt{3}$; for quad meshes can be 2 or $\sqrt{2}$.

Each of the six refinement rules can be easily formulated in terms of mesh connectivity in such a way that the refinement can be applied to an arbitrary polygonal mesh. For ease of understanding, we provide a somewhat informal description. We only specify the set of new vertices and edges, with faces defined implicitly as loops of edges. For primal rules, old vertices are retained, and old edges are discarded. For dual rules, both old vertices and edges are discarded. For each rule, we list how many different types of geometric rules are necessary to construct a subdivision scheme for meshes without boundaries. To handle meshes with boundaries, additional special rules for boundary vertices are necessary.

While triangle-based refinement rules can be applied to any mesh, known geometric rules for such schemes are only formulated for triangle meshes.

Primal triangle rule (TP) of arity 2. This is the rule considered in Section 2: create new vertices for each old edge and split each old edge in two; for each old face onnect new vertices inserted on edges of this face sequentially. Two geometric rules are necessary: one to update control points for old vertices (*vertex rule*) and another to compute positions of new control points (*edge rule*).

Primal triangle (TP) rule of arity $\sqrt{3}$. Create a new vertex for each face; connect old vertices with new vertices for each old face containing the old vertex; connect new vertices for adjacent old faces. Two similar geometric rules (vertex and edge) are needed.

Primal quad rule (QP) of arity 2. Create new vertices for each old edge and face; split old edges in two; for each old face, connect corresponding new vertex with new vertices inserted on edges. Three geometric rules are necessary: one for old vertices (*vertex rule*), one for new vertices corresponding to edges (*edge rule*), and one for new vertices corresponding to faces (*face rule*).

Primal quad rule (QP) of arity $\sqrt{2}$. Create a new vertex for each face; connect old vertices to new vertices for all adjacent faces. Two geometric rules are necessary, similar to the TP rules, the edge rule, and the face rule.

Dual quad rule (QD) of arity 2. For every face, create new vertices for every corner of the face and connect them into a face; connect new vertices corresponding to the same old vertex from adjacent faces. Only one geometric rule is necessary.

Dual quad rule (QD) of arity $\sqrt{2}$. Add a new vertex for each edge; for each face, connect new vertices on edges sequentially. Only one geometric rule is necessary.

The general property of the triangle rules is that it does not increase the number of non-triangular faces in the mesh. The general property of the quad rules is that they do not increase the number of non-quadrilateral faces. Moreover, both primal quad rules and the $\sqrt{3}$ triangle rule make all faces of a mesh triangular after one refinement step.

Classification. For each refinement rule type, there may be many different subdivision schemes depending on the choice of geometric rules. The geometric rules can be further classified by two characteristics: whether they are *approximating* or *interpolating*, and by their support size. Interpolating schemes do not alter the control points at vertices, inherited from the previous refinement level; approximating schemes do. The distinction between approximating and interpolating for schemes with arity no greater than two makes sense only for primal schemes.

With this criteria in place, we can classify most known schemes; in most cases, only one scheme of a given type is known. The reason for this is that only schemes with small support are practical, and additional symmetry considerations considerably reduce the number of degrees of freedom in coefficients. Maximizing smoothness of resulting surfaces on regular grids further restricts the choices, in most cases yielding a known parametric family of schemes.

The table below lists all schemes known to fit into our classification.

Refinement type	Approximating	Interpolating
TP, arity 2	Loop [46,26,9,47,63]	Butterfly [22,92]
TP, arity $\sqrt{3}$	$\sqrt{3}$,[34], composite $\sqrt{3}$ [58]	interpolatory $\sqrt{3}$,[38]
QP, arity 2	Catmull-Clark [11] iterated [91,77]	Kobbelt [32]
QP, arity $\sqrt{2}$	4-8 [81,80]	interpolating $\sqrt{2}$ [27]
QD, arity 2	Doo-Sabin [17,18], iterated [91]	—
QD, arity $\sqrt{2}$	Midedge [61,24]	—

Polygonal meshes with boundaries. The minimal number of geometric rules, ranging from one to three, is sufficient if we require the rules to be invariant with respect to isomorphisms of mask supports and assume the meshes do not have boundaries.

However, in practice it is not sufficient to consider only this class of meshes: in any practical application, the control mesh may have a boundary. Furthermore, the boundary may not be smooth everywhere: it may consist of several smooth pieces, jointed at *corners*. The definition of meshes with boundary is identical to the polygonal mesh definition in Section 2; the only differences are that an edge can be contained only in one face, and the link of a vertex is a chain of edges, with last vertex not connected to the first.

While a boundary edge or vertex is identified unambiguously, corner vertices on the boundary require tags.It turns out that depending on the type of corner (convex or concave); different rules need to be used, so at least two different tags are needed.

We have already seen that subdivision schemes defined on triangular meshes create new vertices only of valence 6 in the interior. On the boundary, the newly created vertices have valence 4. Similarly, on quadrilateral meshes both primal and dual schemes create only vertices of valence 4 in the interior and 3 on the boundary. Hence, after several subdivision steps, most vertices in a mesh will have one of these valences (6 in the interior, 4 on the boundary for triangular meshes, 4 in the interior, 3 on the boundary for quadrilateral). The vertices with these valences are called *regular*, and vertices of other valences are called *extraordinary*. Similarly, faces with 3 and 4 vertices are called regular for triangle and quadrilateral schemes respectively, and faces with a different number of vertices are called extraordinary.

Next, we consider several examples of subdivision schemes. We start with a detailed description of two schemes that are used in most applications: Loop and Catmull-Clark, which use TP and QP refinement rules of arity 2. Then we consider examples of interpolating schemes (Butterfly), dual schemes (Doo-Sabin) and non-arity 2 schemes (Midedge and 4-8 subdivision).

3.2 *Loop Scheme*

The Loop scheme for meshes without boundary was already described in Section 2. The scheme is based on the *three-directional box spline*, which produces C^2-continuous surfaces on the regular meshes. The Loop scheme produces surfaces which are C^2-continuous everywhere except at extraordinary vertices, where they are C^1-continuous. C^1-continuity of this scheme for valences up to 100, including the boundary case, was proved by Schweitzer [74]. The proof for all valences can be found in [89]. In addition to already defined rules for interior vertices, it remains to specify rules for vertices on or near the boundary. The rules we define here were proposed in [9].

A common requirement for rules for boundary vertices is that the control points on level $j + 1$ should only depend on boundary control points on level j. In the case of the Loop scheme, for compatibility with the regular case, we use the standard cubic spline rules, both for edge points and vertex points (Figure 5). If a vertex v is tagged as a corner vertex, a trivial interpolating rule is used: $p^{j+1}(v) = p^j(v)$.

Adding these rules formally completes the definition of the scheme for all possible cases; unfortunately, this set of rules is insufficient to produce limit surfaces which are C^1 continuous at the extraordinary boundary vertices or surfaces with concave corners on the boundary. To achieve this, spatial edge rules are applied at edge points *adjacent* to extraordinary boundary vertices.

For edge points, our algorithm consists of two stages, which, if desired, can be merged, but are conceptually easier to understand separately.

The first stage is a single iteration over the mesh during which we apply the vertex rules and compute initial control points for vertices inserted on edges. All rules used at this stage are shown in Figures 3 and Figure 5. The mask support is the same, but the coefficients are modified. The change in coefficient ensures that the surface is C^1 for boundary vertices. However, the scheme still cannot produce concave corners: The surface develops a “flip” at these vertices; the reason for this, informally, is that the invariant configuration defined by subdominant eigenvectors of subdivision matrix in this case does not have a concave corner; rather, it has a convex one.

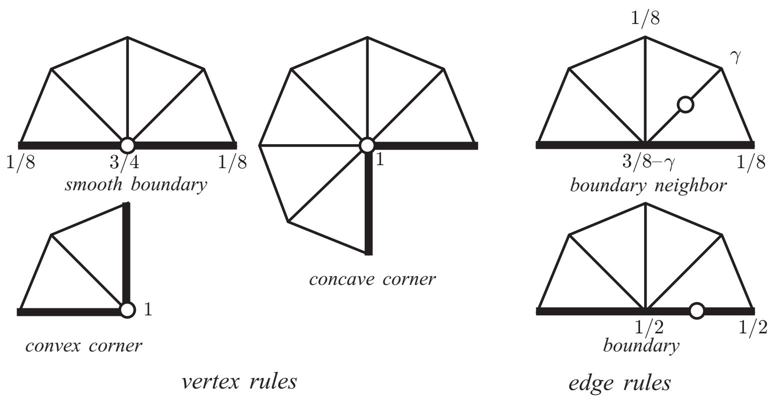

Fig. 5.

The γ is given in terms of parameter θ_k, defined differently for corner and boundary vertices:

$$\gamma\left(\theta_k\right) = 1/2 - 1/4\cos\theta_k$$

For boundary vertices v not tagged as corners, we use $\theta_k = \pi/k$, where k is the number of polygons adjacent to v. For a vertex v tagged as a convex corner, we use $\theta_k = \alpha/k$, where $\alpha < \pi$, and for concave corner we choose $\alpha > \pi$. The parameter α can be either fixed (e.g. $\pi/2$ for convex and $3\pi/2$ for concave) or can be chosen depending on the angle between the vectors from $p^0(v)$ to adjacent boundary control points adjacent to v.

To ensure the correct behavior at the concave corner vertices, an additional step *flatness modification* is required which is defined as follows.

Flatness modification. To avoid the flip problem described above, one needs to ensure that the eigenvalues corresponding to a pair of “correct” eigenvectors, forming a concave corner, are subdominant. The following simple technique proposed in [9] achieves this. We introduce a *flatness parameter* s and modify the subdivision rule to scale all eigenvalues except λ_0 and $\lambda = \lambda_1 = \lambda_2$, corresponding to the desired eigenvectors, by factor $1-s$. The vector of control points p after subdivision in a neighborhood of a point is modified as follows:

$$p^{\text{new}} = (1-s)\,p + s\left(\mathbf{a}_0 x^0 + \mathbf{a}_1 x^1 + \mathbf{a}_2 x^2\right),$$

where, as before, $\mathbf{a}_i = (l^i \cdot p)$, and $0 \leq s \leq 1$. Geometrically, the modified rule blends between control point positions before the flatness modification and certain points in the tangent plane, which are typically close to the projection of the original control point. The limit position $\mathbf{a}_0$ of the center vertex remains unchanged.

The flatness modification is always applied at concave corner vertices; the default values for the flatness parameter is $s = 1 - (1/4)/\lambda_3$, where $\lambda_3 = (1/4)(\cos\pi/k) - \cos(\theta_k)) + 1/2$ (the largest eigenvalue $\neq 1$ of the subdivision matrix before the modification). The modification ensures that the surface is C^1 in this case. In other cases, s can be taken to be 0 by default.

The formulas for limit positions and tangents for all possible cases can be found in [9].

3.3 *Catmull-Clark scheme*

The Catmull-Clark scheme [11] probably is the most widely used subdivision scheme. One of the reasons is it extends tensor-product bicubic B-spline surfaces, the most commonly used type of spline surfaces. This scheme uses the QP refinment rule with arity 2. It produces surfaces that are C^2 everywhere, except at extraordinary vertices, where they are C^1. The tangent plane continuity of the scheme was analyzed in [6], and C^1-continuity in [62].

The masks are shown in Figure 6; for interior vertices, there are three types of masks: for new vertices inserted at edges and faces and for update of control points at old vertices.

If $k = 4$, the masks reduce to subdivision masks for bicubic B-splines. Similar to the Loop scheme, cubic spline rules are applied at the boundary, and at the corner boundary vertices, the trivial interpolating rule is used. Again, just as is the case for the Loop scheme, the minimal set of rules results in surfaces which lack smoothness at extraordinary boundary vertices. A similar technique is used for Catmull-Clark, with parameter γ computed as

$$\gamma\,(\theta_k) = 3/8 - 1/4\cos\theta_k.$$

The parameter θ_k is defined exactly in the same way as for the Loop scheme.

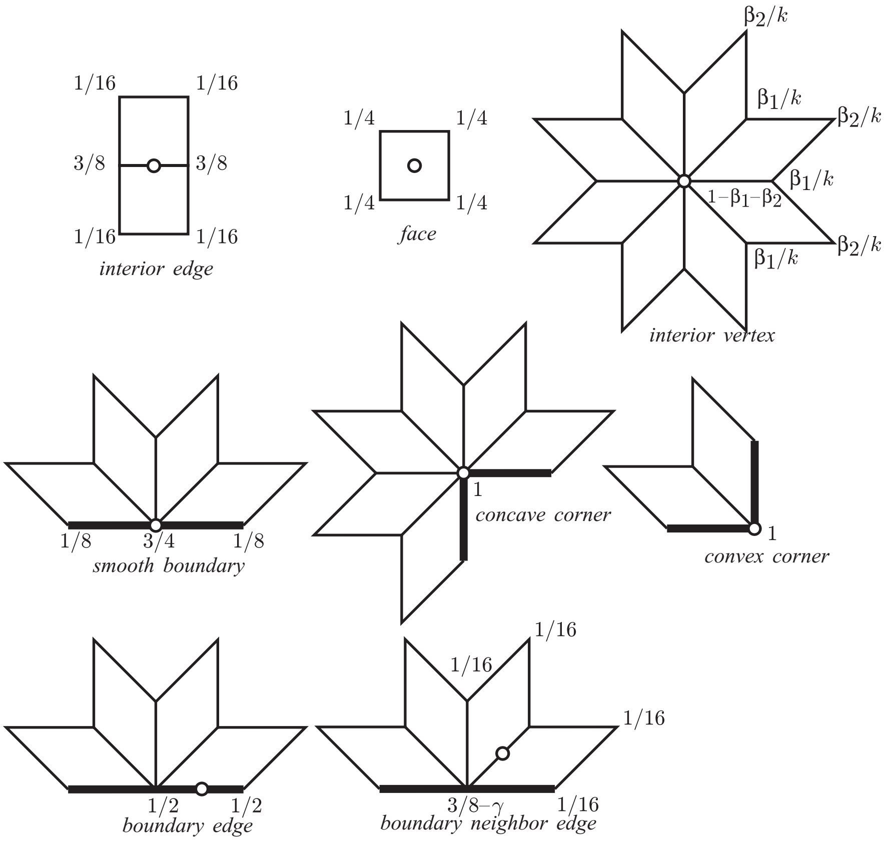

Fig. 6. Catmull-Clark subdivision. Catmull and Clark suggest the following coefficients for rules at extraordinary vertices: $\beta_1 = \frac{3}{2k}$ and $\beta_2 = \frac{1}{4k}$

Finally, a similar extra step is used to ensure correct behaviour at concave corners:

$$p^{\text{new}} = (1-s)\,p + s\left(\mathbf{a}_0 x^0 + \mathbf{a}_1 x^1 + \mathbf{a}_2 x^2\right).$$

The limit position and tangent vector coefficients are listed in [9].

The geometric rules of the Catmull-Clark scheme are defined above for meshes with quadrilateral faces. Arbitrary polygonal meshes can be reduced to a quadrilateral mesh using a more general form of Catmull-Clark rules [11]:

- a face control point for an n-gon is computed as the average of the corners of the polygon;

- an edge control point is the average of the endpoints of the edge and newly computed face control points of adjacent faces;
- the vertex rule can be chosen in different ways; the original formula is

$$p^{j+1}(v) = \frac{k-2}{k} p^j(v) + \frac{1}{k^2} \sum_{i=0}^{k-1} p^j(v_i) + \frac{1}{k^2} \sum_{i=0}^{k-1} p^{j+1}(v_i^f)$$

where v_i are the vertices adjacent to v on level j, and v_i^f are face vertices on level $j+1$ corresponding to faces adjacent to v.

4 Modified Butterfly Scheme

The Butterfly scheme was proposed in [22]. Although the original Butterfly scheme is defined for arbitrary triangular meshes, the limit surface is not C^1-continuous at extraordinary points of valence $k = 3$ and $k > 7$ [89]. The scheme is C^1 on regular meshes.

Unlike approximating schemes based on splines, this scheme does not produce piecewise polynomial surfaces in the limit. In [92] a modification of the Butterfly scheme was proposed, which guarantees that the scheme produces C^1-continuous surfaces for arbitrary meshes as proved in [89]. The scheme is known to be C^1 but not C^2 on regular meshes. The masks for the the scheme are shown in Figure 7.

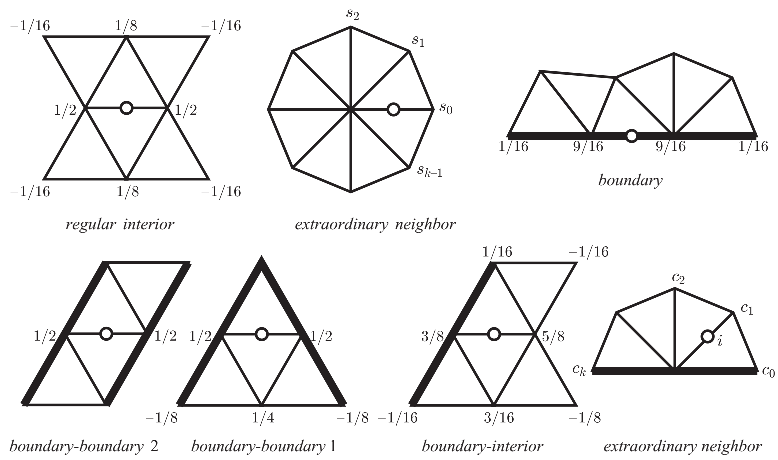

Fig. 7. Modified Butterfly subdivision. The coefficients s_i are $\frac{1}{k}\left(\frac{1}{4} + \cos\frac{2i\pi}{k} + \frac{1}{2}\cos\frac{4i\pi}{k}\right)$ for $k > 5$. For $k = 3$, $s_0 = \frac{5}{12}$, $s_{1,2} = -\frac{1}{12}$; for $k = 4$, $s_0 = \frac{3}{8}$, $s_2 = -\frac{1}{8}$, $s_{1,3} = 0$.

The tangent vectors at extraordinary interior vertices can be computed using the same rules as for the Loop scheme. For regular vertices, the formulas are more complex: in this case, we have to use control points in a 2-neighborhood of a vertex. The masks are shown in Figure 8.

Because the scheme is interpolating, no formulas are needed to compute the limit positions: all control points are on the surface. On the boundary, the four point subdivision scheme is used [21]. To achieve C^1-continuity on the boundary, special coefficients have to be used.

Boundary rules. The rules extending the Butterfly scheme to meshes with boundary are somewhat more complex, because the stencil of the Butterfly scheme is relatively large. A complete set of rules for a mesh with boundary (up to head-tail permutations), includes 7 types of rules: regular interior, extraordinary interior, regular interior-boundary, regular boundary-boundary 1, regular boundary-boundary 2, boundary, and extraordinary boundary neighbor; see Figures 7. To put it all into a system, the main cases can be classified by the types of head and tail vertices of the edge on which we add a new vertex. The following table shows how the type of rule to be applied for computing a *non-boundary* vertex is determined from the valence of the adjacent vertices, and whether they are on the boundary or not. The only case when additional information is necessary, is when both neighbors are regular crease vertices.

Head	**Tail**	**Rule**
regular interior	regular interior	standard rule
regular interior	regular crease	regular interior-crease
regular crease	regular crease	regular crease-crease 1 or 2
extraordinary interior	extraordinary interior	average two extraordinary rules
extraordinary interior	extraordinary crease	same
extraordinary crease	extraordinary crease	same
regular interior	extraordinary interior	interior extraordinary
regular interior	extraordinary crease	crease extraordinary
extraordinary interior	regular crease	interior extraordinary
regular crease	extraordinary crease	crease extraordinary

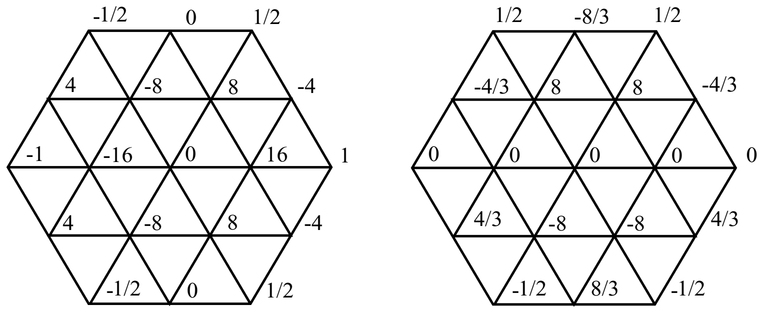

Fig. 8. Tangent masks for regular vertices (Butterfly scheme).

The extraordinary crease rule (Figure 7) uses coefficients c_{ij}, $j = 0 \dots k$, to compute the vertex number i in the ring, when counted from the boundary. Let $\theta_k = \pi/k$. The following formulas define c_{ij} :

$$c_0 = 1 - \frac{1}{k}\left(\frac{\sin\theta_k \sin i\theta_k}{1-\cos\theta_k}\right)$$

$$c_{i0} = -c_{ik} = \frac{1}{4}\cos i\theta_k - \frac{1}{4k}\left(\frac{\sin 2\theta_k \sin 2\theta_k i}{\cos\theta_k - \cos 2\theta_k}\right)$$

$$c_{ij} = \frac{1}{k}\left(\sin i\theta_k \sin j\theta_k + \frac{1}{2}\sin 2i\theta_k \sin 2j\theta_k\right)$$

4.1 *Doo-Sabin scheme*

The Doo-Sabin subdivision is quite simple conceptually: a single mask is sufficient to define the scheme. Special rules are required only for the boundaries, where the limit curve is a quadratic spline. It was observed by Doo that this can also be achieved by replicating the boundary edge, i.e., creating a quadrilateral with two coinciding pairs of vertices. Nasri [53] describes other ways of defining rules for boundaries. The rules for the Doo-Sabin scheme are shown in Figure 9. C^1-continuity for schemes similar to the Doo-Sabin schemes was analyzed in [62].

4.2 *Midedge scheme and other non-integer arity schemes*

A scheme described in [61] is an arity $\sqrt{2}$ QD scheme; two steps of refinement of this type result in Doo-Sabin type scheme.

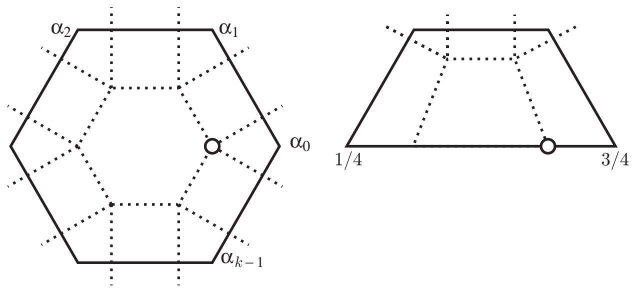

Fig. 9. The Doo-Sabin subdivision. The coefficients are defined by the formulas $\alpha_0 = 1/4 + 5/4k$ and $\alpha_i = (3 + 2\cos(2i\pi/k))/4k$, for $i = 1 \ldots k - 1$

The rules for the simplest version of this scheme are very straightforward: the point inserted on an edge is the average of the endpoints. While the limit surface is smooth for this rule, the quality of the surface is not good for extraordinary faces; the rules can be modified to improve surface quality.

An example of a QP scheme of arity $\sqrt{2}$ is the 4-8 scheme [81,80]. While originally defined in terms of 4-8 refinement, it can be easily reinterpreted in terms of regular quadrilateral grid refinement as shown in Figure 10.

It should be noted that for quadrilateral schemes of non-integer arity; there appears to be no natural treatment for the boundaries: as each quad for the refined mesh has vertices from two quads sharing an edge, it is impossible to construct quads in the same way on the boundary. One needs to introduce special refinement rules on the boundary and corresponding special geometric rules. A set of such rules is described in [81]. The rules are quite complex (six different rules are needed), in contrast to the rules for interior vertices.

On regular grids this scheme produces surfaces of high smoothness (C^4) despite its small support, but at extraordinary vertices, it is still only C^1.

The first TP scheme of arity $\sqrt{3}$ was described in [34]; other schemes were considered in [58].

4.3 *Comparison*

We conclude our survey of subdivision schemes with some comparisons. For sufficiently smooth and fine control meshes, the results for most common schemes are indistinguishable visually. We use relatively simple meshes to demonstrate the differences in clear form; for most meshes used in applications, the differences are less apparent. In our comparison, we consider Loop, Catmull-Clark, Modified Butterfly and Doo-Sabin subdivision.

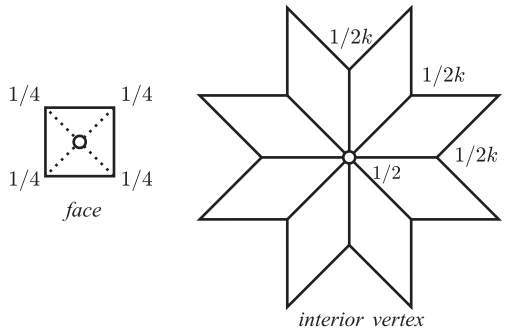

Fig. 10. The 4-8 subdivision scheme rules refinement. As the edges are not refined, only face and vertex rules are necessary.

Figure 11 shows the surfaces obtained by subdividing a cube. Loop and Catmull-Clark subdivision produce surfaces of higher visual quality, as these schemes reduce to C^2 splines on a regular mesh. As all faces of the cube are quads, Catmull-Clark yields the nicest surface; the surface generated by the Loop scheme is more asymmetric because the cube had to be triangulated before the scheme is applied. At the same time, Doo-Sabin and Modified Butterfly reproduce the shape of the cube more closely. The surface quality is worst for the Modified Butterfly scheme, which interpolates the original mesh. We observe that there is a tradeoff between interpolation and surface quality: the closer the surface is to interpolating, the lower the surface quality.

Figure 12 shows the results of subdividing a tetrahedron. Similar observations hold in this case. In addition, we observe extreme shrinking for the Loop and Catmull-Clark subdivision schemes.

Overall, Loop and Catmull-Clark appear to be the best choices for most applications, which do not require exact interpolation of the initial mesh. The Catmull-Clark scheme is most appropriate for meshes with a significant fraction of quadrilateral faces. It might not perform well on certain types of meshes, most notably triangular meshes obtained by triangulation of a quadrilateral mesh (see Figure 13). The Loop scheme performs reasonably well on any triangular mesh, thus, when triangulation is not objectionable, this scheme might be preferable.

More in-depth studies of subdivision surface behavior focusing on curvature can be found in [67,59,30]. Ways to improve surface appearance using coefficient tuning were explored in [7].

5 Smoothness of subdivision surfaces

In this section we review the theory of smoothness of surfaces generated using stationary subdivision. Smoothness is the focus of most of the work in theory of subdivision. The standard goal is to establish conditions on masks of subdivision schemes that ensure that the limit surfaces, for almost all configurations of control points, are in a smoothness class. Most commonly, the classes C^r, for integer values of r are considered.

In the regular case, powerful analysis tools exist. (see e.g. a recent survey [20] or the book [12] as well as [29] for further references). In most cases, subdivision schemes for surfaces are constructed by generalizing relatively simple schemes for regular grids, for which smoothness analysis is relatively straightforward.

Due to locality of subdivision rules, this ensures surfaces are smooth away

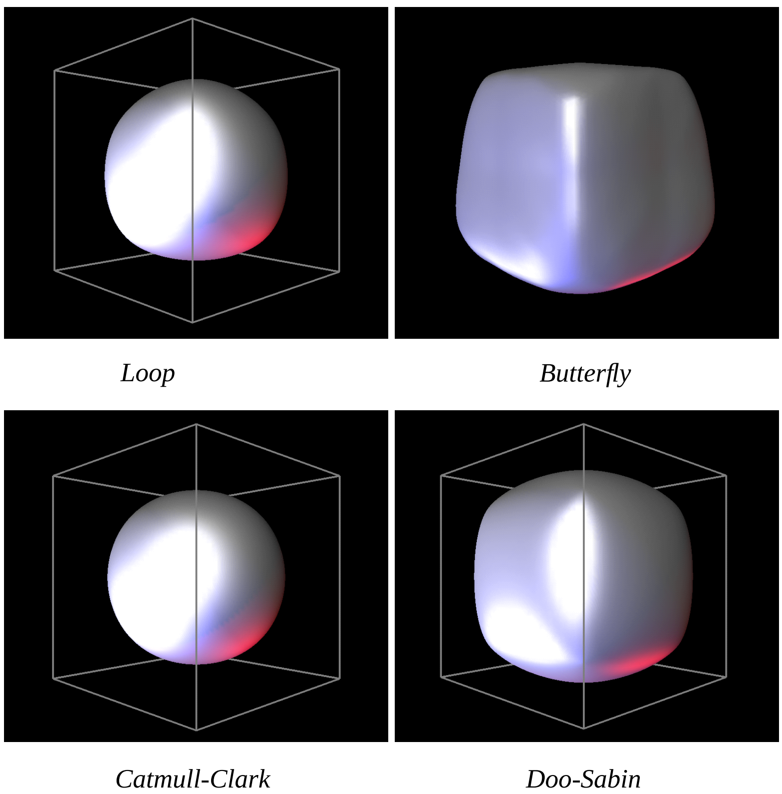

Fig. 11. Results of applying various subdivision schemes to the cube. For triangular schemes (Loop and Butterfly) the cube was triangulated first.

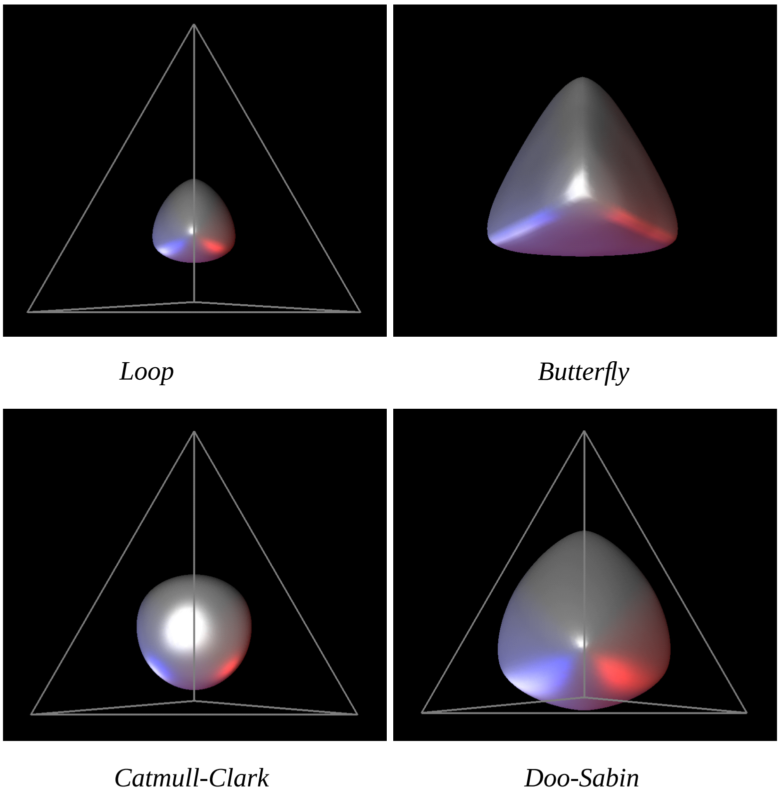

Fig. 12. Results of applying various subdivision schemes to a tetrahedron.

from isolated points, corresponding to vertices or face centers of the initial meshes. To complete the analysis for arbitrary meshes, one needs to analyze behaviour near such points; in this section we concentrate on this topic.

To be able to formulate the criteria for surface smoothness, we precisely define the limit subdivision surfaces and review tangent plane continuity and C^r-continuous surfaces.

5.1 C^r-continuity and tangent plane continuity

There are many different equivalent or nearly equivalent ways to define C^r-surfaces for integer r. A standard approach in differential geometry is to define C^r manifolds, and then define C^r surfaces in $\mathbf{R}^n$ as C^r-continuous *immersions* or *embeddings* of C^r manifolds. However, this approach is not the most convenient for our purposes, as no *a priori* smooth structure exists on the domain of

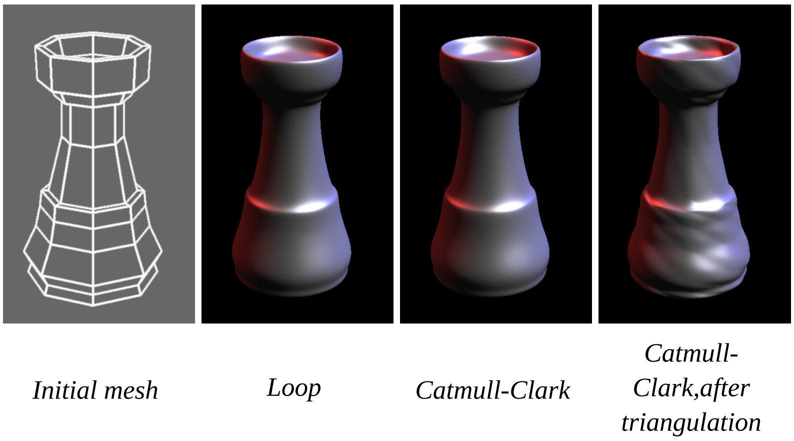

Fig. 13. Applying Loop and Catmull-Clark subdivision schemes to a model of a chess rook. The initial mesh is shown on the left. Before the Loop scheme was applied, the mesh was triangulated. Catmull-Clark was applied to the original quadrilateral model and to the triangulated model; note the substantial difference in surface quality.

subdivision surfaces. Thus, we take a somewhat different but equivalent approach. We do not require a smooth structure and say that a surface defined on a domain, for which only topological structure exists, is C^r if there is a C^r-continuous local reparameterization for a neighborhood of any point. More formally, we use the following definition.

Definition 5.1: A surface $f : M \to \mathbf{R}^n$, where M is a topological 2D manifold, is C^r**-continuous**, for $r \geq 1$, if for every point $x \in M$ there exists an open neighborhood U_x in M of x, and a regular parameterization $\pi : D \to f(U_x)$ of $f(U_x)$ over an open unit disk D in the plane, A **regular parameterization** π is one that is r-times continuously differentiable, one-to-one, and has a Jacobi matrix of maximum rank, i.e. if (s, t) is a choice of coordinates on D $\partial_s \pi$ and $\partial_t \pi$ for any choice of coordinates on D are independent.

We call a subdivision scheme C^r continuous if for any complex K and almost any choice of control points p for vertices of this complex, resulting limit surfaces are C^r-continuous. In practice, however, it is difficult to prove this for arbitrary complexes, and additional restrictions have to be imposed.

The condition that the Jacobi matrix of p has maximum rank is necessary to make sure that there no degeneracies, i.e., f represents a surface, not a curve or point.

In our constructions, it is useful to consider a weaker definition of surface smoothness at a point. This definition captures the intuitive idea that the tangent plane to a surface changes continuously, and is applicable only for an isolated point, i.e. we assume that the surface is C^r-continuous everywhere excluding a point. We first define a tangent plane continuous surface in $\mathbf{R}^3$. Note that if the surface is C^1-continuous in $\mathbf{R}^3$ in a neighborhood of a point, there is a well-defined normal at that point given for a choice of coordinates (s, t) by $\partial_s \pi \times \partial_t \pi$.

Definition 5.2: A surface $f : M \rightarrow \mathbf{R}^3$ is **tangent plane continuous** at $x \in M$ if and only if it is C^1-continuous in a neighborhood of x, and there exists a limit of normals at x.

An example of a surface which is tangent plane continuous but not C^1-continuous is $(x = s^2 - t^2, y = 2 * s * t, z = s^3)$.

We will also need the definition of tangent plane continuity in higher dimensions; for $n > 3$, the appropriate generalization of the cross product is the exterior (wedge) product, $\mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^{n(n-1)/2}$; for two vectors v, w, their product $v \wedge w$ has components $v_i w_j - v_j w_i$, $0 \leq i < j \leq n$. The exterior product is linear in each argument and antisymmetric ($v \wedge w = -w \wedge v$). From antisymmetry, it follows that $v \wedge v = 0$. For $n = 3$, the exterior product is identical to the cross product. The exterior product $v \wedge w$ defines a plane in n dimensions spanned by vectors v and w just as normal $v \times w$ defines the plane in 3D. In higher dimensions, the definition of tangent plane continuity is identical to 3D, with exterior product $\partial_s \wedge \partial_t$ considered instead of the normal.

The following fact can be easily proved: if a surface is tangent plane continuous at a point and the projection of the surface onto the tangent plane at that point is one-to-one for a neighborhood of the point, the surface is C^1.

The definition of tangent plane continuity for a subdivision scheme is similar to the definition of C^r-continuity.

5.2 *Universal surfaces*

We present an approach to establishing smoothness criteria for subdivision schemes described in [90]. We do not derive the necessary and sufficient conditions in full generality, as required algebraic machinery is relatively complicated and obscures the main ideas. Instead, we derive conditions similar to Reif's originally proposed sufficient condition [65]. We use the more general approach based on the universal surfaces over Reif's original derivation since in author's view it provides better geometric intuition for tangent plane continuity and C^1 continuity. Most statements are presented without proof. For more complete analysis and proofs, we refer the reader to [64,88,90].

It is intuitively clear that to verify that a subdivision scheme with finitely supported masks produces smooth surfaces for almost all configurations of control points, it is sufficient to consider behavior of a part of the surface on a 1-neighborhood of an extraordinary vertex v $|N_1(v)|$. We further assume that the control mesh for $|N_1(v)|$ contains a single extraordinary vertex and is an invariant neighborhood. This is, in fact, a limiting assumption; however, all known analysis techniques rely on this assumption, as verification of smoothness of subdivision schemes in a more general setting so far is not possible. This problem is discussed in greater detail in [90].

In this restricted setting, we can regard a regular k-gon U centered at zero in $\mathbf{R}^2$, as the domain of the patch of the subdivision surface in which we are interested. Let S be the subdivision matrix, and p^j vectors of control points for U at subdivision levels j, $p^{j+1} = Sp^j$. Let N be the number of points in p. An important observation following from the construction of the limit subdivision surface is that $p^1 = Sp$ is the vector of control points for the scaled domain $(1/2)U$, and in general, $p^j = S^j p^0$ is the vector of control points for $(1/2^j)U$; in other words, the limit function $f[p]$ evaluated on $(1/2)U$ satisfies

$$f[p](y/2) = f[S^j p](y) \tag{5.1}$$

Consider a basis $e_1, \ldots e_N$; then $p = \sum_i p_i e_i$. By linearity of subdivision, we can write $f[p] = f[\sum_i p_i e_i] = \sum_i p_i f[e_i]$. We introduce the map $\psi : U \to \mathbf{R}^N$, defined as $f = (f[e_1], f[e_2], \ldots f[e_N])$. This surface (*the universal surface*)defined by this map is defined uniquely up to a nonsingular linear transformation.

For any vector of control points p, we can regard the subdivision surface $f[p]$ as a linear map of the universal surface to three-dimensional space give by

$$f[p](y) = (p \cdot \psi(y))$$

It immediately follows from (5.1) that ψ satisfies

$$\psi(y/2) = S^T \psi(y) \tag{5.2}$$

Furthermore, we can verify by direct computation that the normal to the subdivision surface $f[p](y)$ at points y where it is f is differentiable can be computed as

$$\partial_1 f[p] \wedge \partial_2 f[p] = N(y) = ((p^y \wedge p^z) \cdot w(y), (p^z \wedge p^x) \cdot w(y), (p^x \wedge p^y) \cdot w(y)) \tag{5.3}$$

where $w(y) = \partial_1\psi(y) \wedge \partial_2\psi(y)$, i.e. the analog of the normal for the universal surface. We also note that f is differentiable everywhere on U except at edges of triangles of U. Furthermore, one-sided derivative limits exist at edges, excluding the center of U, i.e. zero. One can show that using one-sided limits of derivatives on either side of the edge yields the same vector $w(y)$, so it is defined everywhere.

This surface has the following important property.

Theorem 5.1: A subdivision scheme is tangent plane continuous (C^r) at vertices of a given valence if and only if the universal surface for this valence is tangent plane continuous, assuming that the universal surface is C^1-continuous away from zero. The universal surface is C^r-continuous if and only if the subdivision scheme is C^r continuous.

This theorem allows us to replace analysis of all possible surfaces generated using a subdivision scheme with analysis of a single surface in higher-dimensional space for each valence. The assumption of the theorem about C^1-continuity away from zero typically follows from the analysis of the regular case and *the characteristic map* as explained below.

To analyze whether the universal surface is tangent plane continuous, we need to look at the behavior of the vectors $w(y)$ (the generalized normals) as $y \to 0$. For any linear transform A, $Aw \wedge Av = (\Lambda A)(w \wedge v)$ defines a natural extension of A to the space of exterior products. Thus, taking derivatives and wedge products, we obtain

$$w(y/2) = \partial_1\psi(y/2) \wedge \partial_2\psi(y/2) = 4(\Lambda S^T)\partial_1\psi(y) \wedge \partial_2\psi(y) = 4(\Lambda S^T)w(y) \tag{5.4}$$

i.e. the vector $w(y)$ satisfies a scaling relation, but with a different matrix.

As a result, our problem is reduced to the following: under which conditions does the direction of $A^j w(y)$, where $A = 4\Lambda S^T$, converge to a unique limit for $j \to \infty$ and for any choice of y?

5.3 *Sufficient smoothness criteria*

So far, our discussion has been completely general. Without any assumptions on the matrix S, the conditions for convergence to a unique limit can be quite complex and require analysis of the Jordan normal form of the subdivision matrix. The main ideas can be easily understood if we consider the special case when S satisfies the conditions of Section 2: the matrix has a basis of eigenvectors, λ_1 and λ_2 are real positive, $\lambda_0 = 1 > \lambda_1 \geq \lambda_2 > |\lambda_3|$, if the eigenvalues are ordered by magnitude, and each is repeated once for each of its eigenvectors.

In the case of such matrices, the matrix ΛS also has a simple structure. First, we observe that if x_i and x_j are independent eigenvectors of S, with eigenvalues λ_i and λ_j, then $\Lambda S(x_i \wedge x_j) = Sx_i \wedge Sx_j = \lambda_i \lambda_j x_i \wedge x_j$, i.e. $x_i \wedge x_j$ is an eigenvector with eigenvalue $\lambda_i \lambda_j$. There are $N(N-1)/2$ such eigenvectors, and these eigenvectors are independent. We conclude that ΛS also has a complete system of eigenvectors, with eigenvalues equal to $\lambda_i \lambda_j$, with $i < j$.

This observation allows us to understand the behavior of $A^j w(y)$. Suppose $w(y) = \sum_i \alpha_i x_i$ where x_i are eigenvectors of A. Then, the direction of $A^j w(y)$ converges to the direction of x_i, where x_i is the the eigenvector with largest eigenvalue such that $\alpha_i \neq 0$.

We observe that we can define $\psi = \sum c_i f_i$, where $f_i = f[x_i]$ is the eigenbasis function corresponding to eigenvalue λ_i; in particular, by affine invariance, f_0 is a constant.

$$w(y) = \sum_{i<j} (c_i \wedge c_j)(\partial_1 f_i \partial_2 f_j - \partial_2 f_i \partial_1 f_j) = \sum_{i<j} (c_i \wedge c_j) J[f_i, f_j]$$

where $J[f_i, f_j]$ denotes the Jacobian of two functions.

We note that the terms corresponding to $i = 0$ vanish because f_0 is a constant. Thus, the largest eigenvalue which may have a nonzero term in this decomposition is $\lambda_1 \lambda_2$.

If we assume that for any y, $J[f_i, f_j](y) \neq 0$, we see that the limit direction of $A^j w(y) = w(y/2^j)$ is always $c_1 \wedge c_2$, i.e. all these sequences converge to the same limit. With more careful analysis, one can easily establish that the limit is the same for any sequence $w(y_j)$, with $y_j \to 0$.

We obtain the following *sufficient* condition for tangent plane continuity:

Theorem 5.2: Suppose for a valence k, the subdivision matrix is non-defective and has eigenvalues satisfying $\lambda_0 = 1 > \lambda_1 \geq \lambda_2 > |\lambda_3|$, when ordered in non-increasing order, each eigenvalue repeated according to its multiplicity. Suppose the eigenbasis functions corresponding to eigenvalues λ_1 and λ_2 satisfy $J[f_1, f_2] \neq 0$ everywhere on the regular k-gon $U \setminus \{0\}$. Then, the scheme produces tangent plane continuous surfaces on U for almost any choice of control points.

The pair of functions (f_1, f_2) defines a map $U \to \mathbf{R}^2$. This planar map is called the *characteristic map*[a].

[a] Reif's original definition is somewhat different: only the restriction of (f_1, f_2) to an annular region around zero is included.

This condition is not necessary, even given the assumptions on the scheme: e.g. the Jacobian of f_1 and f_2 can be zero everywhere, but the scheme can still be tangent plane continuous if e.g. the Jacobian of f_1 and f_3 does not vanish. Theorem 5.2 is a weaker form of Reif's criterion; note that we do not obtain C^1 continuity, only tangent plane continuity. However, we note that the stronger C^1-continuity criterion immediately follows from combining Theorem 5.2 with the observation from Section 5.1 that C^1 continuity is equivalent to tangent plane continuity and injectivity of projection to the tangent plane.

We observe that in the coordinate system with basis vectors c_i, the projection of the universal surface to the tangent plane is equivalent to simply discarding all components except f_1 and f_2; this projection is one-to-one, if the map $(f_1, f_2) : U \rightarrow \mathbf{R}^2$ is one-to-one, i.e. the characteristic map is injective. This yields the following criterion.

Corollary 5.3: *(Reif's criterion) If the assumptions of Theorem 5.2 are satisfied, and in addition the characteristic map is injective, the scheme produces C^1-continuous surfaces on U for almost any choice of control points p.*

Higher-order smoothness. The general conditions for higher order smoothness have quite elaborate form and are beyond the scope of this tutorial. We only state a necessary and sufficient condition for C^2-continuity, which are of greatest practical relevance, for a limited class of schemes since the conditions have simple and intuitive form:

Proposition 5.4: *Suppose a scheme satisfies conditions of Corollary 5.3 and has equal subdominant eigenvalues $\lambda = \lambda_1 = \lambda_2$. Then the scheme produces C^2 continuous surfaces if and only if for any eigenvalue $\mu \neq \lambda$, $\mu \neq 1$, either $|\mu| < \lambda^2$ or $\mu = \lambda_2$, and the corresponding eigenbasis function is a homogeneous quadratic function of f_1 and f_2.*

This condition shows a serious limitation of stationary subdivision: the simplest approach to constructing C^2 schemes is to ensure that all non-subdominant eigenvalues are sufficiently small. This can be easily achieved by manipulation of coefficients, as was shown in [9,63], but results in surfaces, which have zero quadratic approximants at extraordinary vertices, i.e. zero curvature. To obtain non-zero curvature, we need to satisfy a much more difficult condition on the eigenbasis functions. In fact, it was demonstrated in [66] that this is impossible to achieve for schemes based on low degree splines.

6 Approximation properties of subdivision surfaces

While smoothness of subdivision surfaces with arbitrary control meshes has received a lot of attention, much less is known about approximation properties: to the best of our knowledge, there is a single published work on the topic [3]. Given that subdivision bases for refined grids coincide with spline surfaces almost everywhere, one would expect similar approximation behavior. However, available estimates do not fully confirm this. At the same time, subdivision surfaces are used by many authors as a practical approximation tool [49,43,71,78,51,2] with good results, which highlights the need for more thorough theoretical exploration of this aspect of subdivision.

In this section we review the main concepts of approximation of surfaces and state the estimates obtained in [3] as well as some results of [87].

6.1 *Functional spaces on surfaces*

A typical form of the approximation estimates for finite element and splines spaces is

$$\|g - \tilde{g}\|_{H^s(\Omega)} < Ch^{t-s}\|g\|_{H^t(\Omega)},$$

where $g \in H^t$ is the approximated function, $\tilde{g}$ is the best approximation (the closest point from the approximating space), h is a parameter characterizing the approximation space (e.g. element size for finite elements, or support size of individual basis functions), and $H^s(\Omega)$ and $H^t(\Omega)$ are Sobolev spaces on the domain Ω.

Our goal is to derive similar estimates for subdivision surfaces. The task is complicated by the fact that the domains on which subdivision bases are defined (polygonal complexes) do not have an intrinsic smoothness structure, and it is impossible to define Sobolev spaces on these domains without introducing such structure.

On the other hand, one can observe that subdivision itself can be used to introduce a smoothness structure on the domain using the characteristic maps. Before we explain the construction, we review the needed general concepts: $C^{r,1}$ manifolds and Sobolev spaces on these manifolds in the context of polygonal complexes.

First, we recall the definition of spaces $H^s(\Omega)$ for an open domain Ω in $\mathbf{R}^n$. Consider a function f in $C_0^\infty(\Omega)$, the space of compactly supported smooth functions on Ω. For an integer s, define the seminorm $|f|_{H^s(\Omega)}$ as the L_p norm of the s-th differential of f on Ω (recall that the s-th differential is a multilinear form in

s variables with coefficients equal to partial derivatives of total order s). The norm $\|f\|_{H^s(\Omega)}$ is defined as $\|f\|_{L_p(\Omega)} + |f|_{H^s(\Omega)}$.

A $C^{r,1}$ 2D manifold structure on a subset M of Euclidean space is an *atlas*, which is a collection of *charts* (χ_i, Ω_i), $\chi_i : \Omega_i \to M$, and Ω_i is an open domain in $\mathbf{R}^2$. Charts satisfy several conditions: (1) the union of $\chi_i(\Omega_i)$ is M; (2) the *transition maps* $\chi_j^{-1} \circ \chi_i$ are of smoothness class $C^{r,1}$, i.e. of r-times differentiable functions with Lipshitz r-th derivatives. We need to consider $C^{r,1}$ smoothness structures, rather than simply C^r because they allow us to construct a broader range of functional spaces on manifolds with smoothness structure defined by subdivision.

Let ρ_i be a partition of unity subordinated to the atlas χ_i, that is, the support of each ρ_i is contained in the range of χ_i (note that the support of a function is defined to be the closure of the set where the function is not zero, and therefore, the distance from $\operatorname{supp} \rho_i$ to $\partial\Omega$ is positive.) For a $C^{r,1}$ and $s \leq r+1$ function $f : M \to \mathbf{R}$ define the norm

$$\|f\|_{H^s(\Omega_i)} = \sum_i \|(\rho_i f) \circ \chi_i\|_{H^s \Omega_i}$$

The space $H^s(M)$ is the completion of $C^{k,1}(M)$ with respect to this norm. It is straightforward to show that the norms defined with respect to different atlases or partitions of unity M are equivalent; thus, the definition of the space $H^s(M)$ does not depend on the atlas or the partition of unity. The fact that $f \circ g$ is in H^s if f is in H^s and g is in $C^{s-1,1}$ is necessary to prove invariance [52].

In this definition, the norm is not invariant with respect to the change of atlas: while the spaces stay the same, the norms may change.

These definitions allow us to formulate approximation estimates for bases defined on manifolds. It remains to define a sufficiently smooth structure on the domain of subdivision surfaces.

6.2 *Manifold structure defined by subdivision*

As in the previous section, we restrict our attention to TP schemes of arity 2, i.e. schemes similar to the Loop scheme.

Suppose for regular grids a subdivision scheme yields functions which are $C^{r,1}$, for example it is based on B-splines of degree $r+1$. We note that these splines reproduce polynomials of degree $r+1$, and therefore, have approximation order $r+2$ for functions from the space H^{r+2}.

We also assume that the characteristic map is regular, in the sense defined in the previous section, and one-to-one. For each vertex v of a complex K, consider the inverse of the composition of a piecewise linear map from $|N_1(K)|$ to the regular k-gon U_k, with the characteristic map $\Phi_k : U_k \to \mathbf{R}^2$. We denote this map χ_v. We use the interior of images $\Phi_k(U_k)$ as the domains of the charts, and χ_v as the chart maps for out atlas.

One can easily show that for any two adjacent vertices v and w for any interior point of $|N_1(v)| \cap |N_2(w)|$, the composition $\chi_v^{-1} \circ \chi_w$ is $C^{r,1}$, i.e. the structure defined by these charts is $C^{r,1}$.

We conclude that for a scheme producing $C^{r,1}$-continuous surfaces for regular control grid and with regular and one-to-one characteristic maps, we can define smoothness spaces up to order $k+1$ on arbitrary meshes. This result is somewhat unsatisfactory as we cannot consider functions of higher smoothness $k+2$, for which the scheme has the best approximation rate in the regular case.

Now we can state the result obtained in [3].

Theorem 6.1: Consider bases defined by the Loop subdivision scheme on complexes K^0, K^1, $\ldots K^j$ $\ldots$, obtained by quadrisection refinement of the initial complex $K^0 = K$. Then, the $C^{2,1}$ manifold structure and Sobolev spaces $H^t(|K|)$ for $t \leq 3$ are defined on $|K|$ as described above, and the best approximation $\tilde{f}$ by subdivision basis functions of a function $f \in H^t(|K|)$ satisfies

$$\|f - \tilde{f}\|_{H^s(|K|)} < C\lambda_{max}^{t-s}\|f\|_{H^t(|K|)}$$

for any $s \leq 2$, where λ_{max} is the maximal subdominant eigenvalue for all valences of vertices in K.

Comparing with what is known for quartic box splines, on which Loop scheme is based, we see that this statement is limiting in several ways. First, the maximal exponent is 3 rather than 4; this is due to the limited smoothness of the chosen manifold structure on $|K|$. The order of approximation is further reduced by having to use λ_{max}, which can be as high as $5/8$ instead of $1/2$, as the scale parameter.

Finally, the norm on the left-hand side can be at most H^2. This is due to the fact that the basis functions are C^1, and therefore, are in H^2 but not higher smoothness spaces.

Given that the basis produced by subdivision almost everywhere coincides with the basis in the regular case, one expects that better estimates should be possible. Indeed, one can show that by choosing a different smoothness structure on $|K|$, one can obtain the following estimate [87]

$$\|f - \tilde{f}\|_{L_2(|K|)} < C(1/2)^t\|f\|_{H^t(|K|)}$$

for $t \leq 4$. While exactly matching splines for $L_2 = H^2$ norms on the right-hand side, the choice of smoothness structure results in the loss of estimates for H^1 and H^2.

The optimal choice of structure for estimates of this type remains open.

7 Conclusions

The survey we have presented is far from exhaustive. We did not discuss many important theoretical and algorithmic topics related to stationary subdivision on meshes. There are a few important extensions. Examples include variational subdivision [33] and PDE-based schemes [83,84,82], subdivision schemes in higher dimensions [5,50] and schemes for arbitrary mesh refinement [23]. Multiresolution surfaces based on subdivision were not considered either.

While there was a rapid progress in subdivision theory in the late 90s, few questions were resolved conclusively. While smoothness criteria exist and are well established, applying these criteria remains difficult, especially for parametric families of subdivision schemes and requires extensive computations. There are no criteria directly relating smoothness of limit surfaces to easy-to-verify conditions on mask coefficients; although, recent work by Prautzsch and Umlauf [79] is a promising step in this direction. Analysis approximation properties and fairness of limit surfaces is even further from completion.

In contrast, an increasing number of applications use subdivision as the surface representation of choice, and applications appear in other areas (e.g. subdivision-based finite elements). We hope that the needs of practical applications will encourage further theoretical advances.

References

1. Marc Alexa. Refinement operators for triangle meshes. *Computer Aided Geometric Design*, 19(3):169–172, March 2002.
2. B. Allen, B. Curless, and Z. Popovic. Articulated body deformation from range scan data. In *SIGGRAPH '02: 29th International Conference on Computer Graphics and Interactive Techniques*, volume 21, pages 612–19. 2002.
3. Greg Arben. *Approximation Properties of Subdivision Surfaces*. PhD thesis, University of Washnigton, 2001.
4. C. Bajaj, S. Schaefer, Joe Warren, and Guoliang Xu. A subdivision scheme for hexahedral meshes. *Visual Computer*, 18(5-6):343–56, 2002.
5. Chandrajit Bajaj, Scott Schaefer, Joe Warren, and Guoliang Xu. A subdivision scheme for hexahedral meshes. *The Visual Computer*, 18(5/6):343–356, 2002.
6. A. A. Ball and D. J. T. Storry. Conditions for tangent plane continuity over recursively generated B-spline surfaces. *ACM Transactions on Graphics*, 7(2):83–102, 1988.

7. Loic Barthe and Leif Kobbelt. Subdivision scheme tuning around extraordinary vertices. *Computer Aided Geometric Design*, 21(6):561–583, 2004.
8. Henning Biermann, Daniel Kristjansson, and Denis Zorin. Approximate boolean operations on free-form solids. In *Proceedings of SIGGRAPH 2001*, pages 185–94. 2001.
9. Henning Biermann, Adi Levin, and Denis Zorin. Piecewise smooth subdivision surfaces with normal control. In *Proceedings of SIGGRAPH'00: 27th International Conference on Computer Graphics and Interactive Techniques Conference*, pages 113–20. 2000.
10. Henning Biermann, Ioana Martin, Fausto Bernardini, and Denis Zorin. Cut-and-paste editing of multiresolution surfaces. In *SIGGRAPH '02: 29th International Conference on Computer Graphics and Interactive Techniques*, volume 21, pages 312–21. 2002.
11. Ed Catmull and James Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-Aided Design*, 10(6):350–355, 1978.
12. Alfred S. Cavaretta, Wolfgang Dahmen, and Charles A. Micchelli. Stationary subdivision. *Mem. Amer. Math. Soc.*, 93(453):vi+186, 1991.
13. F. Cirak, M. Ortiz, and Peter Schröder. Subdivision surfaces: a new paradigm for thin-shell finite-element analysis. *International Journal for Numerical Methods in Engineering*, 47(12):2039–72, 2000.
14. F. Cirak, M. J. Scott, E. K. Antonsson, M. Ortiz, and Peter Schröder. Integrated modeling, finite-element analysis, and engineering design for thin-shell structures using subdivision. *Computer Aided Design*, 34(2):137–48, 2002.
15. J. Claes, K. Beets, and F. Van Reeth. A corner-cutting scheme for hexagonal subdivision surfaces. In *Proceedings SMI. Shape Modeling International 2002*, pages 13–20. 2002. 17-22 May 2002.
16. N.A. Dodgson. An heuristic analysis of the classification of bivariate subdivision schemes. Technical Report 611, University of Cambridge Computer Laboratory, December 2004.
17. D. Doo. A subdivision algorithm for smoothing down irregularly shaped polyhedrons. In *Proceedings on Interactive Techniques in Computer Aided Design*, pages 157–165, Bologna, 1978.
18. D. Doo and M. Sabin. Analysis of the behaviour of recursive division surfaces near extraordinary points. *Computer-Aided Design*, 10(6):356–360, 1978.
19. Nira Dyn and David Levin. The subdivision experience. *Wavelets, images, and surface fitting (Chamonix-Mont-Blanc, 1993)*, pages 229–244, 1994.
20. Nira Dyn and David Levin. Subdivision schemes in geometric modelling. *Acta Numerica*, 11:73–144, 2002.
21. Nira Dyn, David Levin, and John A. Gregory. A 4-point interpolatory subdivision scheme for curve design. *Computer-Aided Geometric Design*, 4(4):257–68, 1987.
22. Nira Dyn, David Levin, and John A. Gregory. A butterfly subdivision scheme for surface interpolation with tension control. *ACM Transactions on Graphics*, 9(2):160–9, 1990.
23. Igor Guskov, Wim Sweldens, and Peter Schröder. Multiresolution signal processing for meshes. In *Proceedings of SIGGRAPH 99*, Computer Graphics Proceedings, Annual Conference Series, pages 325–334, August 1999.
24. Ayman Habib and Joe Warren. Edge and vertex insertion for a class of C^1 subdivision surfaces. *Computer-Aided Geometric Design*, 16(4):223–47, 1999.

25. Bin Han. Classification and construction of bivariate subdivision schemes. In A. Cohen, J.-L. Merrien, and L. L. Schumaker, editors, *Curves and Surfaces*, pages 187–197, Saint-Malo, 2003.
26. Hugues Hoppe, Tony DeRose, Tom Duchamp, Mark Halstead, Hubert Jin, John McDonald, Jean Schweitzer, and Werner Stuetzle. Piecewise smooth surface reconstruction. In *Proceedings of SIGGRAPH 94*, Computer Graphics Proceedings, Annual Conference Series, pages 295–302, July 1994.
27. Ioannis Ivrissimtzis, Neil Dodgson, Mohamed Hassan, and Malcolm Sabin. On the geometry of recursive subdivision. *International Journal Shape Modeling*, 8(1):23–42, June 2002.
28. Ioannis Ivrissimtzis, Neil Dodgson, and Malcolm Sabin. A generative classification of mesh refinement rules with lattice transformations. *Computer Aided Geometric Design*, 21(1):99–109, 2004.
29. Qingtang T. Jiang and Peter Oswald. Triangular $\sqrt{3}$-subdivision schemes: the regular case. *Journal of Computational and Applied Mathematics*, 156(1):47–75, 2003.
30. K. Karciauskas, J. Peters, and U. Reif. Shape characterization of subdivision surfaces – case studies. *Computer-Aided Geometric Design*, 21(6):601–614, july 2004. http://authors.elsevier.com/sd/article/S0167839604000627.
31. Andrei Khodakovsky and Peter Schröder. Fine level feature editing for subdivision surfaces. In *Proceedings of ACM Solid Modeling '99*, pages 203–211, 1999.
32. Leif Kobbelt. Interpolatory subdivision on open quadrilateral nets with arbitrary topology. In *European Association for Computer Graphics 17th Annual Conference and Exhibition. EUROGRAPHICS '96*, volume 15, pages C409–20, C485. 1996.
33. Leif Kobbelt. A variational approach to subdivision. *Computer-Aided Geometric Design*, 13(8):743–761, 1996.
34. Leif Kobbelt. square root 3-subdivision. In *Proceedings of SIGGRAPH'00: 27th International Conference on Computer Graphics and Interactive Techniques Conference*, pages 103–12. 2000.
35. Leif Kobbelt, J. Vorsatz, U. Labsik, and Hans-Peter Seidel. A shrink wrapping approach to remeshing polygonal surfaces. In *European Association for Computer Graphics 20th Annual Conference. EUROGRAPHICS'99*, volume 18, pages C119–29, C405. 1999.
36. P. Krysl, Eitan Grinspun, and Peter Schröder. Natural hierarchical refinement for finite element methods. *International Journal for Numerical Methods in Engineering*, 56(8):1109–24, 2003.
37. T. Kurihara. Interactive surface design using recursive subdivision. In *Proceedings of Computer Graphics International '93*, pages 228–43. 1993.
38. U. Labsik and G. Greiner. Interpolatory square root 3-subdivision. In *European Association for Computer Graphics. 21st Annual Conference. EUROGRAPHICS*, volume 19, pages C131–8, 528. 2000.
39. A. Lee, H. Moreton, and Hughes Hoppe. Displaced subdivision surfaces. In *Proceedings of SIGGRAPH'00: 27th International Conference on Computer Graphics and Interactive Techniques Conference*, pages 85–94. 2000.
40. C. K. Lee. Automatic metric 3d surface mesh generation using subdivision surface geometrical model. 1.construction of underlying geometrical model. *International Journal for Numerical Methods in Engineering*, 56(11):1593–614, 2003.

41. C. K. Lee. Automatic metric 3d surface mesh generation using subdivision surface geometrical model. 2. mesh generation algorithm and examples. *International Journal for Numerical Methods in Engineering*, 56(11):1615–46, 2003.
42. L. Linsen, V. Pascucci, M. A. Duchaineau, B. Hamann, and K. I. Joy. Hierarchical representation of time-varying volume data with $\sqrt{2}^4$ subdivision and quadrilinear B-spline wavelets. In *Proceedings 10th Pacific Conference on Computer Graphics and Applications*, pages 346–55. 2002.
43. Nathan Litke, Adi Levin, and Peter Schröder. Fitting subdivision surfaces. In *Proceedings VIS 2001. Visualization 2001*, pages 319–568. 2001.
44. Nathan Litke, Adi Levin, and Peter Schröder. Trimming for subdivision surfaces. *Computer-Aided Geometric Design*, 18(5):463–81, 2001.
45. Yong-Jin Liu, Matthew Ming-Fai Yuen, and S. Xiong. A feature-based approach for individualized human head modeling. *Visual Computer*, 18(5-6):368–81, 2002.
46. Charles Loop. Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, Department of Mathematics, 1987.
47. Charles Loop. Bounded curvature triangle mesh subdivision with the convex hull property. *Visual Computer*, 18(5-6):316–25, 2002.
48. Michael Lounsbery, Tony DeRose, and Joe Warren. Multiresolution analysis for surfaces of arbitrary topological type. *ACM Transactions on Graphics*, 16(1):34–73, 1997.
49. Weiyin Ma, Xiaohu Ma, and Shiu-Kit Tso. A new algorithm for loop subdivision surface fitting. In *Proceedings of Computer Graphics Imaging*, pages 246–51. 2001.
50. Ron MacCracken and Kenneth I. Joy. Free-form deformations with lattices of arbitrary topology. In *Proceedings of SIGGRAPH 96*, Computer Graphics Proceedings, Annual Conference Series, pages 181–188, August 1996.
51. Martin Marinov and Leif Kobbelt. Optimization techniques for approximation with subdivision surfaces. In *ACM Symposium on Solid Modeling and Applications*, pages 113–122, 2004.
52. Vladimir G. Maz'ja. *Sobolev spaces*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.
53. Ahmad H. Nasri. Polyhedral subdivision methods for free-form surfaces. *ACM Transactions on Graphics*, 6(1):29–73, January 1987.
54. Ahmad H. Nasri. Curve interpolation in recursively generated B-spline surfaces over arbitrary topology. *Computer-Aided Geometric Design*, 14(1):13–30, 1997.
55. Ahmad H. Nasri. An algorithm for interpolating intersecting curves by recursive subdivision surfaces. In *Proceedings Shape Modeling International '99. International Conference on Shape Modeling and Applications*, pages 130–7, 274. 1999.
56. Ahmad H. Nasri. Interpolating an unlimited number of curves meeting at extraordinary points on subdivision surfaces. *Computer Graphics Forum*, 22(1):87–97, 2003.
57. R. Ohbuchi, Y. Kokojima, and S. Takahashi. Blending shapes by using subdivision surfaces. *Computers & Graphics*, 25(1):41–58, 2001.
58. Peter Oswald and Peter Schröder. Composite primal/dual $\sqrt{3}$-subdivision schemes. *Computer-Aided Geometric Design*, 20(3):135–164, 2003.
59. J. Peters and U. Reif. Shape characterization of subdivision surfaces – basic principles. *Computer-Aided Geometric Design*, 21(6):585–599, july 2004.

60. Jörg Peters and Ahmad H. Nasri. Computing volumes of solids enclosed by recursive subdivision surfaces. In *EUROGRAPHICS 97. The European Association for Computer Graphics 18th Annual Conference*, volume 16, pages C89–94. 1997.
61. Jörg Peters and Ulrich Reif. The simplest subdivision scheme for smoothing polyhedra. *ACM Transactions on Graphics*, 16(4):420–31, 1997.
62. Jörg Peters and Ulrich Reif. Analysis of algorithms generalizing b-spline subdivision. *SIAM Journal on Numerical Analysis*, 35(2):728–748 (electronic), 1998.
63. Hartmut Prautzsch and Georg Umlauf. A G^1 and a G^2 subdivision scheme for triangular nets. *International Journal of Shape Modeling*, 6(1):21–35, 2000.
64. U. Reif. Analyse und konstruktion von subdivisionsalgorithmen für freiformflächen beliebiger topologie, 1998. Habilitation.
65. Ulrich Reif. A unified approach to subdivision algorithms near extraordinary vertices. *Computer-Aided Geometric Design*, 12(2):153–74, 1995.
66. Ulrich Reif. A degree estimate for subdivision surfaces of higher regularity. *Proc. Amer. Math. Soc.*, 124(7):2167–2174, 1996.
67. M. A. Sabin, N. A. Dodgson, M. F. Hassan, and I. P. Ivrissimtzis. Curvature behaviours at extraordinary points of subdivision surfaces. *Computer-Aided Design*, 35(11):1047–1051, September 2003.
68. Malcolm A. Sabin. Interrogation of subdivision surfaces. *Handbook of computer aided geometric design*, pages 327–341, 2002.
69. Malcolm A. Sabin. Subdivision surfaces. *Handbook of computer aided geometric design*, pages 309–325, 2002.
70. S. Schaefer, J. Warren, and D. Zorin. Lofting curve networks using subdivision surfaces. In *Eurographics/SIGGRAPH Symposium on Geometry Processing*, pages 103–114, 2004.
71. V. Scheib, J. Haber, M. C. Lin, and Hans-Peter Seidel. Efficient fitting and rendering of large scattered data sets using subdivision surfaces. In *23rd Annual Conference of the Eurographics Association*, volume 21, pages 353–62, 630. 2002.
72. Peter Schröder. Subdivision, multiresolution and the construction of scalable algorithms in computer graphics. *Multivariate approximation and applications*, pages 213–251, 2001.
73. Peter Schröder. Subdivision as a fundamental building block of digital geometry processing algorithms. In *15th Toyota Conference: Scientific and Engineering Computations for the 21st Century - Methodologies and Applications*, volume 149, pages 207–19. 2002.
74. J. E. Schweitzer. *Analysis and Application of Subdivision Surfaces*. PhD thesis, University of Washington, Seattle, 1996.
75. S. Skaria, E. Akleman, and F. I. Parke. Modeling subdivision control meshes for creating cartoon faces. In *Proceedings International Conference on Shape Modeling and Applications*, pages 216–25. 2001.
76. Jos Stam. Exact evaluation of catmull-clark subdivision surfaces at arbitrary parameter values. In *Proceedings of SIGGRAPH 98: 25th International Conference on Computer Graphics and Interactive Techniques*, pages 395–404. 1998.
77. Jos Stam. On subdivision schemes generalizing uniform B-spline surfaces of arbitrary degree. *Computer-Aided Geometric Design*, 18(5):383–96, 2001.
78. H. Suzuki, S. Takeuchi, and T. Kanai. Subdivision surface fitting to a range of points.

In *Proceedings. Seventh Pacific Conference on Computer Graphics and Applications*, pages 158–67, 322. 1999.

79. Georg Umlauf. A technique for verifying the smoothness of subdivison schemes. In M.L. Lucian and M. Neamtu, editors, *Geometric Modeling and Computing*, Seattle, 2003.
80. Luiz Velho. Quasi 4-8 subdivision. *Computer-Aided Geometric Design*, 18(4):345–57, 2001.
81. Luiz Velho and Denis Zorin. 4-8 subdivision. *Computer Aided Geometric Design*, 18(5):397–427, June 2001.
82. Joe Warren and Henrik Weimer. *Subdivision Methods for Geometric Design: A Constructive Approach*. Morgan Kaufmann, 2001.
83. Henrik Weimer and Joe Warren. Subdivision schemes for thin plate splines. In *EUROGRAPHICS '98. 19th Annual Conference*, volume 17, pages C303–13, C392. 1998.
84. Henrik Weimer and Joe Warren. Subdivision schemes for fluid flow. In *Proceedings of SIGGRAPH 99: 26th International Conference on Computer Graphics and Interactive Techniques*, pages 111–20. 1999.
85. Zheng Xu and K. Kondo. Fillet operations with recursive subdivision surfaces. In *Proceedings of 6th IFIP Working Conference on Geometric Modelling: Fundamentals and Applications*, pages 269–84. 2001.
86. Hongxin Zhang and Guojin Wang. Honeycomb subdivision. *Journal of Software*, 13(7):1199–207, 2002.
87. D. Zorin. Approximation on manifolds by bases locally reproducing polynomials. Unpublished manuscript, 2003.
88. Denis Zorin. *Subdivision and Multiresolution Surface Representations*. PhD thesis, Caltech, Pasadena, 1997.
89. Denis Zorin. A method for analysis of C^1-continuity of subdivision surfaces. *SIAM Journal on Numerical Analysis*, 37(5):1677–708, 2000.
90. Denis Zorin. Smoothness of stationary subdivision on irregular meshes. *Constructive Approximation*, 16(3):359–397, 2000.
91. Denis Zorin and Peter Schröder. A unified framework for primal/dual quadrilateral subdivision schemes. *Computer-Aided Geometric Design*, 18(5):429–54, 2001.
92. Denis Zorin, Peter Schröder, and Wim Sweldens. Interpolating subdivision for meshes with arbitrary topology. In *Proceedings of 23rd International Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'96)*, pages 189–92. 1996.
93. Denis Zorin, Peter Schröder, and Wim Sweldens. Interactive multiresolution mesh editing. In *Proceedings of SIGGRAPH 97*, Computer Graphics Proceedings, Annual Conference Series, pages 259–268, August 1997.