The Development of Musical Tuning Systems

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0 INTRODUCTION

When I was learning elementary music theory my teacher told me that the intervals of the diatonic major scale go "tone, tone, semitone, tone, tone, tone semitone" and I asked the question "Why?". With the benefit of hind sight I am grateful that I did not receive a satisfactory answer. My subsequent search for the origins of the diatonic scale has led to a life long interest in the fascinating subject of musical scale structures and tuning systems.

I returned to the subject in 1984 when I was using a Sinclair ZX81 microcomputer and a Memotech 64K RAM pack together with some custom electronics to make an echo-harmonizer for electric guitar. My problem at that time was figuring out why the device I had constructed did not transpose all intervals with the accuracy I wanted to hear. That prompted the question of what exactly were the intervals I wanted to hear. My search led me to Zarlino's system of just intonation. Although I was not aware of Zarlino's progression of harmonic numbers in the range 90 to 180 I discovered that all the intervals of his just intonation could all be expressed as exact numbers of 360ths, the number of degrees in a circle. Seen as angles on a circle the intervals of just intonation provided all the points required to construct regular 3, 4, 5 and 6 sided figures. At the time this seemed to be of mystical significance, perhaps in much the way that the initial discovery of proportion in musical consonance struck Pythagoras or the discovery of both arithmetic and harmonic means in musical intervals struck Zarlino. No doubt anyone who delves into the subject sufficiently deeply will be rewarded with such apparent insights even though, like the Madlebrot Set, they arise purely from the properties of numbers.

Again I returned to the subject in 1996 when as a mature student I had the opportunity to complete a masters degree in software engineering. For my thesis I undertook the construction of a mathematical model of musical scale structure and tuning systems using the formalisms of software engineering mathematics. Subsequently I pursued this further by constructing a computer program which made it possible to compare the various tunings. Initially the software played only pure tones so I extended it first to include an additive harmonic oscillator working something like a drawbar organ and then added dual AM and FM synthesis with four independent drawbar operators each having their own envelope. The result is the **Midicode Synthesizer** presented on the home page of this web site.

In this essay on tuning systems I seek to share what I have learnt of the subject. I trace the development of western tuning systems from their origins in ancient Greece and Babylon up to the development of equal temperament. I have not ventured into modern tuning systems using the seventh and higher harmonics nor into division of the tonal spectrum other than into octaves of 12 semitones (though my software supports such features).

There are many other excellent web sites on the subject of microtonal tuning systems and I have included links to just some of these on my links page. It is often the case that a different form of words can clarify some aspect of this sometimes complex subject and I hope that my contribution may help. I have included some elementary theory of both music and physical acoustics so that no prior knowledge is assumed. As a teacher, I have tried to make the subject as accessible as possible to students and those new to the subject.

My approach is from a mathematical perspective and I have given derivations for several of the major tuning systems. Wherever possible I present information in tabular form and have included tabular analysis of all possible diatonic and chromatic intervals for some of the classic tunings.

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I hope to extend this web site in the future. In the mean time I have other partially complete material and unreleased software for analysing tuning systems. If you do not find what you are looking for, or wish to point out any errors, please email me.

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1 PHYSICAL ACOUSTICS OF TUNING SYSTEMS

1.1 Oscillations, Period and Frequency

Most musical notes are produced by a mechanical or electronic system which vibrates or oscillates. If the vibrations repeat regularly the *period* of the oscillation is the length of time required for one such vibration and the *frequency* of the oscillation is the number of times the vibration is repeated per second.

1.2 Fundamentals, Pitch, Harmonics and Harmonic Series

All periodic oscillations are either pure tones (sine waves) or are a set of pure tones all sounding together. The most prominent pure tone is known as the *fundamental* and determines the *fundamental frequency* or *pitch* of the tone. The others are known as *harmonics* and have frequencies related to the frequency of the fundamental by integer (whole number) factors.

The process of decomposing a periodic tone into its constituent harmonics is known as *harmonic analysis*. A description of the process was given by Jean Baptiste Joseph Fourier (1768 - 1830) in his thesis *Analytical Theory of Heat*, 1822. The equations now known as the Fourier Series were first published by Daniel Bernoulli (1700 - 1782) in 1728. [EB62]

An *harmonic series* contains all the integral multiples of a pitch for as far as the series extends. For example, a fundamental frequency F has harmonics of frequencies 2F, 3F, 4F, 5F, 6F, etc. The fundamental, F, is also referred to as the first harmonic, 2F the second harmonic and so on. This makes it possible to consider the properties of an harmonic series without having to treat the fundamental separately. Sometimes harmonics other than the fundamental are referred to as *overtones* or *upper partials*: the first overtone is the second harmonic, the second overtone is the third harmonic and so on.

Typically, the strength of the harmonics decreases with harmonic number. The relative strengths of the harmonics determines the *timbre* of the tone.

A set of harmonics does not necessarily include all of the harmonics in a series. For example, the set of odd harmonics includes only odd numbered harmonics.

1.3 Frequency and Intervals

If two notes with a fixed ratio between their pitch are sounded together then they are perceived as being the same distance apart regardless of how high or how low in the audio spectrum they both occur. This is because human perception of pitch is logarithmic in nature (like most human senses) so a ratio of frequencies is perceived as a fixed size step in the pitch scale.

For example, the lowest note of the orchestral compass, that produced by a 32 foot organ pipe, has a frequency of about 16 cycles per second. The note an octave higher has a frequency of about 32 cycles per second. Higher up the audible frequency spectrum the highest note of a piccolo has a frequency of about 4,000 cycles per second and the note an octave lower has a

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frequency of about 2,000 cycles per second. So an octave is represented by a doubling of frequency rather than a separation by some number of cycles per second.

The ratio of the frequency of one note to the frequency of another is the physical basis of an *interval* in music. For example, an octave is the interval 2 / 1. Musical intervals are usually measured from the lower note to the higher in which case numerically the interval must be greater than or equal to 1. Conversely, descending intervals must be numerically less than 1.

1.4 Pitch Class and Octaves

If two notes are sounded together, one having twice the frequency of the other, they are perceived as being 'the same note but different'. The interval between them is one *octave* and has the ratio 2 / 1. All notes having frequencies which are successive doublings of the same fundamental are referred to as being of the same *pitch class* and are one or more octaves apart.

If one note has a fundamental frequency F then its harmonics will have the frequencies 2F, 3F, 4F, 5F, 6F, etc. If another note has a fundamental an octave higher, 2F, its harmonics will have frequencies 4F, 6F, 8F, 10F, 12F, etc. These exactly match every other harmonic of the first series, hence the perceived similarity of notes in the same pitch class and the physical basis of the octave.

1.5 Transposing by an Interval

In numerical terms an interval is the ratio between two pitches. If the pitch of a tone is multiplied by an interval then the pitch changes. This is the mathematical basis of *transposition* in music. For example to transpose a pitch up by an octave it is multiplied by the interval 2 / 1, i.e. it is doubled.

To transpose down the interval is *inverted* or reversed. For example, to transpose down an octave multiply by 1 / 2, i.e. halve the pitch.

1.6 Simple and Compound Intervals

Intervals of less than one octave (less than a doubling of pitch or frequency) are referred to as *simple intervals*. Intervals greater than one octave can be composed of one or more octaves plus a simple interval and are called *compound intervals*.

1.7 The 3rd Harmonic and the Interval of a Fifth

The relationship of a frequency, F, and its double, 2F, is an octave, 4F a further octave and so on. But what of a frequency three times that of the fundamental, 3F? This is about an octave and a half above the fundamental and is the interval known to musicians as a *twelfth*, or one octave plus a *fifth* (because it is that many steps in the conventional musical scale). This interval is the second most important interval after the octave.

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The interval of a twelfth is represented by the ratio 3 / 1. To obtain the interval of a fifth we must reduce this by an octave, 1 / 2, so the interval of a fifth is represented by the ratio 3 / 2, three times the fundamental but transposed down an octave. A fifth is the interval which exists between the 2nd and 3rd harmonics of an harmonic series.

1.8 Inversion of an Interval and the Interval of a Fourth

The process of subtracting an interval from an octave is called *inversion* of the interval. Because of the logarithmic nature of the scale of pitch the process of 'subtraction' is in fact one of division. For example, to subtract a fifth, 3 / 2 from an octave, 2 / 1, we divide (cross multiply) 2 / 1 by 3 / 2 giving 4 / 3.

In musical terminology intervals are counted in steps of the scale including both end points. The octave is 8 steps. A fifth is 5 steps. To invert a fifth, i.e. subtract it from an octave, yields a fourth or 4 steps because the end points are counted in each case.

The interval of a fourth has the ratio 4 / 3 and like the fifth, 3 / 2, is of crucial importance to the construction of musical scales. It is the interval that exists between the 3rd and 4th harmonics.

Fifths and fourths are called *perfect fifths* and *perfect fourths* because they occur in both the major and minor scale.

1.9 The 5th Harmonic and the Intervals of a Third

The 1st, 2nd, 4th and 8th harmonics all represent successive doublings of pitch and are therefore octaves,. The 3rd harmonic gives rise to the intervals of a twelfth and a fifth and has been discussed above. The 6th harmonic is an octave above the 3rd.

Another harmonic having a direct bearing on the construction of musical tuning systems is the 5th harmonic which is about two and a quarter octaves above the fundamental. In fact it is two octaves plus the interval known as a *major third*. The harmonic major third is thus the interval 5 / 4, the 5th harmonic, 5 / 1, transposed down 2 octaves, 1 / 4. It is the interval that exists between the 4th and 5th harmonics.

The interval between the 5th and 6th harmonics, 6 / 5 is that known as a *minor third*. A minor third, 6 / 5 plus a major third, 5 / 4, is a fifth, 3 / 2 (6 / 5 x 5 / 4 = 30 / 20 = 3 / 2).

1.10 Consonance, Dissonance and Harmonic Intervals

Most tones are not pure tones and therefore contain higher harmonics. If two tones are sounded together they will sound *consonant* or harmonious if there is a good match between the pitches of their upper partials. They will sound *dissonant* or inharmonious if there is a poor match between the frequencies of their upper partials. Consonance and dissonance are relative terms in a spectrum of sound perception influenced by the subjective expectations of the listener.

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An *harmonic interval* is an interval that occurs between two members of an harmonic series. For example in the series 1F, 2F, 3F, 4F, 5F, 6F, etc. the intervals 2 / 1, 4 / 1, 3 / 2, 4 / 3 and others are all harmonic intervals. Because the harmonics of harmonics are themselves harmonics any two notes related by an harmonic interval will be at least to some extent consonant. Because the relative strength of the harmonics typically decreases with harmonic number it is the lower numbered harmonics that have the greatest influence on consonance.

The closest possible consonance is *unison*, 1 / 1, the consonance between two identical notes. The octave, 2 / 1, is the next most consonant. The fifth, 3 / 2, and the fourth, 4 / 3, come next. These are the only intervals classified as *perfect concords*. A musical scale constructed on such consonant intervals should itself be consonant.

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2 ANCIENT GREEK ORIGINS OF THE WESTERN MUSICAL SCALE

2.1 Proportion and Harmony

The ancient Greek philosopher Pythagoras (?580 - ?500 B.C.) is generally credited with having discovered that musical intervals which are recognized as concordant are related by small integer ratios, an idea he may have acquired from Babylon. [ABRAHAM] It is likely that he determined this result using a monochord, a single stringed instrument having a moveable bridge by means of which the string can be divided into two parts of variable proportion.

Strings with lengths in the ratio 2 : 1 produced the interval of an octave known to the ancient Greeks as *diapason,* Those in the proportion 3 : 2 produced the interval of the fifth, known to the Greeks as *diapente*. Strings of equal tension with length in the proportion 4 : 3 produced the interval of a fourth known to the Greeks as *diatessaron.* The Greek word *dia* meant between, through or across.

All of these intervals are present between strings with relative lengths 2, 3 and 4. Thus the most harmonious of intervals are contained in the number progression 1 : 2 : 3 : 4. This reinforced the concept of spacial and musical harmony being related and the belief that the harmony of the entire universe was inherent in the mystical power of numbers.

Pythagoras himself left no written record of his work so it was via his pupil Philolaus that these observations have been passed on. The first record of the use of a monochord to demonstrate this phenomena was by Euclid (c. 300 B.C.).

2.2 Tetrachord

The basic musical scale unit of ancient Greece was the *tetrachord* meaning literally four strings. The first and fourth notes of the tetrachord were always tuned to the interval of a diatessaron (fourth) but the tuning of the other strings depended on the genus and mode of the music. In the ancient Greek system notes of a scale were arranged in descending order.

In the diatonic genus the tuning of the other intervals comprised two tones and a semitone. The chromatic genus comprised a minor third (three semitones) and two semitones. The enharmonic mode comprised a major third (two tones) and two quarter tones. [EB]

Prior to Pythagoras there appears to be little evidence of a theoretical basis for the tuning of musical scales. Pythagoras was involved with the science of harmonics which was separate from the practical art of music. In the absence of a theoretical basis for the tuning of scales the actual tuning can only have been empirical and probably varied widely.

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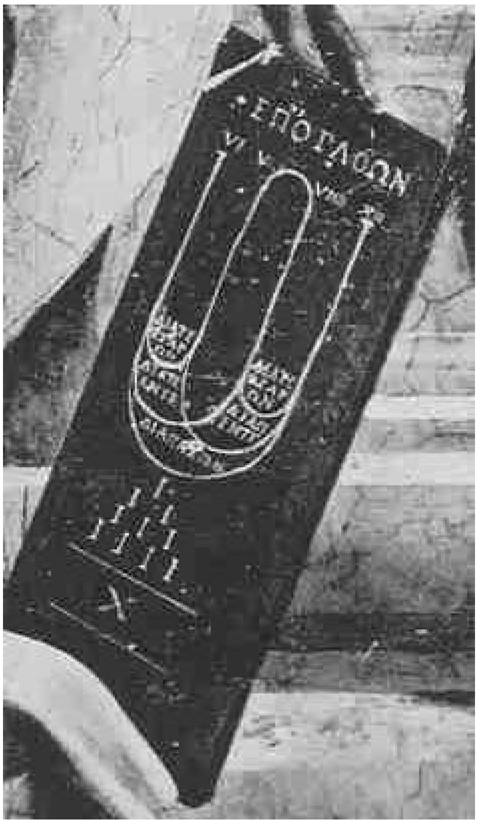
2.3 Diatonic Division of an Octave

The intervals of diatessaron or fourth (4 / 3) and diapente or fifth (3 / 2) when combined (multiplied) give the interval of a diapason or octave (2 / 1). The interval between them (divide) is 9 / 8 which is the interval recognised as a *whole tone*. Indeed, this division of the octave provides a theoretical basis for the tuning of a tone. Furthermore, the division of the octave into two tetrachords separated by a tone is the basis of the *diatonic scale*, *dia tonic*, having a tone between the two tetrachords which are then referred to as *diatonic tetrachords*.

This principle is illustrated in a detail from the painting *School of Athens* by Raphael (1483 - 1520) on the wall of the *Stanza della Segnatura* in the Vatican. In a diagrammatic representation of a lyra the strings are tuned in the relative proportions VI, VIII, VIIII, XII giving the intervals of an octave (diapason) between VI and XII, fifths (diapente) between VI, VIIII and VII, XII and fourths (diatessaron) between VI, VIII and VIIII, XII. The interval of a tone is present between VIII and VIIII. [WITTKOWER]



Detail from Raphael's *School of Athens*

	TONE		
VI	VIII	VIIII	XII
DIATESSARON		DIATESSARON	
DIAPENTE			
	DIAPENTE		
DIAPASON			

2.4 Greater Perfect System

Possibly the most common division of the tetrachord was into two tone steps plus what was left, a half a tone or *semitone* step, in descending order. Using modern note names this might be represented as E - D - C - B (descending order). If a second *disjunct* tetrachord separated from the first by a tone was added then a *diatonic scale* of one octave resulted: E - D - C - B A - G - F - E.

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This was further extended above and below by the addition of two *conjunct* tetrachords which each shared one note with the existing tetrachords. Theorists added a further tone to the bottom of the series to complete a two octave span. This two octave *disdiapason* was called the *Greater Perfect System*. [EB]

			TETRACHORD				TETRACHORD							
A	G	F	E	D	C	B	A	G	F	E	D	C	B	A
TETRACHORD										TETRACHORD				

Greater Perfect System

2.5 Ancient Greek Modes

The scale presented above has semitone intervals between C and B and also F and E. All the rest of the intervals are whole tones. A different scale pattern in terms of the position of tone and semitone intervals may thus be produced by traversing the scale from different starting points.

The ancient Greeks distinguished *harmoniai* which were scale patterns in the greater perfect system from *tonoi* which were the modes used in tuning and performance of a stringed instrument. Tonos meant to stretch or tension.

The seven possible harmoniai are given in the following table. [EB] Unspaced letters show semitone intervals.

E D CB A G FE	Dorian
D CB A G FE D	Phrygian
CB A G FE D C	Lydian
B A G FE D CB	Mixolydian
A G FE D CB A	Hypodorian
G FE D CB A G F	Hypophrygian
FE D CB A G F	Hypolydian

Ancient Greek Modes

2.6 Intervals in the Greater Perfect System

The number series 4 : 6 : 8 : 9 :12 :16 is often associated with the greater perfect system and the theory of proportion. [WITTKOWER] The intervals between members of this series are fifth, fourth, tone, fourth and fourth. These intervals give the tuning for the fixed intervals of the tetrachords of the greater perfect system.

A	G	F	E	D	C	B	A	G	F	E	D	C	B	A
4				6			8	9			12			16

Intervals of the Greater Perfect System

The proportions of the greater perfect system are illustrated on the frontispiece of *Theorica Musice* published in 1492 by the musical theorist Franchino Gafurio (1451 - 1522). The

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blacksmith's hammers, bells, glasses, weighted strings and pipes all bear the numbers 16, 12, 9, 8, 6 and 4.

Frontispiece to *Theorica Musice*, Franchino Gafurio, 1492

The upper left illustration depicts Jubal, the biblical father of music, and six blacksmiths with differing size hammers striking an anvil. This relates to the story that the young Pythagoras was first moved to investigate musical intervals on hearing the notes produced by different size hammers at a blacksmith's shop. The upper right illustration depicts Pythagoras testing the interval of an octave between bells of size 16 and 8 and between glasses filled in the proportion 16 and 8. The lower left illustration shows Pythagoras testing intervals on a stringed instrument and the lower right illustration shows Pythagoras and his pupil Philolaus testing intervals by means of flutes.

Actually, the hammers, bells, glasses and flutes would produce an ascending series of intervals when arranged in the order 16, 12, 9, 8, 6, 4 whereas the weighted strings would produce a descending series. The pitch of a string is proportional to the square root of its tension. [JOHNSTON]

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2.7 Pythagorean Tuning

Pythagoras is credited with having devised a system of tuning based solely upon the interval of a fifth, the next most consonant interval after unison and the octave. One way in which he may have done so was by constructing a series of both descending and ascending fifths from the same starting point. [JOHNSTON]

From a starting point 1 / 1 the ascending fifths are produced by successively multiplying by 3 / 2 to give 3 / 2, 9 / 4, 27 / 16. The descending fifths are produced by successive multiplication by 2 / 3 to give 2 / 3, 4 / 9, 16 / 27. The process amounts to raising the numbers 2 and 3 to successively higher powers and the sequence is built upon powers of just these two numbers. Pythagoras believed that it should not be necessary to go beyond the third power (cube) of a number as one cannot go beyond length, width and depth. [WITTKOWER] This construction of a tuning system is consistent with such a view; schemes in which the fifths are either all ascending or all descending are not.

Although Pythagoras was primarily concerned with the theoretical science of harmonics this scheme is also consistent with practical tuning of an eight stringed instrument in the ancient Greek Phrygian mode. This mode has symmetrical tone, semitone, tone intervals within each tetrachord. The manner of tuning may be illustrated with modern note names. First tune an octave between the two outer strings, D D. The other end of each tetrachord is a fifth from one of these or a fourth from the other, D A G D (descending). Two further notes lie a fourth from these middle notes, D C A G E D. The final two notes lie a fifth from these D CB A G FE D. Unspaced letters indicate semitone steps.

In the table below left the first column illustrates the descending and ascending series of fifths. In order to keep all the pitches within the same (descending) octave it is necessary to transpose them up or down an octave or two if they would otherwise be outside the octave. This is illustrated in the second column. The table below right shows the fifths sorted into descending order of pitch with a corresponding modern note name. The third of this scale is what we now call a *minor third* comprising one tone and one semitone.

Original fifth	Transposed fifth
27 / 8	27 / 32
9 / 4	9 / 16
3 / 2	3 / 4
1 / 1	1 / 1
2 / 3	2 / 3
4 / 9	8 / 9
8 / 27	16 / 27

Descending and Ascending Fifths

Sorted fifth	Note name	Interval to next
1 / 1	D	8 / 9
8 / 9	C	243 / 256
27 / 32	B	8 / 9
3 / 4	A	8 / 9
2 / 3	G	8 / 9
16 / 27	F	243 / 256
9 / 16	E	8 / 9
1 / 2	D	8 / 9

Thus was born the mathematical basis for musical tuning systems.

Analysis of all possible intervals within this tuning scale is given in the table below. Colour is used to indicate similar intervals. The scale is shown in descending order and the intervals are indicated as descending intervals in keeping with ancient Greek custom. The first main

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column of the table repeats the tunings from the table above right. Each main column of the table gives the interval from the note named at the head of the column (using modern note names) to the next lower note named at the left of each row.

	D	C	B	A	G	F	E	D
D	1 / 1	9 / 16	16 / 27	2 / 3	3 / 4	27 / 32	8 / 9	1 / 2
C	8 / 9	1 / 1	128 / 243	16 / 27	2 / 3	3 / 4	64 / 81	8 / 9
B	27 / 32	243 / 256	1 / 1	9 / 16	81 / 128	729 / 1024	3 / 4	27 / 32
A	3 / 4	27 / 32	8 / 9	1 / 1	9 / 16	81 / 128	2 / 3	3 / 4
G	2 / 3	3 / 4	64 / 81	8 / 9	1 / 1	9 / 16	16 / 27	2 / 3
F	16 / 27	2 / 3	512 / 729	64 / 81	8 / 9	1 / 1	128 / 243	16 / 27
E	9 / 16	81 / 128	2 / 3	3 / 4	27 / 32	243 / 256	1 / 1	9 / 16
D	1 / 2	9 / 16	16 / 27	2 / 3	3 / 4	27 / 32	8 / 9	1 / 1

Intervals of Pythagorean Tuning - Ancient Greek Phrygian Mode (Descending)

For a key to the colours using modern interval names, click the button.

All possible five note steps in the scale are true harmonic fifths (2 / 3), except that between F and the B below, which is a semitone smaller. All possible four note steps are true harmonic fourths (3 / 4), except that between B and the F below. which is a semitone larger. The size of these two intervals differs by a small amount known as the *Pythagorean comma*, a matter of great importance to which we shall return.

The two note steps are all either whole tones (8 / 9) or semitones (243 / 256). The three note steps are all either tone plus semitone (27 / 32) or two tones (64 / 81), an interval called *ditone* (major third). All the six note steps are either four tones plus a semitone (16 / 27) or four tones (81 /128). All seven note steps are either five tones (9 / 16) or five tones plus a semitone (128 / 243). The tuning of all these intervals is completely consistent throughout.

To Pythagoras, the discovery that musical intervals were related to arithmetic was of mystical significance and symptomatic of an underlying harmony throughout the cosmos. Observing that there were the same number of heavenly bodies as there were notes in the musical scale he ascribed to the heavenly bodies orbits with musical proportions which he described as the *Music of the Spheres.*

Later Greek philosophers and writers including Plato (?427 - ?347 B.C.), Aristotle (384 - 322 B.C.) and Aristoxenus (c. 320 B.C.) were all influenced by ideas originating from the Pythagorean school. Aristoxenus, however, believed that musical intervals should be judged by ear and not by mathematical ratio.

2.8 Ptolemy

An account of ancient Greek contributions to musical tuning would not be complete without mentioning the later Greek scientist Ptolemy (2nd c. A.D.). He proposed an alternative musical tuning system which included the interval of the major third based on that between the 4th and 5th harmonics, 5 / 4. This system of tuning was ignored during the entire Medieval

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period and only re-surfaced with the development of polyphonic harmony.

2.9 Pentatonic Scale

If the last descending and ascending fifths are omitted from the Pythagorean tuning sequence the resulting scale is a mode of the *pentatonic scale*. This scale type occurs and predominates in many types of music including ancient Chinese, far eastern, and European folk music, for example Celtic music.

The form which arises most naturally from both ascending and descending fifths involves only two steps in each direction. It is shown here in ascending order (reading down the table). The thirds of the scale are minor thirds and, in the mode given, alternate with tone intervals and are symmetrically placed.

There are five modes of the pentatonic scale, one starting on each degree. Using unspaced letters to indicate tone steps and spaced letters to indicate minor thirds the modes are DE GA C, E GA CD, GA CDE , A CDE G, CDE GA.

Sorted fifth	Step in sequence of fifths	Note name	Interval to next
1 / 1	0	D	9 / 8
9 / 8	2	E	32 / 27
4 / 3	-1	G	9 / 8
3 / 2	1	A	32 / 27
16 / 9	-2	C	9 / 8

Pentatonic Scale

In a sense the two members of the Pythagorean tuning which are absent are the two least satisfactory members of that series. In Pythagorean tuning the awkward and inharmonious *tritone* (three tones) arises between the notes B and F (using modern note names). As these members are not present in the Pentatonic scale there is no tritone. All the fourths and fifths are perfect.

	D	E	G	A	C
D	1 / 1	16 / 9	3 / 2	4 / 3	9 / 8
E	9 / 8	1 / 1	27 / 16	3 / 2	81 / 64
G	4 / 3	32 / 27	1 / 1	16 / 9	3 / 2
A	3 / 2	4 / 3	9 / 8	1 / 1	27 / 16
C	16 / 9	128 / 81	4 / 3	32 / 27	1 / 1

Intervals of the Pentatonic Scale

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3 MEDIEVAL THEORY AND PRACTICE

3.1 The Greek Heritage

Knowledge of the ancient Greek musical system was available to medieval theorists mainly through the writings of Anicius Manlius Severinus Boethius (?480 - ?524 A.D.). Boethius was possibly the last Roman writer who understood Greek. Most of the musical theory he passed on was concerned with Pythogoreanism. [JOHNSTON]

In his book *De Institutione Musica* Boethius presents music as a numerical science in which consonance and the intervals to be permitted in melody and tuning are strictly determined by numerical ratios. [GROUT] His work was a primary influence on the Medieval era.

Although Boethius was familiar with the Greeks musical letter notation he used Roman letters in *De Institutione Musica*. To further confuse the issue he used A to represent the lowest note whereas in Greek notation alpha represented the highest note. [ABRAHAM] This may possibly have led to a mistaken belief that Greek scales were of ascending rather than descending order.

3.2 Medieval Pythagorean Tuning

From the knowledge that Pythagoras devised a system of tuning based solely on perfect fifths, medieval theorists set about constructing a tuning on that basis. Such a scheme is documented in 9th and 10th century organ building guides. [SCHULTER]

The scheme usually presented as the mathematical basis for Pythagorean tuning is as follows. Starting from 1 / 1, the series of ascending fifths may be constructed arithmetically by successive multiplication of 3 / 2 as illustrated in the first column of the table below left. To keep all the notes in the same octave, some of them need to be transposed down by a number of octaves as illustrated in the second and third columns. The table below right shows the resultant series of fifths sorted by ascending order of pitch together with modern note names.

Original fifth	Octaves to drop	Transposed fifth
1 / 1	0	1 / 1
3 / 2	0	3 / 2
9 / 4	1	9 / 8
27 / 8	1	27 / 16
81 / 16	2	81 / 64
243 / 32	2	243 / 128
729 / 64	3	729 / 512

Series of Fifths

Sorted fifth	Note name	Interval to next
1 / 1	F	9 / 8
9 / 8	G	9 / 8
81 / 64	A	9 / 8
729 / 512	B	256 / 243
3 / 2	C	9 / 8
27 / 16	D	9 / 8
243 / 128	E	256 / 243
2 / 1	F	9 / 8

Sorted Series of Fifths

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To arrive at such a tuning in practice one would alternately tune upward fifths and downward fourths so as to remain within the same octave.

This is probably not the method actually used by Pythagoras for at least two reasons. First, Pythagoras believed that it should not be necessary to go beyond the third power of any number (see section 2.7 above) yet this method involves the fourth, fifth and sixth powers of 2 and 3. Second, the method does not yield a strict diatonic scale in the sense that the interval in the middle is a semitone not a tone (*dia tonic*) and the first four note group is not a tetrachord as it does not span a perfect fourth. Using a descending series of fifths leads to similar problems. Furthermore, the system of both ascending and descending fifths mentioned in section 2.7 yields the ancient Greek Phrygian mode which is equivalent to the Medieval Dorian mode. This is listed as Mode I in the Gregorian system, possibly because it arises first from the ancient Greek tuning system.

Another significant difference between a sequence of both descending and ascending fifths and a sequence of only ascending fifths is that the former method generates a third scale degree of a minor third whilst the latter generates a major third. Both of these scale degrees differ from equivalent intervals in the series of harmonics, 6 / 5 for the minor third and 5 / 4 for the major third, by 81 / 80, roughly an eighth of a tone, an interval known as the *syntonic comma* or *comma of Didymus*.

The intervals of the scale are of two distinct sizes, 9 / 8 and 256 / 243. The logarithm of the interval 9 / 8 (0.117783) is very roughly twice the logarithm of the interval 256 / 243 (0.052116) so the interval 9 / 8 sounds like a step twice the size of the step 256 / 243. These are *tone* and *semitone* steps.

3.3 Modes

Because the diatonic scale has two semitone steps and five tone steps, different patterns of intervals arise by traversing it from different starting points. These different starting points are referred to as *modes* of the scale.

Gregorian modes are identified by number and are named after Pope Gregory I (?540 - 604 A.D.). The ancient Greek mode names were re-applied by some authors in the 10th century. Possibly because they misinterpreted Boethius, the names of the modes are not consistent with the ancient Greek usage. [GROUT]

Modal music of the Middle Ages and later has a final note for each mode with which the melody (usually) ends. In authentic modes the final is the last note of the note range. In plagal modes the final note is the fourth of the scale. The Hypomixolydian mode, having no counterpart in the Greek system, differs from the Dorian mode only in its final.

In the following table the final is indicated in bold. Unspaced letters show semitone intervals.

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Authentic		Plagal	
I Dorian	**D** EF G A BC D	II Hypodorian	A BC **D** EF G A
III Phrygian	**E**F G A BC D E	IV Hypophrygian	BC D **E**F G A B
V Lydian	**F** G A BC D EF	VI Hypolydian	C D E**F** G A BC
VII Mixolydian	**G** A BC D EF G	VIII Hypomixolydian	D EF **G** A BC D

Medieval Modes

The only four and five note steps in the diatonic scale which are not true harmonic fourths and fifths are those between F - B and B - F. This interval, equal to three whole tones and called a *tritone*, is regarded as most inharmonious. In order to avoid this interval the note B was sometimes lowered by a semitone to a note called B flat (indicated in this article as Bb). An alternative solution was to raise the F by a semitone to F sharp (F#).

The use of Bb, particularly in the Lydian and Dorian modes actually created two new modes. The Lydian mode with Bb is equivalent to the modern major mode, as is the Mixolydian mode with F#. Dorian mode with Bb is equivalent to the modern natural minor mode. However, for a long time they were not regarded as new modes.

3.4 Guido of Arezzo

Guido of Arezzo (?995 - ?1050), a Benedictine monk for parts of his life, was a musician and very influential musical theorist. He is credited with having greatly enhanced the emerging system of written musical notation and of having devised a musical scale structure known as the *hexachord*.

The hexachord was primarily conceived to facilitate the teaching of music and according to Guido it enabled a pupil to learn in five months what might previously have taken ten years. The syllables *ut, re, mi, fa, sol, la* comprised a hexachord having the intervals tone, tone, semitone, tone, tone. For example, it might correspond with the note sequence C D EF G A. An important characteristic was that only a single semitone interval was included and this was in the middle between two three note groups.

Because of its simplified interval structure the hexachord was easy to sing. The names of the scale degrees were derived from the first six lines of a Latin hymn to St. John the Baptist. In a musical setting which may have been created by Guido for the purpose, each line started on the appropriate pitch for that note. At a later time the seventh syllable *si* was added based on the initial letters of the last line.

Ut queant laxis *re*sonare fibris
*Mi*ra gestorum *fa*muli tuorum,
*Sol*ve polluti *la*bii reatum,
Sancte Joannes.

That thy servants may freely sing
forth the wonders of thy deeds,
remove all stain of guilt from their
unclean lips, O Saint John.
[GROUT]

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The notation is still in use in France and Italy for note names and persists in England in the song '*Do, a deer, a female deer...*' used to teach children in the musical *Sound of Music*. The tune may have changed but the utility of the hexachord structure as a teaching aid has stood the test of time.

The hexachord could be found at three places in the existing scale, G A BC D E, C D EF G A or F G ABb C D. The natural form of B was called 'hard' and the flattened B was called 'soft' so the hexachord starting on G was called 'hard' (durum) and the hexachord starting on F was called 'soft' (molle). These later became the names of the major (dur) and minor (moll) modes of today. The hexachord starting on C was called 'natural'.

The lowest note on which a hexachord might start was bass G which was identified by the Greek letter gamma. This hexachord was called gamma ut which is the origin of the word *gamut*, now taken to mean an entire range or compass. A more narrow interpretation of the word gamut is the set of notes comprising the white notes plus Bb, the notes of *musica recta*. In order to move beyond the six notes of the hexachord it was necessary to transfer to a new hexachord starting on the fourth or fifth degree of the old one. This process, called *mutation*, involved treating a single note as belonging to both hexachords, a process analogous to modern key modulation.

3.5 Henricus Glareanus

Modal theory was revised by Heinrich Loris, also known as Henricus Glareanus, in his publication *Dodecachordon* published in 1547. Glareanus added four new modes which were equivalent to the modified Dorian and Lydian modes with a Bb. The new Aeolian and Ionian modes provided the same interval structure but did not require the use of Bb. These were the precursors of the modern minor and major scale modes.

Glareanus also mentions the purely theoretical Lochrian and Hypolocrian modes. Because these would require B as their final they were unusable on account of the tritone from B to F.

Authentic		Plagal	
IX Aeolian	**A** BC D EF G A	X Hypoaeolian	EF G **A** BC D E
(Locrian)	(**B**C D EF G A B)	(Hypolochrian)	(F G A **B**C D EF)
XI Ionian	**C** D EF G A BC	XII Hypoionian	G A B**C** D EF G

Extended Modes

The older modal scales which were mainly applicable to melodic music are also called *church modes* or *ecclesiastical modes*.

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3.6 Pythagorean Tuning of the Diatonic Major Scale

Suppose the system of tuning based on ascending fifths is transposed to start a fourth lower, starting with C rather than F. This can be achieved by multiplying all the intervals by 2 / 3. The notes F, G, A and B are also transposed up an octave (multiply by 2 / 1).

The fourth of the scale is a true fourth and there are two identical tetrachords separated by a tone, a strict diatonic scale. The sequence of intervals in each tetrachord, tone, tone, semitone, is the same as the ancient Greek Lydian modes except that the scale is reversed, ascending not descending. The pattern also matches the intervals of Guido of Arezzo's hexachord and the Ionian mode described by Henricus Glareanus. It is the scale now known as the *diatonic major scale*.

Original fifth	Transposed fifth	Note name	Interval to next step	
3 / 2	1 / 1	C	9 / 8	tetrachord
27 / 16	9 / 8	D	9 / 8	
243 / 128	81 / 64	E	256 / 243	
1 / 1	4 / 3	F	9 / 8	
9 / 8	3 / 2	G	9 / 8	tetrachord
81 / 64	27 / 16	A	9 / 8	
729 / 512	243 / 128	B	256 / 243	
	2 / 1	C	9 / 8	

Diatonic Major Scale - Pythagorean Tuning

In the following table all possible intervals of the diatonic major scale with Pythagorean tuning are analysed. Colour is used to indicate similar intervals. The first main column of the table repeats the tunings from the table above. Each main column of the table gives the interval from the note named at the head of the column to the next higher note named at the left of each row.

	C	D	E	F	G	A	B	C
C	1 / 1	16 / 9	128 / 81	3 / 2	4 / 3	32 / 27	256 / 243	1 / 1
D	9 / 8	1 / 1	16 / 9	27 / 16	3 / 2	4 / 3	32 / 27	9 / 8
E	81 / 64	9 / 8	1 / 1	243 / 128	27 / 16	3 / 2	4 / 3	81 / 64
F	4 / 3	32 / 27	256 / 243	1 / 1	16 / 9	128 / 81	1024 / 729	4 / 3
G	3 / 2	4 / 3	32 / 27	9 / 8	1 / 1	16 / 9	128 / 81	3 / 2
A	27 / 16	3 / 2	4 / 3	81 / 64	9 / 8	1 / 1	16 / 9	27 / 16
B	243 / 128	27 / 16	3 / 2	729 / 512	81 / 64	9 / 8	1 / 1	243 / 128
C	2 / 1	16 / 9	128 / 81	3 / 2	4 / 3	32 / 27	256 / 243	1 / 1

Analysis of Pythagorean Tuning

Like Pythagorean tuning of the ancient Greek Phrygian mode all possible five note steps in the scale are true harmonic fifths (3 / 2), except that between B and the F, which is a semitone

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smaller. All possible four note steps are true harmonic fourths (4 / 3), except that between F and the B which is a semitone larger.

The two note steps are all either whole tones (9 / 8) or semitones (256 / 243). The three note steps are all either two tones (81 / 64) or tone plus semitone (32 / 27) . All the six note steps are either four tones (128 / 81) or four tones plus a semitone (27 / 16) . All seven note steps are either five tones (16 / 9) or five tones plus a semitone (243 / 128). The tuning of all these intervals is completely consistent throughout.

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4 TUNING INTO THE RENAISSANCE

4.1 The Emergence of Polyphony

Possibly the earliest form of music in which notes were sounded together other than in unison or at octaves was the practice of *organum* in which the melody was followed by an accompaniment, a practice dating from about the 9th and 10th centuries. In *parallel organum* the interval between the parts was a fixed fourth or fifth. In *free organum* the interval varied but the consonant intervals fourth, fifth, octave and unison were used at key points in the melodic phrase. In England the practice of using major thirds in organum developed in the late 12th century.

Development of the motet, a form of sacred song usually for three voices, helped carry the development of polyphony through the 13th to 15th centuries. The birth of *Ars Nova*, the new art of the 14th century renaissance saw an increasing shift in musical emphasis from melody to counterpoint and harmony. With this shift came an increasing use of the imperfect consonances of thirds and sixths. But it was not until the 16th century that the efficacy of Pythagorean tuning was called into question.

4.2 Chromatic Scale

Extensions of the modal system to include Bb and F# have been mentioned above (section 3.3). Further exploration of the modal system gradually introduced other semitone steps, particularly Eb, C# and G#. Treating these new notes as modifications to existing notes enables a scale to still include all the note names but with one or more modified by an *accidental*, for example F G A Bb C D E F. In *musica ficta* these accidentals (other than Bb) were understood by convention rather than notation as composers were reluctant to use notes outside of the system authorised by Guido of Arezzo. [GROUT]

These semitone modifications to the tonal scale led to a division of the octave into 12 semitone steps, a *chromatic* scale. The theorist Jacobus of Liege remarked circa 1325 that 'keyboards now have all the whole-tones divided into their unequal semitones'. [SCHULTER]

Pythagorean tuning was still in use and the logical way to tune the new notes was as a fifth from existing notes, so Bb may be tuned as fifth down from the F above and F# as a fifth up from B below.

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Transposed fifth	Note name	Interval to next step
1 / 1	C	9 / 8
9 / 8	D	9 / 8
81 / 64	E	256 / 243
4 / 3	F	2187 / 2048
729 / 512	F#	256 / 243
3 / 2	G	9 / 8
27 / 16	A	256 / 243
16 / 9	Bb	2187 / 2048
243 / 128	B	256 / 243
2 / 1	C	9 / 8

Pythagorean Diatonic Tuning with Added Bb and F#

This results in two slightly different sizes for the semitone step. The steps from A to Bb or F# to G have the value 256 / 243, the same as the existing semitone steps between E and F or B and C. This interval is called a *Pythagorean diatonic semitone*. The semitone steps between Bb and B or F and F#, which do not represent steps of any diatonic scale and would rarely be sounded successively in music of the time, have the value 2187 / 2048. This interval is called a *Pythagorean chromatic semitone* or *apotome*. The two intervals differ by 531441 / 524288, a small interval known as the *Pythagorean comma*, about an eighth of a tone. This is the amount by which a full cycle of 12 fifths differs from 7 octaves, a matter to which we shall return in the section on temperament.

The remaining semitones of a full chromatic scale may be added. Eb may be derived by tuning down a fifth from Bb. C# may be derived by tuning a fifth up from F# and transposing down an octave. G# may be derived by transposing up a fifth from C#.

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Transposed fifth	Note name	Interval to next step
1 / 1	C	2187 / 2048
2187 / 2048	C#	256 / 243
9 / 8	D	256/ 243
32 / 27	Eb	2187 / 2048
81 / 64	E	256 / 243
4 / 3	F	2187 / 2048
729 / 512	F#	256 / 243
3 / 2	G	2187 2048
6561 / 4096	G#	256 / 243
27 / 16	A	256 / 243
16 / 9	Bb	2187 / 2048
243 / 128	B	256 / 243
2 / 1	C	9 / 8

Early Pythagorean Chromatic Tuning (Eb x G#)

A problem with this tuning is that the fifth between G# and Eb is 262144 / 177147 which differs from a true fifth of 3 / 2 by 531441 / 524288, the Pythagorean comma. This fifth is known as the *wolf fifth* because when played on Gothic organs it reminded listeners of howling wolves. The tuning is known as Eb x G# from the position of the rogue interval.

From around 1400 Pythagorean tuning was probably modified to treat all the chromatic notes as flats. [SCHULTER] There is still a wolf fifth, now between B and F# but the tuning is better suited to musical styles employing more thirds and sixths. Arithmetically, these more harmonious intervals are reflected in the smaller numbers of the ratios of the revised tunings implying agreement of harmonics at an earlier point in the series. This tuning is known as F# x B. [SCHULTER]

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Transposed fifth	Note name	Interval to next step
1 / 1	C	256 / 243
256 / 243	Db	2187 / 2048
9 / 8	D	256/ 243
32 / 27	Eb	2187 / 2048
81 / 64	E	256 / 243
4 / 3	F	256 / 243
1024 / 729	Gb	2187 / 2048
3 / 2	G	256 / 243
128 / 81	Ab	2187 / 2048
27 / 16	A	256 / 243
16 / 9	Bb	2187 / 2048
243 / 128	B	256 / 243
2 / 1	C	9 / 8

Late Pythagorean Chromatic Tuning (F# x B)

4.3 Just Intonation

Although almost all the fourth and fifths of Pythagorean tuning correspond exactly with harmonic ratios the thirds and sixths do not. As mentioned above (section 3.2), the Pythagorean major third, 81 / 64, differs from the true harmonic major third, 5 / 4, by a small amount, 81 / 80, known as the syntonic comma or comma of Didymus. The Pythagorean minor third, 32 / 27, differs from the true harmonic minor third, 6 / 5, by exactly the same amount. The same holds true for major and minor sixths.

As early as around 1300 the English theorist Walter Odington pointed out that major and minor thirds have intervals close to 5 / 4 and 6 / 5 and that singers tend towards these intervals. [SCHULTER] In 1482 the Spanish theorist Bartolomé Ramos de Pareja proposed that tuning should be modified to incorporate more harmonious thirds and sixths. The scheme that he proposed was a form of mean tone temperament which is discussed in the next section (5.2).

The influential theorist Franchino Gaffurio (1451 - 1522) uncovered the tuning system devised by the ancient Greek Ptolemy which used natural harmonic thirds. Gaffurio himself was a conservative and opposed to the introduction of a new tuning system. The new scheme was championed primarily by Lodovico Fogliano in *Musica Theorica* of 1529 and Gioseffo Zarlino (1517 - 1590) in *Institutioni Armoniche* published in 1558. [GROUT]

The new tuning is usually referred to a *just tuning* or *just intonation.* More generally the term just intonation is taken to mean any tuning system based on (small) natural numbers and thus includes Pythagorean tuning. In the system championed by Zarlino, the major third becomes 5 / 4, the major sixth becomes 5 / 3 and the major seventh becomes 15 / 8 all based on the relation of the 5th harmonic to other harmonics. Zarlino expressed the series as an harmonic progression of numbers from 180 to 90. [WITTKOWER]

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Interval from tonic	Note name	Interval to next step	Zarlino's harmonic series
1 / 1	C	9 / 8	180
9 / 8	D	10 / 9	160
5 / 4	E	16 / 15	144
4 / 3	F	9 / 8	135
3 / 2	G	10 / 9	120
5 / 3	A	9 / 8	108
15 / 8	B	16 / 15	96
2 / 1	C	9 / 8	90

Just Tuning - Zarlino

This system involves two different sized tone steps, 9 / 8 as in previous tunings and the smaller 10 / 9 which is called a *just minor tone*. The difference between them is the syntonic comma, 81 / 80. The semitone has become 16 / 15 and is called a *just diatonic semitone*. It is very slightly smaller than a Pythagorean chromatic semitone.

In the following table all possible intervals of the diatonic major scale with Zarlino's just tuning are analysed. Colour is used to indicate similar intervals. The first main column of the table repeats the tunings from the table above. Each main column of the table gives the interval from the note named at the head of the column to the next higher note named at the left of each row.

	C	D	E	F	G	A	B	C
C	1 / 1	16 / 9	8 / 5	3 / 2	4 / 3	6 / 5	16 / 15	1 / 1
D	9 / 8	1 / 1	9 / 5	27 / 16	3 / 2	27 / 20	6 / 5	9 / 8
E	5 / 4	10 / 9	1 / 1	15 / 8	5 / 3	3 / 2	4 / 3	5 / 4
F	4 / 3	32 / 27	16 / 15	1 / 1	16 / 9	8 / 5	64 / 45	4 / 3
G	3 / 2	4 / 3	6 / 5	9 / 8	1 / 1	9 / 5	8 / 5	3 / 2
A	5 / 3	40 / 27	4 / 3	5 / 4	10 / 9	1 / 1	16 / 9	5 / 3
B	15 / 8	5 / 3	3 / 2	45 / 32	5 / 4	9 / 8	1 / 1	15 / 8
C	2 / 1	16 / 9	8 / 5	3 / 2	4 / 3	6 / 5	16 / 15	1 / 1

Analysis of Just Diatonic Tuning

This tuning system compromises one of the fourths and fifths, that between D and A. All of the major thirds and minor sixths are consistent but one of the minor thirds, D to F, and the corresponding major sixth, F to D are also compromised. As there are two different sized tone steps so also there are two different minor sevenths.

So long as one remains in the Ionian mode, equivalent to the modern major scale, or the Aeolian mode, equivalent to the modern minor scale, then this tuning works well. Perhaps this is why Zarlino gave precedence to the Ionian mode and listed it first whereas Glareanus, using Pythagorean tuning, had seen it as the least satisfactory and listed it last.

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4.4 Chromatic Just Tuning

Just tuning may readily be extended to the full chromatic scale by adding a minor third (6 / 5), minor sixth (8 / 5) and minor seventh (9 / 5) all derived from the 5th harmonic. The minor second (16 / 15) is the interval which already exists between the just third and the fourth. F# is calculated as a fifth above the just B. All of these tunings use smaller numbered ratios than the corresponding Pythagorean tunings and therefore have coincident harmonics lower in the series. The semitone steps, however, are of four different sizes.

Interval from tonic	Note name	Interval to next step
1 / 1	C	16 / 15
16 / 15	Db	135 / 128
9 / 8	D	16 / 15
6 / 5	Eb	25 / 24
5 / 4	E	16 / 15
4 / 3	F	135 / 128
45 / 32	F#	16 / 15
3 / 2	G	16 / 15
8 / 5	Ab	25 / 24
5 / 3	A	27 / 25
9 / 5	Bb	25 / 24
15 / 8	B	16 / 15
2 / 1	C	9 / 8

Chromatic Just Tuning

A full chromatic analysis of just tuning is given in the table below. Colour is used to indicate similar intervals (the colours are not the same as those used for the diatonic charts). The first main column of the table gives the tunings as tabled previously. Each main column of the table gives the interval from the note named at the head of the column to the next higher note named at the left of each row.

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	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	1 / 1	15 / 8	16 / 9	5 / 3	8 / 5	3 / 2	64 / 45	4 / 3	5 / 4	6 / 5	10 / 9	16 / 15	1 / 1
Db	16 / 15	1 / 1	256 / 135	16 / 9	128 / 75	8 / 5	1024/675	64 / 45	4 / 3	32 / 25	32 / 27	256 / 225	16 / 15
D	9 / 8	135 / 128	1 / 1	15 / 8	9 / 5	27 / 16	8 / 5	3 / 2	45 / 32	27 / 20	5 / 4	6 / 5	9 / 8
Eb	6 / 5	9 / 8	16 / 15	1 / 1	48 / 25	9 / 5	128 / 75	8 / 5	3 / 2	36 / 25	4 / 3	32 / 25	6 / 5
E	5 / 4	75 / 64	10 / 9	25 / 24	1 / 1	15 / 8	16 / 9	5 / 3	25 / 16	3 / 2	25 / 18	4 / 3	5 / 4
F	4 / 3	5 / 4	32 / 27	10 / 9	16 / 15	1 / 1	256 / 135	16 / 9	5 / 3	8 / 5	40 / 27	64 / 45	4 / 3
F#	45 / 32	675 / 512	5 / 4	75 / 64	9 / 8	135 / 128	1 / 1	15 / 8	225 / 128	27 / 16	25 / 16	3 / 2	45 / 32
G	3 / 2	45 / 32	4 / 3	5 / 4	6 / 5	9 / 8	16 / 15	1 / 1	15 / 8	9 / 5	5 / 3	8 / 5	3 / 2
Ab	8 / 5	3 / 2	64 / 45	4 / 3	32 / 25	6 / 5	256 / 225	16 / 15	1 / 1	48 / 25	16 / 9	128 / 75	8 / 5
A	5 / 3	25 / 16	40 / 27	25 / 18	4 / 3	5 / 4	32 / 27	10 / 9	25 / 24	1 / 1	50 / 27	16 / 9	5 / 3
Bb	9 / 5	27 / 16	8 / 5	3 / 2	36 / 25	27 / 20	32 / 25	6 / 5	9 / 8	27 / 25	1 / 1	48 / 25	9 / 5
B	15 / 8	225 / 128	5 / 3	25 / 16	3 / 2	45 / 32	4 / 3	5 / 4	75 / 64	9 / 8	25 / 24	1 / 1	15 / 8
C	2 / 1	15 / 8	16 / 9	5 / 3	8 / 5	3 / 2	64 / 45	4 / 3	5 / 4	6 / 5	10 / 9	16 / 15	1 / 1

Analysis of Just Chromatic Tuning

There are quite a number of wolves to be seen. There are wolf fifths between Bb - F and D - A having the value 40 / 27 with a decimal value of about 1.4815, rather less than the ideal 3 / 2 or 1.5. There is also a wolf fifth between F# - C# with the value 1024 / 675 or about 1.5170, larger than the ideal fifth. There are corresponding wolf fourths at A - D, F - Bb and C# - F#.

More disturbingly for a system which seeks to perfect thirds and sixths there are *wolf thirds* between A - C#, B - D#, E - G# and F# to A#. These wolves have the value 32 / 25 or 1.28 and differ significantly from the ideal 5 / 4 or 1.25. There are corresponding wolf minor sixths. Perception of the tuning of thirds and sixths is less critical than that of fifths and fourths but these are getting on for quarter of a tone out.

There are also wolf minor thirds (and major sixths) of two different sizes. The minor thirds of 32 / 27 have a decimal value of about 1.1852 and those of 75 / 64 have a decimal value of about 1.1719. These both differ significantly from the ideal 6 / 5 or 1.2.

Zarlino was undoubtedly aware of the limitations of just tuning and was amongst the earliest theorists to advocate a system of equal temperament. In 1588 he reported that Abbot Girolamo Roselli praised such a symmetrical temperament as "spherical music". [SCHULTER] The mathematical foundation for such a tuning system is based in the theory of logarithms which was not published until a quarter century after Zarlino's death.

Other historical alternatives to just intonation are discussed in the next section of this essay. For a software synthesizer capable of dynamically modifying just intonation during performance, please see elsewhere on this web site.

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4.5 Arithmetic and Harmonic Means

Zarlino observed that the arithmetic mean 3 between 2 and 4 divides an octave into a fifth and a fourth, 2 : 3 : 4. (Or 6 : 9 : 12.) Alternatively, the harmonic mean 8 between 6 and 12 divides the octave into a fourth and a fifth, 6 : 8 : 12. Similarly, the arithmetic mean 5 between 4 and 6 divides a fifth into major and minor thirds, 4 : 5 : 6, whereas the harmonic mean 12 between 10 and 15 divides the fifth into minor and major thirds, 10 : 12 : 15. Furthermore, the arithmetic mean of a major third, 4 : 5 or 8 : 10, divides it into major and minor tones, 8 : 9 : 10. Zarlino saw this result as 'truly miraculous'. [WITTKOWER]

(To calculate an arithmetic mean add the figures and divide by the number of them, for example the arithmetic mean of 6 and 12 is given by (6 + 12) / 2 = 9. To calculate an harmonic mean divide the sum of the reciprocals of the figures by the number of then and take the reciprocal of the result, for example, the harmonic mean of 6 and 12 is given by 2/(1/6 + 1/12) = 2/(3/12) = 24/3 = 8.)

4.6 The Common Chord and Sestina

The first, third and fifth scale degrees when sounded together came to be recognised as the *common chord*. This combination of notes was perhaps first found in the 14th century English *faburden* style of organum in which the middle part was a fourth below the melody and the bass a third below that. Transposing the top note down an octave yields the common chord.

As observed by Zarlino, the intervals of a major third, 5 / 4, and a minor third 6 / 5, when combined (multiply) amount to a fifth, 3 / 2. (The same holds true in Pythagorean tuning.) The common chord may be either a minor third on top of a major third or a major third on top of a minor third.

16th century theorists recognised a *sestina* as a chord comprising the intervals octave, fifth, fourth, major third, minor third, for example C C G C E G, having pitches related in the proportions 1 : 2 : 3 : 4 : 5 : 6. [EB62]

4.7 Subharmonics

The most prominent component of a musical tone is the fundamental. In the same way that a tone may contain higher harmonic frequencies which are multiples of the fundamental it may also contain lower frequencies which are sub-multiples of the fundamental. These are known as *subharmonics*. An alternative view is to regard subharmonics as the series of pitches of which a tone might itself be an harmonic even if they are not present in the tone. In practice they may be present in small proportions.

This implies a series of descending pitches which are reciprocally related to the ascending harmonics, F, F/2, F/3, F/4, F/5, F/6, etc. This series contains the same intervals as the series of ascending harmonics. The interval between the first two members of the series is an octave. The interval between the second and third subharmonics is a perfect fifth (3 / 2) and between the third and fourth the inversion of this, a perfect fourth, (4 / 3). The interval between the fourth and fifth subharmonics, (5 / 4) is a major third and the interval between the fifth and sixth, (6 / 5) is a minor third.

The 4th, 5th and 6th harmonics have the relative frequency ratios 4 / 4 : 5 / 4 : 6 / 4 and provide the tuning for the just major scale. The 4th, 5th and 6th subharmonics have the relative frequency ratios 1 / 4 : 1 / 5 : 1 / 6, or arranged in ascending order over a common denominator, 10 / 60 : 12 / 60 : 15 /

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60. The arithmetic and harmonic means calculated by Zarlino thus have a natural relation to the harmonic and subharmonic series. [MOORE] One could argue that the major scale mode arises from the harmonic series and that the minor scale mode arises from the subharmonic series.

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5 TEMPERAMENT

5.1 Full Circle of Fifths

Suppose the series of fifth on which Pythagorean tuning is based is extended to 12 or more members. This can be achieved by successively multiplying by 3 / 2 as illustrated in the second column of the following table. The process amounts to raising the numbers 2 and 3 to successively higher powers and the entire sequence is built upon powers of just these two numbers.

To keep all the pitches in the same octave it is necessary to transpose most of them down by one or more octaves, i.e. multiply by 1 / 2, 1 / 4, 1 / 8, etc. as illustrated in the third and fourth columns.

Number	Original fifth	Octaves to drop	Transposed fifth	Approximate decimal value
1	1 / 1	0	1 / 1	1
2	3 / 2	0	3 / 2	1.5
3	9 / 4	1	9 / 8	1.125
4	27 / 8	1	27 / 16	1.6875
5	81 / 16	2	81 / 64	1.265625
6	243 / 32	2	243 / 128	1.898437
7	729 / 64	3	729 / 512	1.423828
8	2187 / 128	4	2187 / 2048	1.067871
9	6561 / 256	4	6561 / 4096	1.601806
10	19683 / 512	5	19683 / 16384	1.201354
11	59049 / 1024	5	59049 / 32768	1.802032
12	177147 / 2048	6	177147 / 131072	1.351524
13	531441 / 4096	7	531441 / 524288	1.013643

Extended Series of Fifths

The decimal value of the interval produced by transposing the 13th fifth down 7 octaves differs from unity by just over 1%. From this point the sequence produces a series of decimal values which are all very close to the first 12 then another series of 12 values slightly higher again and so on. This sequence is known as the *circle of fifths*. The amount by which the 13th value differs from unity, 531441 / 524288 is known as the *Pythagorean comma*. This is the small amount of overlap at the end of the circle of fifths.

If these fifths were sorted by ascending pitch then the following chromatic tuning would result. The first seven members of the series give rise to the white notes of a keyboard and the remaining members of the series give rise to black notes all tuned as sharps.

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Number	Transposed fifth	Approximate decimal value	Interval to next step	Approximate decimal step size	Note name
1	1 / 1	1	2187 / 2048	1.067871	F
8	2187 / 2048	1.067871	256 / 243	1.053498	F#
3	9 / 8	1.125	2187 / 2048	1.067871	G
10	19683 / 16384	1.201354	256 / 243	1.053498	G#
5	81 / 64	1.26562	2187 / 2048	1.067871	A
12	177147 / 131072	1.351524	256 / 243	1.053498	A#
7	729 / 512	1.423828	256 / 243	1.053498	B
2	3 / 2	1.5	2187 / 2048	1.067871	C
9	6561 / 4096	1.601806	256 / 243	1.053498	C#
4	27 / 16	1.6875	2187 / 2048	1.067871	D
11	59049 / 32768	1.802032	256 / 243	1.053498	D#
6	243 / 128	1.898437	256 / 243	1.053498	E

Sorted Series of Fifths

The 13th member of the series would produce the note E# which would differ from F by the Pythagorean comma. It is this discrepancy that gives rise to the wolf fifths and fourths of Pythagorean tuning and just tuning.

5.2 Mean Tone Temperament

A mean tone temperament is a system of tuning which seeks to close the overlap in the circle of fifths by reducing the size of most of the fifths. [SCHULTER]

The interval of a Pythagorean major third arises between the 1st and 5th members of the series of fifths, 81 / 64 when transposed to lie in the same octave. It differs from the ideal harmonic major third, 5 / 4 by the *syntonic comma*, 81 / 80 (81/64 ÷ 5/4 = 81/64 x 4/5 = 324/320 = 81/80). The sequence of notes involved, say C - G - D - A - E, includes 4 intervals of a fifth If the size of each interval is reduced by a quarter of the syntonic comma then the third, C - E, will be a true harmonic third. The two tones of which it is comprised, C - D - E, are equally divided, *mean tone*, as D is half way up the series. The resultant tuning is called *quarter comma mean tone tuning*. It was first documented by Pierto Aaron (1490 - 1545) some time after 1500. Other forms of mean tone temperament include 1/3rd comma, 2/7ths comma, 1/5th comma and 1/6th comma.

But what exactly do we mean by quarter of the syntonic comma? The scale of pitch is logarithmic so we need a 'quarter' in logarithmic terms, $(81/80)^{1/4}$. the fourth root of 81 (i.e. 3) divided by the fourth root of 80. Because the size of each fifth needs to be reduced this term must be reciprocated, $(80/81)^{1/4}$. The tuning ratios from a sequence of fifths can be multiplied by the number of quarter commas required, for example 27/16 x $(80/81)^{3/4}$ for the third member of the series. The mean tone itself is adjusted by the square root of the syntonic comma, $(80/81)^{1/2}$. The arithmetic and harmonic

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means of Zarlino's system are now accompanied by a *geometric mean*. It is interesting to note that the decimal value of 80/81 is approximately 0.987654321.

An alternative view of the process is possible. Arithmetically, the four intervals involved amount to $(3/2)^4$ or 81/16 which evaluates to 5.0625. What is required is to make this value equal to 5 giving the 5/4 major third desired. To do so each (logarithmic) step needs to be equal to the fourth root of 5 or $5^{1/4}$. Each following step is some power of this transposed down an octave if necessary, for example the third step is $5^{3/4} / 2$; the mean tone is $5^{1/2}/2$.

Both methods are shown in the table below. They are arithmetically equivalent yielding the same decimal value. F has been calculated as a fourth down from C by reciprocating the ratios. In musical terms, the effect of this tuning is that the fifths have been very slightly narrowed in order to produce precise harmonic major thirds.

Note name	Place in series	Interval from tonic using quarter comma	Interval from tonic using reduced fifth	Approximate decimal value
C	0	1 / 1	1 / 1	1
D	2	$9/8 \times (80/81)^{2/4}$	$5^{2/4} / 2$	1.118034
E	4	$81/64 \times (80/81)1^{4/4} = 5/4$	$5^{4/4} / 4$	1.25
F	-1	$4/3 \times (81/80)^{1/4}$	$2 / 5^{1/4}$	1.337481
G	1	$3/2 \times (80/81)^{1/4}$	$5^{1/4}$	1.495349
A	3	$27/16 \times (80/81)1^{3/4}$	$5^{3/4} / 2$	1.671851
B	5	$243/128 \times (80/81)^{5/4}$	$5^{5/4} / 4$	1.869186
C	-	2 / 1	2 / 1	2

Quarter Comma Mean Tone Temperament

A problem with this method is that it over corrects the fifths. Every 4 fifths are reduced by the syntonic comma so over a full cycle of 12 fifths the total reduction amounts to 3 syntonic commas or $(80/81)^3$ which evaluates to about 0.963418. To truly close the circle of fifths requires a value of 1. There is going to be a very large wolf howling at the end of the series.

Undeterred, we might extend the series in both directions. Continuing the series by upward fifths leads into the series of sharps, F#, C#, G#, D#, A#. Continuing by downward fifths leads into the series of flats Bb, Eb, Ab, Db, Gb. It is very clear that the sharps and flats do not represent the same note at all. They all differ by the same amount, almost quarter of a tone. One approach to mean tone temperament on keyboards with only one set of black note keys is to pick and mix members of both the ascending and descending series so as to obtain the most even semitone steps, Db, Eb, F#, G# A#. This is indicated in bold in the following table.

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Note name	Place in series	Interval from tonic using quarter comma	Interval from tonic using reduced fifth	Approximate decimal value
C	**0**	**1 / 1**	**1 / 1**	**1**
C#	7	$2187/2048 \times (80/81)^{7/4}$	$5^{7/4} / 16$	1.044907
Db	**-5**	$256/243 \times (81/80)^{5/4}$	$8 / 5^{5/4}$	**1.069984**
D	**2**	$9/8 \times (80/81)^{2/4}$	$5^{2/4} / 2$	**1.118034**
D#	9	$19683/16384 \times (80/81)^{9/4}$	$5^{9/4} / 32$	1.168241
Eb	**-3**	$32/27 \times (81/80)^{3/4}$	$4 / 5^{3/4}$	**1.196279**
E	**4**	$81/64 \times (80/81)^{4/4} = 5/4$	$5^{4/4} / 4$	**1.25**
F	**-1**	$4/3 \times (81/80)^{1/4}$	$2 / 5^{1/4}$	**1.337481**
F#	**6**	$729/512 \times (80/81)^{6/4}$	$5^{6/4} / 8$	**1.397542**
Gb	-6	$1024/729 \times (81/80)^{6/4}$	$16 / 5^{6/4}$	1.431084
G	**1**	$3/2 \times (80/81)^{1/4}$	$5^{1/4}$	**1.495349**
G#	**8**	$6561/4096 \times (80/81)^{8/4} = 25/16$	$5^{8/4} / 16$	**1.5625**
Ab	-4	$128/81 \times (81/80)^{4/4}$	$8 / 5^{4/4}$	1.6
A	**3**	$27/16 \times (80/81)^{3/4}$	$5^{3/4} / 2$	**1.671851**
A#	**10**	$59049/32768 \times (80/81)^{10/4}$	$5^{10/4} / 32$	**1.746928**
Bb	-2	$16/9 \times (81/80)^{2/4}$	$4 / 5^{2/4}$	1.788854
B	**5**	$243/128 \times (80/81)^{5/4}$	$5^{5/4} / 4$	**1.869186**
C	**-**	**2 / 1**	**2 / 1**	**2**

Chromatic Quarter Comma Mean Tone Temperament

A convenient analysis of mean tone temperament is facilitated by the modern logarithmic interval measure of a cent.

5.3 Cents

Cents are a logarithmic measure of intervals defined such that 1200 cents comprise an octave.

To convert the decimal value of a tuning ratio (interval) into cents take its logarithm to the base 2 and multiply by 1200. For example, the tuning ratio of B in the table of mean tone temperament above is 1.869186; to convert this to cents calculate $\log_2(1.869186) \times 1200 = 1082.892164$.

If your calculator does not provide logarithms to the base 2 you can make use of the relation $\log_a(x) = \log_b(x) / \log_b(a)$. That is, take the logarithm in some other base, usually *e* the base of natural logarithms or 10, then divide by the log of 2 in that base. For example using natural logarithms, often indicated *ln*, the previous example becomes $\ln(1.869186) \times 1200 / \ln(2)$.

Because the measure is logarithmic the difference between two intervals may be found by subtracting their values rather than dividing as is required when the intervals are expressed as ratios. To combine two intervals measured in cents simply add the values.

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5.4 Some Common Tuning Intervals Measured in Cents

Interval	Ratio	Approximate decimal value	Measure to nearest cent
Octave	2 / 1	2	1200
Harmonic fifth	3 / 2	1.5	702
Harmonic fourth	4 / 3	1.333333	498
Pythagorean major third	81 / 64	1.265625	408
Just major third	5 / 4	1.25	386
Pythagorean minor third	32 / 27	1.185185	294
Just minor third	6 / 5	1.2	316
Tone	9 / 8	1.125	204
Just minor tone	10 / 9	1.111111	182
Pythagorean chromatic semitone	2187 / 2048	1.067871	113
Just diatonic semitone	16 / 15	1.066667	112
Pythagorean diatonic semitone	256 / 243	1.053498	90
Pythagorean comma	531441 / 524288	1.013643	23
Syntonic comma	81 / 80	1.0125	22
Quarter Pythagorean comma	$(531441 / 524288)^{1/4}$	1.003394	6
Quarter syntonic comma	$(81/80)^{1/4}$	1.003110	5

Common Tuning Intervals

5.5 Analysis of Mean Tone Temperament

A full chromatic analysis of mean tone temperament (using the same black notes as those selected in the derivation of mean tone temperament above) is given in the table below. Colour is used to indicate similar intervals (the colours are not the same as those used for the diatonic charts). The first main column of the table gives the tunings of the interval to the nearest cent. Each main column of the table gives the interval (to the nearest cent) from the note named at the head of the column to the next higher note named at the left of each row.

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	C	Db	D	Eb	E	F	F#	G	G#	A	A#	B	C
C	0	1083	1007	890	814	697	621	503	427	310	234	117	0
Db	117	0	1124	1007	931	814	738	620	544	427	351	234	117
D	193	76	0	1083	1007	890	814	696	620	503	427	310	193
Eb	310	193	117	0	1124	1007	931	813	737	620	544	427	310
E	386	269	193	76	0	1083	1007	889	813	696	620	503	386
F	503	386	310	193	117	0	1124	1006	930	813	737	620	503
F#	579	462	386	269	193	76	0	1082	1006	889	813	696	579
G	697	580	504	387	311	194	118	0	1124	1007	931	814	697
G#	773	656	580	463	387	270	194	76	0	1083	1007	890	773
A	890	773	697	580	504	387	311	193	117	0	1124	1007	890
A#	966	849	773	656	580	463	387	269	193	76	0	1083	966
B	1083	966	890	773	697	580	504	386	310	193	117	0	1083
C	1200	1083	1007	890	814	697	621	503	427	310	234	117	0

Analysis of Quarter Comma Mean Tone Temperament

Most of the fifths have the interval 696 cents which sometimes appears as 697 cents due to the process of rounding to the nearest cent. This is just slightly less than the size of a 3 / 2 fifth which is about 702 cents. But there are two fifths of 656 cents, nearly a quarter tone too narrow and three of 737 cents, nearly quarter of a tone too wide. A similar problem exists with the fourths. For fifths and fourths this is a serious discrepancy as perception of the tuning of these intervals is quite sensitive.

There are also some problematic thirds. Four of the major thirds are too wide. Four of the minor thirds are too narrow and one is too wide. Similar problems exist with the sixths.

In practical terms the tuning is fine so long as one stays in or close to the key of C. Modulation to distant keys is unwise.

5.6 The Common Chord and Key Modulation

Jean Philippe Rameau (1683 - 1764) established a theory of tonal harmony based on the crucial fourth and fifth scale degrees. The fifth scale degree is known as the *dominant* and its inverse, the fourth, five degrees down from the tonic is called the *subdominant*. The entire diatonic scale can be constructed as common chords on these scale degrees, for example C E G, F A C, G B D.

Suppose one similarly constructs a new scale on the dominant of the existing scale. The dominant of fifth of C is G so the new scale would contain G B D, C E G, D F# A. This requires the introduction of F#. It is the last note in the scale of G and is called the *leading note* because in melody it leads back to the tonic.

The pattern of intervals in the new scale is exactly the same as that in the old (tone, tone, semitone; tone; tone, tone semitone) so this is not a different mode, it is a different *key*, the same mode but from a different starting note. One can *modulate* to a new key via a *pivot chord* which is present in both the old and new keys.

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Building a new scale on the dominant of G yields D F# A, G B D, A, C#, E, again introducing a new sharp for the leading note. Continuing this process creates the following sequence of sharps for the leading notes in the same way as they arise from the series of fifths.

Tonic	Dominant	Leading note
C	G	B
G	D	F#
D	A	C#
A	E	G#
E	B	D#
B	F#	A#

Introduction of Sharps

Conversely, one can construct a new scale on the subdominant of C, F. This would contain the triads F A C, Bb D F, C E G. The note Bb must be added as the new subdominant of the scale Again, this is a new key. Construction of a new scale on the subdominant of F, Bb, yields the triads Bb D F, Eb G Bb, F A C. Continuing this process creates the following sequence of flats for the new subdominants.

Tonic	Subdominant
C	F
F	Bb
Bb	Eb
Eb	Ab
Ab	Db
Db	Gb

Introduction of Flats

Moving from one key to another is *key modulation*. Common forms of key modulation involve moving to the dominant or subdominant. Because the underlying intervals of a fifth and a fourth are the inversion of each other, a new key formed on the dominant will have the key note of the old key as its subdominant. From the perspective of the new key one has come from the subdominant. Moving in this direction is known as the *plagal* orbit analogous to the Medieval plagal modes in which the final is the fourth note of the range. If one moves to the subdominant then this new key will have the key note of the old key as its dominant. Movement in this direction, from the dominant, is known as the *cadential* orbit.

5.7 Diatonic Intervals

Diatonic intervals are measured as the number of different named notes in the diatonic scale that they include. For example C to G is a fifth, C D E F G. The interval from B to F is also a fifth, B C D E F, even though it is a semitone smaller.

Every fifth in the diatonic major scale involves 7 semitone steps (3 tones and one semitone) except the one from B to F. The fifths comprising 3 tones and a semitone are called *perfect fifths*. The one from B to F is a semitone smaller and is called a *diminished fifth* or *tritone* (3 tones).

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Similarly, every fourth in the scale involves 5 semitone steps (2 tones and a semitone) except the one from F to B. The fourths comprising 2 tones and a semitone are called *perfect fourths*. The one from F to B is a semitone larger and is called an *augmented fourth*. This interval is also the tritone, an interval generally regarded as most inharmonious. Actually, in just tunings, the augmented fourth and diminished fifth differ by the Pythagorean comma, the overlap in the circle of fifths. The tritone thus has two different values which perhaps accounts for its inharmonious and troublesome nature.

Unison, octaves, fifths and fourths are the only perfect intervals. Because the inversion of a fifth is a fourth (subtracting from an octave) the inversion of a perfect interval is also a perfect interval.

The character (size) of all the other intervals depends on their position in the diatonic scale as the size of the interval may vary by one semitone. The smaller of the two possibilities is called a *minor* interval and the larger a *major* interval. The inversion of a minor interval is always a major interval and vice versa.

In general these are the intervals that arise in the major and minor scale modes. All the degrees of a diatonic major scale are related to the tonic by the corresponding major interval. The minor scale always has a major second and sometimes the sixth and seventh degrees are raised to major intervals. In the *melodic minor* scale, suited to melody, the sixth and seventh degrees are raised a semitone to major intervals in an ascending scale but the minor form is used in a descending scale. In the *harmonic minor* scale used to form chords just the seventh degree is raised to a major seventh.

The following table lists all the diatonic intervals with their size in semitone steps and their inversion.

Interval	Semitone steps	Inversion
Unison	0	Octave
Minor second	1	Major seventh
Major second	2	Minor seventh
Minor third	3	Major sixth
Major third	4	Minor sixth
Perfect fourth	5	Perfect fifth
Tritone	6	Tritone
Perfect fifth	7	Perfect fourth
Minor sixth	8	Major third
Major sixth	9	Minor third
Minor seventh	10	Major second
Major seventh	11	Minor second
Octave	12	Unison

Intervals and their Inversions

Other augmented and diminished intervals occasionally arise. For example C# to Db is a diminished second; it is a second because there are two note names but it is diminished because it is a semitone smaller than a minor second. C to A# is an augmented sixth; it is a sixth because there are 6 note names, C D E F G A# but it is a semitone larger than a major sixth. Augmented and diminished forms of all intervals may be found.

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5.8 Well Temperament

Tunings which attempt to eradicate the wolf intervals or minimize their impact on a particular style of music are known as *well temperament*. Such temperaments flourished from the late 17th to the late 19th century, a time when the common chord major and minor key system also prevailed. [SCHULTER]

Andreas Werckmeister (1645 - 1706) proposed several such tuning systems. In the one known as Werckmeister Temperament III the black notes, the furthest members of the series of fifths and the most troublesome notes of mean tone temperament, are left in Pythagorean just intonation whilst the white notes are tempered. Werckmeister's temperament uses the Pythagorean comma, 531441 / 524288, which is the usual basis of well temperament. In the following table I have indicated this somewhat unwieldy term as P. The white note tunings are adjusted by $P^{3/4}$ at most.

Note name	Position in series	Interval from tonic	Approximate decimal value	Measure to nearest cent
C	0	1 / 1	1	0
Db		256 / 243	1.053498	90
D	2	$9/8 \times 1/P^{2/4}$	1.117403	192
Eb		32 / 27	1.185185	294
E	4	$81/64 \times 1/P^{3/4}$	1.252827	390
F		4 / 3	1.333333	498
Gb		1024 / 729	1.404664	588
G	1	$3/2 \times 1/P^{1/4}$	1.494927	696
Ab		128 / 81	1.580247	792
A	3	$27/16 \times 1/P^{3/4}$	1.670436	888
Bb		16 / 9	1.777778	996
B	5	$243/128 \times 1/P^{3/4}$	1.879241	1092
C		2 / 1	2	1200

Andreas Werckmeister Temperament III

A full chromatic analysis of Werckmeister's Well Temperament is given in the table below. Colour is used to indicate similar intervals. The first main column of the table gives the tunings of the interval to the nearest cent from the table above. Each main column of the table gives the interval (to the nearest cent) from the note named at the head of the column to the next higher note named at the left of each row.

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	C	Db	D	Eb	E	F	Gb	G	Ab	A	Bb	B	C
C	0	1110	1008	906	810	702	612	504	408	312	204	108	0
Db	90	0	1098	996	900	792	702	594	498	402	294	198	90
D	192	102	0	1098	1002	894	804	696	600	504	396	300	192
Eb	294	204	102	0	1104	996	906	798	702	606	498	402	294
E	390	300	198	96	0	1092	1002	894	798	702	594	498	390
F	498	408	306	204	108	0	1110	1002	906	810	702	606	498
Gb	588	498	396	294	198	90	0	1092	996	900	792	696	588
G	696	606	504	402	306	198	108	0	1104	1008	900	804	696
Ab	792	702	600	498	402	294	204	96	0	1104	996	900	792
A	888	798	696	594	498	390	300	192	96	0	1092	996	888
Bb	996	906	804	702	606	498	408	300	204	108	0	1104	996
B	1092	1002	900	798	702	594	504	396	300	204	96	0	1092
C	1200	1110	1008	906	810	702	612	504	408	312	204	108	0

Analysis of Andreas Werckmeister Temperament III

All the fifths are either true fifths, ~702 cents, or a quarter (Pythagorean) comma less. Similarly the fourths are either true fourths, ~498 cents, or a quarter comma more. Major thirds vary adopting one of the quarter comma positions between 5 / 4 (about 386 cents) and 81 / 64 (about 408 cents). Minor thirds take up positions between 32 / 27 (about 294 cents) and 6 / 5 (about 316 cents). This scheme is tending towards one in which the intervals are equally spaced.

By providing a spectrum of different sized major and minor thirds and sixths Werckmeister's system not only facilitates modulation to more distant keys but imbues each key with its own character, a part of the musical language of the 18th century.

Other forms of temperament include those proposed by Francesco Antonio Vallotti (1697 - 1780), Anton Bemetzrieder and Margo Schulter. [SCHULTER] Typically, these involve leaving the white notes and Bb in Pythagorean just intonation and tempering the remaining black notes by either a 1/4 or a 1/6th of a Pythagorean comma.

5.9 Equal Temperament

One way to close the overlap in the circle of fifths is to divide seven octaves into twelve equal fifths. If we represent such a fifth as T then $T^{12} = 2^{7}$. Each fifth then becomes $2^{7/12}$ which evaluates to about 1.498307 or exactly 700 cents. By definition an octave is 1200 cents so a fourth is 500 cents by subtracting a fifth. The difference between a fourth and a fifth, a whole tone, is 200 cents. A semitone is 100 cents. A chromatic scale of 12 equal 100 cent steps can be constructed. Indeed, this is the reason for defining an octave as 1200 cents and the basis of the tuning system called *equal temperament*. Such a tuning, or something close to it, was in use on fretted instruments such as lutes by the middle of the 16th century.

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Note name	Interval from tonic	Exact tuning in cents relative to first note
C	1 / 1	0
C# / Db	$1 / 2^{1/12}$	100
D	$1 / 2^{2/12}$	200
D# / Eb	$1 / 2^{3/12}$	300
E	$1 / 2^{4/12}$	400
F	$1 / 2^{5/12}$	500
F# / Gb	$1 / 2^{6/12}$	600
G	$1 / 2^{7/12}$	700
G# / Ab	$1 / 2^{8/12}$	800
A	$1 / 2^{9/12}$	900
A# / Bb	$1 / 2^{10/12}$	1000
B	$1 / 2^{11/12}$	1100
C	$1 / 2^{12/12}$	1200

Equal Temperament

In equal temperament all similar intervals remain constant no matter where in the scale they occur. Intervals are equally in tune (and equally out of tune) in all keys and key modulation is unhindered by the tuning system. The consonance of pure intervals is lost as there are no longer any pure intervals except the octave. All basis for different keys having different character is gone. Utility has triumphed over beauty.

It is open to debate whether J. S. Bach (1685 - 1750) in *Well-tempered Clavier* was seeking to demonstrate the freedom of key modulation afforded by equal temperament or the contrasting key characters offered by well temperament. [SCHULTER]

5.10 Numerical Comparison of Tuning Systems

In the following table the interval sizes from tonic to other scale degrees for Pythagorean and just intonation are compared with equal temperament.

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Note name	Equal temperament		Pythagorean			Just		
	cents	decimal	ratio	cents	decimal	ratio	cents	decimal
C	0	1.0	1 / 1	0	1.0	1 / 1	0	1.0
C# / Db	100	1.0594	256 / 243	90.22	1.0535	16 / 15	111.73	1.0667
D	200	1.1225	9 / 8	203.91	1.125	9 / 8	203.91	1.125
D# / Eb	300	1.1892	32 / 27	294.13	1.1852	6 / 5	315.64	1.2
E	400	1.2599	81 / 64	407.82	1.2656	5 / 4	386.31	1.25
F	500	1.3348	4 / 3	498.04	1.3333	4 / 3	498.04	1.3333
F# / Gb	600	1.4142	1024 / 729	588.27	1.4047	45 / 32	590.22	1.4063
G	700	1.4983	3 / 2	701.96	1.5	3 / 2	701.96	1.5
G# / Ab	800	1.5874	128 / 81	792.18	1.5802	8 / 5	813.69	1.6
A	900	1.6818	27 / 16	905.87	1.6875	5 / 3	884.36	1.6667
A# / BB	1000	1.7818	16 / 9	996.09	1.7778	9 / 5	1017.60	1.8
B	1100	1.8877	243 / 128	1109.78	1.8984	15 / 8	1088.27	1.875
C	1200	2.0	2 / 1	1200.00	2.0	2 / 1	1200.00	2.0

Numerical Comparison of Tuning Systems

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