

17 POLYTOPE SKELETONS AND PATHS

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INTRODUCTION

The k -dimensional skeleton of a d -polytope P is the set of all faces of the polytope of dimension at most k . The 1-skeleton of P is called the graph of P and denoted by $G(P)$. $G(P)$ can be regarded as an abstract graph whose vertices are the vertices of P , with two vertices adjacent if they form the endpoints of an edge of P .

In this chapter, we will describe results and problems concerning graphs and skeletons of polytopes. In Section 17.1 we briefly describe the situation for 3-polytopes. In Section 17.2 we consider general properties of polytopal graphs—subgraphs and induced subgraphs, connectivity and separation, expansion, and other properties. In Section 17.3 we discuss problems related to diameters of polytopal graphs in connection with the simplex algorithm and the Hirsch conjecture, and to directed graphs obtained by directing graphs of polytopes via a linear functional. Section 17.4 is devoted to skeletons of polytopes, connectivity, collapsibility and shellability, empty faces and polytopes with “few vertices,” and the reconstruction of polytopes from their low dimensional skeletons, and finally we consider what can be said about the collections of all k -faces of a d -polytope, first for $k = d - 1$ and then when k is fixed and d is large compared to k .

17.1 THREE-DIMENSIONAL POLYTOPES

GLOSSARY

Convex polytopes and their *faces* (and, in particular their *vertices*, *edges*, and *facets*) are defined in Chapter 13 of this Handbook.

A graph is ***d-polytopal*** if it is the graph of some d -polytope.

The following standard graph-theoretic concepts are used: *subgraphs*, *induced subgraphs*, the *complete graph* K_n on n vertices, *cycles*, *trees*, a *spanning tree* of a graph, *valence* (or *degree*) of a vertex in a graph, *planar* graphs, *d-connected* graphs, *coloring* of a graph, and *Hamiltonian* graphs.

We briefly discuss results on 3-polytopes. Some of the following theorems are the starting points of much research, sometimes of an entire theory. Only in a few cases are there high-dimensional analogues, and this remains an interesting goal for further research.

THEOREM 17.1.1 *Whitney (1931)*

Let G be the graph of a 3-polytope P . Then the graphs of faces of P are precisely the induced cycles in G that do not separate G .

THEOREM 17.1.2 *Steinitz (1916)*

A graph G is a graph of a 3-polytope if and only if G is planar and 3-connected.

Steinitz's theorem is the first of several theorems that describe the tame behavior of 3-polytopes. These theorems fail already in dimension four; see Chapter 13.

The theory of planar graphs is a wide and rich theory. Let us quote here the fundamental theorem of Kuratowski.

THEOREM 17.1.3 *Kuratowski (1930)*

A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.

THEOREM 17.1.4 *Lipton and Tarjan (1979), in a stronger form given by Miller (1986)*

The graph of every 3-polytope with n vertices can be separated, by $2\sqrt{2n}$ vertices forming a circuit in the graph, into connected components of size at most $2n/3$.

It is worth mentioning that the Koebe circle packing theorem gives a new approach to both the Steinitz and Lipton-Tarjan theorems. (See [Zie95, PA95]).

Euler's formula $V - E + F = 2$ has many applications concerning graphs of 3-polytopes; in higher dimensions, our knowledge of face numbers of polytopes (see Chapter 15) applies to the study of their graphs and skeletons. Simple applications of Euler's theorem are:

THEOREM 17.1.5

Every 3-polytopal graph has a vertex of valence at most 5. (Equivalently, every 3-polytope has a face with at most five sides).

THEOREM 17.1.6

Every 3-polytope has either a trivalent vertex or a triangular face.

A deeper application of Euler's theorem is:

THEOREM 17.1.7 *Kotzig (1955)*

Every 3-polytope contains two adjacent vertices the sum of whose valences is at most 13.

For a simple 3-polytope P , denote by $p_k = p_k(P)$ the number of k -sized faces of P .

THEOREM 17.1.8 *Eberhard (1891)*

For every finite sequence (p_k) of nonnegative integers with $\sum_{k \geq 3} (6 - k)p_k = 12$, there exists a simple 3-polytope P with $p_k(P) = p_k$ for every $k \neq 6$.

Eberhard's theorem is the starting point of a large number of results and problems. While no high-dimensional direct analogues are known or even conjectured,

the results and problems on facet-forming polytopes and nonfacets mentioned below seems related.

THEOREM 17.1.9 *Motzkin* (1964)

The graph of a simple 3-polytope whose facets have $0 \pmod{3}$ vertices has, all together, an even number of vertices.

THEOREM 17.1.10 *Barnette* (1966)

Every 3-polytopal graph contains a spanning tree of maximal valence 3.

We will now describe some results and a conjecture on colorability and Hamiltonian circuits.

THEOREM 17.1.11 *Four Color Theorem: Appel and Haken* (1977)

The graph of every 3-polytope is 4-colorable.

THEOREM 17.1.12 *Tutte* (1956)

4-connected planar graphs are Hamiltonian.

Tait conjectured in 1880, and Tutte disproved in 1946, that the graph of every simple 3-polytope is Hamiltonian. This started a rich theory of trivalent planar graphs without large paths.

CONJECTURE 17.1.13 *Barnette*

Every graph of a simple 3-polytope whose facets have an even number of vertices is Hamiltonian.

Finally, there are several exact and asymptotic formulas for the numbers of distinct graphs of 3-polytopes. A remarkable enumeration theory was developed by Tutte and was further developed by several authors. We will quote one result.

THEOREM 17.1.14 *Tutte* (1962)

The number of rooted simplicial 3-polytopes with v vertices is

$$\frac{2(4v - 11)!}{(3v - 7)!(v - 2)!}$$

17.2 GRAPHS OF d -POLYTOPES—GENERALITIES

GLOSSARY

For a graph G , TG denotes any *subdivision* of G , i.e., any graph obtained from G by replacing the edges of G by paths with disjoint interiors.

A d -polytope P is *simplicial* if all its proper faces are simplices. P is *simple* if every vertex belongs to d edges or, equivalently, if the polar of P is simplicial.

P is **cubical** if all its proper faces are cubes.

A simplicial polytope P is **stacked** if it is obtained by repeatedly gluing simplices along facets.

For the definition of the *cyclic polytope* $C(d, n)$, see Chapter 13.

For two graphs G and H (considered as having disjoint sets V and V' of vertices)

$G + H$ denotes the graph on $V \cup V'$ that contains all edges of G and H together with all edges of the form $\{v, v'\}$ for $v \in V$ and $v' \in V'$.

A graph G is **d -connected** if G remains connected after the deletion of any set of at most $d - 1$ vertices.

An **empty simplex** of a polytope P is a set S of vertices such that S does not form a face but every proper subset of S forms a face.

A graph G whose vertices are embedded in \mathbb{R}^d is **rigid** if every small perturbation of the vertices of G that does not change the distance of adjacent vertices in G is induced by an affine rigid motion of \mathbb{R}^d . G is **generically d -rigid** if it is rigid with respect to “almost all” embeddings of its vertices into \mathbb{R}^d . (Generic rigidity is thus a graph theoretic property, but no description of it in pure combinatorial terms is known for $d > 2$; cf. Chapter 49.)

We say that a set A of vertices of a graph G is **totally separated** by a set B of vertices, if A and B are disjoint and every path between two distinct vertices in A meets B .

A graph G is an **ϵ -expander** if, for every set A of at most half the vertices of G , there are at least $\epsilon \cdot |A|$ vertices not in A that are adjacent to vertices in A .

Neighborly polytopes and *(0, 1)-polytopes* are defined in Chapter 13.

The **polar dual** P^Δ of a polytope P is defined in Chapter 13.

SUBGRAPHS AND INDUCED SUBGRAPHS

THEOREM 17.2.1 *Grünbaum* (1965)

Every d -polytopal graph contains a TK_{d+1} .

THEOREM 17.2.2 *Kalai* (1987)

The graph of a simplicial d -polytope P contains a TK_{d+2} if and only if P is not stacked.

One important difference between the situation for $d = 3$ and for $d > 3$ is that K_n , for every $n > 4$, is the graph of a 4-dimensional polytope (e.g., a cyclic polytope). Simple manipulations on the cyclic 4-polytope with n vertices show:

PROPOSITION 17.2.3 *Perles*

- (i) *Every graph G is a spanning subgraph of the graph of a 4-polytope.*
- (ii) *For every graph G , $G + K_n$ is a d -polytopal graph for some n and some d .*

This proposition extends easily to higher dimensional skeletons in place of graphs. It is not known what the minimal dimension is for which $G + K_n$ is d -polytopal, nor even whether $G + K_n$ (for some $n = n(G)$) can be realized in some bounded dimension uniformly for all graphs G .

CONNECTIVITY AND SEPARATION

THEOREM 17.2.4 *Balinski* (1961)

The graph of a d -polytope is d -connected.

A set S of d vertices that separates P must form an empty simplex; in this case, P can be obtained by gluing two polytopes along a simplex facet of each.

THEOREM 17.2.5 *Larman* (1970)

Let G be the graph of a d -polytope. Let $e = \lfloor (d+1)/3 \rfloor$. Then for every two disjoint sequences (v_1, v_2, \dots, v_e) and (w_1, w_2, \dots, w_e) of vertices of G , there are e vertex-disjoint paths connecting v_i to w_i , $i = 1, 2, \dots, e$.

PROBLEM 17.2.6 *Larman*

Is the last theorem true for $e = \lfloor d/2 \rfloor$?

THEOREM 17.2.7 *Cauchy, Dehn, Alexandrov, Whiteley, ...*

- (i) *If P is a simplicial d -polytope, $d \geq 3$, then $G(P)$ (with its embedding in \mathbb{R}^d) is rigid.*
- (ii) *For a general d -polytope P , let G' be a graph (embedded in \mathbb{R}^d) obtained from $G(P)$ by triangulating the 2-faces of P without introducing new vertices. Then G' is rigid.*

COROLLARY 17.2.8

For a simplicial d -polytope P , $G(P)$ is generically d -rigid. For a general d -polytope P and a graph G' (considered as an abstract graph) as in the previous theorem, G' is generically d -rigid.

The main combinatorial application of the above theorem is the Lower Bound Theorem (see Chapter 15) and its extension to general polytopes. Note that Corollary 17.2.8 can be regarded also as a strong form of Balinski's theorem. It is easy to see that generic d -rigidity implies d -connectivity. If the graph G of a general d -polytope P can be separated into two parts (say a red part and a blue part) by deleting $d-1$ vertices, then it is possible to triangulate the 2-faces of P without introducing a blue-red edge, and hence the resulting triangulation is not $(d-1)$ -connected and therefore not generically d -rigid.

Let $\mu(n, d) = f_{d-1}(C(d, n))$ be the number of facets of a cyclic d -polytope with n vertices, which, by the Upper Bound Theorem, is the maximal number of facets possible for a d -polytope with n vertices.

THEOREM 17.2.9 *Klee (1964)*

The number of vertices of a d -polytope that can be totally separated by n vertices is at most $\mu(n, d)$.

EXPANSION

Expansion properties for the graph of the d -dimensional cube are known and important in various areas of combinatorics. By direct combinatorial methods, one can obtain expansion properties of duals to cyclic polytopes. There are a few positive results and several interesting conjectures on expansion properties of graphs of large families of polytopes.

THEOREM 17.2.10 *Kalai (1992)*

Graphs of duals to neighborly d -polytopes with n facets are ϵ -expanders for $\epsilon = O(n^{-4})$.

CONJECTURE 17.2.11 *Mihail and Vazirani*

Graphs of $(0, 1)$ -polytopes P have the following expansion property: For every set A of at most half the vertices of P , the number of edges joining vertices in A to vertices not in A is at least $|A|$.

It is also conjectured that graphs of polytopes cannot have very good expansion properties:

CONJECTURE 17.2.12 *Graphs of polytopes are not too good expanders*

Let d be fixed and set $r = \lfloor d/2 \rfloor$. The graph of every simple d -polytope with n vertices can be separated into two parts, each having at least $n/3$ vertices, by removing $O(n^{(r-1)/r})$ vertices.

CONJECTURE 17.2.13 *Expansion properties of random polytopes*

A random simple d -polytope with n facets is an $O(1/(n-d))$ -expander.

CONJECTURE 17.2.14 *There are only a "few" graphs of polytopes*

The number of distinct (isomorphism types) of graphs of simple d -polytopes with n vertices is at most C_d^n , where C_d is a constant depending on d .

It is even possible that the same constant applies for all dimensions and that the conjecture holds even for graphs of general polytopes.

OTHER PROPERTIES**CONJECTURE 17.2.15** *Barnette*

Every graph of a simple d -polytope, $d \geq 4$, is Hamiltonian.

THEOREM 17.2.16 *Goodman and Onishi (1984)*

For a simple d -polytope P , $G(P)$ is 2-colorable if and only if $G(P^\Delta)$ is d -colorable.

17.3 DIAMETERS OF POLYTOPAL GRAPHS

GLOSSARY

A ***d*-polyhedron** is the intersection of a finite number of halfspaces in \mathbb{R}^d .

$\Delta(d, n)$ denotes the maximal diameter of the graphs of *d*-dimensional polyhedra *P* with *n* facets.

$\Delta_b(d, n)$ denotes the maximal diameter of the graphs of *d*-polytopes with *n* vertices.

Given a *d*-polyhedron *P* and a linear functional ϕ on \mathbb{R}^d , we denote by $G^{\rightarrow}(P)$ the directed graph obtained from $G(P)$ by directing an edge $\{v, u\}$ from *v* to *u* if $\phi(v) \leq \phi(u)$. *v* $\in P$ is a **top vertex** if ϕ attains its maximum value in *P* on *v*.

Let $H(d, n)$ be the maximum over all *d*-polyhedra with *n* facets and all linear functionals on \mathbb{R}^d of the maximum length of a minimal monotone path from any vertex to a top vertex.

Let $M(d, n)$ be the maximal number of vertices in a monotone path over all *d*-polyhedra with *n* facets and all linear functionals on \mathbb{R}^d .

For the notions of *simplicial complex*, *polyhedral complex*, *pure simplicial complex*, and the *boundary complex* of a polytope, see Chapter 15.

Given a pure $(d-1)$ -dimensional simplicial (or polyhedral) complex *K*, the **dual graph** $G^\Delta(K)$ of *K* is the graph whose vertices are the facets ($(d-1)$ -faces) of *K*, with two facets *F, F'* adjacent if $\dim(F \cap F') = d - 2$.

A pure simplicial complex *K* is **vertex-decomposable** if there is a vertex *v* of *K* such that $\text{lk}(v) = \{S \setminus \{v\} \mid S \in K, v \in S\}$ and $\text{ast}(v) = \{S \mid S \in K, v \notin S\}$ are both vertex-decomposable. (The complex $K = \{\emptyset\}$ consisting of the empty face alone is vertex-decomposable.)

It is a long-outstanding open problem to determine the behavior of the function $\Delta(d, n)$. In 1957, Hirsch conjectured that $\Delta(d, n) \leq n - d$. Klee and Walkup showed that the Hirsch conjecture is false for unbounded polyhedra. The Hirsch conjecture for bounded polyhedra is still open. The special case asserting that $\Delta_b(d, 2d) = d$ is called the ***d*-step conjecture**, and it was shown by Klee and Walkup to imply that $\Delta_b(d, n) \leq n - d$. Another equivalent formulation is that between any pair of vertices *v* and *w* of a polytope *P* there is a non-revisiting path, i.e., a path $v = v_1, v_2, \dots, v_m = w$ such that for every facet *F* of *P*, if $v_i, v_j \in F$ for $i < j$ then $v_k \in F$ for every $k, i \leq k \leq j$.

THEOREM 17.3.1 *Klee and Walkup (1967)*

$$\Delta(d, n) \geq n - d + \min\{\lfloor d/4 \rfloor, \lfloor (n - d)/4 \rfloor\}.$$

THEOREM 17.3.2 *Adler (1974)*

$$\Delta_b(d, n) > \lfloor n - d - (n - d)/\lfloor 5d/4 \rfloor \rfloor.$$

THEOREM 17.3.3 Larman (1970)

$$\Delta(d, n) \leq n2^{d-3}.$$

THEOREM 17.3.4 Kalai and Kleitman (1992)

$$\Delta(d, n) \leq n \cdot \binom{\log n + d}{d} \leq n^{\log d + 1}.$$

THEOREM 17.3.5 Provan and Billera (1980)

Let G be the dual graph that corresponds to a vertex-decomposable $(d-1)$ -dimensional simplicial complex with n vertices. Then the diameter of G is at most $n - d$.

It is known that this theorem does not imply the Hirsch conjecture (for polytopes) since there are simplicial polytopes whose boundary complexes are not vertex-decomposable. Yet, such examples are not so easy to come by.

Some special classes of polytopes are known to satisfy the Hirsch bound or to have upper bounds for their diameters that are polynomial in d and n .

THEOREM 17.3.6 Naddef (1989)

The graph of every $(0, 1)$ d -polytope has diameter at most d .

Balinski proved the Hirsch bound for dual transportation polytopes, Dyer and Frieze showed a polynomial upper bound for unimodular polyhedra, Kalai observed that if the ratio between the number of facets and the dimension is bounded above for the polytope and all its faces then the diameter is bounded above by a polynomial in the dimension, Kleinschmidt and Onn proved extensions of Naddef's results to integral polytopes, and Deza and Onn found upper bounds for the diameter in terms of lattice points in the polytope.

The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed for Dantzig's simplex algorithm for linear programming with any pivot rule. However, it is still an open problem to find pivot rules where each pivot step can be computed with a polynomial number of arithmetic operations in d and n such that the number of pivot steps needed comes close to the upper bounds for $\Delta(d, n)$ given above. It is worth noting, however, that by using linear programming it is possible to find a path in a polytope P that obeys the upper bounds given above such that the number of arithmetic operations is bounded by the size of the path times a polynomial in the input size.

The upper bounds for $\Delta(d, n)$ mentioned above apply even to $H(d, n)$. Klee and Minty considered a certain geometric realization of the d -cube to show that

THEOREM 17.3.7 Klee and Minty (1972)

$$M(d, 2d) \geq 2^d.$$

Recent far-reaching extensions of the Klee-Minty construction were found by Amenta and Ziegler.

Objective functions allow us to direct the graph of the polytope so that every edge is directed toward the vertex with the higher value of the objective function. Let us consider objective functions that give different values to adjacent vertices, and call the digraph resulting from such an objective function a **polytopal digraph**. An important property of such directed graphs is that there is always one sink (and

one source). This property is inherited for induced subgraphs on vertices of any face of the polytope.

The h -vector of a simplicial polytope P has a simple and important interpretation in terms of the directed graph that corresponds to the polar of P . The number $h_k(P)$ is the number of vertices v of P^Δ of outdegree k . (Recall that every vertex in a simple polytope has exactly d neighboring vertices.) Switching from ϕ to $-\phi$, one gets the Dehn-Sommerville relations $h_k = h_{d-k}$ (including the Euler relation for $k = 0$); see Chapter 15.

17.4 SKELETONS OF POLYTOPES

GLOSSARY

A pure polyhedral complex K is **strongly connected** if its dual graph is connected.

A **shelling order** of the facets of a polyhedral $(d-1)$ -dimensional sphere is an ordering of the set of facets F_1, F_2, \dots, F_n so that the simplicial complex K_i spanned by $F_1 \cup F_2 \cup \dots \cup F_i$ is a simplicial ball for every $i < n$. A polyhedral complex is **shellable** if there exists a shelling order of its facets.

A simplicial polytope is **extendably shellable** if any way to start a shelling can be continued to a shelling.

An **elementary collapse** on a simplicial complex is the deletion of two faces F and G so that F is maximal and G is a codimension 1 face of F that is not included in any other maximal face. A polyhedral complex is **collapsible** if it can be reduced to the void complex by repeated applications of elementary collapses.

A d -dimensional polytope P is **facet-forming** if there is a $(d+1)$ -dimensional polytope Q such that all facets of Q are combinatorially isomorphic to P . If no such Q exists, P is called a **nonfacet**.

A **rational polytope** is a polytope whose vertices have rational coordinates. (Not every polytope is combinatorially isomorphic to a rational polytope; see Chapter 13.)

A d -polytope P is **k -simplicial** if all its faces of dimension at most k are simplices. P is **k -simple** if its polar dual P^Δ is k -simplicial.

Zonotopes are defined in Chapters 13 and 15.

Let K be a polyhedral complex. An **empty simplex** S of K is a minimal non-face of K , i.e., a subset S of the vertices of K with S itself not in K but every proper subset of S in K .

Let K be a polyhedral complex and let U be a subset of its vertices. The **induced subcomplex** of K on U , denoted by $K[U]$, is the set of all faces in K whose vertices belong to U . An **empty face** of K is an induced polyhedral subcomplex of K that is homeomorphic to a polyhedral sphere. An empty 2-dimensional face is called an **empty polygon**. An **empty pyramid** of K is an induced subcomplex of K that consists of all the proper faces of a pyramid over a face of K .

CONNECTIVITY AND SUBCOMPLEXES
THEOREM 17.4.1 *Grünbaum* (1965)

The i -skeleton of every d -polytope contains a subdivision of $\text{skel}_i(\Delta^d)$, the i -skeleton of a d -simplex.

THEOREM 17.4.2 *Folklore*

- (i) For $i > 0$, $\text{skel}_i(P)$ is strongly connected.
- (ii) For every face F , let $U_i(F)$ be the set of all i -faces of P containing F . Then if $i > \dim F$, $U_i(F)$ is strongly connected.

Part (ii) follows at once from the fact that the faces of P containing F correspond to faces of the quotient polytope P/F . However, properties (i) and (ii) together are surprisingly strong, and all the known upper bounds for diameters of graphs of polytopes rely only on properties (i) and (ii) for the dual polytope.

THEOREM 17.4.3 *van Kampen and Flores* (1935)

For $i \geq \lfloor d/2 \rfloor$, $\text{skel}_i(\Delta^{d+1})$ is not embeddable in S^{d-1} (and hence not in the boundary complex of any d -polytope).

(This extends the fact that K_5 is not planar.)

CONJECTURE 17.4.4 *Lockeberg*

For every partition of $d = d_1 + d_2 + \cdots + d_k$ and two vertices v and w of P , there are k disjoint paths between v and w such that the i th path is a path of d_i -faces in which any two consecutive faces have $(d_i - 1)$ -dimensional intersection.

SHELLABILITY AND COLLAPSIBILITY
THEOREM 17.4.5 *Bruggesser and Mani* (1970)

Boundary complexes of polytopes are shellable.

The proof of Bruggesser and Mani is based on starting with a point near the center of a facet and moving from this point to infinity, and back from the other direction, keeping track of the order in which facets are seen. This proves a stronger form of shellability, in which each K_i is the complex spanned by all the facets that can be seen from a particular point in \mathbb{R}^d . It follows from shellability that

THEOREM 17.4.6

Polytopes are collapsible.

THEOREM 17.4.7 *Ziegler* (1992)

There are d -polytopes, $d \geq 4$, whose boundary complexes are not extendably shellable.

FACET-FORMING POLYTOPES AND SMALL LOW-DIMENSIONAL FACES
THEOREM 17.4.8 *Perles and Shephard (1967)*

Let P be a d -polytope such that the maximum number of k -faces of P on any $(d-2)$ -sphere in the skeleton of P is at most $(d-1-k)/(d+1-k)f_k(P)$. Then P is a non-facet.

THEOREM 17.4.9 *Schulte (1985)*

The cuboctahedron and the icosidodecahedron are nonfacets.

PROBLEM 17.4.10

Is the icosahedron facet-forming?

For all other regular polytopes the situation is known. The simplices and cubes in any dimension and the 3-dimensional octahedron are facet-forming. All other regular polytopes with the exception of the icosahedron are known to be non-facets.

Next, we try to understand if it is possible for all the k -faces of a d -polytope to be isomorphic to a given polytope P . The following conjecture asserts that if d is large with respect to k , this can happen only if P is either a simplex or a cube.

CONJECTURE 17.4.11 *Kalai*

For every k there is a $d(k)$ such that every d -polytope with $d > d(k)$ has a k -face that is either a simplex or combinatorially isomorphic to a k -dimensional cube.

For simple polytopes, it follows from the next theorem that if $d > ck^2$ then every d -polytope has a k -face F such that $f_r(F) \leq f_r(C_k)$. (Here, C_k denotes the k -dimensional cube.)

THEOREM 17.4.12 *Nikulin (1981)*

The average number of r -dimensional faces of a k -dimensional face of a simple d -dimensional polytope is at most

$$\binom{d-r}{d-k} \cdot \left(\binom{\lfloor d/2 \rfloor}{r} + \binom{\lfloor (d+1)/2 \rfloor}{r} \right) / \left(\binom{\lfloor d/2 \rfloor}{k} + \binom{\lfloor (d+1)/2 \rfloor}{k} \right).$$

THEOREM 17.4.13 *Kalai (1989)*

Every d -polytope for $d \geq 5$ has a 2-face with at most 4 vertices.

THEOREM 17.4.14 *Meisinger (1994)*

Every rational d -polytope for $d \geq 9$ has a 3-face with at most 150 vertices.

The last two theorems and the next one are proved using the linear inequalities for flag numbers that are known via intersection homology of toric varieties; see Chapter 15. One can study in a similar fashion also quotients of polytopes.

CONJECTURE 17.4.15 *Perles*

For every k there is a $d'(k)$ such that every d -polytope with $d > d'(k)$ has a k -dimensional quotient that is a simplex.

As was mentioned in the first section, $d'(2) = 3$. The 24-cell, which is a regular 4-polytope all whose faces are octahedra, shows that $d'(3) > 4$.

THEOREM 17.4.16 *Meisinger (1994)*

Every d -polytope with $d \geq 9$ has a 3-dimensional quotient that is a simplex.

PROBLEM 17.4.17

For which values of k and r are there d -polytopes other than the d -simplex that are both k -simplicial and r -simple?

It is known that this can happen only when $k+r \leq d$. There are infinite families of $(d-2)$ -simplicial and 2-simple polytopes, and some examples of $(d-3)$ -simplicial and 3-simple d -polytopes.

RECONSTRUCTION

THEOREM 17.4.18 *Extension of Whitney's theorem*

d -polytopes are determined by their $(d-2)$ -skeletons.

THEOREM 17.4.19 *Perles (1973)*

Simplicial d -polytopes are determined by their $\lfloor d/2 \rfloor$ -skeletons.

This follows from the following theorem (here, $\text{ast}(F, P)$ is the complex formed by the faces of P that are disjoint to all vertices in F).

THEOREM 17.4.20 *Perles (1973)*

Let P be a simplicial d -polytope.

- (i) *If F is a k -face of P , then $\text{skel}_{d-k-2}(\text{ast}(F, P))$ is contractible in $\text{skel}_{d-k-1}(\text{ast}(F, P))$.*
- (ii) *If F is an empty k -simplex, then $\text{ast}(F, P)$ is homotopically equivalent to S^{d-k} ; hence, $\text{skel}_{d-k-2}(\text{ast}(F, P))$ is not contractible in $\text{skel}_{d-k-1}(\text{ast}(F, P))$.*

THEOREM 17.4.21 *Blind and Mani (1987)*

Simple polytopes are determined by their graphs.

THEOREM 17.4.22 *Kalai and Perles (1988)*

Simplicial d -polytopes are determined by the incidence relations between i - and $(i+1)$ -faces for every $i > \lfloor d/2 \rfloor$.

THEOREM 17.4.23 *Bjorner, Edelman, and Ziegler (1990)*

Zonotopes are determined by their graphs.

In all instances of the above theorems except the single case of the Blind-Mani theorem, the proofs give reconstruction algorithms that are polynomial in the data. It is an open question if a polynomial algorithm exists to tell a simple polytope from its graph.

PROBLEM 17.4.24

Is there an e -dimensional polytope other than the d -cube with the same graph as the d -cube?

If we go beyond the class of polytopes and consider polyhedral spheres, then

THEOREM 17.4.25 Babson, Billera, and Chan

For every $e \geq 4$ there is an e -dimensional cubical sphere with 2^d vertices whose $\lfloor e/2 \rfloor$ -skeleton is combinatorially isomorphic to the $\lfloor e/2 \rfloor$ -skeleton of a d -dimensional cube.

Another issue of reconstruction for polytopes that was studied extensively is the following: In which cases does the combinatorial structure of a polytope determine its geometric structure (up to projective transformations)? Such polytopes are called *projectively unique*, and the major unsolved problem is

PROBLEM 17.4.26

Are there only finitely many projectively unique polytopes in each dimension?

McMullen constructed projectively unique d -polytopes with $3^{d/3}$ vertices.

EMPTY FACES AND POLYTOPES WITH FEW VERTICES**THEOREM 17.4.27** Perles (1970)

Let $f(d, k, b)$ be the number of combinatorial types of k -skeletons of d -polytopes with $d + b + 1$ vertices. Then, for fixed b and k , $f(d, k, b)$ is bounded.

This follows from

THEOREM 17.4.28 Perles (1970)

The number of empty i -pyramids for d -polytopes with $d + b$ vertices is bounded by a function of i and b .

For a d -polytope P , let $e_i(P)$ denote the number of empty i -simplices of P .

PROBLEM 17.4.29

Characterize the sequence of numbers $(e_1(P), e_2(P), \dots, e_d(P))$ arising from simplicial d -polytopes and from general d -polytopes.

CONJECTURE 17.4.30 Kalai, Kleinschmidt, and Lee

For all simplicial d -polytopes with prescribed h -vector $h = (h_0, h_1, \dots, h_d)$, the number of i -dimensional empty simplices is maximized by the Billera-Lee polytopes $P_{BL}(h)$.

$P_{BL}(h)$ is the polytope constructed by Billera and Lee (see Chapter 15) in their proof of the sufficiency part of the g -theorem. It is quite possible that the conjecture applies also to general polytopes.

17.5 CONCLUDING REMARKS AND EXTENSIONS TO MORE GENERAL OBJECTS

The reader who compares this chapter with other chapters on convex polytopes may notice the sporadic nature of the results and problems described here. Indeed, it seems that our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the right questions. Another feature that comes to mind (and is not unique to this area) is the lack of examples, methods of constructing them, and means of classifying them.

We have considered mainly properties of general polytopes and of simple or simplicial polytopes. There are many classes of polytopes that are either of intrinsic interest from the combinatorial theory of polytopes, or that arise in various other fields, for which the problems described in this chapter are interesting.

Most of the results of this chapter extend to much more general objects than convex polytopes. Finding combinatorial settings for which these results hold is an interesting and fruitful area. On the other hand, the results described here are not sufficient to distinguish polytopes from larger classes of polyhedral spheres, and finding delicate combinatorial properties that distinguish polytopes is an important area of research. Few of the results on skeletons of polytopes extend to skeletons of other convex bodies, and relating the combinatorial theory of polytopes with other aspects of convexity is a great challenge.

17.6 SOURCES AND RELATED MATERIAL

FURTHER READING

Grünbaum [Grü75] is a survey on polytopal graphs with many references (most references for the results in Section 17.1 and related results can be found there). More material on the topic of this chapter can also be found in [Grü67], [Zie95], [KK96], and [BL93]. The references in Ziegler's book [Zie95] and their electronic updates cover most of the results described in this chapter. For a survey on the Hirsch conjecture and its relatives, see [KK87]. More recent results can be found in Kleinschmidt's chapter in [BMS⁺94] and in [Zie95, Section 3.3]. Related results on simplex algorithms are mentioned in Chapter 38 of this Handbook.

Martini's Chapter in [BMS⁺94] is on the regularity properties of polytopes (a topic not covered here; cf. Chapter 16), and contains references on facet-forming polytopes and nonfacets. The original papers on facet-forming polytopes and nonfacets contain many more results, and describe relations to questions on tiling spaces with polyhedra. Other chapters of [BMS⁺94] are also relevant.

RELATED CHAPTERS

- Chapter 13: Basic properties of convex polytopes
- Chapter 15: Face numbers of polytopes and complexes
- Chapter 38: Linear programming in low dimensions
- Chapter 39: Mathematical programming

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