The word problem for computads

by M. Makkai *
McGill University

(Revised May 16, 2005)

Contents:

* Supported by NSERC Canada

Introduction

(A) The origins of the paper

Computads were introduced by Ross Street, for dimension 2 as early as 1976 in [S1] and in general, in [S3]. Albert Burroni's paper [Bu] calls computads "polygraphs". Jacques Penon's paper [Pe] is important for us, since it contains the full syntactical definition of computads (reproduced with small changes in section 7 below), which will be used to formulate the problem in the title of the present paper.

My interest in computads stems from their role in the definition and theory of weak higher-dimensional categories. This role came to be realized as an afterthought.

In [He/M/Po], the definition of "opetopic set" introduced by John Baez and James Dolan [Bae/D] is reworked into what we called "multitopic set". In [He/M/Po], it was shown, among other, that the category of multitopic sets is, up to equivalence, the same as the category of presheaves on a category called the category of *multitopes*. In [M3], inspired by the second part of the Baez/Dolan definition of "opetopic category", but also following my earlier work [M1], [M2] on logic with dependent sorts, I proposed a definition of "the large multitopic category of all small multitopic categories"; the small multitopic categories constitute the zero-cells in said large multitopic category.

Already at the time of our joint work with Claudio Hermida and John Power, we had the feeling that multitopic sets were related to computads, in fact, that they were essentially identical with the "many-to-one" computads, ones whose indeterminates (free generating cells) have codomains that are themselves indeterminates (although, I must confess, at the time I did not really understand the notion of computad). The paper [Ha/M/Z] established this result, in the form of a pair of adjoint functors between the category of multitopic sets on the one hand, and the category of small ω-categories on the other, under which the left adjoint functor, from the first of the above categories to the second, being faithful and full on isomorphisms, has, as its essential and full-on-isomorphisms image, the category of many-to-one computads.

This result represented an advance inasmuch the fairly complicated, albeit combinatorially explicit, original definition in [He/M/Po] of multitopic sets became a conceptually simple one. On the other hand, it is to be noted that the fact that the category of many-to-one computads is a presheaf category, and the implied equivalent concept to the notion of multitope, do not become obvious by merely looking at many-to-one computads. At the present stage of our knowledge, said fact needs, for its proof, the detour via the original theory of multitopic sets in [He/M/Po].

The basic perspective of the present paper is a reversal of the above chronology. The notions "multitopic set" and "multitope" are seen here as the result of a combinatorial/algebraic analysis of the notion of many-to-one computad. The paper attempts to extend said analysis to *all* computads.

(B) Computads as the algebraic notion of higher-dimensional diagram.

The notion of *computad* is, as far as I am concerned, nothing but *the* precise notion of *higher-dimensional categorical diagram*. To explain this, I start earlier, with an informal introduction to the notion of (strict!) ω*-category*. (In this paper, no "weak" category theory appears at all.)

Consider the following ordinary categorical diagram:

$$\begin{array}{ccccc} X_3 & \xrightarrow{f_3} & X_6 & \xrightarrow{f_6} & X_9 \\ {\scriptstyle g_2}\uparrow & 2 & {\scriptstyle g_4}\uparrow & 4 & \uparrow{\scriptstyle g_6} \\ X_2 & \xrightarrow[f_2]{} & X_5 & \xrightarrow[f_5]{} & X_8 \\ {\scriptstyle g_1}\uparrow & 1 & {\scriptstyle g_3}\uparrow & 3 & \uparrow{\scriptstyle g_5} \\ X_1 & \xrightarrow[f_1]{} & X_4 & \xrightarrow[f_4]{} & X_7 \end{array}$$

consisting of objects X_i and arrows f_j , g_k in some category. The reader will agree when we say this:

(*) **if** the four small squares 1, 2, 3, 4 commute,

then, as a consequence, the big outside square will commute as well.

Having agreed on this, one may ask what are the *general laws* behind this, and countless other similar and/or more complicated facts. The answer is: the laws codified in the definition of notion of " *ω-category*".

To motivate that definition, the starting point is to adopt the position that "there is no bare equality": every equality is mediated by some data that we -- conveniently or not -- forget when we simply assert the fact of an equality. (At the Minneapolis (IMA) meeting on higher-dimensional categories in June 2004, John Baez gave, as the introductory talk to the conference, a brilliant lecture with this theme.) That position dictates that we, abstractly and theoretically, introduce data that are responsible for the commutativity of the four numbered squares above, in the way of fillers, 2-dimensional cells, or 2-arrows, as follows:

$$\begin{array}{ccccc} X_3 & \xrightarrow{f_3} & X_6 & \xrightarrow{f_6} & X_9 \\ g_2\uparrow & a_2\searrow & g_4\uparrow & a_4\searrow & \uparrow g_6 \\ X_2 & \xrightarrow[f_2]{} & X_5 & \xrightarrow[f_5]{} & X_8 \\ g_1\uparrow & a_1\searrow & g_3\uparrow & a_3\searrow & \uparrow g_5 \\ X_1 & \xrightarrow[f_1]{} & X_4 & \xrightarrow[f_4]{} & X_7 \end{array} \qquad (1)$$

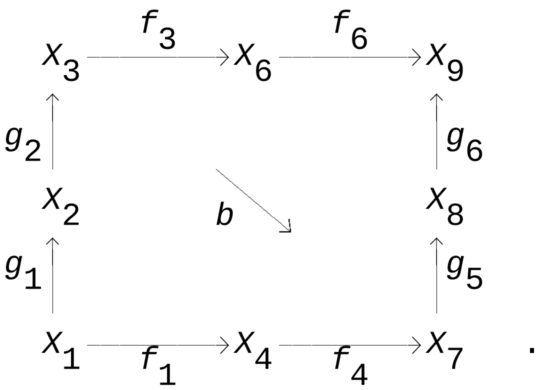
We think e.g. of a_2 as a (2-)*arrow* with *domain* $g_2 f_3$, and *codomain* $f_2 g_4$. (We use "geometric order"; $g_2 f_3$ is what usually is denoted by $f_3 \circ g_2$, or also $g_2 \# f_3$.) We even have given up the symmetry in the idea of the equality $g_2 f_3 = f_2 g_4$, and think of $g_2 f_3$ being transformed into $f_2 g_4$ in some general way, that way being denoted by a_2 .

A 2-arrow must have a domain and a codomain that are ordinary (1-) arrows, which are parallel: they share their domain and their codomain, which are 0-*cells*. The idea here is that a 2-arrow as a transformation does not have any effect on 0-cells: it must leave them alone; transforming 0-cells is the responsibility of 1-cells.

The "if-then" statement (*) above becomes an *operation* that, applied to the four arguments a_1, a_2, a_3, a_4 results in a transformation, say b , of $g_1 g_2 f_3 f_6$ into $f_1 f_4 g_5 g_6$:

$$b\colon g_1 g_2 f_3 f_6 \longrightarrow f_1 f_4 g_5 g_6 \qquad (2)$$

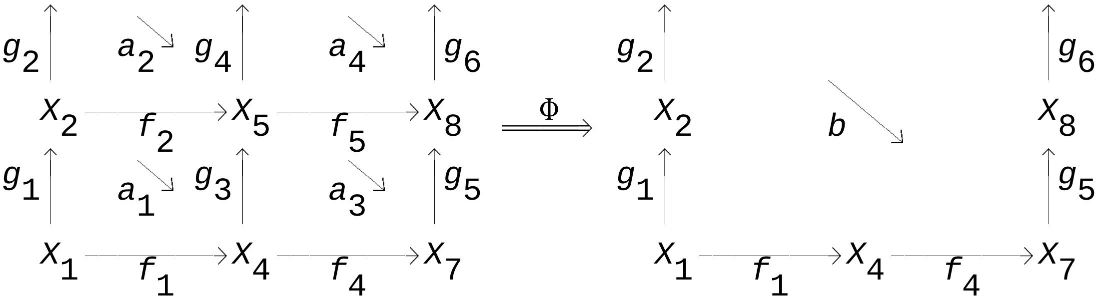
depicted as



The above procedure of introducing 2-dimensional arrows into diagrams to represent evidence, or proof, of a commutativity is closely related to the similar procedures in proof theory, especially categorical proof theory; see for example [L/Sc], section I.1, "Propositional calculus as a deductive system".

The concept of ω-category (" ω- " here anticipates the need for passing to ever higher dimensions after 0, 1 and 2 that have appeared so far) will have, on the one hand, some algebraically codified *primitive operations* that let us obtain b out of a_1, a_2, a_3, a_4 by repeatedly applying those operations, and on the other, certain *laws* that ensure that no matter in what order we apply the primitive operations to the four arguments, the result is always the same: b is *well-defined* as *the* composite of a_1, a_2, a_3, a_4 without any further qualification.

Just as the commutativity of 1-dimensional diagrams, that is, the equality of composite 1-arrows, has "given rise" to 2-dimensional diagrams, the contemplation of the equality of composite 2-cells (possible 2-commutativities) gives rise to 3-arrows, and 3-dimensional diagrams. For instance, the fact that the composite of a_1, a_2, a_3, a_4 *equals* b is mediated by a 3-cell



(3)

Of course, the process does not stop at dimension 3, and we see the need for a concept of *ω-category* in which there are arrows (cells) of arbitrary non-negative integer dimensions (but none of dimension ω or ∞).

Since the examples like the ones we considered clearly encompass a large variety, especially when one contemplates arbitrarily high dimensions, it is a highly non-obvious fact that a satisfactory concept of ω-category is possible at all. It is not a priori clear that there is a neatly defined set of primitive operations whose combinations account for all the desired compositions of cells; and it is not a priori clear that there is a neat set of laws that ensure facts like the one above of b being well-defined as *the* composite of a_1, a_2, a_3, a_4. It is therefore a kind of miracle that in fact we do have a good notion of ω-category. It is the basic general aim of the present paper and its projected sequels to investigate the ways and means of this "miracle".

There is another, perhaps even more convincing, way of approaching the concept of ω-category. This argues that the totality of (small) n-(dimensional) categories, properly construed, is an $(n+1)$-category; therefore, if we want to freely form "arbitrary totalities", we need n-categories for all n. (Let me note that the process stops at ω: the totality of (small) ω-categories is, in a natural way, an ω-category again, not an $(\omega+1)$-category.) However, in this second argumentation, when carried out with proper care, we find a similar step of replacing an *equality* by a *transformation*. In fact, this latter thinking, when followed to its logical conclusion, gives rise to the notion of *weak ω-category*, a concept that we do not discuss in this paper. The present paper sticks to the formal or *syntactical* role of higher dimensional diagrams, and it does not need the consideration of "totalities".

In an ω-category in which the diagram (3) lives, there are many cells (infinitely many if we consider the identities of all dimensions required by the concept of ω-category). In particular, we have the composite 1-cell g_2f_3 "on the same level" as the generating arrows g_2, f_3, etc. The concept that makes the distinction between "generating cell" and "composite cell" is the concept of *computad*. This is a conceptually very simple notion; it can be stated as *levelwise free ω-category*.

Imagine a structure, a typical computad, that can be taken to be essentially identical with the diagram (3). We want the *elements* of this structure to be exactly the named items in the diagram: the 0-cells X_i ($i=1, \ldots, 9$), the 1-cells f_j ($j=1, \ldots, 6$), g_k ($k=1, \ldots, 6$), the 2-cells a_l (l=l, ..., 4), b , and the 3-cell Φ . However, to account for the structure itself, we need to consider various composites of the elements. We decide to form the composites *freely*.

To begin with, we take the free category $\mathbf{X}^1$ on the ordinary graph consisting of said 0-cells and 1-cells. To incorporate the 2-cells and their composites, we need the operation of *freely adjoining* the mentioned 2-cells as *indeterminates* to $\mathbf{X}^1$, *with the appropriate preassigned domains and codomains* given as certain (composite) 1-cells in the category $\mathbf{X}^1$.

This process of free adjunction is very familiar from algebra. The ring of polynomials $R[X, Y, \ldots]$ is obtained from the ring R , by freely adjoining the indeterminates $X, Y, \ldots$. The definition, via a universal property, is too familiar to be quoted here. The 2-category $\mathbf{X}^2$ obtained by the free adjunction of the appropriate 2-cells to $\mathbf{X}^1$, with the specified domains and codomains in $\mathbf{X}^1$, is defined by a similar universal property. The only additional complication is that the adjoined 2-cell a_2 , to have an example, is constrained to have the specified domain $g_2 f_3$ and codomain $f_2 g_4$ given in $\mathbf{X}^1$ already. In the section I.5, "Polynomial categories", of [L/Sc], we find a similar situation in which an arrow with preassigned domain and codomain is freely adjoined to Cartesian closed category. (The definition of computad is given in section 5 , based on section 4.)

When we adjoin an indeterminate u to an ω-category $\mathbf{X}$ in which we have specified $\mathrm{d}u$ and $\mathrm{c}u$ in $\mathbf{X}$, to get $\mathbf{X}[u]$, we usually assume that $\mathrm{d}u$ and $\mathrm{c}u$ are parallel: they have the same domain and codomain. However, this is only a "reasonability assumption". The definition through the appropriate universal property works without this assumption. The canonical map $F:\mathbf{X}\to\mathbf{X}[u]$ will naturally produce the equality $F(\mathrm{dd}u)=F(\mathrm{dc}u)=\mathrm{d}_{\mathbf{X}[u]}(u)$. Thus, F can be injective only if said parallelism condition is satisfied. As we will see, in that case, F is indeed injective.

The composite of a_1, a_2, a_3, a_4 will be a definite 2-arrow $\vec{a}$ in $\mathbf{X}^2$. Of course, this is a major point of the construction, and it has to be ascertained specifically. That is, we have to define, using the primitive operations of " ω-category", a specific 2-cell that we will take, by definition, to be the composite of a_1, a_2, a_3, a_4 . This we will not do here, since we do not have the formalism of ω-category yet. However, once we have done this, the resulting 2-cell

$\vec{a}$ will have domain and codomain as b does in (2); in other words, $\vec{a}$ will be parallel to b.

Finally, the whole structure -- a computad -- is obtained by freely adjoining the 3-dimensional indeterminate Φ to X^2, with the stipulation that $d(\Phi)$ is to be the composite $\vec{a}$, and $c(\Phi)$ is b.

We have outlined the definition of a particular ω-category, in fact, a computad, that we take to be the structure representing the pasting diagram (3). It gives a good idea of the general notion of *computad*.

It turns out (see sections 4 and 5) that, when we define a computad to be an ω-category without additional data as we did in the example, we are able to recover the indeterminates in the computad from its ω-category structure as the elements that are *indecomposable* in a natural sense. Thus, it is not necessary to carry the indeterminates as data for the structure.

I consider the notion of computad as being *identical* to the notion of higher dimensional diagram, or pasting diagrams.

An analysis, using combinatorial, algebraic or geometric means, may provide descriptions amounting to equivalent definitions of smaller or larger classes of computads. In fact, such descriptions are one of the main areas of the theory of computads. As I mentioned above, the paper [Ha/M/Z] is part of this area.

The notion of a diagram being pastable (composable), the focus of the attention in the theory of pasting diagrams, is implicit in the concept of computad, since a computad always carries within itself all possible (free) compositions of the indeterminates (elements of the diagram). Of course, this does not mean that the problem of pastability of given candidates of pasting diagrams, given in some combinatorial or other manner, is solved automatically by using computads. The value of computads is mainly in their ability to provide mathematically satisfactory definitions of intuitive concepts -- such as pastability --, which then can be analyzed in any manner that comes to mind.

The important papers [J], [Po1], [Po2], [Ste]) give with various combinatorial, algebraic and geometric definitions of classes of pasting diagrams. They make connections to computads to varying degrees. In ongoing and future work (e.g. [M4]), I revisit the results of the existing theory of pasting diagrams in the spirit of computads.

(C) Concrete presheaf categories.

In part (A) above, we mentioned two results, both asserting the equivalence of certain categories. The first said that $\mathtt{MltSet}$, the category of multitopic sets, is equivalent of $\mathtt{Mlt}^{\wedge}=\mathtt{Set}^{\mathtt{Mlt}^{\mathrm{op}}}$, with $\mathtt{Mlt}$ the category of multitopes. The second said that $\mathtt{MltSet}$ is equivalent to $\mathtt{Comp}_{\mathtt{m/1}}$, the category of many-to-one computads.

It turns out that in both cases, what is proved is stronger than what is stated. In both cases, we have equivalences of *concrete* categories.

A *concrete category* is a category $\boldsymbol{A}$ together with an "underlying-set" functor $|-|:\boldsymbol{A}\to\mathtt{Set}$. *Equivalence* of concrete categories $(\boldsymbol{A},\ |-|_{\boldsymbol{A}})$, $(\boldsymbol{B},\ |-|_{\boldsymbol{B}})$ is equivalence of categories compatibly with the underlying-set functors: we require the existence of a functor $\Phi:\boldsymbol{A}\to\boldsymbol{B}$ that is an equivalence of categories, such that the following diagram of functors:

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\quad\Phi\quad} & \boldsymbol{B} \\ & \searrow \overset{\cong}{} \swarrow & \\ |-|_{\boldsymbol{A}} & \mathtt{Set} & |-|_{\boldsymbol{B}} \end{array}$$

commutes up to an isomorphism: there is an isomorphism $\varphi:\ |-|_{\boldsymbol{A}}\xrightarrow{\cong}|-|_{\boldsymbol{B}}\circ\Phi$.

All three of the above-mentioned categories $\mathtt{MltSet}$, $\mathtt{Mlt}^{\wedge}$, $\mathtt{Comp}_{\mathtt{m/1}}$ are equipped with canonical underlying-set functors.

A multitopic set, an object of $\mathtt{MltSet}$, consists of *n-cells*, for each $n\in\mathbb{N}$; we have an a priori underlying-set functor $|-|:\mathtt{MltSet}\longrightarrow\mathtt{Set}$. (In the notation of [He/M/Po] , Part 3, p. 83, for the multitopic set S, $|S|=\bigsqcup_{k\in\mathbb{N}}C_k$; the elements of C_k are called the *k-cells* of S. In [Ha/M/Z], §1 gives an alternative, possibly more conceptual introduction to multitopic sets. On p.51 loc.cit., it is pointed out that in dimension 0, a detail in the definition in [He/M/Po] is to be corrected.)

For any small category $\boldsymbol{C}$, we take the presheaf category $\hat{\boldsymbol{C}} = \mathtt{Set}^{\boldsymbol{C}^{\mathrm{op}}}$ to be equipped with the underlying-set functor $|\text{-}| : \hat{\boldsymbol{C}} \to \mathtt{Set}$ defined as $|A| = \coprod_{U \in \mathrm{Ob}(\boldsymbol{C})} A(U)$.

We have the underlying-set functor $|\text{-}| : \mathtt{Comp} \to \mathtt{Set}$ which assigns to each computad $\boldsymbol{X}$ the set $|\boldsymbol{X}|$ of all indeterminates of $\boldsymbol{X}$. $\mathtt{Comp}_{\mathtt{m/1}}$ is a concrete category with the underlying-set functor the restriction of that for $\mathtt{Comp}$.

It turns out that both equivalences

$$\mathtt{MltSet} \simeq \mathtt{Mlt}^{\wedge}$$

$$\mathtt{Comp}_{\mathtt{m/1}} \simeq \mathtt{MltSet}$$

are in fact concrete, that is, compatible with said underlying-set functors. As a corollary, we have the concrete equivalence

$$\mathtt{Comp}_{\mathtt{m/1}} \simeq \mathtt{Mlt}^{\wedge} .$$

This statement is meaningful even if we do not know the precise definition of $\mathtt{Mlt}$; it says that $\mathtt{Comp}_{\mathtt{m/1}}$ is (concrete-equivalent to) a *concrete presheaf category*.

A concrete presheaf category is, of course, in particular, a presheaf category; thus it is a very "good" category. Let us see what the "concreteness" in the equivalence says, in addition.

The concrete equivalence of $(\boldsymbol{A}, |\text{-}|)$ and $\hat{\boldsymbol{C}}$ means that we have an equivalence functor $F : \boldsymbol{A} \xrightarrow{\simeq} \hat{\boldsymbol{C}}$, and a natural bijection

$$|A| \cong \coprod_{U \in \mathrm{Ob}(\boldsymbol{C})} (FA)(U) . \qquad (A \in \boldsymbol{A}) .$$

Thus, up to a natural bijection, we have a *classification* of the elements of an object A (the elements of the set $|A|$) of $\boldsymbol{A}$ into mutually disjoint classes $(FA)(U)$, the classes being labelled with a fixed set of *types*, the objects U of $\boldsymbol{C}$. This classification is functorial: it is compatible with the arrows of $\boldsymbol{A}$. Moreover, we have arrows between the types, the *type-arrows*, that account for the complete structure of the category $\mathbf{A}$: an arrow $A \xrightarrow{f} B$, essentially a natural transformation, is given by a system of maps $(FA)(U) \xrightarrow{f_U} (FB)(U)$, one for each type U , that are jointly compatible with the type-arrows.

It turns out that the equivalence type a concrete presheaf category $\hat{\mathbf{C}}$ determines $\mathbf{C}$ up to *isomorphism*; we say that $\mathbf{C}$ is the *shape category* of any concrete category that is concretely equivalent to $\hat{\mathbf{C}}$.

Given a concrete category $\mathbf{A}=(\mathbf{A}, |\text{-}|)$, we identify a category, denoted by $\mathbf{C}^*[\mathbf{A}]$, which *is* the shape category of $\mathbf{A}$ in case $\mathbf{A}$ turns out to be a concrete presheaf category. Here is the definition.

$\mathrm{El}(\mathbf{A})$ denotes the category of elements of the functor $|\text{-}|:\mathbf{A}\to\mathrm{Set}$; its objects are pairs $(A, a) = (A\in\mathbf{A}, a\in |A|)$, and they are called *elements* of $\mathbf{A}$.

An element (A, a) is said to be *principal* if it is A is *generated* by a , in the sense that whenever $f:(B, b)\to(A, a)$ is an arrow in $\mathrm{El}(\mathbf{A})$ such that $f:A\to B$ is a monomorphism in $\mathbf{A}$, then f is an isomorphism. The element (A, a) is *primitive* if it is principal, and for any principal (B, b) , any arrow $f:(B, b)\to(A, a)$ must be an isomorphism.

The shape category $\mathbf{C}^*[\mathbf{A}]$ has objects that are in a bijective correspondence with the isomorphism types of primitive elements (A, a) . Moreover, if the primitive elements (A, a) , (B, b) are (represent) objects of $\mathbf{C}^*[\mathbf{A}]$, then an arrow $(A, a)\to(B, b)$ in $\mathbf{C}^*[\mathbf{A}]$ is the same as an arrow $A\to B$ in the category $\mathbf{A}$. Thus, there is a full and faithful forgetful functor $\mathbf{C}^*[\mathbf{A}]\to\mathbf{A}$.

Furthermore, we can spell out a set of conditions, some of them involving the primitive elements of $\mathbf{A}$, that are jointly necessary and sufficient for $\mathbf{A}$ to be a concrete presheaf category.
The first group, (i), of the conditions says that $\mathbf{A}$ is small cocomplete, $|\text{-}|:\mathbf{A}\to\mathrm{Set}$

preserves small colimits, and reflects isomorphisms.

The second group contains four conditions.

The first, (ii)(a), says that the set of isomorphism types of primitive elements is (indexed by a) small (set).

The second, (ii)(b), says that every element is the *specialization of a primitive element*:

for every element (A, a) of $\mathbf{A}$, there is a primitive element (U, u) together with a map $f:(U, u)\longrightarrow(A, a)$ in $\mathrm{El}(\mathbf{A})$.

Here, (U, u) is said to be a *type for* (A, a), f a *specializing map* for (A, a).

The third condition, (ii)(c), says that, for any element (A, a), with any given primitive (U, u), there is at most one specializing map $(U, u)\rightarrow(A, a)$.

Finally, the last one, (ii)(d), says that if the primitive elements (U, u), (V, v) are both types for (A, a), then they are isomorphic: $(U, u)\cong(V, v)$.

All the above facts concerning concrete presheaf categories are established as parts of standard category theory; they are easy, but form a basic setting for the first of the two main lines of inquiry in the paper, the investigation of the category $\mathtt{Comp}$ and certain of its full subcategories as to which of the above conditions are satisfied in them. $\mathtt{Comp}_{\mathtt{m/1}}$ is one of those full subcategories, and, by what we know from previous work, it satisfies every one of said conditions.

It is relatively easy to show that $\mathtt{Comp}$ itself satisfies (i) and (ii)(a); see the work leading up to section 6. One of the main results of the paper that $\mathtt{Comp}$ satisfies (ii)(b) ; *every element of* $\mathtt{Comp}$ *has at least one type*. The proof of this result requires the more substantial tools of the paper developed in sections 8 , 9 and 11. An easy example shows that (ii)(c) fails in $\mathtt{Comp}$ (see section 6). I do not know if (ii)(d) is satisfied or not by $\mathtt{Comp}$.

Let $\mathcal{C}$ be a *sieve* in $\mathtt{Comp}$, that is a full subcategory of $\mathtt{Comp}$ for which if B is in $\mathcal{C}$, and $A\rightarrow B$ is any arrow, then A is in $\mathcal{C}$. ($\mathtt{Comp}_{\mathtt{m/1}}$ is an example for a sieve in $\mathtt{Comp}$). $\mathbf{C}$ is regarded as a concrete category with the underlying-set functor inherited from $\mathtt{Comp}$. It is then immediate that the notions of principal element, primitive element, and type for an element for $\mathcal{C}$ become the direct restrictions of those for $\mathtt{Comp}$. More precisely, for (A, a) in $\mathtt{El}(\mathcal{C})$, (A, a) is principal resp. primitive for the concrete category $\mathcal{C}$ just in case it is principal resp. primitive for $\mathtt{Comp}$. Moreover, obviously, for an element (A, a) of $\mathcal{C}$, any (U, u) is a type of (A, a) in the sense of the concrete category $\mathtt{Comp}$ if and only if (U, u) is a type of (A, a) in the context of the concrete category $\mathcal{C}$.

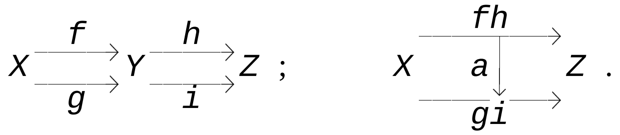
Thus, for a sieve $\mathcal{C}$ in $\mathtt{Comp}$, to say that it is a concrete presheaf category, is to say that it satisfies (i) -- which is ensured by assuming that $\mathcal{C}$ is closed under colimits in $\mathtt{Comp}$ --, and that the conditions (ii)(c) and (ii)(d) are satisfied by primitive elements of $\mathtt{Comp}$ that belong to $\mathcal{C}$.

An additional simplification is provided by the fact that a principal element (A, a) of $\mathtt{Comp}$ is determined by the underlying computad A; a is the unique indeterminate of maximal dimension in A; it is denoted by m_A. We call A a *computope* if (A, m_A) is primitive. If $\mathcal{C}$ is a sieve in $\mathtt{Comp}$, and as a concrete category is a concrete presheaf category, then its shape category $\mathbf{C}^*[\mathcal{C}]$ is the skeletal category of the computopes that are in $\mathcal{C}$. Furthermore, it is a *one-way category* (all non-identity arrows $A \to B$ have $\mathtt{dim}(A) < \mathtt{dim}(B)$), which makes it amenable to the manipulations of logic with dependent sorts ([M1], [M2]).

In particular, multitopes can be identified with many-to-one computopes: an elegant, albeit fairly abstract, definition of "multitope".

Here is an example illustrating the role of computopes.

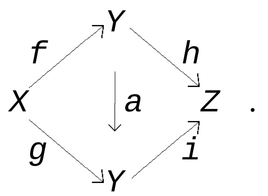
Consider the diagram



(4)

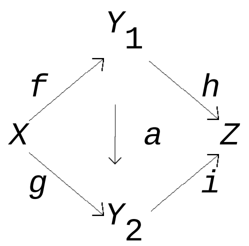
(recall the use of geometric order in compositions). It is clear how to interpret (4) as a computad: once again, the elements (indeterminates) of the computad are exactly the distinct elements named by single letters in the figure. (4) is a principal computad; its main cell ($m_{(4)}$) is a.

In drawing the diagram, we had the inconvenience of Y being in the way of placing the 2-cell a ; this made us repeat parts and denote some composites (the last should not be done ...). We would do better drawing the same as follows:



(5)

This repeats the 0-cell Y , but this is "all right". It seems right to say that (5) shows the real shape of the diagram (4). Of course, as a computad, (5) is identical to (4). However, we have the diagram -- computad --



(6)

"without repetition" of indeterminates; in fact, it is easy to see that (6) is a computope. We also have the obvious computad map $f:(6)\to(5)$ that, in particular, collapses Y_1 and Y_2 to Y . f is a specializing map for (5) (using the terminology introduced above), and (6) is *the* type for (5). In this case, it is easy to see that the type is unique up to isomorphism (condition (ii)(d) above), and it is obvious that the specializing map is unique (condition (ii)(c) above). We are inclined to say that (6) is in fact *the shape* of (5) (and (4)). (5) is obtained from the shape by *labelling*, in particular, labelling the spots Y_1 , Y_2 both by the same item Y .

"Computope" is the mathematical concept of shape of (principal, in particular finite) higher-dimensional diagrams. The specializing maps are the labellings of shapes to get the general diagrams.

I should note that the fact that $\mathrm{Comp}_{m/1}$ is a concrete presheaf category does not become obvious by what has been said above: although we know, by previous work, that conditions (ii)(c) and (d) hold true for many-to-one computads, I do not have a direct proofs of these facts. Despite this circumstance, I think it is be possible, by further developing the methods of this paper, to show that further significant categories of computads are concrete presheaf categories.

Perhaps it is not superfluous to state that my interest in higher-dimensional diagrams, hence, in computads in general, stems from the view that they should constitute the language for talking in a flexible way about matters *within* weak higher dimensional categories. Although the many-to-one computads are sufficient for defining a suitable concept of higher-dimensional weak category, a flexible language to develop mathematics in the context of a suitable weak higher-dimensional category, in analogy to mathematics developed in a topos, one needs higher-dimensional diagrams in general.

(D) The word problem for computads

For a fully explicit, computationally adequate, implementation of higher-dimensional diagrams -- that is, computads -- we need a notational system to represent, not only the indeterminates, but also the *pasting diagrams*, or *pd*'s, i.e., all composite cells, in the computad. After all, we must input the information about the domain and the codomain (arbitrary pd's in general) of each indeterminate.

The formalism of ω-categories provides such a notational system; as usual with free constructions, we can denote all cells of the ω-category freely generated by indeterminates by using a system of *words* derived directly from said formalism. This method is familiar from algebra, for instance, in the study of free groups, or more generally, groups given by presentations. In the case of computads, there is a new element, namely, the necessity to consider the condition of a word being *well-formed*. This becomes clear on the conditional nature of composition: one needs the precondition that a domain be equal to a codomain for the composite to be well-formed. However, having realized that we have to talk about well-formedness, the system of words is naturally defined. In this paper, this is done in section 7, following Jacques Penon's system in [Pe].

Similarly to what happens in the algebra of groups, the pd's in a computad will be identified with *equivalence classes* of words, rather than with words simply; the laws of ω-category will make certain pairs of words *equivalent*, that is, denote the same pd in the computad. The question how to see if two words are equivalent naturally arises, and one wants to know if the *word problem is solvable*: whether or not there is an decision method, efficient if possible, to decide for any two words if they are equivalent. Only in possession of such a decision method can we hope to have a reasonably general way of handling higher-dimensional diagrams computationally.

One of the main results of this paper is that *the word problem for computads* in general *is solvable*. After preparations, the main part of the work of the proof is done in section 10.

The motivation for this result also came from the situation of the many-to-one computads. In [Ha/M/Z] and independently, in [Pa], there is a description of the ω-category, in fact, a typical many-to-one computad, generated by a multitopic set, in which the general cells of the ω-category are given as multitopic pd's of the multitopic set "with niches". (In [Ha/M/Z], this is given as the left-adjoint of a pair of adjoint functors between `MltSet` and ω`Cat`, the right adjoint of which is a *multitopic nerve* functor). The construction provides a *normal form* for words denoting the pd's of the many-to-one computad. Starting from any many-to-one word, its normal form is computable, and two words are equivalent iff their normal forms are identical; the word problem of many-to-one computads is solvable as a consequence.

The solution of the word problem for general computads starts in a similar manner, with reducing an arbitrary well-formed word to a "pre"-normal form. The question of equivalence of pre-normal words is still non-trivial, but it is simpler than that for raw words, and it is eventually manageable, although the decision procedure as it stands at present uses searches through fairly large finite sets, and therefore it is quite unfeasible.

(E) The contents and the methods of the paper

The paper separates into two parts, one that uses, and the other that does not use, words. Sections 7 and 10 use, and are about, words. The other sections do not mention words, or use results based on words, at all.

The elementary theory of equivalence to a concrete presheaf category is explained in section 1.

Here, and elsewhere, the proofs that were found boring or less than easily readable were put into appendices. On the other hand, the paper, taken as a whole, is more than usually self-contained.

Sections 2 and 3 contain the generally accepted definitions of ω-graph and ω-category. Compare [Str2].

Sections 4 and 5 contain the concepts underlying the definition of "computad", and the basic results concerning these concepts. The approach is leisurely and the proofs are mostly routine. Section 4 explores the operation of adjoining indeterminates to a general ω-category, and the iteration of this operation. Section 5 defines computad as an ω-category obtained by iterated adjunctions of indeterminates to the empty ω-category. The emphasis in section 5 is on the properties of the category $\mathtt{Comp}$ of all computads, and the way this category resembles "good" categories such as presheaf categories.

The one element of section 5 that seems to be novel is the concept of the *content* of a pd in a computad: this is a multiset of the indeterminates occurring in the pd, counting the multiplicity (number of occurrences) of each indeterminate.

The definition of the content function was a non-trivial matter, and in fact, it is not entirely successful. One of the main intuitive requirements would be that in case of a computope A, the multiplicity of each occurring indeterminate in m_A is equal to 1. Our definition of the content function definitely does not satisfy this; and I do not know if it is possible to give such a definition, also having the other desired properties.

Despite its drawbacks, the content function is an efficient tool for the main purposes of the paper. One needed property is its invariance under equivalence. Its verbal description sounds as if it is defined for words, by a direct count of occurrences. However, such a definition would not give something that is invariant under equivalence of word, that is well-defined for pd's. As a matter of fact, the definition of the content is not done using words at all.

Another crucial property of "content" is its "linear" behaviour under maps of computads; see 5.(12)(ix).

Section 6 was fairly completely described under (C).

Section 7 displays the system of words for computads in complete (and straight-forward) detail.
Sections 8 and 9 contain the main mathematical novelty in the paper. I propose a kind of normal form, the *expanded form*, for compound expressions (words) in the language of

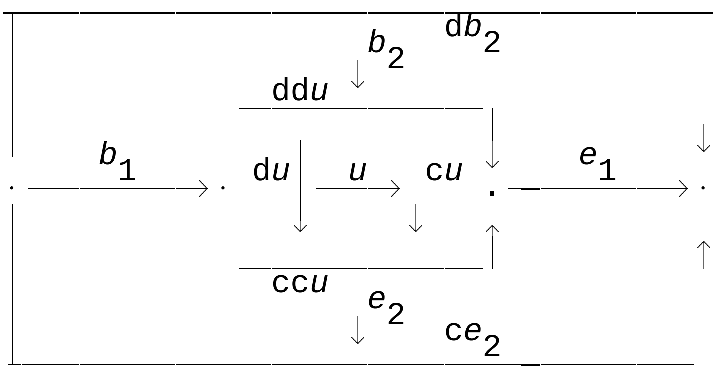
ω-categories. The expanded form is constrained in two ways. The first is that it admits only restricted instances of the ω-category operations. Specifically, the operation $a\#_k b$ is allowed for cells a of dimension m and b of dimension n only if $k=\min(m,n)-1$. Since k is determined by a and b, its notation is not necessary; we write $a \cdot b$ for $a\#_k b$.

The second constraint is that the expanded form allows the operations only in a certain order. For instance, denoting a 4-cell using a single indeterminate 4-cell u, is allowed only in the form of an *atom*

$$b_3 \cdot (b_2 . (b_1 \cdot u \cdot e_1) \cdot e_2) \cdot e_3$$

where b_i and e_i are cells of dimensions i, $i=1, 2, 3$. Of course, it is required that the composites be well-defined. "Bigger" 4-cells are obtained in the form of *molecules*, which are $\cdot$-composites of atoms.

The picture for a 3-atom $b_2 . (b_1 \cdot u \cdot e_1) \cdot e_2$ is



The success of the expanded form to account for all expressions rests with certain features of the constrained dot-operations. Section 8 shows that the operations obey laws that are of a nature that is more familiar from algebra than the laws in the generally accepted definition of ω-category. In particular, we have an associative law involving three variables (similarly to the usual definition), a distributive law, also involving three variables, and a "commutative law" involving two variables. It is shown that the usual operations, with their laws, are recoverable from the dot-operations with their postulated laws, effectively providing a new definition of ω-category, equivalent to the original one. In particular, the distributive law let's one distribute a lower dimensional cell over the composite of higher-dimensional cells as in

$$a\cdot(b\cdot e) = (a\cdot b)\cdot(a\cdot e)\ ,$$

where $\dim(a)<\dim(b)$, $\dim(a)<\dim(e)$, and the expressions are well-defined. It is mainly this that allows to reduce an arbitrary word to the form of a molecule.

The definition of ω-category through the dot-operations and the expanded form are very natural, and they readily come to mind when one discusses examples. For instance, the composite of the diagram in (1) has two molecular forms, both shown in

$$\begin{array}{ccccc}
 & & g_1g_2f_3f_6 & & \\
 & & \downarrow\ g_1a_2f_6 & & \\
 & a_1g_4f_6 \swarrow & g_1f_2g_4f_6 & \searrow g_1g_2a_4 & \\
f_1g_3g_4f_6 & & & & g_1f_2f_5g_6 \\
 & f_1g_3a_4 \searrow & f_1g_3g_5g_6 & \swarrow a_1f_5g_6 & \\
 & & \downarrow\ f_1a_3g_6 & & \\
 & & f_1f_4g_5g_6 & &
\end{array}$$

as the two equal composites from top to bottom.

The expanded form is used in section 9, the heart of the paper. This provides a reduction of the structure of an $(n+1)$-dimensional computad to that of a "collapsed" n-dimensional one, whereby the only thing, beyond the n-computad, left to discuss for the description of the $(n+1)$-computad is the effect of the commutative law on interchanging $(n+1)$-dimensional atoms.

I note that the results of section 8 and 9 are stated without referring to words. They have immediate variants involving words, which are stated and used as the main tools for the solution of the word problem in section 9. The same results, without reference to words, are used to establish certain finiteness lemmas, which are needed, in a natural fashion, to limit certain searches to finite sets, and to establish the decision procedure for the equivalence of words in section 10. In both sections 9 and 10, the content function of section 5 is crucial.

Acknowledgements

I thank Bill Boshuck, Victor Harnik and, especially, Marek Zawadowski for ideas and inspiring conversations, taking place over several years, about higher-dimensional categories in general and computads in particular. The counter-examples of section 6 came out of joint work with Marek Zawadowski.

I also thank the participants of the McGill Category Seminar for their interest in, and their unfailing tolerance for my often tiring talks about, these subjects.

1. Concrete presheaf categories

A *concrete category* is a category $\boldsymbol{A}$ with small hom-sets, together with a (*forgetful*) functor $|-|_{\boldsymbol{A}} = |-| : \boldsymbol{A} \to \mathrm{Set}$. Usually, the forgetful functor $|-|$ has various good properties such as faithfulness, etc., but at this point we make no additional assumptions.

$\mathrm{El}(\boldsymbol{A})$ denotes the category of elements of the functor $|-| : \boldsymbol{A} \to \mathrm{Set}$: its objects are the pairs $(A \in \mathrm{Ob}(\boldsymbol{A}), a \in |A|)$, an arrow $(A, a) \to (B, b)$ is $f : A \to B$ such that $|f|(a) = b$.

Let $\boldsymbol{A} = (\boldsymbol{A}, |-|_{\boldsymbol{A}})$, $\boldsymbol{B} = (\boldsymbol{B}, |-|_{\boldsymbol{B}})$ be concrete categories. We say that they are *equivalent* if there is a functor $\Phi : \boldsymbol{A} \to \boldsymbol{B}$ that is an equivalence of categories such that the following diagram of functors:

$$\begin{array}{ccc} \boldsymbol{A} & \xrightarrow{\Phi} & \boldsymbol{B} \\ & \cong & \\ |-|_{\boldsymbol{A}} \searrow & \mathrm{Set} & \swarrow |-|_{\boldsymbol{B}} \end{array}$$

commutes up to an isomorphism: there is an isomorphism $\varphi\colon |\text{-}|_{\boldsymbol{A}} \xrightarrow{\cong} |\text{-}|_{\boldsymbol{B}} \circ \Phi$.

If the concrete categories $\boldsymbol{A}$, $\boldsymbol{B}$ are equivalent, by the equivalence (Φ, φ) , then the ordinary categories $\mathrm{El}(\boldsymbol{A})$, $\mathrm{El}(\boldsymbol{B})$ are also equivalent, by the equivalence functor

$$\begin{array}{ccc} \mathrm{El}(\boldsymbol{A}) & \xrightarrow{\Psi} & \mathrm{El}(\boldsymbol{B}) \\ (A, a) & \longmapsto & (\Phi A, \varphi_A(a)) \end{array} .$$

A (*full*) *subcategory* of a concrete category is a (full) subcategory in the usual sense, with the forgetful functor the restriction of the given one.

We wish to regard *presheaf categories* as concrete categories.

Let $\boldsymbol{C}$ be a small category, let $\hat{\boldsymbol{C}} = \mathrm{Set}^{\boldsymbol{C}^{\mathrm{op}}}$, the corresponding presheaf category. U, V, ... denote objects of $\boldsymbol{C}$; A, B, ... objects of $\hat{\boldsymbol{C}}$.

We view $\hat{\boldsymbol{C}}$ as a concrete category, with $|\text{-}| = |\text{-}|_{\boldsymbol{C}} : \hat{\boldsymbol{C}} \to \mathrm{Set}$, the forgetful functor, defined by $|A| = \coprod_{U \in \boldsymbol{C}} A(U)$, and, for $F\colon A \to B$, $|F| \overset{\text{def}}{=} \coprod_{U \in \boldsymbol{C}} |F_U| \; : |A| \to |B|$.

It is obvious that the construction $\boldsymbol{C} \mapsto (\hat{\boldsymbol{C}}, |\text{-}|_{\boldsymbol{C}})$ respects *isomorphism* of categories, but it is equally obvious that it does *not* respect *equivalence* of categories. In fact, we have

(1) Proposition If $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{D}}$ are *equivalent* as concrete categories, then $\boldsymbol{C}$ and $\boldsymbol{D}$ are necessarily *isomorphic*.

For the (elementary) proof, see the Appendix.

This is to be contrasted with the corresponding situation of the ordinary equivalence of presheaf categories $\hat{\boldsymbol{C}}$, $\hat{\boldsymbol{D}}$, which happens if and only if the Cauchy (idempotent splitting) completions of $\boldsymbol{C}$ and $\boldsymbol{D}$ are *equivalent*.

Our interest is in questions of the form whether or not a certain specific concrete category $\boldsymbol{A}$ is equivalent to some concrete presheaf category $\hat{\boldsymbol{C}}$. If the answer is "yes", we say, somewhat abbreviatedly, that $\boldsymbol{A}$ *is a concrete presheaf category*. Further, if the answer is "yes", we call the category $\boldsymbol{C}$ in question, which we know to be determined up to isomorphism, the *shape category* of $\boldsymbol{A}$.

Starting with any concrete category $\boldsymbol{A}$, we will construct two particular categories, $\boldsymbol{C}[\boldsymbol{A}]$ and $\boldsymbol{C}^*[\boldsymbol{A}]$, such that, if $\boldsymbol{A}$ is a concrete presheaf category, then the shape category of $\boldsymbol{A}$ is isomorphic to both $\boldsymbol{C}[\boldsymbol{A}]$ and $\boldsymbol{C}^*[\boldsymbol{A}]$. The second one, $\boldsymbol{C}^*[\boldsymbol{A}]$, is the more "concrete" construction.

Consider the concrete category $\boldsymbol{A}=\hat{\boldsymbol{C}}$, with $|\text{-}|:\boldsymbol{A}\to\mathrm{Set}$ defined above. The Yoneda lemma translates into the statement that $\mathrm{El}(\boldsymbol{A})$ is the disjoint union (coproduct) of full subcategories $\boldsymbol{E}_U$, one for each $U\in\mathrm{Ob}(\boldsymbol{C})$, and the object $(\hat{U}, 1_U)\in\hat{U}(U))$ is an initial object of $\boldsymbol{E}_U$. Here, we have used the notation $\hat{U}=\boldsymbol{C}(\text{-}, U)\in\mathrm{Ob}(\hat{\boldsymbol{C}})$.

Let $\mathcal{E}$ be any category. A *partial initial object* (PIO) of $\mathcal{E}$ is an object that is initial in the connected component of $\mathcal{E}$ (regarded as a full subcategory of $\mathcal{E}$) to which it belongs. Obviously, the property being a PIO is invariant under isomorphism inside $\mathcal{E}$, and is preserved by an equivalence of categories.

For the concrete category $\hat{\boldsymbol{C}}$, the objects $(\hat{U}, 1_U)$ are PIO's; these we call the *standard* PIO's. Further, the standard PIO's form a *precise* set of representatives of the isomorphic classes of PIO's: every PIO is isomorphic to exactly one standard one.

Let $\boldsymbol{A}$ be a concrete category. We construct the category $\boldsymbol{C}=\boldsymbol{C}[\boldsymbol{A}]$ as follows. We pick a precise class $\mathcal{U}$ of representatives of PIO's in $\mathrm{El}(\boldsymbol{A})$: every PIO of $\mathrm{El}(\boldsymbol{A})$ is isomorphic to exactly one member of $\mathcal{U}$. We let $\mathrm{Ob}(\boldsymbol{C})$ be the class (in good cases, a set) $\mathcal{U}$. For (U, u), (V, v) in $\mathcal{U}$, an arrow $(U, u)\longrightarrow(V, v)$ in $\boldsymbol{C}$ is an arrow $U\longrightarrow V$ in $\boldsymbol{A}$ (*without* any reference to the elements u and v). $\boldsymbol{C}$ has the forgetful functor $(U, u)\mapsto U$

to $\boldsymbol{A}$, and this functor is full and faithful.

(2) Proposition Let $\mathbf{A}=(\boldsymbol{A}, |\text{-}|_{\boldsymbol{A}}:\boldsymbol{A}\to\mathrm{Set})$ be a concrete category. Assume that $\boldsymbol{A}$ is cocomplete, $|\text{-}|_{\boldsymbol{A}}:\boldsymbol{A}\to\mathrm{Set}$ preserves all (small) colimits, and reflects isomorphisms. (It follows that $|\text{-}|$ is faithful and reflects colimits). Assume, moreover, that

(*) $\mathrm{El}(\mathbf{A})$ is the disjoint union of a small set of full subcategories, each of which has an initial object.

Then $\mathbf{A}$ is a concrete presheaf category, with shape category isomorphic to $\boldsymbol{C}[\mathbf{A}]$.

For the proof, which is elementary category theory, see the Appendix.

Let $(\boldsymbol{A}, |\text{-}|)$ be a concrete category. An element of $\mathbf{A}$, that is, an object of $\mathrm{El}(\mathbf{A})$, (A, a), is *principal* if a generates A: iff for any $f:(B, a)\to(A, a)$, if $f:B\to A$ is a monomorphism, f is an isomorphism. (A, a) is *primitive* if it is principal, and for all principal (B, b), any arrow (in $\mathrm{El}(\mathbf{A})$) $(B, b)\longrightarrow(A, a)$ is necessarily an isomorphism.

Of course, "principal" and "primitive" are isomorphism-invariant properties of objects of $\mathrm{El}(\mathbf{A})$. Notice that any morphism between primitive elements of $\mathbf{A}$ is necessarily an isomorphism.

(3) Proposition Suppose that the concrete category $(\boldsymbol{A}, |\text{-}|)$ satisfies condition (*) in (1). Then an element (A, a) of $\mathbf{A}$ is primitive if an only if it is a PIO.

Proof. Assume first that (U, u) is initial in the component $\boldsymbol{E}$ of $\mathrm{El}(\mathbf{A})$.

(U, u) is principal: suppose $f:(A, a)\to(U, u)$, with $f:A\to B$ a monomorphism. Since there is an arrow between (U, u) and (A, a), (A, a) must belong to $\boldsymbol{E}$. Since (U, u) is initial in $\boldsymbol{E}$, there is a right inverse $r:(U, u)\to(A, a)$ to f, $fr=1_U$. Since f is mono, f is an isomorphism.

Next, (U, u) is primitive: assume (A, a) is principal and $f:(A, a)\to(U, u)$. Again, we have a right inverse $r:(U, u)\to(A, a)$ of f. But then r is a split mono, and thus, since (A, a) is principal, r is an isomorphism. It follows that f is an isomorphism.

Conversely, assume (A, a) is primitive. Let (U, u) be an initial object of the component of $\mathrm{El}(\boldsymbol{A})$ containing (A, a). We have $f:(U, u)\to(A, a)$. Since (U, u) is principal (see above), it follows that f is an isomorphism. (A, a), being isomorphic to the partial initial (U, u), is itself partial initial. This completes the proof.

In view of (3), we modify the construction $\boldsymbol{C}[\boldsymbol{A}]$ above to $\boldsymbol{C}^*[\boldsymbol{A}]$, by changing the references to PIO's to references to primitive elements. Of course, if $\boldsymbol{A}$ is a concrete presheaf category, then, by (3), $\boldsymbol{C}[\boldsymbol{A}]$ and $\boldsymbol{C}^*[\boldsymbol{A}]$ are isomorphic.

The following is a summary.

(4) Theorem Let $(\boldsymbol{A}, |-|)$ be a concrete category. The following conditions are jointly necessary and sufficient for $(\boldsymbol{A}, |-|)$ to be a concrete presheaf category.

(i) $\boldsymbol{A}$ is cocomplete, $|-|:\boldsymbol{A}\to\mathrm{Set}$ preserves all (small) colimits, and reflects isomorphisms.

(ii) **(a)** The collection of the isomorphism classes of primitive elements of $\boldsymbol{A}$ is small.

(b) For every element (A, a) of $\boldsymbol{A}$, there is a primitive element (U, u) with a morphism $(U, u)\longrightarrow(A, a)$.

(c) Whenever (U, u) is primitive, and $(U, u)\underset{g}{\overset{f}{\rightrightarrows}}(A, a)$ in $\mathrm{El}(\boldsymbol{A})$, we have $f=g$.

(d) Whenever (U, u) and (V, v) are primitive, and there are arrows $(U, u)\longrightarrow(A, a)\longleftarrow(V, v)$ in $\mathrm{El}(\boldsymbol{A})$, we have $(U, u)\cong(V, v)$.

If $\boldsymbol{A}$ is a concrete presheaf category, then its shape category is isomorphic to $\boldsymbol{C}^*[\boldsymbol{A}]$.

In this paper, I will show that the concrete category `Comp` of small computads satisfies conditions (4)(i), (ii)(a), (ii)(b) , and *does not* satisfy condition (ii)(c). I do not know whether or not (ii)(d) holds in `Comp` .

By [H/M/P] , the concrete category of many-to-one computads satisfies all conditions in (4). In future work, I hope to isolate significant other concrete full subcategories of `Comp` that satisfy all conditions in (4).

In section 6, after the basics concerning computads have been established, we return to the subject of this section, specialized to full subcategories on `Comp` .

2. ω-graphs.

An *ω-graph* $\boldsymbol{X}$ is given by a sequence of sets $\boldsymbol{X}_n$, $n\in\mathbb{N}\cup\{-1\}$, together with maps

$$\boldsymbol{X}_n \underset{\mathrm{c}}{\overset{\mathrm{d}}{\rightrightarrows}} \boldsymbol{X}_{n-1}$$

(we have abbreviated d_n to d, c_n to c) for each $n\geq 0$, such that always $\boldsymbol{X}_{-1}$ is a singleton, $\boldsymbol{X}_{-1}=\{*\}$, and such that we have the following "globularity" conditions satisfied: $\mathrm{dd}=\mathrm{dc}$, $\mathrm{cd}=\mathrm{cc}$ (where, again, subscripts have been suppressed; they are to be restored in all meaningful ways to obtain an infinity of commutativity conditions; this kind of abbreviation in the notation of arrows will be practiced in other contexts as well). Elements of $\boldsymbol{X}_n$ are the *n-cells* of $\boldsymbol{X}$.

Morphisms of ω-graphs are defined in the natural way. The thus-obtained category of small ω-graphs, $\omega\mathrm{Graph}$, is, clearly, the presheaf category $\mathrm{Set}^{(\mathrm{gph}_\omega)^{\mathrm{op}}}$, with gph_ω the category generated by the (ordinary) graph

$$x_0 \underset{\gamma}{\overset{\delta}{\rightrightarrows}} x_1 \underset{\gamma}{\overset{\delta}{\rightrightarrows}} \;\ldots\; \underset{\gamma}{\overset{\delta}{\rightrightarrows}} x_{n-1} \underset{\gamma}{\overset{\delta}{\rightrightarrows}} x_n \cdots$$

subject to the relations $\delta\delta=\gamma\delta$, $\delta\gamma=\gamma\gamma$.

For convenience, for an ω-graph $\boldsymbol{X}$, we assume that the sets $\boldsymbol{X}_n$ are pairwise disjoint, and write $\|\boldsymbol{X}\| = \coprod_{n\in\mathbb{N}\cup\{-1\}} \boldsymbol{X}_n = \dot{\bigcup}_{n\in\mathbb{N}\cup\{-1\}} \boldsymbol{X}_n$. This assumption entails no serious loss of generality, since, obviously, every ω-graph is isomorphic to one with said property.

We write $\mathrm{dim}(x)=n$ for $x\in\boldsymbol{X}_n$.

The notation $\|\boldsymbol{X}\|$ is avoided whenever possible; e.g., we write $x\in\boldsymbol{X}$ for $x\in\|\boldsymbol{X}\|$.

Compared to the usual formulation, we have "formally" added a cell $*$ of dimension -1 and declared that $\mathrm{d}X=\mathrm{c}X=*$ for all $X\in\boldsymbol{X}_0$. We say that a and b are *parallel*, in notation

$a \| b$, if $\mathrm{d}a = \mathrm{d}b$ and $\mathrm{c}a = \mathrm{c}b$; any two 0-cells are parallel.

We use the notation $X_{\leq n}$ to mean $\dot{\bigcup}_{-1 \leq m \leq n} X_m$. If d or c is applied to $* \in X_{-1}$, it should mean $* : \mathrm{d}(*) = \mathrm{c}(*) = *$.

For $a \in X_n$ and $k \leq n$, we write $\mathrm{d}^{(k)}a$ for $\mathrm{d}^{n-k}(a) = \underset{1}{\underset{\uparrow}{\mathrm{d}}} \ldots \underset{n-k}{\underset{\uparrow}{\mathrm{d}}}(a)$; similarly for c in place of d . Note that $\dim(\mathrm{d}^{(k)}a) = \dim(\mathrm{c}^{(k)}a) = k$.

$X_n \times_k X_n$ denotes the pullback in

$$\begin{array}{ccc} X_n \times_k X_n & \xrightarrow{\pi_1} & X_n \\ \pi_0 \downarrow & & \downarrow \mathrm{d}^{(k)} \\ X_n & \xrightarrow[\mathrm{c}^{(k)}]{} & X_k \end{array}$$

In other words, $X_n \times_k X_n = \{(a, b) \in X_n \times X_n : \mathrm{c}^{(k)}a = \mathrm{d}^{(k)}b\}$. When $\mathrm{c}^{(k)}a = \mathrm{d}^{(k)}b$, we say that $a \wedge_k b$ is well-defined (abbreviated as $a \wedge_k b \downarrow$) and equals $\mathrm{c}^{(k)}a = \mathrm{d}^{(k)}b$.

An n-*graph*, for $n \in \mathbb{N}$, is like an ω-graph except it only has m-cells for $m = -1, 0, \ldots, n$ only. Every n-graph, for $n \in \mathbb{N} \dot{\cup} \{\omega\}$, has, for every $m < n$, its m-*truncation*, an m-graph.

3. ω-categories

An ω-*category* is an ω-graph X , together with the partial -- or better: *conditional* -- operations:

$$1_{(-)} : X_n \longrightarrow X_{n+1} \qquad (n \geq 0)$$

$$\begin{aligned}\#_k : \mathbf{X}_n \times_k \mathbf{X}_n &\longrightarrow \mathbf{X}_n \qquad (n>k\geq 0)\\ (a, b) &\longmapsto a\#_k b\\ a\wedge_k b\downarrow &\end{aligned}$$

satisfying the conditions given below.

Let us write, recursively, $1_a^{(n)}$ for $1_{1_a^{(n-1)}}$ for $a\in\mathbf{X}_k$ and $n>k$, with $1_a^{(k)} \overset{\text{def}}{=} a$; we have $1_a^{(n)}\in\mathbf{X}_n$. Any cell of the form $1_a^{(n)}$ with $a\in\mathbf{X}_k$ is a *k-to-n identity cell.*

(Thus, $\#_k$ is composition in the "geometric" order of arguments; we may write $b\circ_k a$ for $a\#_k b$. However, the "geometric" $\#_k$ notation is preferred, and when below the juxtaposition ab, or the form $a\cdot b$ occurs, it will stand for $1_a^{(n)}\#_k 1_b^{(n)}$ with $n=\max((\dim(a),\dim(b))$ and $k = \min(\dim(a),\dim(b))-1$.)

The axioms on the operations are as follows; throughout, $n>k\geq 0$ and a, b, e, $f\in\mathbf{X}_n$ are arbitrary.

Domain/codomain laws:

$$\mathrm{d}(1_a) = \mathrm{c}(1_a) = a \ ;$$

$$\mathrm{d}(a\#_k b) = \begin{cases} \mathrm{d}a & \text{if } k=n-1\\ (\mathrm{d}a)\#_k(\mathrm{d}b) & \text{if } k<n-1\end{cases}$$

$$\mathrm{c}(a\#_k b) = \begin{cases} \mathrm{c}b & \text{if } k=n-1\\ (\mathrm{c}a)\#_k(\mathrm{c}b) & \text{if } k<n-1\end{cases}$$

(Remark: note that if $a\wedge_k b$ $(a, b\in\mathbf{X}_n)$ is well-defined, that is, $\mathrm{c}^{(k)}a=\mathrm{d}^{(k)}b$, and $k<n-1$, then $\mathrm{c}^{(k)}\mathrm{d}a=\mathrm{d}^{(k)}\mathrm{d}b$, by the laws of ω-graphs; i.e., $(\mathrm{d}a)\wedge_k(\mathrm{d}b)$ is well-defined and equals $a\wedge_k b$.)

Left unit law:

$$1^{(n)}_{d^{(k)}b} \#_k b = b$$

Right unit law:

$$a \#_k 1^{(n)}_{c^{(k)}a} = a \;.$$

Two-sided unit law:

$$1_a \#_k 1_b = 1_{a\#_k b}$$

provided that $a\#_k b$ is well-defined.

Associative law:

$$(a\#_k b)\#_k e = a\#_k (b\#_k e)$$

provided that $a\#_k b$ and $b\#_k e$ are well-defined.

(Remark: note that if $a\#_k b$ and $b\#_k e$ are well-defined, then

$$c^{(k)}(a\#_k b) = c^{(k)}b = d^{(k)}e =$$

and

$$c^{(k)}a = d^{(k)}b = d^{(k)}(b\#_k e) \;,$$

thus both sides of the associativity identity are well-defined. In other words, under the conditions for the associative law,

$$(a\#_k b)\wedge_k e \;=\; b\wedge_k e \qquad \text{and} \qquad a\wedge_k(b\#_k e) \;=\; a\wedge_k b\ .$$

Note, moreover, that, even before we know that they are equal, the two sides are seen to be parallel.)

(**Middle Four**) **Interchange law**:

$$(a\#_k b)\#_\ell(e\#_k f) \;=\; (a\#_\ell e)\#_k(b\#_\ell f)$$

provided that $k\neq\ell$, and the four "simple composites" involved are well-defined.

(Remark: we assume that $a\wedge_k b$, $e\wedge_k f$, $a\wedge_\ell e$, $b\wedge_\ell f$ are well-defined; in other words,

$$\mathrm{c}^{(k)}a=\mathrm{d}^{(k)}b\ ,\ \ \mathrm{c}^{(k)}e=\mathrm{d}^{(k)}f\ , \tag{1}$$
$$\mathrm{c}^{(\ell)}a=\mathrm{d}^{(\ell)}e\ ,\ \ \mathrm{c}^{(\ell)}b=\mathrm{d}^{(\ell)}f\ . \tag{2}$$

Because of the obvious symmetry in the interchange identity, we may assume that, e.g., $k<\ell$. It then follows that

$$\mathrm{c}^{(k)}a = \mathrm{c}^{(k)}\mathrm{c}^{(\ell)}a = \mathrm{c}^{(k)}\mathrm{d}^{(\ell)}e = \mathrm{c}^{(k)}e\ ,$$

and similarly, $\mathrm{d}^{(k)}(b) = \mathrm{d}^{(k)}(f)$. Thus

$$\mathrm{c}^{(k)}a = \mathrm{c}^{(k)}e = \mathrm{d}^{(k)}(b) = \mathrm{d}^{(k)}(f)\ . \tag{3}$$

Since $k<\ell$, we have

$$c^{(\ell)}(a\#_k b) \;=\; \mathrm{c}^{(\ell)}(a)\#_k\mathrm{c}^{(\ell)}(b)\ ,$$
$$\mathrm{d}^{(\ell)}(e\#_k f) \;=\; \mathrm{d}^{(\ell)}(e)\#_k\mathrm{d}^{(\ell)}(f)$$

which are equal by (2), hence, the left-hand side of the interchange identity is well-defined.

Since

$$c^{(k)}(a\#_\ell e) = c^{(k)}(a) = d^{(k)}(b) = d^{(k)}(b\#_\ell f)$$

(by (3)), the right-hand side of the interchange identity is well-defined.

For future reference, let us record some facts just verified. Under the conditions of the interchange law, when $k<\ell$, we have

$$(A \overset{=}{\text{def}}) \; a\wedge_k b = e\wedge_k f = (a\#_\ell e)\wedge_k(b\#_\ell f)$$

and

with $\varphi \overset{=}{\text{def}} a\wedge_\ell e$, $\psi \overset{=}{\text{def}} b\wedge_\ell f$, we have

$$(a\#_k b)\wedge_\ell(e\#_k f) = \varphi\#_k\psi \; .$$

We can show by induction that, assuming interchange in lower dimensions, the two sides of the interchange identity are parallel.)

The notion of n-category is the obvious truncated version of that of ω-category. An n-category $\mathbf{X}$ has m-cells for m up to and including n; the operation of identity $a\mapsto 1_a$ is defined for $a\in\mathbf{X}_{<n}-\{*\}$. Every $n(\le\omega)$-category has its m-truncation for any $m<n$.

A *morphism* of $n(\le\omega)$-categories is a morphism of the underlying n-graphs that preserve, in the direct and strict sense, all the n-category operations. Given $n\le\omega$, we have the (ordinary) category $n\mathrm{Cat}$ of small n-categories. We have the truncation functors $(-)\upharpoonright m : n\mathrm{Cat}\longrightarrow m\mathrm{Cat}$ $(m\le n)$.

Inspecting the definition of "ω-category", we see that it is given by a finite-limit sketch $\mathcal{S}_{\omega\text{-cat}}$ so that an ω-category is, in essence (up to isomorphism), the same as a Set-model of $\mathcal{S}_{\omega\text{-cat}}$. The morphisms of ω-cats are the same as morphisms of models of $\mathcal{S}_{\omega\text{-cat}}$. Therefore, $\omega\mathrm{Cat}$, as the category $\mathrm{Mod}(\mathcal{S}_{\omega\text{-cat}})$ of models of $\mathcal{S}_{\omega\text{-cat}}$, is an essentially algebraic, that is, locally finitely presentable, category [A/R]. It also follows that (small) limits and filtered colimits in $\omega\mathrm{Cat}$ are computed "pointwise". That is, limits and filtered colimits in $\omega\mathrm{Cat}$ are created jointly by the functors $(-)_k:\omega\mathrm{Cat}\to\mathrm{Set}$ for $k\in\mathbb{N}$.

Analogous statements can be made for $n\mathtt{Cat}$ ($n\in\mathbb{N}$).

The functor $(\text{-})\upharpoonright n:\omega\mathtt{Cat}\longrightarrow n\mathtt{Cat}$ has a left adjoint (for the simple reason that it is a limit-preserving functor between essentially algebraic categories), call it $(\text{-})^{(\omega)}:n\mathtt{Cat}\longrightarrow\omega\mathtt{Cat}$, which is easy to describe. For $\boldsymbol{X}\in n\mathtt{Cat}$, $\boldsymbol{X}^{(\omega)}$ has its n-truncation equal to $\boldsymbol{X}$; for all $k>n$, the k-cells of $\boldsymbol{X}^{(\omega)}$ are all n-to-k identity cells; their composition law is the only possible one.

Because of the innocence of the functor $(\text{-})^{(\omega)}:n\mathtt{Cat}\to\omega\mathtt{Cat}$, it is often the case that we regard the n-category $\boldsymbol{X}$ as identical to the corresponding ω-category $\boldsymbol{X}^{(\omega)}$.

For any $m<n\in\mathbb{N}$, we have the truncation functor $(\text{-})\upharpoonright m:n\mathtt{Cat}\to m\mathtt{Cat}$, and its left adjoint $(\text{-})^{(n)}$, with properties analogous to the the above.

4. Adjoining indeterminates

Let $\boldsymbol{X}$ be an ω-category. Let U be a set, and $u\mapsto \mathrm{d}u$, $u\mapsto \mathrm{c}u$ two functions $U\longrightarrow\|\boldsymbol{X}\|$ such that, for each $u\in U$, $\mathrm{d}u\|\mathrm{c}u$. The elements of U are regarded as "indeterminate" elements, each $u\in U$ of dimension $n+1$ if $\dim(\mathrm{d}u)=\dim(\mathrm{c}u)=n$, waiting to be adjoined to $\boldsymbol{X}$ as a new element, fitted into the slot given by $\mathrm{d}u$ and $\mathrm{c}u$ as $u:\mathrm{d}u\to\mathrm{c}u$. The pair (d,c) of functions is sometimes referred to as the *attachment* of U to $\boldsymbol{X}$.

Suppose $\boldsymbol{X}$ and $U=(U,d,c)$ are given as above.

Let's say that the triple $(\boldsymbol{Y},\boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{Y},U\xrightarrow{\Lambda}\|\boldsymbol{Y}\|)$, with an arbitrary ω-category $\boldsymbol{Y}$, morphism Γ and set-map Λ as shown, also satisfying the commutativity

$$\begin{array}{ccc} U & \xrightarrow{\quad\Lambda\quad} & \boldsymbol{Y} \\ {\scriptstyle\langle \mathrm{d},\,\mathrm{c}\rangle}\big\downarrow & \circ & \big\downarrow{\scriptstyle\langle \mathrm{d},\,\mathrm{c}\rangle} \\ \|\boldsymbol{X}\|\times\|\boldsymbol{X}\| & \xrightarrow[\|\Gamma\|\times\|\Gamma\|]{} & \|\boldsymbol{Y}\|\times\|\boldsymbol{Y}\| \end{array}$$

is an *extension of* $\boldsymbol{X}$ *by* U . In an extension of $\boldsymbol{X}$ by U , we have the elements of U "realized" as real cells, with domain and codomain that are given by what the (d, c)-data on U and the "injection" of $\boldsymbol{X}$ into the extension say they should be.

Extensions of $\boldsymbol{X}$ by U form a natural category $\mathtt{Ext}(\boldsymbol{X}; U)$: an arrow

$$(\boldsymbol{Y}, \boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{Y}, U\xrightarrow{\Lambda}\|\boldsymbol{Y}\|) \longrightarrow (\hat{\boldsymbol{Y}}, \boldsymbol{X}\xrightarrow{\hat{\Gamma}}\hat{\boldsymbol{Y}}, U\xrightarrow{\hat{\Lambda}}\|\hat{\boldsymbol{Y}}\|)$$

is a morphism $\boldsymbol{Y}\longrightarrow\hat{\boldsymbol{Y}}$ that makes the following two diagrams commute:

$$\begin{array}{ccc} & \Gamma\nearrow & \boldsymbol{Y} \\ \boldsymbol{X} & & \downarrow \\ & \hat{\Gamma}\searrow & \hat{\boldsymbol{Y}} \end{array} \qquad\qquad \begin{array}{ccc} & \Lambda\nearrow & \|\boldsymbol{Y}\| \\ U & & \downarrow \\ & \hat{\Lambda}\searrow & \|\hat{\boldsymbol{Y}}\| \end{array}$$

A *free* extension $\boldsymbol{X}$ by U is an initial object of $\mathtt{Ext}(\boldsymbol{X}; U)$. It easy to see that $\mathtt{Ext}(\boldsymbol{X}; U)$ is an essentially algebraic category. Therefore, the free extension of $\boldsymbol{X}$ by U exists and is determined up to isomorphism. It is denoted

$$(\boldsymbol{X}[U], \boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{X}[U], U\xrightarrow{\Lambda}\|\boldsymbol{X}[U]\|) .$$

Extensions in our present sense may be regarded in a slightly different way. The data $(\boldsymbol{X}; U)$ -- meaning $(\boldsymbol{X};U,\mathtt{d},\mathtt{c})$ as above -- form, with all parameters varying, a category $\mathcal{F}$ of "(extension) frames": an arrow

$$(\boldsymbol{X}; U) \longrightarrow (\boldsymbol{Y}; V)$$

is meant to be a pair $(\boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{Y},\ U\xrightarrow{\Lambda}V)$ making the diagram

$$\begin{array}{ccc} U & \xrightarrow{\Lambda} & V \\ {\scriptstyle\langle d, c\rangle}\downarrow & \circ & \downarrow{\scriptstyle\langle d, c\rangle} \\ \|X\|\times\|X\| & \xrightarrow[\|\Gamma\|\times\|\Gamma\|]{} & \|Y\|\times\|Y\| \end{array}$$

commute; composition in $\mathcal{F}$ is the evident one. Note that every $Y\in\omega\mathtt{Cat}$ gives rise to the "tautological" frame $\tau(Y)\overset{=}{_{\mathrm{def}}}(Y;(\|Y\|-\{*\},\mathrm{d},\mathrm{c}))$, with the d and c maps those given by the ω-category Y. We see that an extension $(Y, X\xrightarrow{\Gamma}Y, U\xrightarrow{\Lambda}|Y|)$ of X by U to Y is the same as a map $(X; U)\longrightarrow\tau(Y)$. All this amounts to saying that "free extension is left adjoint to tautological frame":

$$\mathcal{F}\ \underset{\mathcal{E}}{\overset{\tau}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}}\ \omega\mathtt{Cat}\quad ,$$

$$(X;U)\longmapsto X[U]$$

the pair $(X\xrightarrow{\Gamma}X[U], U\xrightarrow{\Lambda}\|X[U]\|)$ being the component at $(X; U)$ of the unit of the adjunction $\mathcal{E}\dashv\tau$. This is useful: we see that, for any map $(\Gamma, \Lambda):(X; U)\to(Y; V)$ of frames, we have the corresponding "canonical" map $\mathcal{E}(\Gamma, \Lambda):X[U]\to Y[V]$. Thus, for instance, if we have two sets $U\subset V$ of indeterminates for X, with the d and c functions on U the restrictions of those for V, we have the canonical map $X[U]\longrightarrow X[V]$, given as $\mathcal{E}(\mathrm{Id}_X, \mathrm{incl}_{U\subset V})$.

We collect some plausible, and mostly easy, facts about free extensions.

Suppose we have frames $(X; U)$ and $(X; V)$, with the same underlying ω-category X. We can do two things. On the one hand, we can consider $X[U]$, and consider V as indeterminates in $X[U]$, by using, for $\mathrm{d}:V\longrightarrow\|X[U]\|$, the composite $V\xrightarrow{\mathrm{d}}\|X\|\xrightarrow{\Gamma}\|X[U]\|$, and similarly for c. This gives rise to the free extension $X[U][V]$. On the other hand, we may look at $U\dot\cup V$ (assuming, of course, that U and V are disjoint), with the obvious d and c on this set, as a new set of indeterminates for X; this gives rise to $X[U\dot\cup V]$. The claim is that

(1) $\boldsymbol{X}[U][V]$ and $\boldsymbol{X}[U\dot{\cup}V]$ are canonically isomorphic.

For instance, one way of seeing this is to see that $\boldsymbol{X}[U\dot{\cup}V]$ has the universal property of $\boldsymbol{X}[U][V]$. More precisely, we have the canonical arrow $F:\boldsymbol{X}[U]\longrightarrow\boldsymbol{X}[U\dot{\cup}V]$ as explained above; and we have $\Lambda\circ i:V\xrightarrow{i}U\cup V\xrightarrow{\Lambda}\|\boldsymbol{X}[U\dot{\cup}V]\|$, with i the inclusion; we can show that

$$(\boldsymbol{X}[U\dot{\cup}V],\boldsymbol{X}[U]\xrightarrow{F}\boldsymbol{X}[U\dot{\cup}V],V\xrightarrow{\Lambda\circ i}\|\boldsymbol{X}[U\dot{\cup}V]\|)$$

is initial in $\mathrm{Ext}(\boldsymbol{X}[U],V)$.

(2) The canonical morphism $\Gamma:\boldsymbol{X}\longrightarrow\boldsymbol{X}[U]$ is an injection, and the images of Γ and $\Lambda:U\longrightarrow\|\boldsymbol{X}[U]\|$ are disjoint. Moreover, a composite of two elements in $\boldsymbol{X}[U]$ belong to the image of Γ only if both factors belong to the image of Γ.

(Later we'll see that $\Lambda:U\longrightarrow\|\boldsymbol{X}[U]\|$ is injective too.)

For the proof, see the appendix.

A *subωcategory* of an ω-cat $\boldsymbol{Y}$ is an ω-cat $\boldsymbol{S}$ for which $\|\boldsymbol{S}\|\subseteq\|\boldsymbol{Y}\|$, and the inclusion mapping $i:\|\boldsymbol{S}\|\to\|\boldsymbol{Y}\|$ induces a (unique) morphism of ω-cats. The subωcategories of $\boldsymbol{Y}$ are in a bijective correspondence with subsets S of $\|\boldsymbol{Y}\|$ which are *closed*, that is closed under the operations of domain, codomain, identity, and (well-defined) compositions in $\boldsymbol{Y}$.

For a morphism $F:\boldsymbol{X}\to\boldsymbol{Y}$ of ω-cats which is a monomorphism (equivalently (!), injective on all cells), the concept of image of F is well-defined: we can take the subset $S=\{Fa:a\in|\boldsymbol{X}|\}$ of $\|\boldsymbol{Y}\|$, and define the ω-cat operations domain, codomain, identity and compositions on S compatibly with both $\boldsymbol{X}$ and $\boldsymbol{Y}$, making up $\boldsymbol{S}$, a subcategory of $\boldsymbol{Y}$.

We have that, for any monomorphism $F:\boldsymbol{X}\to\boldsymbol{Y}$ of ω-categories, there is a unique factorization $F=j\circ i$ such that j is an isomorphism, and i is an inclusion of a subωcategory.

We can combine assertions (1) and (2) into

(3) Given ω-category $\boldsymbol{X}$, and sets U and V of indeterminates attached to $\boldsymbol{X}$, the attachment of U being the restriction of that for V , the canonical map $\boldsymbol{X}[U]\longrightarrow\boldsymbol{X}[V]$ is an injection.

The reason is that $\boldsymbol{X}[V]=\boldsymbol{X}[U\cup(V-U)]$ is, by (1), the same $\boldsymbol{X}[U][V-U]$, and $\boldsymbol{X}[U]\longrightarrow\boldsymbol{X}[U][V-U]$ is, by (2), an injection.

We also have the following corollary of (1) and (2), which is something one really cannot do without:

(4) The canonical map $\Lambda:U\to\|\boldsymbol{X}[U]\|$ is an injection.

Proof. Let u be any fixed element of U . By (2), we may regard $\boldsymbol{X}[U]$ as $\boldsymbol{X}[U-\{u\}][\{u\}]$. We have the canonical maps $\Lambda_1:\{u\}\longrightarrow\boldsymbol{X}[U-\{u\}][\{u\}]$ and $\Gamma_1:\boldsymbol{X}[U-\{u\}]\longrightarrow\boldsymbol{X}[U-\{u\}][\{u\}]$. By (2), the images of Λ_1 and Γ_1 are disjoint. It is clear $\Lambda(u)=\Lambda_1(u)$ and $\Lambda\restriction(U-\{u\})$ factors through the map $|\Gamma_1|$. It follows that $\Lambda(u)\notin\Lambda(U-\{u\})$ (direct image). Since $u\in U$ was arbitrary, the assertion follows.

It is important that $\boldsymbol{Y}=\boldsymbol{X}[U]$, initially given by an "externally attached" set U , can in fact be written as $\boldsymbol{X}[V]$ where $V=\Lambda(U)$, the direct image of U under Λ : V is a set of cells in $\boldsymbol{Y}$, and its attachment to $\boldsymbol{X}$ -- which is, or rather, may assumed to be, a subωcategory of $\boldsymbol{Y}$ (see (2)) -- is given by the "internal" domain and codomain functions of $\boldsymbol{Y}$. This is true because of (4).

Next, I am going to reformulate (3) as the statement saying that if, in a given ω-cat of the form $\boldsymbol{X}[V]$, I take a subset U of V , and form the subcategory $\boldsymbol{X}\langle U\rangle$ generated by $\boldsymbol{X}\cup U$, then $\boldsymbol{X}\langle U\rangle$ is in fact $\boldsymbol{X}[U]$, the extension of $\boldsymbol{X}$ by U . However, I will do it carefully.

Let $\boldsymbol{Y}$ be an ω-category, $\boldsymbol{X}$ a subωcategory of $\boldsymbol{Y}$, and U a set of cells in $\boldsymbol{Y}$. Let $\boldsymbol{X}\langle U\rangle$ the least subset of $\|\boldsymbol{Y}\|$ that contains $\|\boldsymbol{X}\|\cup U$ and closed under the operations of taking identities and well-defined composites. Note that, for any fixed n , $\boldsymbol{X}\langle U\rangle_n$ is the least set Z such that $\boldsymbol{X}_n\subseteq Z$, $U_n\subseteq Z$ $(U_n\overset{\mathrm{def}}{=}U\cap\boldsymbol{Y}_n)$, $b\in\boldsymbol{X}\langle U\rangle_{n-1}\Rightarrow 1_b\in Z$, and $a, b\in Z$, $a\#_k b\downarrow \Longrightarrow a\#_k b \in Z$. If it is the case that for all $u\in U$, we have $\mathrm{d}u, \mathrm{c}u\in\boldsymbol{X}$, then it is easy to see that $\boldsymbol{X}\langle U\rangle$ becomes also closed under "domain" and "codomain", and thus, it is a subωcategory of $\boldsymbol{Y}$.

This last situation takes place when $\boldsymbol{Y}$ is the free extension $\boldsymbol{Y}=\boldsymbol{X}[V]$, with V a set of indeterminates internally attached to $\boldsymbol{X}$ (in particular, $\|\boldsymbol{X}\|,V\subseteq\|\boldsymbol{Y}\|$) and U is a subset of V . What we just said applies, and $\boldsymbol{X}\langle U\rangle$ is a subωcategory of $\boldsymbol{Y}$. I claim that, in fact, $\boldsymbol{X}\langle U\rangle=\boldsymbol{X}[U]$, meaning that the inclusion $\boldsymbol{X}\to\boldsymbol{X}\langle U\rangle$ has the universal property of the free extension $\Gamma:\boldsymbol{X}\to\boldsymbol{X}[U]$.

Consider an abstract instance of the free extension $\Gamma:\boldsymbol{X}\to\boldsymbol{X}[U]$. The universal property of Γ gives us a map $\hat{\Gamma}:\boldsymbol{X}[U]\to\boldsymbol{X}[V]$ that is the identity on the set $\|\boldsymbol{X}\|\cup U$. By (3), $\hat{\Gamma}$ is injective. Its image is clearly the same as $\boldsymbol{X}\langle U\rangle$. Therefore, $\hat{\Gamma}$ induces an isomorphism $\hat{\Gamma}:\boldsymbol{X}[U]\xrightarrow{\cong}\boldsymbol{X}\langle U\rangle$. We have shown that $\boldsymbol{X}[U]\cong\boldsymbol{X}\langle U\rangle$ as promised.

We have shown:

(5) Given a free extension $\boldsymbol{Y}=\boldsymbol{X}[V]$, with internal indeterminates V , then for any subset U of V , $\boldsymbol{X}\langle U\rangle$ is the free extension $\boldsymbol{X}[U]$ of $\boldsymbol{X}$ by U . Moreover, $\boldsymbol{X}[V] = \boldsymbol{X}[U][V-U]$.

We now consider finite iterations $\boldsymbol{X}[U^1]...[U^n]$, and infinite iterations $\boldsymbol{X}[\vec{U}] = \boldsymbol{X}[U^1]...[U^n]...$ of the operation of forming free extensions.

Let $\mathbf{X}$ be an ω-cat, and assume we have the following (in what follows, superscripts such as in $\mathbf{X}^n$ do not mean exponentiation):

the ω-categories $\mathbf{X}^n$ for $n\in\mathbb{N}$, with $\mathbf{X}^0=\mathbf{X}$;

for $n\in\mathbb{N}-\{0\}$, the set U^n of indeterminates attached to $\mathbf{X}^{n-1}$ (by "parallel" maps $U^n \underset{c}{\overset{d}{\rightrightarrows}} \|\mathbf{X}^{n-1}\|$) such that $\mathbf{X}^n = \mathbf{X}^{n-1}[U^n]$.

We then have the injective maps (see (2) and (4)) $\Gamma^n:\mathbf{X}^{n-1}\longrightarrow\mathbf{X}^n$, $\Lambda^n:U^n\longrightarrow|\mathbf{X}^n|$. We can form the directed colimit

$$\mathbf{X}[\vec{U}] \overset{=}{\text{def}} \operatorname*{colim}_{n\in\mathbb{N}} \mathbf{X}^n .$$

Filtered colimits in $\omega\mathtt{Cat}$ are created by the forgetful functor to $\mathtt{Set}$. It follows that the colimit coprojections $\varphi_n:\mathbf{X}_n\to\mathbf{X}[\vec{U}]$ are injective.

For convenience, we assume that the sets U^n are pairwise disjoint. We let $U = \dot{\bigcup}_{n\in\mathbb{N}-\{0\}} U^n$. By iterating (4), we get that the induced map $\psi:U\to\|\mathbf{X}[\vec{U}]\|$ is injective. Let $V=\psi(U)$, the direct image of ψ. Thus, we have that $\mathbf{Y}=\mathbf{X}[\vec{U}]$ can also be written as $\mathbf{Y}=\mathbf{X}[\vec{U}]=\mathbf{X}[\vec{V}]$, with the obvious meaning for $\vec{V}=\langle V^n\rangle_{n\in\mathbb{N}-\{0\}}$; the attachment of $\vec{V}$ is internal to $\mathbf{Y}$ (that is, the attachment values $\mathrm{d}v$ and $\mathrm{c}v$ are the same as $\mathrm{d}v$ and $\mathrm{c}v$ in the sense of $\mathbf{Y}$).

Let $\mathbf{Y}$ be an ω-cat, $\mathbf{X}$ a subωcat. Consider a subset U^n of $\|\mathbf{Y}\|$, for each $n\geq 1$, such that $u\in U^n$ implies that $\mathrm{d}u,\mathrm{c}u\in\mathbf{X}\langle U^{\leq n-1}\rangle$ (where $U^{\leq n-1}=\bigcup_{m\leq n-1} U^m$); in this case we say that the system $\vec{U}=\langle U^n\rangle_n$ is *self-contained*. If so, then we have that, with $U=\bigcup_n U^n$, $\mathbf{X}\langle U\rangle$ is a subωcat of $\mathbf{Y}$; also, $\mathbf{X}\langle U\rangle = \bigcup^{\uparrow}_n \mathbf{X}\langle U^{\leq n}\rangle$ (directed union).

Note that if $\mathbf{Y}=\mathbf{X}[\vec{V}]$, an internal iterated free extension, then we have that $\vec{V}$ is self-contained and $\mathbf{Y}=\mathbf{X}\langle V\rangle$; moreover the system $\vec{V}$ is *disjoint*: $V^m\cap V^n=\emptyset$ for $m\neq n$ (see

(2)).

(6) Let $\mathbf{Y}=\mathbf{X}[\vec{V}]$ be an internal iterated free extension, $V=\bigcup_n V^n$, $\vec{U}$ a disjoint self-contained system of elements in $\mathbf{Y}$ with total set $U=\bigcup_n U^n=V$. Then $\mathbf{Y}$ is the iterated free extension $\mathbf{X}[\vec{U}]$ of $\mathbf{X}$.

Proof. Using (5), by induction on n, we prove that $\mathbf{X}\langle U^{\le n}\rangle=\mathbf{X}[U^1]\ldots[U^n]$, with the obvious canonical inclusions. Passing to the colimit makes the assertion clear.

Note that (6) allows us to write $\mathbf{X}[\vec{V}]$ as $\mathbf{X}[V]$, with V the total set of $\vec{V}$, since $\mathbf{X}[\vec{V}]$ depends really only on V and $\mathbf{X}$.

We have the following generalization of (5):

(7) Let $\mathbf{Y}=\mathbf{X}[\vec{V}]=\mathbf{X}[V]$, an internal iterated free extension as explained above, $V=\bigcup_n V^n$, and let, for each $n\ge 1$, $U^n\subseteq V$, and assume that $\vec{U}=\langle U^n\rangle_n$ is disjoint and self-contained. For the total set U of $\vec{U}$, $\mathbf{X}\langle U\rangle$ is the iterated free extension $\mathbf{X}[\vec{U}]=\mathbf{X}[U]$ of $\mathbf{X}$ by U. Moreover, $\mathbf{X}[V]=X[U][V-U]$.

Note that we did *not* assume $U^n\subseteq V^n$, only $U^n\subseteq V$.

Proof. First, we show the assertion under the stronger assumption $U^n\subseteq V^n$ for all n. Secondly, we reduce the general case to said special case as follows. Given the data as in (6), we construct a disjoint self-contained system $\vec{W}$ ($W_n\cap W_m=\emptyset$ for $n\neq m$) with total set $W=V$ such that, for each n, $U^n\subseteq W^n$. This construction is left as an exercise. By (6), $\mathbf{Y}=\mathbf{X}[\vec{W}]$, and we have made the promised reduction.

We will need the operation of freely adjoining n-dimensional indeterminates in a set U to an $(n\text{-}1)$-category $\mathbf{X}$, to obtain the n-category $\mathbf{X}[U]$. This is essentially a special case of our construction above, since $\mathbf{X}$ may be regarded to be an ω-category, namely $\mathbf{X}^{(\omega)}$. In fact, it is not necessary to bring in the object $\mathbf{X}^{(\omega)}$, since everything we want has an obvious, direct expression in terms of $(n\text{-}1)$-categories and n-categories.

We want to state a result to the effect that, from $\mathbf{X}[U]$ as a mere ω-category, under certain conditions we can recover $\mathbf{X}$ and U.

Let $\mathbf{Y}$ be any $\leq\omega$-category.

Recall the notation $1_a^{(n)}$. For a k-cell a, $k<n$, we say that $1_a^{(n)}$ is a k-to-n *identity* cell.

Let $n\in\mathbb{N}$. We consider the following condition (C_n) on $\mathbf{Y}$:

(C_n) Whenever $\ell<n$, $k<n$, $x, y\in Y_n$, and $x\#_\ell y$ is well-defined, if $x\#_\ell y$ is a k-to-n identity, then both x and y are k-to-n identities.

Condition (C_n) says that $\mathbf{Y}$ is the "opposite" to being an n-groupoid.

Let $n\in\mathbb{N}$ and $x\in Y_n$. Let us say that x is *indecomposable* if the following hold:

(i) $x\neq 1_y$ for all $y\in Y_{n-1}$; and

(ii) whenever $y, z\in Y_n$, $k<n$ and $x=y\#_k z$, we have that either y or z is a k-to-n identity (and $x=z$, respectively, $x=y$).

(8) Proposition For any $(n\text{-}1)$-category $\mathbf{X}$ satisfying (C_{n-1}), and any set U of n-indeterminates attached to $\mathbf{X}$, we have the following:

(8.1) $\boldsymbol{X}[U]$ satisfies (C_n) .

(8.2) The canonical map $\Lambda : U \longrightarrow (\boldsymbol{X}[U])_n$ is one-to-one.

(8.3) The image of Λ consists exactly of the indecomposable n-cells of $\boldsymbol{X}[U]$.

(8.4) The canonical inclusion $\Gamma:\mathbf{X}\to\mathbf{X}[U]$ is an isomorphism onto the $(n\text{-}1)$-truncation of $\mathbf{X}[U]$.

(The conclusions (8.2) and (8.4) are already known, under more general conditions.)

For the **proof**, which is similar to that of (2) but more complicated, see the appendix.

Let us note that without assuming (C_{n-1}) for $\mathbf{X}$, even when we drop (8.1) from the assertion, (8) becomes false: take the example when $n=2$, and $\mathbf{X}$ is a groupoid.

5. Computads

A computad is an ω-category of the form $\emptyset[\vec{U}]$, that is, an iterated free extension of the empty (initial) ω-category $\emptyset$ (which still has $*\in\emptyset_{-1}$).

An alternative definition, equivalent to the first one, is as follows.

n-computads, for $n\in\mathbb{N}$, are defined recursively; each n-computad is, in particular, an n-category.

A *0-computad* is a 0-category: a set.

An *n-computad* is any n-category isomorphic to one of the form $\mathbf{X}[U]$, where $\mathbf{X}$ is an $(n-1)$-computad, and U is a set of n-indeterminates attached to $\mathbf{X}$.

A *computad* is an ω-category whose n-truncation is an n-computad, for each $n\in\mathbb{N}$.

It is important to realize that the indeterminates are not "lost" in the wording of the definition. Indeed, as a consequence of 4.(8), the indeterminates of a computad are exactly the indecomposable cells.

To emphasize what we just said, we rephrase the definition as follows.

Let $\mathbf{X}$ be an ω-category. Let $\mathbf{X}\restriction n$ the n-truncation of $\mathbf{X}$, and let U_n denote the set of all n-indecomposables in $\mathbf{X}$, attached to $\mathbf{X}_n$ internally. Then $\mathbf{X}$ is a computad iff, for all $n\geq 1$, $(\mathbf{X}_n, \Gamma:\mathbf{X}_{n-1}\to\mathbf{X}_n, \Lambda:U_n\to|\mathbf{X}_n|)$, with Γ, Λ denoting inclusions, is a free extension of $\mathbf{X}$ by U_n.

A corollary is

(1) If $\mathbf{X}$ is a computad, and $\mathbf{X}'$ is an ω-category isomorphic to $\mathbf{X}$, then $\mathbf{X}'$ is a computad as well. Moreover, any isomorphism $f:\mathbf{X}\xrightarrow{\cong}\mathbf{X}'$ of ω-categories between computads $\mathbf{X}$, $\mathbf{X}'$ takes any indeterminate in $\mathbf{X}$ to an indeterminate in $\mathbf{X}'$.

A *morphism* $F:X\longrightarrow Y$ *of computads* X, Y is a morphism of ω-categories that maps indeterminates to indeterminates. We obtain the category $\mathtt{Comp}$ of small computads, with a non-full inclusion $\Phi:\mathtt{Comp}\to\omega\mathtt{Cat}$. By (2), Φ is full with respect to isomorphisms: the restriction $\Phi^*:\mathtt{Comp}^*\to\omega\mathtt{Cat}^*$ is a full inclusion (for a category C, C^* is its underlying groupoid).

For a computad X, $|X|$ denotes the set of its indeterminates: $|X| = \coprod_{n\in\mathbb{N}} |X|_n = \bigcup_{n\in\mathbb{N}} |X|_n$, where $|X|_n$ is the set of n-indeterminates (indecomposables of dimension n). We have the forgetful functor $|\text{-}|:\mathtt{Comp}\to\mathtt{Set}$.

A special case of 4.(8) is

(2) Let X be a computad, $U\subseteq |X|$; write $U_n=U\cap |X|_n$. Assume that $u\in U_n$ implies that $\mathrm{d}u,\mathrm{c}u\in\varnothing\langle U_{n-1}\rangle$ (for this, we say that U is a *down-closed* set of indeterminates). Then $\varnothing\langle U\rangle$, the sub$\omega$cat of X generated by U, is a computad, and $|\varnothing\langle U\rangle|=U$.

(2) tells us how to generate some of the subobjects of an object of the category $\mathtt{Comp}$. To show that we obtain all subobjects in this way requires more work. To anticipate that result, for any computad X, we call an subωcat of X of the form $\varnothing\langle U\rangle$ with U a down-closed subset of $|X|$ a *subcomputad* of X. By (2), any subcomputad is a computad on its own right.

(3) The category is small-cocomplete, and the functors $\Phi:\mathtt{Comp}\to\omega\mathtt{Cat}$, $|\text{-}|:\mathtt{Comp}\to\mathtt{Set}$ preserve all small colimits.

This is essentially clear from the definitions; for details see the appendix.

(4) The functor $|\text{-}|:\mathtt{Comp}\to\mathtt{Set}$ is faithful and reflects isomorphisms.

Proof: see appendix.

In what follows, we let $\boldsymbol{X}$ be a computad; a, b, ... are arbitrary elements of $\boldsymbol{X}$, u, ..., x, ... are indeterminates of $\boldsymbol{X}$.

The following is an important tool.

(5) Lemma There is a unique function

$$\mathrm{supp}_{\boldsymbol{X}} = \mathrm{supp} : |\boldsymbol{X}| \longrightarrow \mathcal{P}(|\boldsymbol{X}|)$$

satisfying the equations

$$\begin{array}{ll} \mathrm{supp}(*) = \emptyset & \\ \mathrm{supp}(x) = \{x\} \cup \mathrm{supp}(\mathrm{d}x) \cup \mathrm{supp}(\mathrm{c}x) & (x \in |\boldsymbol{X}|) \\ \mathrm{supp}(1_a) = \mathrm{supp}(a) & (a \in \boldsymbol{X}) \\ \mathrm{supp}(a \#_k b) = \mathrm{supp}(a) \cup \mathrm{supp}(b) & (a, b \in \boldsymbol{X}) \end{array}$$

Moreover, for all $a \in \boldsymbol{X}$,

$\mathrm{supp}(\mathrm{d}a)$, $\mathrm{supp}(\mathrm{c}a)$ are subsets of $\mathrm{supp}(a)$;
$\mathrm{supp}(a) \subseteq |\boldsymbol{X}|_{\le \dim(a)}$;
$\mathrm{supp}(a) \subseteq |\boldsymbol{X}|_{\le \dim(a)-1} \Longleftrightarrow a = 1_b$ for some b .
$\mathrm{supp}(a)$ is a finite set.

Proof: see the appendix.

We think of $\mathrm{supp}(a)$, the *support* of a, as the set of indeterminates "occurring" in a.

Before we proceed, let us make a general remark. The fact that $\boldsymbol{X} = \emptyset\langle |\boldsymbol{X}| \rangle$ translates into the following "computad induction" principle. Assume P is a property of elements of $\boldsymbol{X}$,

$P \subseteq |\mathbf{X}|$. Suppose we have the following four conditions satisfied:

(i) $* \in P$;
(ii) for all $x \in |\mathbf{X}|$: $\mathbf{X}_{<\mathtt{dim}(x)} \subseteq P \Longrightarrow x \in P$;
(iii) for all $b \in \mathbf{X}_{\geq 0}$: $b \in P \Longrightarrow 1_b \in P$;
(iv) for all b, e and k : $(b \#_k e\downarrow \,\&\, b \in P \,\&\, e \in P) \Longrightarrow b \#_k e \in P$.

Then $P = |\mathbf{X}|$.

Skeptics may see the appendix.

(6) **(i)** $\mathtt{supp}(a)$ is a down-closed subset of $|\mathbf{X}|$ and $a \in \varnothing\langle \mathtt{supp}(a)\rangle$.

(ii) Given $F{:}\mathbf{X} \longrightarrow \mathbf{Y}$ in $\mathtt{Comp}$, let $a \in \mathbf{X}$. Then the direct image of $\mathtt{supp}_{\mathbf{X}}(a)$ under F is $\mathtt{supp}_{\mathbf{Y}}(Fa)$. In other words, F induces a surjective map $\mathtt{supp}_{\mathbf{X}}(a) \longrightarrow \mathtt{supp}_{\mathbf{X}}(Fa)$.

Proof: straight-forward computad induction; see the appendix.

(7) Suppose $\mathbf{X}$ is a subcomputad of $\mathbf{Y}$.

(i) For $a \in \mathbf{X}$, $\mathtt{supp}_{\mathbf{X}}(a) = \mathtt{supp}_{\mathbf{Y}}(a)$.

(ii) If $a \in \mathbf{X}$, then $\mathtt{supp}_{\mathbf{Y}}(a) \subseteq |\mathbf{X}|$. Hence, $\mathtt{supp}_{\mathbf{Y}}(a)$ is the least down-closed subset U of $|\mathbf{Y}|$ for which $a \in \varnothing\langle U\rangle$.

(iii) A subset U of $|\mathbf{Y}|$ is down-closed iff for all $u \in U$, we have $\mathtt{supp}_{\mathbf{Y}}(u) \subseteq U$.

(iv)(!) Whenever $a, b \in \mathbf{Y}$, $a \#_k b$ is well-defined and $a \#_k b \in \mathbf{X}$, then a and b both belong to $\mathbf{X}$.

Proof. (i) is a special case of 6.(ii). (ii) and (iii) follow from (i) . To prove (iv), assume the assumptions. $\mathtt{supp}_{\mathbf{X}}(a \#_k b) = \mathtt{supp}_{\mathbf{Y}}(a \#_k b) = \mathtt{supp}_{\mathbf{Y}}(a) \cup \mathtt{supp}_{\mathbf{Y}}(b)$, hence, $\mathtt{supp}_{\mathbf{Y}}(a) \subseteq \mathtt{supp}_{\mathbf{X}}(a \#_k b) \subseteq |\mathbf{X}|$. Since $a \in \varnothing\langle \mathtt{supp}_{\mathbf{Y}}(a)\rangle$, we have $a \in \varnothing\langle |\mathbf{X}| \rangle = \mathbf{X}$. Similarly, $b \in \mathbf{X}$.

(8) Let $F:\mathbf{X}\longrightarrow\mathbf{Y}$ be a map of computads.

(i) $F(|\mathbf{X}|)$, the direct image of $|\mathbf{X}|$ under F, is a down-closed set of indeterminates of $\mathbf{Y}$.

(ii) For any down-closed subset V of $|\mathbf{Y}|$, the inverse image of V, $|f|^{-1}(V)=\{x\in|\mathbf{X}|:f(x)\in V\}$ is down-closed in $\mathbf{X}$. (In (10)(ii) below, we'll see that, in fact, $|f|^{-1}(V)=f^{-1}(V)$.)

(iii) Pullbacks of diagrams in $\mathtt{Comp}$ in which one of the arrows is a monomorphism are preserved by the forgetful functor $|-|:\mathtt{Comp}\to\mathtt{Set}$.

(iv) Small colimits in $\mathtt{Comp}$ are stable under pullbacks *along monomorphisms*.

Proof. (i) and (ii) follow from (6) and (7); (iii) follows from (ii). (iv) follows from (iii) and (3) and (4).

(9) Let $F:\mathbf{X}\longrightarrow\mathbf{Y}$ be a map of computads.

(i) F is factored in the category $\mathtt{Comp}$ uniquely as $F=i\circ P$ where i is the inclusion map of a subcomputad of $\mathbf{Y}$, and $|P|$ is surjective.

(ii) F is a monomorphism in $\mathtt{Comp}$ iff it (that is, $\Phi(F)$ for the inclusion $\Phi:\mathtt{Comp}\to\omega\mathtt{Cat}$) is a mono in $\omega\mathtt{Cat}$ iff $|F|$ is injective.

(iii) F is an epimorphism in $\mathtt{Comp}$ iff $|F|$ is surjective.

(iv) The subobjects of a computad $\mathbf{X}$, in the sense of the category $\mathtt{Comp}$, are the same as (are in a bijective correspondence with) the subcomputads of $\mathbf{X}$.

Proof: see the appendix.

(10) Let $F:\mathbf{X}\to\mathbf{Y}$ be a morphism of computads, $n\in\mathbb{N}$, $a\in\mathbf{X}_n$. Then

(i) $a=1_{da} \iff Fa=1_{Fda}$

(ii) a is an indeterminate $\iff$ Fa is an indeterminate.

Proof. The left-to-right implications are clear.

For $\Leftarrow$ in (i): If $Fa=1_{Fda}$, then $\mathrm{supp}(Fa)\subseteq|\boldsymbol{Y}|_{n-1}$; by (7), it follows that $\mathrm{supp}(a)\subseteq|\boldsymbol{X}|_{n-1}$, hence, by (5), the third "moreover" statement, $a=1_{da}$.

For (ii), first of all note that by (i), if $b=Fa$, and $b=1_f^{(n)}$, a k-to-n identity, then $a=1_e^{(n)}$, for a suitable e, and $Fe=f$. Assume a is *not* an indeterminate. We use 4.(8): we have that a is not indecomposable, i.e., $a=a_1\#_k a_2$ where neither a_1 nor a_2 is a k-to-n identity. Then, by what we just said, neither Fa_1 nor Fa_2 is a k-to-n identity, and $Fa=Fa_1\#_k Fa_2$ is not indecomposable, i.e., not an indeterminate.

Let us call a computad $\boldsymbol{X}$ *finite* if $|\boldsymbol{X}|$ is a finite set.

Let $a\in\boldsymbol{X}$. Since $\mathrm{supp}(a)$ is down-closed, $\mathrm{Supp}(a) \underset{\mathrm{def}}{=} \emptyset\langle\mathrm{supp}(a)\rangle$ is a subcomputad of $\boldsymbol{X}$. Since $|\mathrm{Supp}(a)|=\mathrm{supp}(a)$, $\mathrm{Supp}(a)$ is a finite computad.

Given two finite subcomputads $\emptyset\langle U\rangle$, $\emptyset\langle V\rangle$, defined by the finite down-closed sets $U, V\subseteq|\boldsymbol{X}|$, $U\cup V$ is finite and down-closed (obviously, any union of down-closed sets of indets is down-closed). We can form the finite subcomputad $\emptyset\langle U\cup V\rangle$. This shows that the set $\mathcal{S}_{\mathrm{fin}}(\boldsymbol{X})$ of finite subcomputads of $\boldsymbol{X}$ ordered by inclusion is directed.

The union of all the elements of $\mathcal{S}_{\mathrm{fin}}(\boldsymbol{X})$ is $\boldsymbol{X}$, and this union is a colimit (see (3)). We have shown that every computad is a filtered colimit of finite computads.

It is easy to see that a finite computad is finitely presentable (fp) object of Comp. In fact, since a retract, in fact, any subcomputad, of a finite computad is finite, the finite computads are exactly the fp ones.

Since, by (3), Comp has all small filtered colimits, we have shown that Comp is an $\aleph_0$-accessible category. Since it is small-cocomplete ((3)), it is a locally finite presentable (lfp) category. In particular, Comp is small-complete.

Unlike in most of the lfp categories appearing in practice, in $\mathtt{Comp}$, it is not the limit structure, but the colimit structure, that is familiar. The limit structure is complicated, and, to a large extent, "unknown". The terminal computad is "large"; its set of indeterminates is countably infinite. Although, as we later show, it is a structure with a recursively solvable word problem, it is "very complicated".

(11) Arbitrary intersections and unions of down-closed sets of indets are again down-closed. The down-closed sets of the form $\mathrm{supp}(x)$, x an indet are join irreducible: $\mathrm{supp}(x)=\bigcup_i U_i$, each U_i down-closed, imply that there is i such that $\mathrm{supp}(x)=U_i$. All down-closed sets are unions of ones of the form $\mathrm{supp}(x)$, x an indet. The subobject lattice of $\boldsymbol{X}$ is a completely distributive lattice.

We like to call arbitrary elements (cells) of a computad *pasting diagrams* (pd's).

Next, we introduce a concept of "multiplicity" of an indet in a pd a . Let $\boldsymbol{X}$ be a computad. The elements of $\mathbb{Z}^{|\boldsymbol{X}|}$ are the *multisets* of indeterminates. The elements of the subset $|\boldsymbol{X}|\cdot\mathbb{Z}$ of $\mathbb{Z}^{|\boldsymbol{X}|}$ are the vectors (functions) $\vec{m}=\langle m_x\rangle_{x\in|\boldsymbol{X}|}$ ($m_x=\vec{m}(x)$) for which only finitely many m_x is non-zero; these are the *finite* multisets of indets. Multisets form an Abelian group under componentwise addition $+$; finite multisets form a subgroup.

Obviously, the Abelian group $|\boldsymbol{X}|\cdot\mathbb{Z}$ is the free Abelian group generated by the elements of $|\boldsymbol{X}|$; accordingly, we may write $\langle m_x\rangle_{x\in|\boldsymbol{X}|}$ as $\sum_{x\in|\boldsymbol{X}|} m_x\cdot x$. Still, we prefer the functional notation $\binom{x}{m}$ to $m\cdot x$.

The multiset $\vec{m}$ is *non-negative* if $m_x\geq 0$ for all x . We also use the partial order $\vec{m}\leq\vec{n} \iff \vec{n}-\vec{m}$ is non-negative.

In the next proposition, we define the *content* $[a]$ of a pd a in a computad $\boldsymbol{X}$. $[a]$ is intended to be the multiset of all the indeterminates in a , with each indet counted with the proper multiplicity. The definition given here *does not* have a certain expected property:

(*?) When $\boldsymbol{X}$ is a computope, $[m_{\boldsymbol{X}}]_{\boldsymbol{X}}(u)=1$ for all $u\in|\boldsymbol{X}|$

(computopes will be explained in the next section). I do not know if there is a content function with (*) and all the properties (v) to (x) in the next proposition.

(12) Proposition There is a unique function

$$[-]=[-]_{\boldsymbol{X}}:\boldsymbol{X}_{\geq -1}\longrightarrow |\boldsymbol{X}|\cdot\mathbb{Z}$$

satisfying the following equalities:

(i) $[*]=0$

(ii) $[x]=\binom{x}{1}+[dx]+[cx]$ $\quad(x\in|\boldsymbol{X}|)$

(iii) $[1_a]=[a]$ $\quad(a\in\boldsymbol{X}_{\geq 0})$

(iv) $[a\#_k b]=[a]+[b]-[a\wedge_k b]$ $\quad(a,b\in\boldsymbol{X}_{\geq 1},\ a\#_k b\downarrow)$

Moreover, we have

(v) $[a]\geq 0$

(vi) $[da],[ca]\leq[a]$

(vii) $[a],[b]\leq[a\#_k b]$

(viii) $[a](x)>0\iff x\in\mathrm{supp}(a)$

(ix) For any $F:\boldsymbol{X}\longrightarrow\boldsymbol{Y}$ in Comp and any $a\in\boldsymbol{X}_{\geq -1}$, $y\in|\boldsymbol{Y}|$,

$$[Fa]_{\boldsymbol{Y}}(y)=\sum_{\substack{x\in|\boldsymbol{X}|\\ Fx=y}}[a]_{\boldsymbol{X}}(x)\ .$$

(x) $x\in\mathrm{supp}(a)\implies[x]\leq[a]$.

Proof: see the appendix.

6. Multitopes and computopes

As an application of 1.(4), we discuss multitopes and multitopic sets.

Let $\boldsymbol{X}$ be a computad. An indeterminate $x\in|\boldsymbol{X}|$ is *many-to-one* if $\mathtt{c}x$ is an indeterminate. $\boldsymbol{X}$ itself is *many-to-one* if all indeterminates in $\boldsymbol{X}$ are so.

The full subcategory of $\mathtt{Comp}$ on the many-to-one computads is denoted $\mathtt{Comp}_{\mathtt{m/1}}$. As usual with subcategories of $\mathtt{Comp}$, we regard $\mathtt{Comp}_{\mathtt{m/1}}$ as a concrete category, with $|\boldsymbol{X}|$ the set of all indets of $\boldsymbol{X}$ ($\boldsymbol{X}\in\mathtt{Comp}_{\mathtt{m/1}}$).

Our starting point is a result that is a consequence of theorems proved in the papers [H/M/P] and [H/M/Z].

(1) Theorem $\mathtt{Comp}_{\mathtt{m/1}}$ is a concrete presheaf category.

The shape category of $\mathtt{Comp}_{\mathtt{m/1}}$ is called the category of *multitopes*; it is denoted by $\mathtt{Mlt}$.

In some detail, the reasons for the truth of the last theorem are as follows.

In [H/M/P], the concept of *multitopic set* is introduced. The category $\mathtt{MltSet}$ of multitopic sets is defined, the category $\mathtt{Mlt}$ of multitopes is defined, and it is proved that $\mathtt{MltSet}$ is equivalent to $\widehat{\mathtt{Mlt}}=\mathtt{Set}^{\mathtt{Mlt}^{\mathtt{op}}}$. Although it is not stated explicitly, it is implicit in [H/M/P] that, in fact, $\mathtt{MltSet}$ is equivalent as a concrete category to $\widehat{\mathtt{Mlt}}$, where the forgetful functor $|\text{-}|:\mathtt{MltSet}\longrightarrow\mathtt{Set}$ is defined as explained in the Introduction, part (C).

On the other hand, in [H/M/Z] it is shown, among others, that $\mathtt{MltSet}$ is equivalent to $\mathtt{Comp}_{\mathtt{m/1}}$. Again, it is not explicitly stated, but it is implicit that $\mathtt{MltSet}$ and $\mathtt{Comp}_{\mathtt{m/1}}$ are equivalent as concrete categories, with the same forgetful functors as before.

Combining the two facts, we get that $\mathtt{Comp}_{\mathtt{m/1}}$ and $\hat{\mathtt{Mlt}}$ are equivalent as concrete categories.

In section 1, we gave a characterization of concrete categories equivalent to a concrete presheaf category (see 1.(4)); this will give a characterization of the multitopes mentioned above. Keeping that characterization in mind, let us call a (not necessarily many-to-one) computad $\boldsymbol{X}$ a *computope* if there is $x\in|\boldsymbol{X}|$ such that $(\boldsymbol{X}, x)$ is a primitive element (see section 1) of $\mathtt{Comp}$.

Note that the element $(\boldsymbol{X}, x)$ of $\mathtt{Comp}$ is principal (see section 1) iff $\boldsymbol{X}=\mathtt{Supp}_{\boldsymbol{X}}(x)$.

Such an x in $|\boldsymbol{X}|$ as in the last sentence, if it exists, is unique, since, for any computad $\boldsymbol{X}$ and $x\in|\boldsymbol{X}|$, $\mathtt{Supp}_{\boldsymbol{X}}(x)$ has a single indeterminate of the dimension equal to that of $\mathtt{Supp}_{\boldsymbol{X}}(x)$, namely x itself; hence, $\mathtt{Supp}_{\boldsymbol{X}}(x)=\mathtt{Supp}_{\boldsymbol{X}}(y)$ for $x, y\in|\boldsymbol{X}|$ implies $x=y$. The indet x such that $(\boldsymbol{X}, x)$ is principal is called the *main indet* of $\boldsymbol{X}$, and denoted $m_{\boldsymbol{X}}$. We are justified in using the adjective "principal" in referring to a computad, rather than a pointed computad (an element of $\mathtt{Comp}$). Every computope is principal.

For $\boldsymbol{X}$ a computope, any self-map $\boldsymbol{X}\to\boldsymbol{X}$ (in $\mathtt{Comp}$) is necessarily an isomorphism: we know that any map $(\boldsymbol{X}, m_{\boldsymbol{X}})\to(\boldsymbol{X}, m_{\boldsymbol{X}})$ in $\mathtt{El(Comp)}$ is an iso, and we just saw that for any $f:\boldsymbol{X}\to\boldsymbol{X}$, we must have $|f|(m_{\boldsymbol{X}})=m_{\boldsymbol{X}}$.

Obviously, a principal computad is finite as a computad. For any finite computad $\boldsymbol{X}$, $\mathtt{dim}(\boldsymbol{X})$ is defined as $\mathtt{max}\{\mathtt{dim}(x):x\in|\boldsymbol{X}|\}$. For a principal computad $\boldsymbol{X}$, $\mathtt{dim}(\boldsymbol{X})=\mathtt{dim}(m_{\boldsymbol{X}})$.

It is clear that if $\boldsymbol{X}\to\boldsymbol{Y}$ is any map of finite computads, then $\mathtt{dim}(\boldsymbol{X})\leq\mathtt{dim}(\boldsymbol{Y})$.

Note the obvious facts that a computad map $f:A\to B$ of principal computads A, B is an epi iff f is surjective (on indets) iff $f(m_A)=m_B$ iff $\mathtt{dim}(A)=\mathtt{dim}(B)$.

Computopes can be equivalently described as those computads $\boldsymbol{X}$ which are principal, and for which any epimorphism $\boldsymbol{Y}\to\boldsymbol{X}$ from a principal $\boldsymbol{Y}$ to $\boldsymbol{X}$ is necessarily an isomorphism.

We let $\texttt{Comtope}$ be the full subcategory of $\texttt{Comp}$ whose objects are the computopes. $\texttt{Comtope}^*$ is defined to be the skeleton of $\texttt{Comtope}$: any skeletal full subcategory of $\texttt{Comtope}$ for which the inclusion $\texttt{Comtope}^* \to \texttt{Comtope}$ is an equivalence.

We isolate four properties of a variable full subcategory $\mathcal{C}$ of $\texttt{Comp}$; we will point out that each property is shared by $\mathcal{C} = \texttt{Comp}_{\texttt{m/1}}$.

(a) $\mathcal{C}$ is a *sieve* in $\texttt{Comp}$; if $\boldsymbol{X} \to \boldsymbol{Y}$ is a map of computads, and $\boldsymbol{Y}$ is in $\mathcal{C}$, then so is $\boldsymbol{X}$.

(b) $\mathcal{C}$ is closed under small colimits in $\texttt{Comp}$.

(c) If $f: A \to B$ in $\texttt{Comp}$ is surjective, and $A \in \mathcal{C}$, then $B \in \mathcal{C}$.

(d) $\mathcal{C}$ is a concrete presheaf category.

Note that (d) implies (b), by the (obvious) necessity condition (i) in 1.(4), and the same condition holding for $\texttt{Comp}$.

5.(10) immediately implies that (a) is satisfied by $\texttt{Comp}_{\texttt{m/1}}$.

If $F: \boldsymbol{X} \to \boldsymbol{Y}$ is a map of computads, and $x \in |\boldsymbol{X}|$ is many-to-one, then $F(x) \in |\boldsymbol{Y}|$ is also many-to-one. Therefore, if $\{F_i : \boldsymbol{X}_i \to \boldsymbol{Y}\}_{i \in I}$ is a jointly surjective (on indets) family of morphisms of computads, and each $\boldsymbol{X}_i$ is many-to-one, then $\boldsymbol{Y}$ is many-to-one. It follows that $\texttt{Comp}_{\texttt{m/1}}$ satisfies (b) and (c).

Let $\mathcal{C}$ be an, otherwise arbitrary, full subcategory of $\texttt{Comp}$ satisfying (a) above. We observe that

(1.1) for $\boldsymbol{X}$ in $\mathcal{C}$, and $x \in |\boldsymbol{X}|$, to say that $(\boldsymbol{X}, x)$ is a principal, resp. primitive, element of $\mathcal{C}$ is equivalent to saying that $(\boldsymbol{X}, x)$ is a principal, resp. primitive, element of $\texttt{Comp}$, i.e., that $\boldsymbol{X}$ is a principal computad and $x = m_{\boldsymbol{X}}$, respectively that $\boldsymbol{X}$ is a computope and $x = m_{\boldsymbol{X}}$.

We have noted that, for principal computads, in particular computopes, $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X}\cong\mathbf{Y}$ in $\mathtt{Comp}$ iff $(\mathbf{X}, m_{\mathbf{X}}) \cong (\mathbf{Y}, m_{\mathbf{Y}})$ in $\mathtt{El(Comp)}$. This means that the category $\mathbf{C}^{*}[\mathtt{Comp}]$, constructed in section 1, is *isomorphic* to $\mathtt{Comtope}^{*}$.

For any sieve $\mathcal{C}$ in $\mathtt{Comp}$, we let $\mathtt{Comtope}^{*}_{\mathcal{C}}$ be the intersection $\mathtt{Comtope}^{*}\cap\mathcal{C}$, a sieve (in particular, a full subcategory) in $\mathtt{Comtope}^{*}$.

A *one-way* category $\mathbf{L}$ is a category in which there are no "descending" infinite chains

$$K_0 \xleftarrow{f_0} K_1 \xleftarrow{f_1} K_2 \ldots \xleftarrow{f_n} K_{n+1} \ldots$$

consisting entirely of non-identity arrows. It follows that $\mathbf{L}$ is skeletal, and we can define a dimension-function $\mathtt{dim}$ on objects, taking values that are natural numbers, such that the presence of a non-identity arrow $f:A\to B$ implies that $\mathtt{dim}(A)<\mathtt{dim}(B)$.

A category $\mathbf{L}$ is said (here) to be *finitary* if, for each object B, the sieve $\{f\in\mathtt{Arr}(\mathbf{L}):\mathtt{cod}(f)=B\}$ is a finite set. In section 11, we will prove

(2) The skeletal category $\mathtt{Comtope}^{*}$ of computopes is finitary.

The quality of being both one-way and finitary is that makes $\mathtt{Mlt}^{\mathtt{op}}$ a FOLDS-signature, in the sense of [M1]. [M2]. This property of $\mathtt{Mlt}$ was evident on its definition already, but here we see that this is a "necessary quality" of $\mathtt{Mlt}$.

One of the main results of this paper is that the condition 1.(4)(ii)(b) holds in the category $\mathbf{A}=\mathtt{Comp}$.

(3) Theorem If $\mathbf{X}$ is any computad, $x\in|\mathbf{X}|$, then there exist a computope $\mathbf{X}$ and a morphism $f:\mathbf{X}\to A$ in $\mathtt{Comp}$ such that $|f|(m_{\mathbf{X}})=x$.

The proof will be given in section 11.

The following theorem is a summary, obtained from 1.(4) when we take into account what we have said in this section so far.

(4) Theorem Suppose that the full subcategory $\mathcal{C}$ of $\mathtt{Comp}$ is a sieve in $\mathtt{Comp}$ (satisfies (a)). Then $\mathcal{C}$ is a concrete presheaf category if and only if $\mathcal{C}$ satisfies conditions (o), (i) and (ii) below.

(o) $\mathcal{C}$ is closed under small colimits in $\mathtt{Comp}$ (satisfies (b)).

(i) For any computad $\boldsymbol{Z}$ in $\mathcal{C}$:

(i)$_{\boldsymbol{Z}}$ if $\boldsymbol{X}$ is a computope, and $\boldsymbol{X} \underset{g}{\overset{f}{\rightrightarrows}} \boldsymbol{Z}$ are maps in $\mathtt{Comp}$ such that $|f|(m_{\boldsymbol{X}}) = |g|(m_{\boldsymbol{X}})$, then $f = g$. In particular, every self-map of a computope in $\mathcal{C}$ is the identity.

(ii) For any computad $\boldsymbol{Z}$ in $\mathcal{C}$:

(ii)$_{\boldsymbol{Z}}$ whenever $\boldsymbol{X}$ and $\boldsymbol{Y}$ are computopes, and $\boldsymbol{X} \xrightarrow{f} \boldsymbol{Z} \xleftarrow{g} \boldsymbol{Y}$ are maps in $\mathtt{Comp}$ such that $|f|(m_{\boldsymbol{X}}) = |g|(m_{\boldsymbol{Y}})$, then $\boldsymbol{X} \cong \boldsymbol{Y}$.

If so, the shape category of $\mathcal{C}$ is isomorphic to $\mathtt{Comtope}^{*}_{\mathcal{C}}$, and $\mathtt{Comtope}^{*}_{\mathcal{C}}$ is a finitary, one-way category.

When $\mathcal{C}$ is $\mathtt{Comp}_{\mathtt{m/1}}$, we have all of (a), (b), (c) and (d) satisfied. We conclude that

the category $\mathtt{Mlt}$ *of multitopes, the shape category of* $\mathtt{Comp}_{\mathtt{m/1}}$, *is isomorphic to the skeletal category of all many-to-one computopes.*

The theorem offers some hope for a new and softer proof of the fact that $\mathtt{Comp}_{\mathtt{m/1}}$ is a concrete presheaf category.

Inspired by the example of $\mathtt{Comp}_{\mathtt{m/1}}$, we use the following terminology. Let $\boldsymbol{Z}$ be a computad, z an indeterminate in $\mathbf{Z}$. If $\mathbf{X}$ is a computope, and $f:(\mathbf{X}, m_{\mathbf{X}}) \to (\mathbf{Z}, z)$ (in $\mathrm{El(Comp)}$), we say that $\mathbf{X}$ is a *type for* z, z is a *specialization* of $m_{\mathbf{X}}$, and f is a *specializing map for* z (or, *for* $\mathbf{Z}$ if $\mathbf{Z}$ is principal). The condition (4)(i)$_{\mathbf{Z}}$ says that, once the type is fixed, the specializing map of any $z\in |\mathbf{Z}|$ is unique. (4)(ii)$_{\mathbf{Z}}$ says that the type of any $z\in |\mathbf{Z}|$ is unique up to isomorphism (which, of course, is the most we can expect). We know (see (3)) that every indet in every computad has at least one type.

One might wish to talk about a type of an arbitrary pd in a computad, not just that of an indet. However, this is not really more general. Let $\mathbf{Z}$, $\mathbf{X}$ be computads, a a pd in $\mathbf{Z}$, α a pd in $\mathbf{X}$, both of dimension n; assume $\mathbf{Z}=\mathrm{Supp}_{\mathbf{Z}}(a)$, $\mathbf{X}=\mathrm{Supp}_{\mathbf{X}}(\alpha)$. Let x, ξ be "new" indeterminates, both of dimension $n+1$, x attached to $\mathbf{Z}$ by $\mathrm{d}x=\mathrm{c}x=a$, ξ to $\mathbf{X}$ by $\mathrm{d}\xi=\mathrm{c}\xi=\alpha$. Then we have a bijection of maps, depicted as

$$\frac{\mathbf{X}[\xi] \longrightarrow \mathbf{Z}[x]}{\mathbf{X} \longrightarrow \mathbf{Z} \;::\; \alpha \mapsto a} \;.$$

This says that talking about a type for a is the same as talking about a type for the "new" indeterminate $x:a \to a$.

In particular,

every hereditarily many-to-one indeterminate (an indet in a many-to-one computad) *has a unique type, and a unique specializing map.*

In the logical language of FOLDS (see [M1], [M2]), corresponding to many-to-one computads, we have structures of a fixed *FOLDS signature*, namely $\mathrm{Mlt}^{\mathrm{op}}$; corresponding to the indeterminates in a computad, we have the elements of the structure; corresponding to the type of an indeterminate, we have the *kind* of the element; corresponding to the type together with the specializing map, we have the *dependent sort* of the element.

Coming to the example when $\mathcal{C}$ is chosen to be Comp itself, (4)(i) fails; in particular, Comp is not a concrete presheaf category. To show this, first we make some remarks of an elementary nature.

We remind the reader of the Eckmann-Hilton identity. Suppose that, in an ω-category, X is a 0-cell, and u and v are 2-cells, both of the form $1_X \to 1_X$. Then all of $u\#_0 v$, $v\#_0 u$, $u\#_1 v$, $v\#_1 u$ are well-defined, and they are all equal. Thus, $\mathrm{hom}(1_X, 1_X)$, the set of all 2-cells of the form $1_X \to 1_X$, is a commutative monoid, and both compositions, $\#_0$ and $\#_1$, for 2-cells in $\mathrm{hom}(1_X, 1_X)$ coincide with the monoid operation.

Conversely, any commutative monoid M can be turned into a 2-category $\boldsymbol{Z}$ with a single 0-cell, a single 1-cell (the identity), and 2-cells the elements of M, with both compositions of 2-cells given by the monoid operation.

Let now $\boldsymbol{Y}$ be a computad such that $\boldsymbol{Y}$ has no 1-indet: $|\boldsymbol{Y}|_1=\emptyset$. It follows that all 1-pd's (= 1-cells) in $\boldsymbol{Y}$ are identities, every 2-pd (= 2-cell) in $\boldsymbol{Y}$ is of the form $1_X \to 1_X$, and $\mathrm{hom}(1_X, 1_X)$ is the *free* commutative monoid on the set of indets in $\mathrm{hom}(1_X, 1_X)$ as free generators. (The universal property of the computad $\boldsymbol{Y}$ played against the 2-category derived from the appropriate free commutative monoid will give that $\mathrm{hom}(1_X, 1_X)$ is free as a commutative monoid).

Let $\boldsymbol{X}$ be the computad generated by the 0-indet X, the distinct 2-indets u, v and w, all of the form $1_X \to 1_X$, and the 3-indet $u\cdot v \xrightarrow{\varphi} w$. I claim $\boldsymbol{X}$ is a computope. First, it is clearly principal, $\boldsymbol{X}=\mathrm{Supp}(\varphi)$.

To show that $(\boldsymbol{X},\varphi)$ is primitive, let $\boldsymbol{Y}$ be a principal computad, $\boldsymbol{Y}=\mathrm{Supp}(\hat{\varphi})$ with a 2-cell $\hat{\varphi}$, and let $F:\boldsymbol{Y}\to\boldsymbol{X}$ be a morphism; $F(\hat{\varphi})=\varphi$. There is no 1-indet in $\boldsymbol{Y}$, since there is none in $\boldsymbol{X}$; what we said above about such computads $\boldsymbol{Y}$ applies. In particular, for some 0-cell $\hat{X}$, $\mathrm{c}\hat{\varphi}$ and $\mathrm{d}\hat{\varphi}$ are elements of the free commutative monoid $\hat{M}=\mathrm{hom}(1_{\hat{X}}, 1_{\hat{X}})$. F induces a monoid map $F:\hat{M}\to M$ for $M=\mathrm{hom}(1_X, 1_X)$ in $\boldsymbol{X}$, which, in addition, maps free generators to free generators. Therefore, since $F(\mathrm{c}\hat{\varphi})=\mathrm{c}\varphi=w$ and $F(\mathrm{d}\hat{\varphi})=\mathrm{d}\varphi=u\cdot v$, we must have that $\mathrm{c}\hat{\varphi}=\hat{w}$, $\mathrm{d}\hat{\varphi}=\hat{u}\cdot\hat{v}$ for indets $\hat{w}, \hat{u}, \hat{v}\in\hat{M}$.

We see that $\mathrm{supp}_{\boldsymbol{Y}}(\hat{\varphi})=\{\hat{X}, \hat{u}, \hat{v}, \hat{w}, \hat{\varphi})$ and that F induces an isomorphism $\mathrm{Supp}_{\boldsymbol{Y}}(\hat{\varphi}) \xrightarrow{\cong} \boldsymbol{X}$. Since $\boldsymbol{Y}$ is principal, $\boldsymbol{Y}=\mathrm{Supp}_{\boldsymbol{Y}}(\hat{\varphi})$, and $F:\boldsymbol{Y}\to\boldsymbol{X}$ is an isomorphism as needed for the claim. We've proved that $(\boldsymbol{X},\varphi)$ is primitive.

There is the non-trivial automorphism $\alpha:\boldsymbol{X}\to\boldsymbol{X}$ that switches u and v, and leaves all other indets the same: this works precisely because of the Eckmann-Hilton identity. We have found a computope with a non-trivial automorphism, as promised.

In fact, the same computope $\boldsymbol{X}$ can be used to show that Comp is not equivalent, even in the "abstract", non-concrete sense, to any presheaf category. To make the argument below more interesting, we again make some general observations concerning when a full subcategory of Comp is, abstractly, a presheaf category.

In any category $\mathcal{C}$, we call *small* the objects A of $\mathcal{C}$ for which $\mathcal{C}(A, -):\mathcal{C}\to\mathcal{C}$ commutes with colimits. The small objects of $\mathcal{C}$, under the assumption $\mathcal{C}\simeq\hat{\boldsymbol{C}}$, are the ones that correspond under the equivalence to retracts of the representable functors $\boldsymbol{C}(-, X)$ in $\hat{\boldsymbol{C}}$. We also have that every object in $\mathcal{C}$ is a colimit of small objects; this is inherited from the presheaf category $\hat{\boldsymbol{C}}$.

Recall (1.1) above, about the use of the words "principal" and "primitive".

(5) Let $\mathcal{C}$ be a full subcategory of Comp. Assume that $\mathcal{C}$ is a sieve in Comp (satisfies (a)), $\mathcal{C}$ is closed under colimits in Comp ((b)), and, on its own right, $\mathcal{C}$ is a presheaf category. Then

(i) Every $\mathcal{C}$-small object of $\mathcal{C}$ is principal, in particular, small in the sense of Comp;

(ii) Every primitive object of $\mathcal{C}$ is $\mathcal{C}$-small.

If, in addition, $\mathcal{C}$ satisfies (c) above, then

(iii) An object in $\mathcal{C}$ is $\mathcal{C}$-small if and only if it is Comp-small.

For the proof, see the appendix.

Let us return to our particular computad $\boldsymbol{X}$ introduced above. $\boldsymbol{X}$ is not small in $\mathtt{Comp}$, as it is shown now.

Let $\boldsymbol{Y}$ be the computad, a variant of $\boldsymbol{X}$, which is generated by the 0-indet X, the distinct 2-indets u and w, both of the form $1_X \to 1_X$, and the 3-indet $u\cdot u \xrightarrow{\varphi} w$. Let $\boldsymbol{Z}$ be $\mathtt{Supp}_{\boldsymbol{X}}(u)$. We have the diagram of $\mathtt{Comp}$-morphisms

$$\boldsymbol{Z} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{X} \xrightarrow{H} \boldsymbol{Y} \qquad (6)$$

where F, G, H are determined by $F(u)=u$, $G(u)=v$, $H(u)=H(v)=u$. (6) is a colimit diagram in $\mathtt{Comp}$, since upon applying $|\text{-}|$ to it results in the diagram

$$\{X,\ u\} \underset{u\mapsto v}{\overset{u\mapsto u}{\rightrightarrows}} \{X,\ u,\ v,\ w\} \xrightarrow[w\mapsto w]{u\mapsto u,\quad v\mapsto u} \{X,\ u,\ w\}$$

which is a coequalizer diagram in $\mathtt{Set}$. When we "hom" into (6) from $\boldsymbol{X}$, that is, consider the diagram

$$\mathrm{hom}(\boldsymbol{X},\boldsymbol{Z}) \underset{\mathrm{hom}(\boldsymbol{X},G)}{\overset{\mathrm{hom}(\boldsymbol{X},F)}{\rightrightarrows}} \mathrm{hom}(\boldsymbol{X},\boldsymbol{X}) \xrightarrow{\mathrm{hom}(\boldsymbol{X},H)} \mathrm{hom}(\boldsymbol{X},\boldsymbol{Y}) \ ,$$

we get

$$\emptyset \rightrightarrows \{1_{\boldsymbol{X}},\ \alpha\} \longrightarrow \{H\}$$

(here, α is the non-trivial automorphism of $\boldsymbol{X}$ found above), which, of course, is not a colimit diagram.

$\boldsymbol{X}$ is not small, but it is primitive in $\mathtt{Comp}$ as we saw before. Therefore, by (5)(ii) applied to $\mathcal{C}=\mathtt{Comp}$, $\mathtt{Comp}$ cannot be a presheaf category. More generally, no $\mathcal{C}$ satisfying the hypotheses of (5) can contain $\boldsymbol{X}$ as an object.

7. Words for computads

This section is devoted to a precise formulation of the word problem for computads. It is the natural formulation, following directly the definition of ω-category given in section 2.

The definition here is but an inessential variant of Jacques Penon's word-oriented, syntactical, definition of computads (polygraphs) in [Pe].

At the end of the section we give the precise statement of the theorem saying that the word problem for computads is solvable. The definition of the word-problem for computads is given here in its most direct manner in order to make the statement of that theorem as natural as possible. As a matter of fact, in the course of the proof of the theorem later in the paper, we will have to reformulate the word problem itself into a different, albeit equivalent, form.

The syntax for the word problem for computads is more complicated than the analogous syntax for free groups, and free constructions in general for algebraic structures of the usual kind, since the condition of being well-defined for a formal expression denoting a cell of a higher-dimensional category is non-trivial: it is defined in parallel with the essential equivalence of expressions.

First, we give "global" definitions for words, their well-definedness and essential equivalence. The relevant concepts for particular computads will be obtained by taking appropriate restrictions of the global ones.

Let us mention one important choice made in the definition that may not be a priori the obvious one. This is that the operations of domain and codomain do not have direct symbolic representations; rather, "domain" and "codomain" become operations on words (this feature is also present in Penon's approach).

In this section, no proofs of assertions are given. With the exception of the proof of the theorem at the end of the section, to be given later, they are all routine.

Words with prescribed *dimensions* are defined "absolutely freely" in the following inductive definition. We write W_m for the set of words of dimension m ; here, $m\in\mathbb{N}\cup\{-1\}$.

$$W_{-1} = \{*\} \ .$$

For $n\in\mathbb{N}$:

$$W_n = W_n^0 \dot\cup W_n^1 \dot\cup W_n^2 \ ,$$

with

$$W_n^0 = \{(0, n, \xi, a, b):\ \xi\in\mathbf{V},\ a, b\in W_{n-1}\}\ ,$$

$$W_n^1 = \begin{cases} \varnothing & \text{if } n=0 \\ \{(1, n, a): a\in W_{n-1}\} & \text{if } n\ge 1 \end{cases}$$

$$W_n^2 = \{(2, n, k, a, b): 0\le k<n,\ a, b\in W_n\}$$

(In other words, $W=\dot\bigcup_{m\ge -1} W_m$ is the least class for which the above equalities hold. The parentheses indicate ordered tuples (quintuples, triples, quintuples, respectively) in the set-theoretic sense. $\mathbf{V}$ is the universe of all sets.)

The elements of W_n^0 are the *pre-indeterminates*, or pre-indets, of dimension n . The "pre-" is there because in order for a pre-indet to be a real indet it will have to be well-defined, according to the definition given below. The tuple $x=(0, n, \xi, a, b)$ contains the reference to the fact that we now are talking about an indeterminate (the zero up front); n gives the dimension; next it has an arbitrary *name* ξ , to ensure that we have an unlimited supply of indets; a is to be the domain of x , b the codomain. Of course, the "problem" is that, eventually, a and b will have to be well-formed and parallel for x to be a (real) indet.

We write $W^0 \underset{\text{def}}{=} \dot\bigcup_{n\in\mathbb{N}} W_n^0$. For $x\in W^0$, we denote the ingredients of x in the following way: $x = (0, \mathrm{dim}(x), |x|, \mathrm{d}x, \mathrm{c}x)$. When $\mathrm{dim}(x)=0$, we have $\mathrm{d}x=\mathrm{c}x=*$.

The element $(1, n, a)$ of W_n^1 ($n\ge 1$) will be written as 1_a , intended as the word standing for the identity cell with domain and codomain a .

The element $(2, n, k, a, b)$ of W_n^2 ($n\ge 1$) will be written as $a\#_k b$, since it stands for the appropriate composite cell. There is a "problem" with this in the same way as with pre-indets, since the composite $a\#_k b$ is defined only if a and b "match" each other in a prescribed manner.

The words $\mathrm{d}a$ (*domain* of a) and $\mathrm{c}a$ (*codomain* of a) are defined recursively, for all $a\in W_{\geq 0}$ as follows; always, $a\in W_n$ implies $\mathrm{d}a, \mathrm{c}a\in W_{n-1}$.

For $a\in W_0$, $\mathrm{d}a=\mathrm{c}a=*$.

For $x\in W^0$, $\mathrm{d}x$ and $\mathrm{c}a$ were defined above.

For $1_a\in W^1$, $\mathrm{d}(1_a) = \mathrm{c}(1_a) = a$;

For $n\geq 1$, $a\#_k b\in W_n^2$:

$$\mathrm{d}(a\#_k b) = \begin{cases} \mathrm{d}a & \text{if } k=n-1 \\ (\mathrm{d}a)\#_k(\mathrm{d}b) & \text{if } k<n-1 \end{cases}$$

$$\mathrm{c}(a\#_k b) = \begin{cases} \mathrm{c}b & \text{if } k=n-1 \\ (\mathrm{c}a)\#_k(\mathrm{c}b) & \text{if } k<n-1 \end{cases}$$

We have the words $\mathrm{d}^{(k)}a$, $\mathrm{c}^{(k)}a$ whenever $a\in W_n$ and $0\leq k<n$ defined in the expected way.

The subset $W{\downarrow}$ of W (read $a\in W{\downarrow}$ as " a is well-formed") and the binary relation $\approx$ on W (read $a\approx b$ as " a and b are well-formed and define the same element") are defined inductively and simultaneously as the least pair of relations on W satisfying the following clauses ($W{\downarrow}_n$ is $W{\downarrow}\cap W_n$):

(i) $* \in W{\downarrow}_{-1}$.

(ii) $n\geq 0$ & $a, b\in W{\downarrow}_{n-1}$ & $\xi\in \mathbf{V}$ & $\mathrm{d}a\approx\mathrm{d}b$ & $\mathrm{c}a\approx\mathrm{c}b \Longrightarrow (0, n, \xi, a, b)\in W{\downarrow}_n$.

(iii) $a\in W{\downarrow}_n \Rightarrow 1_a\in W{\downarrow}_{n+1}$.

(iv) $n>k\geq 0$ & $a, b\in W{\downarrow}_n$ & $\mathrm{c}^{(k)}a \approx \mathrm{d}^{(k)}b \Longrightarrow (a\#_k b) \in W{\downarrow}_n$.

(v) $\approx$ restricted to $W{\downarrow}$ is an equivalence relation: for $a, b, e\in W{\downarrow}$,

$a\approx a$,

$a\approx b \Longrightarrow b\approx a$,

$a\approx b$ & $b\approx e \Longrightarrow a\approx e$.

(vi) $\approx$ restricted to $W{\downarrow}$ is a congruence: for $a, b, e, f\in W{\downarrow}$,

$$n\geq 0 \ \&\ a,b,e,f\in W{\downarrow}_{n-1} \ \&\ \xi\in \boldsymbol{V} \ \&\ \mathrm{d}a\approx\mathrm{d}b \ \&\ \mathrm{c}a\approx\mathrm{c}b \ \&\ \mathrm{d}e\approx\mathrm{d}f \ \&\ \mathrm{c}e\approx\mathrm{c}f \ \&\ a=e \ \&\ b\approx f \Longrightarrow (0,n,\xi,a,b)\approx(0,n,\xi,e,f) \ ;$$

$$a\approx b \Longrightarrow 1_a\approx 1_b \ ;$$

$$\mathrm{c}^{(k)}a\approx\mathrm{d}^{(k)}b \ \&\ \mathrm{c}^{(k)}e\approx\mathrm{d}^{(k)}f \ \&\ a\approx e \ \&\ b\approx f \Longrightarrow a\#_k b\approx e\#_k f \ .$$

(vii) the relation $\approx$ restricted to $W{\downarrow}$ obeys the five laws of identity and composition for ω-categories: for $a, b, e, f\in(W{\downarrow})_n$,

$$1^{(n)}_{\mathrm{d}^{(k)}b}\#_k b\approx b \ ,$$

$$a\#_k 1^{(n)}_{\mathrm{c}^{(k)}a}\approx a \ ,$$

$$\mathrm{c}^{(k)}a\approx\mathrm{d}^{(k)}b \Longrightarrow 1_a\#_k 1_b\approx 1_{a\#_k b} \ ,$$

$$\mathrm{c}^{(k)}a\approx\mathrm{d}^{(k)}b \ \&\ \mathrm{c}^{(k)}b\approx\mathrm{d}^{(k)}e \Longrightarrow (a\#_k b)\#_k e\approx a\#_k(b\#_k e) \ ,$$

$$\mathrm{c}^{(k)}a\approx\mathrm{d}^{(k)}b \ \&\ \mathrm{c}^{(k)}e\approx\mathrm{d}^{(k)}f \ \&\ \mathrm{c}^{(\ell)}a\approx\mathrm{d}^{(\ell)}e \ \&\ \mathrm{c}^{(\ell)}d\approx\mathrm{d}^{(\ell)}f \Longrightarrow (a\#_k b)\#_\ell(e\#_k f)\approx(a\#_\ell e)\#_k(b\#_\ell f) \ .$$

(1) Lemma. (i) $n\geq 0 \ \&\ a\in W{\downarrow}_n \Longrightarrow \mathrm{d}a, \mathrm{c}a\in W{\downarrow}_{n-1}$.

(ii) $n\geq 1 \ \&\ a\in W{\downarrow}_n \Longrightarrow \mathrm{dd}a\approx\mathrm{dc}a \ \&\ \mathrm{cd}a\approx\mathrm{cc}a$.

(iii) $a\in W \ \&\ b\in W \ \&\ a\approx b \Longrightarrow a\in W{\downarrow} \ \&\ b\in W{\downarrow}$.

(iv) $a\in W{\downarrow}_{\geq 0} \ \&\ b\in W{\downarrow}_{\geq 0} \ \&\ a\approx b \Longrightarrow \mathrm{d}a\approx\mathrm{d}b \ \&\ \mathrm{c}a\approx\mathrm{c}b$.

Part (iii) is the result of having made sure that an instance $a\approx b$ of the relation $\approx$ is generated by the clauses only if it has been ensured that a and b are well-formed. Note that, to some extent, this is an optional feature: in a different treatment, we may have arranged, for instance, that $\approx$ be reflexive on the whole of W .

By an *indeterminate* (indet) we mean an element of the class $W{\downarrow}^0 \underset{\text{def}}{=} W{\downarrow}\cap W^0$: an indet is a well-formed pre-indet.

Next, we give the versions of the definitions that are restricted to a fixed but arbitrary set of (pre-)indets.

Let I be any class of pre-indets, $I\subseteq W^0$. $W[I]$ denotes the class of words that involve only the (pre-)indets from I . $W[I]$ is defined by the following inductive definition:

$$W_{-1}[I] = \{*\} .$$

For $n\geq 0$, $W_n[I] = W_n^0[I] \dot{\cup} W_n^1[I] \dot{\cup} W_n^2[I]$,

with

$$W_n^0[I] = \{x\in I_n : \mathrm{d}x, \mathrm{c}x\in W_{n-1}[I]\} ,$$

$$W_n^1[I] = \begin{cases} \emptyset & \text{if } n=0 \\ \{1_a : a\in W_{n-1}[I]\} & \text{if } n\geq 1 \end{cases}$$

$$W_n^2[I] = \{a\#_k b : 0\leq k<n, \ a, b\in W_n\}$$

The change is in the clause for $W_n^0[I]$ where we have insisted that the pre-indet has to belong to the pre-assigned class I , and also, that its domain and codomain should be "defined from I ".

Let $I\subset W^0$.

We define $W{\downarrow}[I]$ as $W{\downarrow}\cap W[I]$. $W{\downarrow}[I]$ is the class of well-formed words defined from indets in I . The definition of $W{\downarrow}[I]$, together with the restriction of the relation $\approx$ to $W{\downarrow}[I]$, may be given equivalently by repeating the simultaneous definition of $W{\downarrow}$ and $\approx$, with clause (ii) replaced by the variant

$$(0, n, \xi, a, b) \in I \ \& \ a, b \in W{\downarrow}[I]_{n-1} \ \& \ \mathrm{d}a\approx\mathrm{d}b \ \& \ \mathrm{c}a\approx\mathrm{c}b \Longrightarrow (0, n, \xi, a, b) \in W[I]{\downarrow}_n .$$

and. of course, by replacing $W{\downarrow}$ everywhere by $W{\downarrow}[I]$.

We define $I^{@} \underset{\text{def}}{=} I \cap W[I]$ and $I{\downarrow} \underset{\text{def}}{=} I \cap W{\downarrow}[I] = W^{0} \cap W{\downarrow}[I]$. $I^{@}$ is the subclass of I consisting of those pre-indets that refer, in their domain and codomain, to (lower dimensional) pre-indets in I only. $I{\downarrow}$, the class of (well-formed) indets in $W{\downarrow}[I]$, is a subclass of $I^{@}$. We have $W[I] = W[I^{@}]$ and $W{\downarrow}[I] = W{\downarrow}[I{\downarrow}]$, and, as consequences, $I^{@@}=I^{@}$, $I^{@}{\downarrow}=I{\downarrow}$, $I{\downarrow}{\downarrow}=I{\downarrow}$.

Note that $I{\downarrow}$ is not necessarily the same as $I \cap W{\downarrow}^{0}$: the former may be a proper subclass of the latter, since if $x \in I$ has a domain or codomain not in $W[I]$, then $x \notin W[I]$.

I is *separated* if for any $x, y \in I$, if $|x| = |y|$, then $x=y$ (ordinary equality of words). As a consequence, for a separated I , for any $x, y \in I$, $x \approx y$ implies $x=y$: no two formally different indeterminates in I get identified. Of course, if I is separated, then so is any subset of it, and in particular, $I{\downarrow}$ is separated too.

(2) Proposition For any subset I of W^{0} , the set $<I> \underset{\text{def}}{=} W{\downarrow}[I]/{\approx}$ of all equivalence classes of the relation $\approx$ restricted to $W{\downarrow}[I]$ form an ω-category under the evident operations. $<I>$ is a computad, with indeterminates the equivalence classes $[x]_{\approx}$ for elements x of $I{\downarrow}$. If I is separated, then the indets of $<I>$ are in a bijective correspondence with the elements of $I{\downarrow}$. $\langle I \rangle$ is identical to $\langle I{\downarrow} \rangle$.

Conversely, every computad is isomorphic to $<I>$ for some $I \subset W^{0}$, which can be chosen to be separated.

For the purposes of questions of decidability, we restrict words to ones in which the names of indeterminates are natural numbers. Let W^{e} ("e" for "effective") be the subset of W given by the definition of W modified by replacing the clause " $\xi \in \mathbf{V}$ " with " $\xi \in \mathbb{N}$ ". We have $W^{e}{\downarrow}=W{\downarrow}^{e}$, expressing the obvious fact that the set $W{\downarrow}^{e}=W{\downarrow} \cap W^{e}$ can also be obtained by repeating the definition of $W{\downarrow}$ with W^{e} replacing W .

The set W^e is obviously a decidable (recursive) subset of $\mathbb{HF}$, the set of hereditarily finite sets. It is also clear that $W^e{\downarrow}$ and the relation $\approx$ restricted to $W^e{\downarrow}$ are semi-recursive (recursively enumerable). We will prove that, in fact,

(3) Theorem. $W^e{\downarrow}$ and the relation $\approx$ restricted to $W^e{\downarrow}$ are decidable (recursive). As a consequence, for any decidable subset I of $(W^e)^0$, the relation $\approx$ restricted to the set $W{\downarrow}[I]$ is decidable.

The last fact is the precise expression of the solvability of the word problem for the computad $\langle I\rangle$, for any decidable set I of indeterminates.

Let, in particular, $\dot{I}$ be the set of preindets that use, in the hereditary sense, the single name 0 only. Formally, we define $\dot{W}=\dot{W}^0\dot{\cup}\dot{W}^1\dot{\cup}\dot{W}^2$ as W was defined at the outset, but with " $\xi=0$ " replacing " $\xi\in\mathbf{V}$ ", and define $\dot{I} \underset{\text{def}}{=} \dot{W}^0$. Clearly, $\dot{I}$ is a decidable subset of $(W^e)^0$. It is easy to see that that $\langle\dot{I}\rangle$ is the terminal computad.

(4) Corollary The word problem for the terminal computad is solvable.

Let us make some comments on the supp and content ($[-]$) functions defined on words.

Having defined these functions on pd's in a computad, (2) can be used to define them for words, by $\mathrm{supp}(a) = \mathrm{supp}([a]_{\approx})$, and similarly for the content function. Alternatively, one can copy the recursive definitions of these functions, and apply them to words; one can prove directly that the functions are invariant under $\approx$, i.e., well-defined on equivalence classes.

The supp function on words is the direct notion of occurrence: $x\in\mathrm{supp}(a)$ iff x occurs, in the usual syntactical sense, in the word a. However, the content $[a]$ does not have such a direct meaning. For instance, $[a](x)$ is not the same as the number of occurences of x

in a . The reason is that the number of occurences of a fixed x is not invariant under $\approx$. This is most obviously seen on the two sides of the distributive law, explained in the next section.

8. Another set of primitive operations for ω-categories

In an ω-category, let a be an m-cell, b an n-cell. Let us denote the number $\mathrm{min}(m, n)-1$ by $k[a, b]$, now k for short. Let's write $N=\mathrm{max}(m, n)$.
Assume that $0\le k$ and

$$c^{(k)}a=d^{(k)}b \quad . \tag{1}$$

Then the composite

$$1_a^{(N)}\#_k 1_b^{(N)} \tag{2}$$

is well-defined since $c^{(k)}a = c^{(k)}1_a^{(N)}$, and similarly for the other factor. Note that at least one of the two "identities" is just the corresponding original cell a or b . since either $N=m\ge n$ or $N=n\ge m$. We denote the composite (2) by $a\cdot b$, or even just ab . The main thing is that there is no need to carry k in the notation since k is given by a and b : $k=k(a, b)$.

The "new primitive" operation we are proposing is the -- conditional (partial) -- binary operation $(a, b)\longmapsto a\cdot b$.

According to the new definition (which will be seen to be equivalent to the original), an ω-category is an ω-graph X with an identity operation $a\mapsto 1_a$ as before, and the conditional binary operation

$$(a, b)\ \text{[subject to } 0\le k=k(a, b) \text{ and (1)]} \longmapsto a\cdot b \ ,$$

required to satisfy the conditions below. To simplify writing, we agree to use m, n, p and q

for the dimensions of the cells a, b, e and f, respectively. I'll write $a \wedge b$ for $c^{(k)}a = d^{(k)}b$, with $k = k(a, b)]$, assuming that the equality does indeed hold. We may even write $a \wedge b \downarrow$ (" $a \wedge b$ is well-defined") for the condition (1).

Dimension:

(i) $\dim(1_a) = a$

(ii) $\dim(a \cdot b) = \max(m, n)$

\Assumption: $a \wedge b \downarrow$.)

Domain/codomain laws:

(i) $d(1_a) = c(1_a) = a$

(ii) $$d(a \cdot b) = \begin{cases} (da) \cdot b & \text{if } m > n \\ a \cdot (db) & \text{if } m < n \\ da & \text{if } m = n \end{cases}$$

$$c(a \cdot b) = \begin{cases} (ca) \cdot b & \text{if } m > n \\ a \cdot (cb) & \text{if } m < n \\ cb & \text{if } m = n \end{cases}$$

\Assumption: $a \wedge b \downarrow$.

(Remark: the left-hand sides being assumed defined, so are the (various) right-hand sides: for each of the four composites, call it $e \cdot f$, we have $e \wedge f \downarrow \&= a \cdot b$ under the suitable precondition.)

Unit laws:

(i) $$1_a \cdot b = \begin{cases} b & \text{if } n \geq m+1 \\ 1_{a \cdot b} & \text{if } n < m+1 \end{cases}$$

\Assumption: $1_a \wedge b \downarrow$.

(Remark: $1_a \wedge b = a$ if $n \geq m+1$, and $1_a \wedge b = a \wedge b$ if $n < m+1$.)

(ii) $$a \cdot 1_b = \begin{cases} a & \text{if } m \geq n+1 \\ 1_{a \cdot b} & \text{if } m < n+1 \end{cases}$$

\Assumption: $a \wedge 1_b \downarrow$.

(Remark: $a \wedge 1_b = b$ if $m \geq n+1$, and $a \wedge 1_b = a \wedge b$ if $m < n+1$.)

Associative law:

$$a \cdot (b \cdot e) = (a \cdot b) \cdot e \tag{3}$$

\Assumptions: either $m = n \leq p$, or $m \geq n = p$, or $m = p \leq n$ (4)

and: $b \wedge e$ and $a \wedge b$ are well-defined. (5)

(Remarks: Assuming (5), (4) is equivalent to saying that

$k(a, be) = k(a, b) = k(ab, e) = k(b, e)$.

(4) and (5) together ensure that $a\wedge(be) = a\wedge b$ and $(ab)\wedge e = b\wedge e$ (thus, both sides of (3) are well-defined), and, before we know that (3) is true, the two sides of (3) are parallel.)

Distributive laws:

(i) $$a\cdot(b\cdot e) = (a\cdot b)\cdot(a\cdot e) \tag{6}$$

\Assumptions: $m<n$, $m<p$ (7)

and

$b\wedge e$, $a\wedge b$, $a\wedge e$ are well-defined. (8)

(Remark: (7) is equivalent to saying that $k(a, b) < k(b, e)$. Also, (7) implies that $k(a, b)=k(a, e)=m-1$. Assuming both (7) and (8), we have that $a\wedge(be) = a\wedge b = a\wedge e$ and $(ab)\wedge(ae)$ ↓&= $a\cdot(b\wedge e)$ (!) (thus, both sides of (6) are well-defined), and, before we know the equality in (6), the fact that the two sides are parallel. It is also good to know that, alternatively, if (7) and

$b\wedge e$ and $a\wedge(b\cdot e)$ are well-defined (8')

hold, we again have the distributive identity, since (7)&(8') implies (8).

(ii) $$(a\cdot b)\cdot e = (a\cdot e)\cdot(b\cdot e)$$

\Assumption: $p<n$, $p<m$

and $a\wedge b$, $a\wedge e$, $b\wedge e$ are well-defined.

Commutative law:

$$(a\cdot(\bar{d}b))\cdot((\bar{c}a)\cdot b) = ((\bar{d}a)\cdot b)\cdot(a\cdot(\bar{c}b))$$

\Notation: for $k=k(a,b)$, $\bar{d}=d^{(k)}$, $\bar{c}=c^{(k)}$.

\Assumption: $k\geq 1$ ($\Longleftrightarrow$ $m,n\geq 2$) and $c^{(k-1)}a=d^{(k-1)}b$.

(Remark: Assume the assumption. Let's write $a*b \underset{\text{def}}{=} c^{(k-1)}a=d^{(k-1)}b$. Then

$$a\wedge(\bar{d}b)=(\bar{c}a)\wedge b=(\bar{d}a)\wedge b=a\wedge(\bar{c}b)=\bar{c}a\wedge\bar{d}b=\bar{d}a\wedge\bar{c}b=a*b,$$

and

$$(a\cdot(\bar{d}b))\wedge((\bar{c}a)\cdot b)\ \downarrow\&=\ \bar{c}a\cdot\bar{d}b\ .$$

$$((\bar{d}a)\cdot b)\wedge(a\cdot(\bar{c}b))\ \downarrow\&=\ \bar{d}a\cdot\bar{c}b\ ,$$

Thus, all composites in the identity exist.

Moreover, the two sides of the identity are parallel, even before we know the truth of the identity itself. This is seen directly when, for $r\underset{\text{def}}{=}\max(m,n)-\min(m,n)$, we have $r=1$; and by induction on r in general; in the induction, lower dimensional instances of the commutative law itself are used.)

(**End of the new definition of ω-category**.)

Let $\boldsymbol{X}$ be an ω-category in the new sense. We define operations to show that we have an ω-category in the original sense.

Let $a\in\boldsymbol{X}_m$, $b\in\boldsymbol{X}_n$; as before, $k=k(a,b)=\min(m,n)-1$. Assume $0\leq\ell\leq k$ and $c^{(\ell)}a=d^{(\ell)}b$ (we write $a\wedge_\ell b$ for the joint value if the equality holds). We define $a\#_\ell b$ for $0\leq\ell\leq k$ by recursion on $k-\ell$ as follows. Simultaneously with the recursion, we prove inductively the **generalized commutativity law** saying that

$$(a\cdot(\tilde{d}b))\#_\ell((\tilde{c}a)\cdot b)=((\tilde{d}a)\cdot b)\#_\ell(a\cdot(\tilde{c}b));\qquad(9)$$

with the notation $\tilde{d}=d^{(\ell)}$, $\tilde{c}=c^{(\ell)}$, provided $c^{(\ell-1)}a=d^{(\ell-1)}b$.

When $\ell=k$, we put $a\#_{\ell}b \underset{\text{def}}{=} a\cdot b$; (9) is the (simple) commutativity law. When $\ell<k$, writing $\bar{d}=d^{(\ell+1)}$, $\bar{c}=c^{(\ell+1)}$,

$$a\#_{\ell}b \underset{\text{def}}{=} (a\cdot(\bar{d}b))\#_{\ell+1}((\bar{c}a)\cdot b) \overset{(!)}{=} ((\bar{d}a)\cdot b)\#_{\ell+1}(a\cdot(\bar{c}b)) ,$$

where the the equality marked with (!) is true by the induction hypothesis.

(10) Proposition **(i)** The definition provided gives an ω-category in the original sense.

(ii) Conversely, every ω-category in the original sense is one in the new sense, with the definition of $a\cdot b$ given by (2) above.

(iii) The processes of passing from an ω-category in the old sense to one in the new sense and vice versa are inverses of each other.

For some details of the (straight-forward) proof, see the Appendix.

Next, we explain the pre-normal form mentioned in the introduction. We will call it the *expanded form*. The expanded form is for cells in an ω-category, relative to a given strongly generating set of cells ("strongly generating" means, roughly speaking: "generating, without the use of the domain and codomain operations"; see also below.)

We repeat a definition from section 4.

Let $\boldsymbol{X}$ be an ω-category, G a set of cells of $\boldsymbol{X}$ (of various dimensions). We say that G *strongly generates* $\boldsymbol{X}$ if $\|\boldsymbol{X}\|$ equals the least subset S of $\|\boldsymbol{X}\|$ such that (a) S contains G, (b) $a\in S$ implies $1_a\in S$, and (c) $a, b\in S$ and $a\#_k b$ is well-defined imply that $a\#_k b \in S$. In the notation of section 3, this means that $\boldsymbol{X}=\varnothing\langle G\rangle$. As the main example, we know that if $\boldsymbol{X}$ is a computad, $|\boldsymbol{X}|$, the set of indeterminates of $\boldsymbol{X}$, strongly generates $\boldsymbol{X}$.

Note that, using the new primitive operation $\cdot$, we can replace (c) by the equivalent condition (c)* $a, b \in S$ and $a \cdot b$ is well-defined imply that $a \cdot b \in S$.

Let $n \geq 0$. By a G-*atom* of dimension $n+1$ we mean a well-defined element of the form

$$b_n \cdot (b_{n-1} \cdot (\ldots (b_1 \cdot u \cdot e_1) \ldots) \cdot e_{n-1}) \cdot e_n \ , \tag{11}$$

where b_i, $e_i \in \boldsymbol{X}_i$ $(i=1, \ldots, n-1, n)$, and $u \in G_{n+1}$. A G-molecule of dimension $n+1$ is either an identity cell, or of form $\varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_\ell$ where $\ell \geq 1$, and each φ_i is a G-atom of dimension $n+1$ (because of the associative law, no bracketing is required in writing $\varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_\ell$).

(12) Proposition. Assume that G strongly generates $\boldsymbol{X}$. Then, for every $n \in \mathbb{N}$, every ($n+1$)-cell in $\boldsymbol{X}$ is a G-molecule.

Proof It suffices to show that the set $S = \boldsymbol{X}_{\leq n} \cup M_{n+1}$, where M is the set of all G-$(n+1)$-molecules, satisfies (a), (b) and (c)*. Since every $u \in G_{n+1}$ is a G-atom, and thus a G-molecule itself, (a) is clear. (b) is taken care of explicitly. It remains to see (c)*.

For (c)*, there are the four cases: 1) $a, b \in \boldsymbol{X}_{\leq n}$; 2) $a \in \boldsymbol{X}_{\leq n}$, $b \in M_{n+1}$;
3) $a \in M_{n+1}$, $b \in \boldsymbol{X}_{\leq n}$; 4) $a, b \in M_{n+1}$.

The cases 1) and 4) are clear. 2) and 3) are similar; we deal with 2).

Assume 2). When b is an identity, $b = 1_c$, $c \in \boldsymbol{X}_n$, we have $a \cdot b = 1_{a \cdot c}$, which belongs to S. When $b = \varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_\ell$, then $a \cdot b = (a \cdot \varphi_1) \cdot (a \cdot \varphi_2) \cdot \ldots \cdot (a \cdot \varphi_\ell)$ by the distributive law (together with the associative law). Therefore, it suffices to show that if φ is a G-$(n+1)$-atom, $a \in \boldsymbol{X}_{\leq n}$, and $a \cdot \varphi$ is well-defined, then $a \cdot \varphi$ is again a G-$(n+1)$-atom. In fact, if φ is given by (10), then, again by the distributive and associative laws,

$$a \cdot \varphi = \hat{b}_n \cdot (\hat{b}_{n-1} \cdot (\ldots(\hat{b}_1 \cdot u \cdot \hat{e}_1)\ldots) \cdot \hat{e}_{n-1}) \cdot \hat{e}_n$$

where $\hat{b}_i = a \cdot b_i$ for $i=n, \ldots k$;
$\hat{b}_i = b_i$ for $i=k-1, \ldots, 1$;
$\hat{e}_j = e_j$ for $j=1, \ldots k$;
$\hat{e}_j = a \cdot e_j$ for $j=k+1, \ldots n$.

Next, we introduce a construction that makes an $(n+1)$-category $\mathbf{Z}$ into an n-category, $\bar{\mathbf{Z}}$, called the *collapse* of $\mathbf{Z}$, without losing any cells, by "demoting" $(n+1)$-cells to n-cells. This peculiar construction will be used in the next section.

Let $\mathbf{Z}$ be an $(n+1)$-category. We define the n-category $\bar{\mathbf{Z}}$. The $(n-1)$-truncation of $\bar{\mathbf{Z}}$ agrees with that of $\mathbf{Z}$. $\bar{\mathbf{Z}}_n \overset{\text{def}}{=} \mathbf{Z}_n \dot{\cup} \mathbf{Z}_{n+1}$. Writing $\bar{\mathrm{d}}$ and $\bar{\mathrm{c}}$ for "domain" and "codomain" in $\bar{\mathbf{Z}}$, and leaving d and c for "domain" and "codomain" in $\mathbf{Z}$, we define, for $a \in \mathbf{Z}_{n+1}$, $\bar{\mathrm{d}}a \overset{\text{def}}{=} \mathrm{d}^{(n-1)}a$, $\bar{\mathrm{c}}a \overset{\text{def}}{=} \mathrm{c}^{(n-1)}a$. Thus far, we have an n-graph.

We define the dot-operation for $\bar{\mathbf{Z}}$, denoted $\bar{\cdot}$, as follows (the plain dot is the operation in $\mathbf{Z}$). $a \bar{\cdot} b$ is to be defined under the assumption

$$\bar{\mathrm{c}}^{(\bar{k})}a = \bar{\mathrm{d}}^{(\bar{k})}b \,, \tag{13}$$

where $\bar{k} = \min(\dim_{\bar{\mathbf{Z}}}(a), \dim_{\bar{\mathbf{Z}}}(b)) - 1$.

Note that $\bar{k} = k = \min(\dim_{\mathbf{Z}}(a), \dim_{\mathbf{Z}}(b)) - 1$ unless both a and b belong to $\mathbf{Z}_{n+1}$, in which case $\bar{k} = n-1$ and $k = n$. Thus, unless $a, b \in \mathbf{Z}_{n+1}$, $a \bar{\cdot} b$ is to be defined iff $a \cdot b$ is defined, and when $a, b \in \mathbf{Z}_{n+1}$, $a \bar{\cdot} b$ is to be defined iff $\mathrm{c}^{(n-1)}a = \mathrm{d}^{(n-1)}b$. Accordingly, under the assumption (13), we can make the following definition:

$$\bar{a \cdot b} = \begin{cases} a\#_{n-1}b & \text{if } a, b \in \mathbf{Z}_{n+1} \\ a \cdot b & \text{otherwise} \end{cases}$$

(14) Proposition $\bar{\mathbf{Z}}$ is an n-category.

Proof. The proof is a short calculation, if we apply the present section's definition of n-category (which is the obvious one, implied by the statement of the definition for "ω-category").

The associative law has one new case, not directly contained in $\mathbf{Z}$ being an $(n+1)$-category, the case when all three variables are "new", i.e., elements of $\mathbf{Z}_{n+1}$; and in this case, the law is the associative law for the operation $\#_{n-1}$ applied to n-cells.

The distributive law again has one new case, the one when b, e are new, and a is old; the required equality is the "generalized distributivity" (see Appendix, the proof of 8.(10)) for $\mathbf{Z}$, for the operation $\#_{n-1}$.

The commutative law will be reduced to "generalized commutativity", 8.(9), in $\mathbf{Z}$, for a, b of dimension $n+1$, and $\ell=n-1$ in 8.(9).

The collapse is a functor $\overline{(-)}:(n+1)\mathrm{Cat} \longrightarrow n\mathrm{Cat}$, and it has the flavor of being a forgetful functor. But it does not preserve products, for instance.

9. A construction of the one-step free extension $X[U]$.

In this section, n is a fixed but arbitrary non-negative integer, and X is a fixed but arbitrary n-category. Further, U is a set of $(n+1)$-indets attached to X : with $u\in U$, we have $\mathrm{d}u, \mathrm{c}u\in X_n$, $\mathrm{d}u\|\mathrm{c}u$. We give a particular construction of the free extension $X[U]$ (see section 4 for the basic definitions).

Let us use the symbol $\bar{u}$ for $u\in U$ to denote a new indeterminate of dimension n attached to X such that $\mathrm{d}\bar{u}=\mathrm{d}^{(n-1)}u$, $\mathrm{c}\bar{u}=\mathrm{c}^{(n-1)}u$ (that is, $\bar{u}\|\mathrm{d}u\|\mathrm{c}u$); $\bar{u}$, $\bar{v}$ distinct for distinct u , v . Let $\bar{U}$ be the set of all $\bar{u}$. We start with the n-category $X[\bar{U}]$, the free extension of X by $\bar{U}$, which we take as given. We undertake to explain the $(n+1)$-category $X[U]$ in terms of the n-category $X[\bar{U}]$ with a reasonably simple additional structure. In brief, what we will learn is that, to obtain $X[U]$, the only thing that needs to be added to $X[\bar{U}]$ is the effect of the commutativity law in the highest dimension.

As explained in section 4 , X is a sub-n-category of $X[\bar{U}]$: we take the canonical map $X\longrightarrow X[\bar{U}]$ to be an inclusion. Also, the canonical map $\bar{U}\to\|X[\bar{U}]\|$ is an inclusion.

An *atom with nucleus* $\bar{u}\in\bar{U}$ is, by definition, a (well-defined) n-cell in $X[\bar{U}]$ of the form

$$b_n\cdot(b_{n-1}\cdot(\ldots(b_1\cdot\bar{u}\cdot e_1)\ldots)\cdot e_{n-1})\cdot e_n\ , \qquad (1)$$

where b_i , $e_i\in X_i$ $(i=1,\ldots,n)$.

Note the slight-looking but essential difference to "atom" in the previous section. Whereas in the last section, in 8.(11), u was a cell of dimension $n+1$, in (1) now $\bar{u}$ is dimension n itself. Our intention is that, via the construction to be given in this section, the element (1) should *stand for* the element 8.(11), with u the indeterminate of dimension $n+1$ giving rise formally to $\bar{u}$.

Of course, two atoms obtained from different coefficients b_i , e_i could very well be equal, but at least an atom determines its nucleus $\bar{u}$. Calling the atom (1) φ , by section 4, we can characterize $\bar{u}$ as the unique element $\bar{v}$ of $\bar{U}$ for which φ belongs to $\boldsymbol{X}\langle\bar{v}\rangle$. We write $\varphi[\bar{u}]$ to indicate that the nucleus of the atom φ is $\bar{u}$.

Let φ be the atom in (1). Note that for any $m\leq n$ and any $a\in\boldsymbol{X}_m$ such that $a\cdot\varphi$ is well-defined, $a\cdot\varphi$ is again an atom with the same nucleus; this is seen as the analogous statement was seen in the proof of 8.(12).

Similarly, under the appropriate conditions, $\varphi\cdot f$ is an atom .

Given any $\bar{u}\in\bar{U}$, and any $r\in\boldsymbol{X}[\bar{U}]_n$ such that $r\|\bar{u}$, the universal property of $\boldsymbol{X}[\bar{U}]$ gives us a unique self-map $h:\boldsymbol{X}[\bar{U}]\longrightarrow\boldsymbol{X}[\bar{U}]$, a map of ω-categories, such that h is the identity on $\boldsymbol{X}$, also the identity on $\bar{U}-\{\bar{u}\}$, and $h(\bar{u})=r$. With any element φ of $\boldsymbol{X}[\bar{U}]$ such that $\varphi\|\bar{u}$, we write $\varphi[r/\bar{u}]$ for $h(\varphi)$ with this h (which h we may refer to as $h_{r/u}$). Indeed, $\varphi[r/\bar{u}]$ should be imagined as the result of *substituting* r for $\bar{u}$ in φ .

When $\varphi[\bar{u}]$ is an atom, we write $\varphi[r]$ for $\varphi[r/\bar{u}]$.

Let $\varphi=\varphi[\bar{u}]$ be an atom, and assume the substitution $\varphi[r]$ is well-defined. We have that $(a\cdot\varphi)[r]=a\cdot\varphi[r]$ since, for $h=h_{r/u}$,

$$(a\cdot\varphi)[r]=h(a\cdot\varphi)=ha\cdot h\varphi=a\cdot\varphi[r] \text{ .}$$

because h is the identity on $\boldsymbol{X}$. Similarly,

$$(\varphi\cdot a)[r]=\varphi[r]\cdot a \text{ .}$$

As a consequence, we have, for φ as in (1), that

$$\varphi[r]=b_n\cdot(b_{n-1}\cdot(\ldots(b_1\cdot r\cdot e_1)\ldots)\cdot e_{n-1})\cdot e_n \text{ ;} \qquad (2)$$

but also notice that we could not, with good conscience, give the last formula as a definition, since it is not obvious -- although true as we now know -- that two different expressions for φ of the form (1) give the same value for $\varphi[r]$ via (2).

(2) shows that if $r\in\mathbf{X}$, then $\varphi[r]\in\mathbf{X}$ as well.

We also have $\mathrm{d}(\varphi[r])=\mathrm{d}\varphi$, since

$$\mathrm{d}(\varphi[r])=\mathrm{d}(h(\varphi))=h(\mathrm{d}\varphi)=\mathrm{d}\varphi$$

because h is the identity on $\mathbf{X}$. Similarly, $\mathrm{c}(\varphi[r])=\mathrm{c}\varphi$.

Let $\varphi=\varphi[\bar{u}]$ be an atom. Since $\mathrm{d}u$ and $\mathrm{c}u$ $(\in\mathbf{X}_n)$ are parallel to $\bar{u}$, the n-cells $D\varphi \underset{\mathrm{def}}{=} \varphi[\mathrm{d}u]$, $C\varphi \underset{\mathrm{def}}{=} \varphi[\mathrm{c}u]$ are well-defined and they belong to $\mathbf{X}$. $D\varphi$ and $C\varphi$ are going to be the domain and codomain of φ when we understand φ as an $(n+1)$-cell of $\mathbf{X}[U]$.

The direct formulas for $D\varphi$ and $C\varphi$ are

$$D\varphi = b_n(b_{n-1}(\ldots(b_1\cdot\mathrm{d}u\cdot e_1)\ldots)e_{n-1})e_n \;,$$

$$C\varphi = b_n(b_{n-1}(\ldots(b_1\cdot\mathrm{c}u\cdot e_1)\ldots)e_{n-1})e_n \;.$$

Note the equalities

$$\mathrm{d}D\varphi=\mathrm{d}C\varphi=\mathrm{d}\varphi\,, \qquad \mathrm{c}D\varphi=\mathrm{c}C\varphi=\mathrm{c}\varphi\;;$$

and for $a\in\mathbf{X}$, provided the composites involved are well-defined,

$$D(a\cdot\varphi)=a\cdot D\varphi\,,\ C(a\cdot\varphi)=a\cdot C\varphi\,,\ D(\varphi\cdot a)=D\varphi\cdot a\,,\ C(\varphi\cdot a)=C\varphi\cdot a\,. \quad (3)$$

Below, $\alpha,\beta,\rho,\sigma,\varphi,\psi$ will denote atoms.

We say ρ *matches* σ (not a symmetric relation) if we have $C\rho=D\sigma$.

Suppose $\alpha=\alpha[\bar{u}]$ and $\beta=\beta[\bar{v}]$ are atoms; assume that $\mathrm{c}\alpha=\mathrm{d}\beta$. We derive four further atoms, $\rho[\bar{u}]$, $\sigma[\bar{v}]$, $\varphi[\bar{v}]$, $\psi[\bar{u}]$ from α and β :

$$\rho = \alpha\cdot D\beta \ ; \qquad \sigma = (C\alpha)\cdot\beta \ ; \qquad \varphi = (D\alpha)\cdot\beta \ ; \qquad \psi = \alpha\cdot(C\beta) \ .$$

Since $\mathrm{d}(D\beta) = \mathrm{d}\beta$, ρ is well-defined; similarly the three remaining atoms. φ matches ψ , since

$$C\varphi = C((D\alpha)\cdot\beta) = ((D\alpha)\cdot\beta)[\mathrm{c}u] = D\alpha\cdot(\beta[\mathrm{c}u]) = D\alpha\cdot C\beta \,,$$

and similarly

$$D\psi = D\alpha\cdot C\beta \,,$$

thus $C\varphi=D\psi$. We similarly see that $D\varphi=D\rho$ and $C\psi=C\sigma$.

Let us define the quaternary relation $L(\rho, \sigma, \varphi, \psi)$ on atoms as follows:

$L(\rho, \sigma, \varphi, \psi) \overset{\text{def}}{\Longleftrightarrow}$ there are atoms α and β with $\mathrm{c}\alpha=\mathrm{d}\beta$ such that ρ,σ,φ,ψ are each the so-named atom derived from (α, β) above.

We write $R(\rho, \sigma, \varphi, \psi)$ for $L(\varphi, \psi, \rho, \sigma)$, and $E(\rho, \sigma, \varphi, \psi)$ for $L(\rho, \sigma, \varphi, \psi)\vee R(\rho, \sigma, \varphi, \psi)$.

The motivation for the above definitions is as follows. As we said before, we want the atom $\varphi[\bar{u}]$, an n-cell of the n-category $\boldsymbol{X}[\bar{U}]$, to stand for $\varphi[u]$, an $(n+1)$-cell of the $(n+1)$-category $\boldsymbol{X}[U]$. The relation $L(\rho, \sigma, \varphi, \psi)$ is the description of what it means, in terms of the atoms $\rho[\bar{u}]$, $\sigma[\bar{v}]$, $\varphi[\bar{v}]$, $\psi[\bar{u}]$ in n-category $\boldsymbol{X}[\bar{U}]$, for the equality

$$\rho[u]\cdot\sigma[v] = \varphi[v]\cdot\psi[u]$$

to be an instance of the commutative law in the $(n+1)$-category $\boldsymbol{X}[U]$. $R(\rho, \sigma, \varphi, \psi)$ means that we have an instance with the sides reversed; $E(\rho, \sigma, \varphi, \psi)$ that we have one or the other.

A *molecule* is either the symbol 1_a with a any element of X_n, or, for any positive integer ℓ, a ℓ-tuple $\vec{\varphi} = (\varphi_1, \ldots, \varphi_\ell)$ of atoms φ_i such that, for each $i=1, \ldots, \ell-1$, φ_i matches φ_{i+1}. ℓ is the *length* of $\vec{\varphi}$, $\ell=\ell(\vec{\varphi})$. The *length* of 1_a is 0.

The set of all molecules is denoted $\mathcal{M}$.

For molecules $\vec{\varphi}$ and $\vec{\psi}$, let us write $E(\vec{\varphi}, \vec{\psi})$ meaning that the following conditions hold:

the lengths of $\vec{\varphi}$ and $\vec{\psi}$ are the same, say ℓ, and
there is $i\in\{1, \ldots, \ell-1\}$ such that $E(\varphi_i, \varphi_{i+1}, \psi_i, \psi_{i+1})$,
and for $j\in\{1, \ldots, \ell\}-\{i, i+1\}$, $\psi_j=\varphi_j$.

E is a symmetric relation on molecules. We define $\approx$ to be the reflexive and transitive closure of E on the set $\mathcal{M}$ of all molecules. For length-0 and length-1 molecules, the equivalence $\approx$ is the same as equality.

The fact that $E(\vec{\varphi}, \vec{\psi})$ holds means that a pair of consecutive atoms in $\vec{\varphi}$, $\rho=\varphi_i$, $\sigma=\varphi_{i+1}$ have been replaced by another pair, $\varphi(=\psi_i)$, $\psi(=\psi_{i+1})$, the second pair being in the relation E with the first pair: $E(\rho.\sigma, \varphi, \psi)$. Note that, by what we said above, any transformation as described here produces a well-formed molecule $\vec{\psi}$ from a well-formed molecule $\vec{\varphi}$.

Note also that if $L(\rho, \sigma, \varphi, \psi)$, and the nuclei of ρ and σ are $\bar{u}$ and $\bar{v}$, in this order, than the nuclei in φ and ψ are $\bar{v}$ and $\bar{u}$ in this order. Therefore, one effect of passing from $\vec{\varphi}$ to $\vec{\psi}$ when $E(\vec{\varphi}, \vec{\psi})$ is to switch the order of the ith and $(i+1)$st nuclei in $\vec{\varphi}$. Thus if $\vec{\varphi}\approx\vec{\psi}$, the indets of $\vec{\varphi}$ undergo a permutation when passing to $\vec{\psi}$. In particular, the multiset of the occurrences of indets of the form $\bar{u}$, $u\in U$, in a molecule is invariant under the equivalence $\approx$.

The equivalence class of $\approx$ containing the molecule $\vec{\varphi}$ is denoted by $[\vec{\varphi}]$.

We define an $(n+1)$-category $\mathbf{Y}$ as follows. The n-truncation of $\mathbf{Y}$ is defined to be $\mathbf{X}$. $\mathbf{Y}_{n+1}$ is defined as the set $\mathcal{M}/\approx$ of all equivalence classes $[\vec{\varphi}]$ ($\vec{\varphi}\in\mathcal{M}$).

To avoid confusion in dealing with the domain/codomain operations in $\mathbf{X}[\bar{U}]$ and $\mathbf{Y}$, we write D and C for "domain", respectively "codomain", in $\mathbf{Y}$.

We put $D([1_a])=C([1_a])=a$.

For a molecule $\vec{\varphi}$ of length $\ell\geq 1$, we put $D([\vec{\varphi}])=D\varphi_1$, $C([\vec{\varphi}])=C\varphi_\ell$. By what we saw above, it is clear that D and C are well-defined on equivalence classes. From the facts that, for an atom ρ, $D\rho\|C\rho\|\rho$, and that $C\varphi_i=D\varphi_{i+1}$, we have that $D\varphi_1$ is parallel to $C\varphi_\ell$, i.e., $D[\vec{\varphi}]$ and $C[\vec{\varphi}]$ are parallel. This ensures that our definitions so far give an $(n+1)$-graph.

To define the $(n+1)$-category operations, we need to define 1_a for $a\in\mathbf{X}_n$, and $a\cdot[\vec{\varphi}]$, $[\vec{\varphi}]\cdot a$ and $[\vec{\varphi}]\cdot[\vec{\psi}]$ for $a\in\mathbf{X}_{\leq n}$ and $\vec{\varphi},\vec{\psi}\in\mathcal{M}$ under the appropriate composability conditions.

The identity element 1_a, $a\in\mathbf{X}_n$, is, of course, defined as $[1_a](=\{1_a\})$.

When $[\vec{\varphi}]=[1_b]$, we put $a\cdot[\vec{\varphi}]=[1_{a\cdot b}]$. Similarly for $[1_b]\cdot f$.

For any molecule $\vec{\varphi}=(\varphi_1,\ldots,\varphi_\ell)$ and any $a\in\mathbf{X}_m$, $1\leq m\leq n$, such that $\mathrm{c}^{(m-1)}a=\mathrm{d}^{(m-1)}\vec{\varphi}$ $(=\mathrm{d}^{(m-1)}\varphi_i$ for any i), we can define

$$a\cdot\vec{\varphi} \overset{=}{\text{def}} (a\cdot\varphi_1,\ldots,a\cdot\vec{\varphi}) ; \tag{4}$$

$a\cdot\vec{\varphi}$ so defined is a molecule.

In fact, we can define

$$a \cdot [\vec{\varphi}] \underset{\text{def}}{=} [a \cdot \vec{\varphi}] \ . \tag{5}$$

This is because if the pair (α, β) gives rise to the quadruple $(\rho, \sigma, \varphi, \psi)$ as described above, i.e., $L(\rho, \sigma, \varphi, \psi)$ holds *via* (α, β), then, clearly, $(a \cdot \alpha, a \cdot \beta)$ gives rise to $(a \cdot \rho, a \cdot \sigma, a \cdot \varphi, a \cdot \psi)$ in the same sense, provided $a \cdot \rho$ is well-defined (equivalently, $a \cdot \varphi$ is well-defined). Thus, if $a \cdot \rho$ is well-defined and $E(\rho, \sigma, \varphi, \psi)$, then $E(a \cdot \rho, a \cdot \sigma, a \cdot \varphi, a \cdot \psi)$. As a consequence, $\vec{\varphi} \approx \vec{\psi}$ implies $a \cdot \vec{\varphi} \approx a \cdot \vec{\psi}$.

The definition of $[\vec{\varphi}] \cdot f$ is analogous.

The definition of $[\vec{\varphi}] \cdot [\vec{\psi}]$ is by concatenation: for $\vec{\varphi} = (\varphi_1, \ldots, \varphi_\ell)$ and $\vec{\psi} = (\psi_1, \ldots, \psi_p)$,

$$[\vec{\varphi}] \cdot [\vec{\psi}] \underset{\text{def}}{=} [\vec{\varphi} \cdot \vec{\psi}] = [(\varphi_1, \ldots, \varphi_\ell, \psi_1, \ldots, \psi_p)] \ ; \tag{6}$$

the assumed condition $C[\vec{\varphi}] = D[\vec{\psi}]$ is exactly the matching condition for φ_ℓ and ψ_1: we have a well-formed molecule. It is obvious that the definition is correct for equivalence classes.

We may write $\vec{\varphi} \cdot \vec{\psi}$ for the concatenation itself, and even,

$$\vec{\varphi} = \varphi_1 \cdot \ldots \cdot \varphi_\ell \ , \tag{7}$$

for the molecule $\vec{\varphi} = (\varphi_1, \ldots, \varphi_\ell)$.

It remains to define $[\vec{\varphi}] \cdot [\vec{\psi}]$ when one or both of $[\vec{\varphi}]$, $[\vec{\psi}]$ are of length 0. In this case, we treat the zero-length molecule as an empty tuple. More precisely, whenever the products are defined,

$$[1_a] \cdot [\vec{\psi}] \underset{\text{def}}{=} [\vec{\psi}] \ , \qquad [\vec{\varphi}] \cdot [1_b] \underset{\text{def}}{=} [\vec{\varphi}] \ .$$

$[1_a] \cdot [1_b]$ is defined only if $a = b$; the value is $[1_a]$.

To see that $\boldsymbol{Y}$ so defined is an $(n+1)$-category, we need to verify those instances of the "new" laws (see section 8) that involve cells of dimension equal to $n+1$; all other instances are, of course, true already in $\boldsymbol{X}$. In addition, much of the remaining laws are, in essence, inherited from $\boldsymbol{X}[\bar{U}]$.

Leaving the verification of all but the last law to the reader, let us look at the commutative law.

First, we take the case

$$(a\cdot D^{(k)}[\vec{\varphi}])\cdot(C^{(k)}a\cdot[\vec{\varphi}]) \overset{?}{=} (D^{(k)}a\cdot[\vec{\varphi}])\cdot(a\cdot C^{(k)}[\vec{\varphi}]) \tag{8}$$

under the condition

$$C^{(k-1)}a = D^{(k-1)}\vec{\varphi} \ ,$$

when one of the two variables, a , is of dimension $m\leq n$, the other, $[\vec{\varphi}]$, with $\vec{\varphi}$ a molecule as in (7), of length $\ell\geq 1$, is of dimension $n+1$. We have $k=k[a,\vec{\varphi}]<n$, and thus $D^{(k)}a=\mathrm{d}^{(k)}a \underset{\mathrm{def}}{=} \bar{\mathrm{d}}a$, and

$$D^{(k)}[\vec{\varphi}] = D^{(k)}\vec{\varphi} = \mathrm{d}^{(k)}\varphi_1 = \mathrm{d}^{(k)}\varphi_i \tag{9}$$

$$C^{(k)}[\vec{\varphi}] = C^{(k)}\vec{\varphi} = \mathrm{d}^{(k)}\varphi_\ell = \mathrm{d}^{(k)}\varphi_i \tag{10}$$

for all $i=1, \ldots, \ell$.

By (4), (5), (9) and (10), (8) reduces to the truth of

$$(a\cdot\mathrm{d}^{(k)}\varphi_i)\cdot(\mathrm{c}^{(k)}a\cdot\varphi_i) = (\mathrm{d}^{(k)}a\cdot\varphi_i)\cdot(a\cdot\mathrm{c}^{(k)}\varphi_i)$$

for every $i=1, .., \ell$; but this is an instance of commutativity in $\boldsymbol{X}[\bar{U}]$.

(We do not need to look at the case when $\vec{\varphi}$ is of length 0 : the commutative law is automatic when one of the factors is an identity.)

Secondly, note that the case of the commutative law:

$$([\alpha]\cdot D[\beta])\cdot(C[\alpha]\cdot[\beta]) = (D[\alpha]\cdot[\beta])\cdot([\alpha]\cdot C[\beta])$$

where α and β are atoms of $\boldsymbol{X}[\bar{U}]$ (thus, $[\alpha]$, $[\beta]$ are of particular types of dimension-(n+1) cells in $\boldsymbol{Y}$), is directly built into the definition of $\boldsymbol{Y}$, in the form

$$(\alpha\cdot D\beta)\cdot(C\alpha\cdot\beta) \approx (D\alpha\cdot\beta)\cdot(\alpha\cdot C\beta) \quad .$$

Thirdly, we make an observation. Suppose we have an "$(n+1)$-category" $\boldsymbol{Y}$ in which all the laws (in the sense of section 8) are known to hold, except commutativity in the case when both variables a and b in the law are of dimension $n+1$. Let's say that the pair (a, b) *is OK* when both a and b are of dimension $n+1$, we have $\mathrm{c}^{(n-1)}a=\mathrm{d}^{(n-1)}b$, and the instance of the commutative law for a and b holds. The claim is that if (a_1, b) and (a_2, b) are OK, the so is $(a_1\cdot a_2, b)$, provided $a_1\cdot a_2$ is well-defined; and the dual statement, involving elements a, b_1, b_2 in the evident way.

The proof is a simple calculation, as follows. We want to show

$$((a_1\cdot a_2)\cdot\mathrm{d}b)\cdot(\mathrm{c}(a_1\cdot a_2)\cdot b) \overset{?}{=} (\mathrm{d}(a_1\cdot a_2)\cdot b)\cdot((a_1\cdot a_2)\cdot\mathrm{c}b)$$

under the assumption that (a_1, b) and (a_2, b) are OK:

LHS = $((a_1\cdot a_2)\cdot\mathrm{d}b)\cdot(\mathrm{c}a_2\cdot b)$
$= ((a_1\cdot\mathrm{d}b)\cdot(a_2\cdot\mathrm{d}b))\cdot(\mathrm{c}a_2\cdot b)$ (distributive law)
$= (a_1\cdot\mathrm{d}b)\cdot((a_2\cdot\mathrm{d}b)\cdot(\mathrm{c}a_2\cdot b))$ (associative law)
$= (a_1\cdot\mathrm{d}b)\cdot((\mathrm{d}a_2\cdot b)\cdot(a_2\cdot\mathrm{c}b))$ ((a_2, b) is OK)
$= ((a_1\cdot\mathrm{d}b)\cdot((\mathrm{c}a_1\cdot b))\cdot(a_2\cdot\mathrm{c}b))$ (associativity; $\mathrm{d}a_2=\mathrm{c}a_1$)
$= ((\mathrm{d}a_1\cdot b)\cdot((a_1\cdot\mathrm{c}b))\cdot(a_2\cdot\mathrm{c}b)$ ((a_1, b) is OK)

$$= ((\mathrm{d}a_1 \cdot b) \cdot ((a_1 \cdot \mathrm{c}b) \cdot (a_2 \cdot \mathrm{c}b))$$
$$= (\mathrm{d}(a_1 \cdot a_2) \cdot b) \cdot ((a_1 \cdot a_2) \cdot \mathrm{c}b) = \text{RHS}$$

Another observation is that, in an " $(n+1)$-category", the pairs $(1_e, b)$, $(a, 1_f)$ are OK, provided $\dim(a)=\dim(b)=n+1$, $\dim(e)=\dim(f)=n$, $e=\mathrm{d}b$, $\mathrm{c}a=f$.

Since in the " $(n+1)$-category" $\boldsymbol{Y}$, every $(n+1)$-cell is a $\cdot$-product of (equivalence classes of) $\boldsymbol{X}[\bar{U}]$-atoms, and suitably matching pairs of atoms are OK, it follows that the commutative law in dimension $n+1$ holds generally.

This completes the proof of the fact that $\boldsymbol{Y}$ is indeed an $(n+1)$-category.

It remains to show that $\boldsymbol{Y}$ is the free extension $\boldsymbol{X}[\mathrm{U}]$. This is where we use the construction of the collapse (see the last section).

We have the inclusion map $\Gamma:\boldsymbol{X}\to\boldsymbol{Y}$. Note that, for $u\in U$, the "bare indet" $\bar{u}$ is an atom, and the equivalence class $[\bar{u}]$ is an element of $\boldsymbol{Y}_{n+1}$, with $D[\bar{u}]=\mathrm{d}u$, $C[\bar{u}]=\mathrm{c}u$, with $\mathrm{d}u$, $\mathrm{c}u$ given in the attachment of U to $\boldsymbol{X}$.

Define $\Lambda:U\to\|\boldsymbol{Y}\|$ by $\Lambda(u)=[\bar{u}]$. Then $(\boldsymbol{Y},\boldsymbol{X}\xrightarrow{\Gamma}\boldsymbol{Y},U\xrightarrow{\Lambda}\|\boldsymbol{Y}\|)$ is an extension of $\boldsymbol{X}$ by U (see section 4). We claim it is a *free* such extension.

To prove the claim, we let $(\boldsymbol{Y},\boldsymbol{X}\xrightarrow{\hat{\Gamma}}\boldsymbol{Z},U\xrightarrow{\hat{\Lambda}}\|\boldsymbol{Z}\|)$ be any extension of $\boldsymbol{X}$ by U . We want a (unique) morphism $F:\boldsymbol{Y}\to\boldsymbol{Z}$ of $(n+1)$-categories, with the diagrams

$$\begin{array}{ccc} & \overset{\Gamma}{\nearrow} & \boldsymbol{Y} \\ \boldsymbol{X} & & \downarrow F \\ & \underset{\hat{\Gamma}}{\searrow} & \boldsymbol{Z} \end{array} \qquad\qquad \begin{array}{ccc} & \overset{\Lambda}{\nearrow} & \|\boldsymbol{Y}\| \\ U & & \downarrow \|F\| \\ & \underset{\hat{\Lambda}}{\searrow} & \|\boldsymbol{Z}\| \end{array} \qquad (11)$$

commutative.

Consider the collapse $\bar{Z}$ of Z. The set $\bar{U}$ of n-dimensional indeterminates, attached to X by $\bar{u} \mapsto (\mathrm{d}^{(n-1}u, \mathrm{c}^{(n-1)}u)$ to X, gives rise to the extension

$$(\bar{Z}, X \xrightarrow{\bar{\Gamma}} \bar{Z}, \bar{U} \xrightarrow{\bar{\Lambda}} \|\bar{Z}\|) \ , \qquad (12)$$

of X by $\bar{U}$, where $\bar{\Gamma}$ is the composite $X \xrightarrow{\Gamma} Z \xrightarrow{\mathrm{incl}} \bar{Z}$, and $\bar{\Lambda}(\bar{u}) = \Lambda(u)$, the latter meant as a "new" element of $\bar{Z}_n = Z_n \dot{\cup} Z_{n+1}$, one in Z_{n+1}. (The compatibility condition involved in the notion of "extension" is satisfied by (12).)

Comparing with the initial extension $X[\bar{U}]$ of X by $\bar{U}$, we have a map $G: X[\bar{U}] \longrightarrow \bar{Z}$ of n-categories such that $G(a) = \bar{\Gamma}(a)$ $(a \in X)$ and $G(\bar{u}) = \hat{\Lambda}(u)$ ($\in Z_{n+1} \subseteq \bar{Z}_n$) for $u \in U$. Let us write $\hat{r}$ for $G(r)$ ($r \in \|X[\bar{U}]\|$).

Note that every atom φ in $X[\bar{U}]$ gets mapped by G into a "new" element, one in Z_{n+1}, of $\bar{Z}$ (since the "new" elements are closed under composition). In other words, $\hat{\varphi}$ is an $(n+1)$-cell of Z.

We continue using the simple dot $\cdot$ for composition in Z, and, if necessary, the barred dot $\bar{\cdot}$ for that in $\bar{Z}$. (Of course, the effects of the two frequently coincide.) Therefore, for $X[\bar{U}]$-atoms φ, ψ, $\hat{\varphi} \cdot \hat{\psi}$ means a composite of $(n+1)$-cells in Z.

We can extend the map G to molecules, and use the notation $(\hat{-})$ for the extension too, by the formulas

$$\hat{\vec{\varphi}} = (\varphi_1, \ldots, \varphi_\ell)^{\wedge} = \hat{\varphi}_1 \cdot \ldots \cdot \hat{\varphi}_\ell$$

(by associativity in Z, there is no need to use parentheses on the right), and

$$(1_a)^{\wedge} = 1_{\hat{a}} \ .$$

I claim that the map $(\hat{-})$ induces the required map $F:\mathbf{Y}\to\mathbf{Z}$, given as $\bar{\Gamma}$ on $\mathbf{Y}\restriction n=\mathbf{X}$, and by

$$F([\vec{\varphi}]) = \hat{\vec{\varphi}}$$

on $(n+1)$-cells of $\mathbf{Y}$.

Let us show that F is well-defined.

As before, $\alpha,\beta,\rho,\sigma,\varphi,\psi$ mean atoms in $\mathbf{X}[\bar{U}]$; $\vec{\varphi}$, $\vec{\psi}$ are molecules.

Start by noting that $\rho=\alpha\cdot D\beta$ implies $\hat{\rho}=\hat{\alpha}\cdot(D\beta)^{\wedge}$. This is because $\alpha,D\beta$ are in $\mathbf{X}[\bar{U}]$, and the mapping $(\hat{-}):\mathbf{X}[\bar{U}]\to\mathbf{Z}\restriction n$ is a map of n-categories.

The just noted fact, with three analogous ones, shows that if $L(\rho,\ \sigma,\ \varphi,\ \psi)$ via $(\alpha,\ \beta)$, then we have

$$\hat{\rho} = \hat{\alpha}\cdot(D\beta)^{\wedge}\ ; \qquad \hat{\sigma} = (C\alpha)^{\wedge}\cdot\hat{\beta}\ ; \qquad \hat{\varphi} = (D\alpha)^{\wedge}\cdot\hat{\beta}\ ; \qquad \hat{\psi} = \hat{\alpha}\cdot(C\beta)^{\wedge},$$

and as a consequence, by the commutative law in $\mathbf{Z}$, for $\hat{\alpha}$ and $\hat{\beta}$ as a and b, we have

$$\hat{\rho}\cdot\hat{\sigma} = \hat{\varphi}\cdot\hat{\psi}. \tag{13}$$

We have proved that $L(\rho,\ \sigma,\ \varphi,\ \psi)$ implies (13). This immediately gives that

$$\vec{\varphi}\approx\vec{\psi} \text{ implies } \hat{\vec{\varphi}} = \hat{\vec{\psi}},$$

which shows that F is well-defined.

Since the only part of the operations on $\mathbf{Y}$ beyond those in $\mathbf{X}[\bar{U}]$ is given by concatenation of atoms, it is clear that $F:\mathbf{Y}\to\mathbf{Z}$ is a map of $(n+1)$-categories. It is also obvious that the commutativities (11) hold true.

The uniqueness of $F:\mathbf{Y}\to\mathbf{Z}$ with the stated properties is easily seen.

This completes the proof that our construction of $\mathbf{X}[U]$ is correct.

Of course, we are interested mainly in the case when $\mathbf{X}$ is an n-computad. We have given a construction of the typical $(n+1)$-computad $\mathbf{X}[U]$ in terms of the n-computad (!) $\mathbf{X}[\bar{U}]$. The main practical conclusions about $\mathbf{X}[U]$ are as follows.

Already from section 8, we know that every $(n+1)$-cell of $\mathbf{X}[U]$ is a molecule, that is, a dot-product of $(n+1)$-dimensional U-atoms. In this section, we have learned two further things.

One is that two U-atom expressions are equal (represent the same $(n+1)$-pd) iff their collapses, obtained by replacing each indet $u\in U$ by the corresponding n-indet $\bar{u}$, are equal. Note that the collapses are n-pd's.

The other thing is a description when two molecule expressions are equal. They are equal if one can be transformed into the other by a finite series of moves, each of which "interchanges" a consecutive pair of atoms in the particular way described by the relation L .

In the next section, in the proof of the decidability of the word problem, we give more precise versions of these remarks.

10. Solution of the word problem

In this section, we prove theorem 7.(3).

The alternative definition of " ω-category" of section 8 gives rise to a syntax of words, in the same way as the original definition of section 2 gave rise to the syntax explained in section 7. There is no need to repeat the definitions for the "new" syntax; they are straight-forward variants of the ones in section 7. Essentially, all that happens is the replacement of the conditional operation $(-)\#_{(-)}(-)$ with another one, $(-)\cdot(-)$.

When the two syntaxes appear in the same context, we use dots to distinguish the "new" one from the "old" one. E.g., $\dot{W}$ is the class of all words in the "new" syntax. However, when we start dealing with the new syntax exclusively, we drop the dotting.

The proof of Proposition 8.(10) gives a translation of the two syntaxes into each other, the main features of which are summarized in the next statement.

(1) **(i)** There is a mapping $(\dot{-}):W\longrightarrow\dot{W}$ having the following properties: for all $a, b\in W$

$$\mathrm{dim}(\dot{a})=\mathrm{dim}(a)$$

$$(\mathrm{d}a)^{\cdot}=\mathrm{d}\dot{a}$$

$$(\mathrm{c}a)^{\cdot}=\mathrm{c}\dot{a}$$

$$(1_a)^{\cdot}=1_{\dot{a}}$$

$$a\in W^{e}\downarrow\iff\dot{a}\in\dot{W}^{e}\downarrow$$

$$a\approx b\iff\dot{a}\dot{\approx}\dot{b}\ .$$

(ii) There is an "inverse" map $(\overset{\circ}{-}):\dot{W}\longrightarrow W$, which is an inverse to $(\dot{-})$ up to $\approx$: $\overset{\circ}{\dot{a}}\approx a\ (a\in W\downarrow)$, $\dot{\overset{\circ}{b}}\mathrel{\dot{\approx}} b\ (b\in\dot{W}\downarrow)$.

(iii) $(\dot{-})$ restricts to a recursive function $(\dot{-}):W^e \longrightarrow W^e$.

Concerning the definition of $(\dot{-})$, we note that, for a pre-indeterminate $u \in W^0$, we keep its name for $\dot{u}$: $|\dot{u}| \underset{\text{def}}{=} |u|$.

(1) and the dotted version of 8.(3) Theorem imply the truth of the original version of 8.(3), which is our goal. We proceed to the proof of the dotted version of 8.(3).

From now on, all words are dot-words; all auxiliary concepts of the syntax (see section 7) are understood in the dotted sense; dots are suppressed (except as the operation symbol).

We single out a particular class, N , of words called *normal*; $N \subseteq W$. We are mainly interested in the well-formed normal words, the elements of the set $N{\downarrow} = N \cap W{\downarrow}$, but it will be useful to keep all of N around. $N^e \underset{\text{def}}{=} N \cap W^e$; $N{\downarrow}^e \underset{\text{def}}{=} N \cap W{\downarrow}^e$.

N is defined recursively.

$$N_{-1} = W_{-1}$$
$$N_0 = W_0$$

For $n \geq 0$, the $(n+1)$-dimensional normal words are the words of the form

$$1_a \tag{2.1}$$

where $a \in N_n$; and

$$\varphi_1 \cdot \ldots \cdot \varphi_\ell \tag{2.2}$$

where $\ell \in \mathbb{N} - \{0\}$, and each φ is of the form

$$b_n \cdot (b_{n-1} \cdot (\ldots(b_1 \cdot u \cdot e_1)\ldots) \cdot e_{n-1}) \cdot e_n \ , \tag{3}$$

where $b_i, e_i \in N_i$ $(i=1, ..., n)$, $u \in W^0_{n+1}$ (u is an $(n+1)$-dimensional pre-indeterminate) and $du, cu \in N_n$.

Note that, because of the associative law, there is no real need to use parentheses in (2), although, if pressed to be precise, we choose associating to the left.

We are going to call words like the one in (3) *pre-atoms*, ones of the form (2.1) or (2.2) *pre-molecules*; well-formed pre-atoms are called *atoms,* well-formed pre-molecules *molecules*.

There is a natural connection - not an identity - between the present terminology and that of sections 8 and 9, given by 7.(2) Proposition, or rather, its "dotted" version.

For the pre-atom φ as in (3), u is the *nucleus* of φ .

We can define a version, denoted $\circ : N \times N \to N$, of the dot-operation on normal words, resulting in normal words again, so that,

(4) For $a, b \in N$, $a \circ b$ is well-formed ($\in N\downarrow$) iff $a \cdot b$ is well-formed, and in that case, $a \circ b \approx a \cdot b$.

In fact, the relevant formulas were already used in the proof of 8.(12); nevertheless, here are the details.

The definition of $(-) \circ (-)$ is recursive. Suppose we have defined $a \circ b$ appropriately whenever $\dim(a), \dim(b) \le n$. The extension of the definition to dimensions $\le n+1$ is done in the following six clauses (4)(i) to (4)(vi).

(4)(i) For $a \in N_{\le n}$ and φ , an $(n+1)$-dimensional pre-atom as in (3), we put

$$a \circ \varphi \underset{\text{def}}{=} \hat{b}_n \cdot (\hat{b}_{n-1} \cdot (...(\hat{b}_1 \cdot u \cdot \hat{e}_1)...) \cdot \hat{e}_{n-1}) \cdot \hat{e}_n$$

where $\hat{b}_i = a \circ b_i$ for $i=n, ...k$;

$$\hat{b}_i = b_i \text{ for } i=k-1, \dots, 1 ;$$
$$\hat{e}_j = e_j \text{ for } j=1, \dots k ;$$
$$\hat{e}_j = a \circ e_j \text{ for } j=k+1, \dots n .$$

Thus, for an $(n+1)$-pre-atom φ and $a \in N_{\le n}$, $a \circ \varphi$ is an $(n+1)$-pre-atom again.

(4)(ii) For a as in (i), and for an $(n+1)$-dimensional pre-molecule as in (2),

$$a \circ (\varphi_1 \cdot \ldots \cdot \varphi_\ell) \overset{=}{\text{def}} (a \circ \varphi_1) \cdot \ldots \cdot (a \cdot \varphi_\ell) .$$

The result is a pre-molecule.

(4)(iii) For a as in (i), and $b \in N_n$,

$$a \circ 1_b \overset{=}{\text{def}} 1_{a \circ b} .$$

(4)(iv) Dually to (ii) and (iii), we define $\mu \circ b$ for μ an $(n+1)$-pre-molecule and $b \in N_{\le n}$.

(4)(v) For $(n+1)$-pre-molecules $\mu = \varphi_1 \cdot \ldots \cdot \varphi_\ell$, $\nu = \psi_1 \cdot \ldots \cdot \psi_m$,

$$\mu \circ \nu \overset{=}{\text{def}} \mu \cdot \nu$$

(more precisely, $\mu \circ \nu$ is defined to be the left-associated product $\varphi_1 \cdot \ldots \cdot \varphi_\ell \cdot \psi_1 \cdot \ldots \cdot \psi_m$).

(4)(vi) For $a, b \in N_n$, μ and ν as in (v),

$$1_a \circ \nu \overset{=}{\text{def}} \mu , \qquad \mu \circ 1_b \overset{=}{\text{def}} \mu$$

and

$$1_a \circ 1_a \overset{=}{\text{def}} 1_a .$$

(*end of definition of* $\circ : N \times N \to N$)

The proof of (4) is essentially contained in the proof of 8.(12).

(5) **(i)** The $\circ$-operation induces a "normalizing" function

$$(a \mapsto \tilde{a}) : W \longrightarrow N \,,$$

with the properties that, for all $a \in W$, we have $a \in W{\downarrow}$ iff $\tilde{a} \in N{\downarrow}$, and if $a \in W{\downarrow}$, we have $a \approx \tilde{a}$.

(ii) $(\tilde{-})$ restricts to a recursive function $(\tilde{-}) : W^e \to N^e$.

Namely,

for $a \in W_{\leq 0}$: $\tilde{a} \overset{=}{\text{def}} a$;
for $a \in W_{\geq 0}$: $(1_a)^{\sim} \overset{=}{\text{def}} 1_{\tilde{a}}$;
for $m, n \geq 1$, $a \in W_m$, $b \in W_n$: $(a \cdot b)^{\sim} \overset{=}{\text{def}} \tilde{a} \circ \tilde{b}$.

Note that the domain and codomain of normal words are, most of the time, not normal. The domain of the atom φ in (3) is

$$\mathrm{d}\varphi = b_n \cdot (b_{n-1} \cdot (\ldots (b_1 \cdot \mathrm{d}u \cdot e_1) \ldots) \cdot e_{n-1}) \cdot e_n \ .$$

On the other hand, we can always take $(\mathrm{d}\varphi)^{\sim} = \tilde{\mathrm{d}}\varphi$ as a normal replacement for $\mathrm{d}\varphi$.

(5) implies that, to prove that $W{\downarrow}^e$ and $\approx$ restricted to $W{\downarrow}^e$ are decidable, it suffices to prove that $N{\downarrow}^e$ and $\approx$ restricted to $N{\downarrow}^e$ are decidable.

To emphasize the dimensions occurring in what follows, we will write $\approx_m$ for the relation $\approx$ restricted to the set $W{\downarrow}_m$.

For proving (6), we use induction on dimension. The main tool in this proof is a "reduction", provided by the last section, of the relation $\approx_{n+1}$ to $\approx_n$. To state this reduction in rigorous terms, we restate much of the terminology and the results of the last section in the present contexts of words. The assertions made are routine translations of results of the last section, by using (the dotted version of) section 2, especially Proposition 7.(2).

Let n be a non-negative natural number.

For an atom φ as in (3), u is the *nucleus* of φ . Let φ an $(n+1)$-atom with nucleus u ; in symbols, $\varphi=\varphi[u]$. Let $\bar{u}$ be any n-indeterminate (element of $W{\downarrow}^0_n$) such that $\bar{u}$ does not occur in φ ($\bar{u}\notin \mathrm{supp}(\varphi)$), and $\mathrm{d}\bar{u}=\mathrm{d}^{(n-1)}u$, $\mathrm{c}\bar{u}=\mathrm{c}^{(n-1)}u$. Define $\varphi[\bar{u}]$ to be the n-dimensional word given as

$$\varphi[\bar{u}] = b_n\cdot(b_{n-1}\cdot(\ldots(b_1\cdot\bar{u}\cdot e_1)\ldots)\cdot e_{n-1})\cdot e_n \ .$$

I call $\varphi[\bar{u}]$ "the" n-(dimensional) *collapse* of $\varphi[u]$.

Since we assumed that φ is well-formed (an atom), the collapse is also well-formed.

The collapse of an $(n+1)$-atom is far from being an n-atom in general; nevertheless, we call the indeterminate $\bar{u}$ the *nucleus* of the collapse $\varphi[\bar{u}]$.

Although $\varphi[\bar{u}]$ depends on the choice of $\bar{u}$, this dependence is not essential. As long as $\bar{u}^1$ and $\bar{u}^2$ are two choices for $\bar{u}$, including the conditions that $\bar{u}^1, \bar{u}^2\notin \mathrm{supp}(\varphi)$, the collapses $\varphi[\bar{u}^1]$, $\varphi[\bar{u}^2]$ are isomorphic in the sense that the two words can be obtained by *renaming* from each other. Here, by "renaming" of a word a I mean setting up a bijection r between the set of all names of indets in a on the one hand, and another set on the other, and replacing in a each occurrence of a name $\xi\in\mathrm{dom}(r)$ with $r(\xi)$.

(6) Reduction Theorem, Part 1 For $(n+1)$-atoms $\varphi=\varphi[u]$, $\psi=\psi[v]$, we have that $\varphi\approx_{n+1}\psi$ only if $u=v$; and for $\varphi=\varphi[u]$, $\psi=\psi[u]$ with the same nucleus u ,

$$\varphi\approx_{n+1}\psi \iff \varphi[\bar{u}] \approx_n \psi[\bar{u}]$$

where $\varphi[\bar{u}]$, $\psi[\bar{u}]$ are n-collapses of φ and ψ , respectively, with the *same* n-indet $\bar{u}$.

For $(n+1)$-atoms ρ,σ,φ,ψ , we write $L(\rho,\sigma,\varphi,\psi)$ to mean

there exist $(n+1)$-atoms α and β such that

$$\rho \approx_{n+1} \alpha\circ\tilde{\mathrm{d}}\beta\,, \quad \sigma\approx_{n+1}\tilde{\mathrm{c}}\alpha\circ\beta\,, \quad \varphi\approx_{n+1}\tilde{\mathrm{d}}\alpha\circ\beta\,, \quad \psi\approx_{n+1}\alpha\circ\tilde{\mathrm{c}}\beta\,. \qquad (7)$$

Note that $\alpha\circ\tilde{\mathrm{d}}\beta \approx_{n+1} \alpha\cdot \mathrm{d}\beta$, and also that $\alpha\circ\tilde{\mathrm{d}}\beta$ is an $(n+1)$-*atom* (see (4)(i) above); similar statements can be made for the three other cases. Thus, in (7), both sides of the $\approx_{n+1}$-relations are $(n+1)$-atoms, and those instances of $\approx_{n+1}$ are covered by (6).

For molecules $\vec{\varphi}=\varphi_1\cdot\ ...\ \cdot\varphi_\ell$, $\vec{\psi}=\psi_1\cdot\ ...\ \cdot\psi_m$, we write $E(\vec{\varphi},\vec{\psi})$ to mean that

$\ell=m$ and there is $i\in\{1,\ ...,\ \ell-1\}$ such that
$L(\varphi_i,\varphi_{i+1},\psi_i,\psi_{i+1})$ or $L(\psi_i,\psi_{i+1},\varphi_i,\varphi_{i+1})$
and for all $j\in\{1,\ ...,\ \ell\}-\{i,i+1\}$, $\varphi_j \approx_{n+1} \psi_j$.

(8) Reduction Theorem, Part 2

(i) For $a,b\in W{\downarrow}_n$, $1_a \approx_{n+1} 1_b \iff a\approx_n b$.

(ii) For $(n+1)$-molecules $\vec{\varphi},\vec{\psi}$ of positive length, $\vec{\varphi}\approx_{n+1}\vec{\psi}$ if and only if ***there exists*** a finite sequence $\vec{\varphi}^1,\ ...,\ \vec{\varphi}^p$ of $(n+1)$-molecules such that $\vec{\varphi}=\vec{\varphi}^1$, $\vec{\psi}=\vec{\varphi}^p$, and for every $i\in\{1,\ ...,\ p-1\}$, we have $E(\vec{\varphi}^i,\vec{\varphi}^{i+1})$.

Knowing that the relation $\approx_n$ is decidable (induction hypothesis), to prove that $\approx_{n+1}$ is decidable, we have to deal with, that is, somehow *bound*, the two unbounded existential quantifiers italicized/underlined above (in (7) and (8)(ii)). This we do by using the content function of section 5, 5.(12) Proposition. The relevant facts for words are given in the following variant of 5.(12).

(9) Define the function

$$[-] : W_{\geq -1} \longrightarrow W \cdot \mathbb{Z}$$

recursively by:

(i) $[*] = 0$

(ii) $[x] = \binom{x}{1} + [dx] + [cx]$ $\quad (x \in W^0)$

(iii) $[1_a] = [a]$ $\quad (a \in W_{\geq 0})$

(iv) $[a \cdot b] = [a] + [b] - [c^{(k)} a]$ $\quad (a, b \in W_{\geq 1}, \ k = k(a, b))$.

We have

(v) $x \notin \mathrm{supp}(a) \Longrightarrow [a](x) = 0$

(vi) The function $[-]$ restricts to $[-] : W{\downarrow}_{\geq -1} \longrightarrow W{\downarrow} \cdot \mathbb{Z}$,
and $a \approx b$ implies $[a] = [b]$;
clause (iv) for the restriction becomes

$$[a \cdot b] = [a] + [b] - [a \wedge_k b] \ .$$

Moreover, we have, for $a, b, a \cdot b \in W{\downarrow}$:

(vii) $[a] \geq 0$

(viii) $[\mathrm{d}a], [\mathrm{c}a] \leq [a]$

(ix) $[a], [b] \leq [a \cdot b]$

(ix) $[a](x) > 0 \Longleftrightarrow x \in \mathrm{supp}(a)$.

Because of (9)(v), $[a]$ may and will be identified with the finite object $[a]\upharpoonright \overset{=}{\text{def}} [a]\upharpoonright\text{supp}(a)$. For $a\in N^e$, $[a]\upharpoonright\in\mathbb{HF}$ ($\mathbb{HF}$ is the set of hereditarily finite sets); and the function $(a\mapsto[a]\upharpoonright):W^e\longrightarrow\mathbb{HF}$ is recursive (once $\mathbb{Z}$ is understood as a subset of $\mathbb{HF}$ in a standard way).

Let S be the (recursive) set of finite functions s whose domain is a finite subset of $(W_{\downarrow}{}^{e})^{0}$, the set of effective indeterminates, and whose values are in $\mathbb{Z}$. We define the (recursive) partial order $s\le t$ on S by the condition that

$$
\begin{array}{ll}
s(a) \le t(a) & \text{for } a\in\text{dom}(s)\cap\text{dom}(t)\,, \\
s(a) \le 0 & \text{for } a\in\text{dom}(s)\text{-dom}(t)\,, \\
0 \le t(a) & \text{for } a\in\text{dom}(t)\text{-dom}(s)\;.
\end{array}
$$

We have $[a]\upharpoonright\le[b]\upharpoonright \iff [a]\le[b]$, the latter meant pointwise from $\mathbb{Z}$.

From now on, we write $[a]$, but actually mean $[a]\upharpoonright$.

A key point is the

(10) Finiteness Lemma For any $s\in S$, the set $N_{\downarrow}(s)$ of all well-formed normal words a for which $[a]\le s$ is finite.

In words: if we require of a *normal* word a to have indeterminates in a preassigned finite set ($\text{dom}(s)$ in the formal context), and moreover, we bound the multiplicity of each indet x by a fixed number ($s(x)$ in the formal context), then we only have finitely many possible a's.

(As a matter of fact, the same is *almost*, but not completely, true without the qualification "normal" (think of words which are long composites of the same identity cell).)

The set $N_{\downarrow}(s)$ is important; for $a\in N_{\downarrow}(s)$ we also say that "a is bounded by s".

The proof of (10) will be contained in the proof of the main theorem, 7.(3), where we need something stronger and more technical. However, the finiteness lemma shows the role of the content function clearly. Note that the $\mathtt{supp}$ function trivially fails to have the same effect: the computad whose indeterminates are the 0-cell X and the 1-cell $f:X\to X$ has infinitely many 1-pd's, namely the powers f^n $(n\in\mathbb{N})$ of f, but all (except 1_X) have the same $\mathtt{supp}$, namely $\{X, f\}$.

Proof of Theorem 7.(3).

(A) In essence, the proof is an induction on the dimension n, the induction statement being

(11)$_n$ $(N{\downarrow}^e)_{\le n}$, and $\approx$ restricted to $(N{\downarrow}^e)_{\le n}$, are decidable.

(11)$_n$ is clearly true for $n=0$.

However, it seems necessary to strengthen the induction statement.

Recall the definition of the set $S\subseteq\mathbb{HF}$ above. Let $S_{\le n}=\{s:\mathrm{dom}(s)\subseteq W{\downarrow}^0_{\le n}\}$; the elements of $S_{\le n}$ have indets in their domain that are all of dimensions $\le n$.

A further induction hypothesis is

(12)$_n$ There is a recursive function $f_{\le n}:S_{\le n}\to\mathbb{HF}$ such that, for each $s\in S_{\le n}$, $f_{\le n}(s)\subseteq N{\downarrow}(s)$, and for all $a\in N{\downarrow}(s)$, there is $b\in f_{\le n}(s)$ with $a\approx b$.

(In particular: $f_{\le n}(s)$ is a finite set of normal words, forming a complete set of representatives, possibly with repetitions, of the equivalence relation $\approx_{\le n}$ restricted to words that are bounded by s.)

Under the assumption (11)$_n$&(12)$_n$, we prove (11)$_{n+1}$&(12)$_{n+1}$. (Expressed more pedantically, our proof consists of the definition of two recursive functions,

$ch_{\approx_n} = ch_{\approx}(n, \ldots)$ and $f_n = f(n, \ldots)$, on $\mathbb{HF}$, one of whose variables is n, together with the proof that the first of these functions coincides with the characteristic function of the relation $\approx = \bigcup_n \approx_n$.)

(B) Assume $(11)_n \& (12)_n$.

The first observation is that $(W{\downarrow}^e)_{\le n+1}$ is decidable: it is decidable whether an $(n+1)$-dimensional word is well-formed. The reason is that the question of well-formedness of an $(n+1)$-dimensional word a is answered by answering questions of the well-formedness of subwords of a of dimension at most n, together with questions whether certain words (possibly repeated domains and codomains of subwords of a) of dimensions at most n are "equal", i.e. in the relation $\approx$; and all these questions are decidable by the induction hypothesis $(11)_n$.

In what follows, all words are in W^e, in fact, most of the time. in $N{\downarrow}^e$. We suppress the superscript e.

Next, we note that (8) immediately implies that the relation $\approx_{n+1}$ restricted to $(n+1)$-atoms is decidable, since it is directly reduced to the decidability of $\approx_n$. One should only point out that, given atoms $\varphi = \varphi[u]$, $\psi = \psi[u]$ with the same nucleus u, we can choose $\bar{u}$ an n-indet chosen outside $s_0 = \mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi)$ "canonically", e.g., with the least natural number as the name $|\bar{u}|$ of $\bar{u}$ that does not occur as a name of any indet in s_0, and then inquire if $\varphi[\bar{u}] \approx_n \psi[\bar{u}]$, a decidable question.

In fact, the same reduction gives the part of $(12)_{n+1}$ for atoms. More precisely: given any $s \in S_{\le n+1}$, let

$r = \mathrm{dom}(s)$;

$r_{n+1} = \{u \in r : \mathrm{dim}(u) = n+1\}$;

$\bar{r}_{n+1} = \{\bar{u} : u \in r_{n+1}\}$ where $\bar{u}$ is chosen to be an n-indet, as usual, with $\mathrm{d}\bar{u} = \mathrm{d}^{(n-1)}u$, $\mathrm{c}\bar{u} = \mathrm{c}^{(n-1)}u$; also, the map $u \mapsto \bar{u}$ is bijective; and $\bar{u} \notin r$ for all $u \in r_{n+1}$.

$\bar{r} = (r - r_{n+1}) \cup \bar{r}_{n+1}$;

$\bar{s}$ is defined as the function for which $\bar{s}(x)=s(x)$ for $x \in r - r_{n+1}$, and $\bar{s}(\bar{u})=s(u)$ for $u \in r_{n+1}$. Also, the map $s \mapsto \bar{s}$ is made recursive ("canonical"), by choosing each $\bar{u}$ such that the names $|\bar{u}|$ for $u \in r_{n+1}$ are the least possible integers not equal to the names of the indets in $r - r_{n+1}$.

The important thing to see is that

(13) if the $(n+1)$-atom $\varphi[u]$ is bounded by s, then $\bar{\varphi}=\varphi[\bar{u}]$ is bounded by $\bar{s}$.

This must be checked by a direct look at the relevant formulas. We have

$$\varphi = b_n \cdot (b_{n-1} \cdot (\ldots(b_1 \cdot u \cdot e_1)\ldots) \cdot e_{n-1}) \cdot e_n ;$$
$$\bar{\varphi} = b_n \cdot (b_{n-1} \cdot (\ldots(b_1 \cdot \bar{u} \cdot e_1)\ldots) \cdot e_{n-1}) \cdot e_n .$$

By using the formulas for $[-]$, and noting that, for $i=1, \ldots, n$,

$$b_i \wedge (b_{i-1} \cdot (\ldots(b_1 \cdot u \cdot e_1)\ldots) \cdot e_{i-1}) \cdot e_i) = b_i \wedge (b_{i-1} \cdot (\ldots(b_1 \cdot \bar{u} \cdot e_1)\ldots) \cdot e_{i-1}) \cdot e_i) ,$$

and the "dual" facts (roughly, there is no change in passing from φ to $\bar{\varphi}$ except in the innermost part), we conclude that

$$[\varphi] - [\bar{\varphi}] = [u] - [\bar{u}] \ .$$

On the other hand,

$$[u] = \binom{u}{1} + [\mathrm{d}u] + [\mathrm{c}u] ,$$

$$[\bar{u}] = \binom{\bar{u}}{1} + [\mathrm{d}\bar{u}] + [\mathrm{c}\bar{u}] = \binom{\bar{u}}{1} + [\mathrm{dd}u] + [\mathrm{cc}u] \ .$$

Since $[ddu] \le [du]$, $[ccu] \le [cu]$, we have that

$$([u]-[\bar{u}])\restriction W^0_{\le n} \ge 0 \text{, and } ([u]-[\bar{u}])\restriction W^0_{n+1} = \binom{u}{1}-\binom{\bar{u}}{1} .$$

(13) follows.

With s as before, define $A(s)$ as the set

$$A(s) = \{\varphi : \varphi=\varphi[u] \text{ an } (n+1)\text{-atom, } u \in r_{n+1} ,\ \varphi[\bar{u}] \in f_{\le n}(\bar{s})\} ,$$

where $f_{\le n}$ is given by $(12)_n$. Then, clearly, by (8) and (13), the $(12)_n$ property of $f_{\le n}(\bar{s})$ translates into the fact that, for $(n+1)$-atoms, $A(s)$ is a complete set of representatives of the equivalence $\approx_{n+1}$ restricted to $(n+1)$-atoms bounded by s.

$A(s)$ is a recursive function of s.

Next, we show that the relation $L(\rho,\sigma,\varphi,\psi)$ on $(n+1)$-atoms ρ,σ,φ,ψ is decidable. Inspect the definition of $L(\rho,\sigma,\varphi,\psi)$ in (7). Suppose we have α and β as there. By (9)(vi)&(viii)&(ix), we have $[\alpha]\le[\rho]$, $[\beta]\le[\sigma]$. Also, α and β can be replaced by any α' and β' such that $\alpha\approx_{n+1}\alpha'$, $\beta\approx_{n+1}\beta'$. It follows that, for the given ρ,σ,φ,ψ, we have that

$$L(\rho,\sigma,\varphi,\psi) \Longleftrightarrow \text{there exist } (n+1)\text{-atoms } \alpha,\beta \textit{ in } A(s) \text{ such that (7);}$$

i.e., the quantifier "there exist α,β" in (7) can be replaced by the *bounded* (in the sense of $\mathbb{HF}$) quantifier "there exist α,β *in* $A(s)$"; here $s=[\rho]\vee[\sigma]\vee[\varphi]\vee[\psi]$. Since the ingredients of the line (7) are already seen to be decidable (the reference to $\approx_{n+1}$ being restricted to arguments that are atoms), we have what we want.

It follows that the relation $E(\vec{\varphi}, \vec{\psi})$, for $(n+1)$-molecules $\vec{\varphi}$ and $\vec{\psi}$, is decidable.

(C) We complete the proof of $(11)_{n+1}$ by considering the relation $\approx_{n+1}$ for $(n+1)$-molecules.

Let $\ell \in \mathbb{N}-\{0\}$, and let s be any element of S_{n+1} (a "bound" for "contents"). Recall the set $N{\downarrow}(s)$ of well-formed words "bounded" by s. Let $N{\downarrow}_{n+1}(\ell, s)$ denote the set of all $(n+1)$-molecules of length ℓ, bounded by s.

For $\vec{\theta}=\theta_1 \cdot \ldots \cdot \theta_\ell$, $\vec{\tau}=\tau_1 \cdot \ldots \cdot \tau_\ell$ any molecules of length ℓ, let's write $\vec{\theta} \approx\approx_{n+1} \vec{\tau}$ for the condition that for each $i=1, \ldots, \ell$, we have $\theta_i \approx_{n+1} \tau_i$ (pointwise equivalence). Note that $\vec{\theta} \approx\approx_{n+1} \vec{\tau}$ implies $\vec{\theta} \approx_{n+1} \vec{\tau}$, and, hence, that $\vec{\theta} \approx\approx_{n+1} \vec{\tau}$ and $\vec{\theta} \in N{\downarrow}_{n+1}(\ell, s)$ imply $\vec{\tau} \in N{\downarrow}_{n+1}(\ell, s)$.

Further, note that if $\vec{\theta} \in N{\downarrow}_{n+1}(\ell, s)$, and we take the atom τ_i, $i=1, \ldots, \ell$, independently of each other, but such that $\theta_i \approx_{n+1} \tau_i$, then $\vec{\tau} \overline{\overline{\mathrm{def}}}\, \tau_1 \cdot \ldots \cdot \tau_\ell$ is a well-defined molecule, and $\vec{\theta} \approx\approx_{n+1} \vec{\tau}$; and hence, also $\vec{\tau} \in N{\downarrow}_{n+1}(\ell, s)$.

Let $M(\ell, s) \subseteq N{\downarrow}_{n+1}(\ell, s)$ be the finite set of all molecules $\vec{\theta}=\theta_1 \cdot \ldots \cdot \theta_\ell$ of length ℓ such that each atom θ_i belongs to the finite set $A(s)$ of $(n+1)$-atoms defined above. It follows from the previous paragraph, by the property of $A(s)$, and by the coarse estimate $[\theta_i] \leq [\vec{\theta}]$ (hence, if the molecule $\vec{\theta}$ is bounded by s, then so is every atom θ_i in it) that

(14) for each $\vec{\theta} \in N{\downarrow}_{n+1}(\ell, s)$, there is $\vec{\tau} \in M(\ell, s)$ such that $\vec{\theta} \approx\approx_{n+1} \vec{\tau}$.

Now, assume $\vec{\varphi}$, $\vec{\psi}$ are molecules; assume that $\vec{\varphi} \in N{\downarrow}_{n+1}(\ell, s)$.
Further, let $\vec{\varphi}^1, \ldots, \vec{\varphi}^p$ be a finite sequence of $(n+1)$-molecules such that

$$\vec{\varphi}=\vec{\varphi}^1, \quad \vec{\psi}=\vec{\varphi}^p, \quad \text{and } E(\vec{\varphi}^i, \vec{\varphi}^{i+1}) \text{ for every } i \in \{1, \ldots, p-1\}.$$

First off, then, for each i, $\vec{\varphi}^i \approx_{n+1} \vec{\varphi}$; thus each $\vec{\varphi}^i$, and $\vec{\psi}$ in particular, is also bounded by s, and of length equal to ℓ: each $\vec{\varphi}^i$ belongs to $N{\downarrow}_{n+1}(\ell, s)$. Let, by (14), $\vec{\theta}^2, \ldots, \vec{\theta}^{p-1}$ be molecules in $M(\ell, s)$ such that $\vec{\varphi}^i \approx\approx_{n+1} \vec{\theta}^i$ $(i=2, \ldots, p-1)$. For uniformity, let $\vec{\theta}^1=\vec{\varphi}$, $\vec{\theta}^p=\vec{\psi}$. Since, as inspection shows, the $E(-,-)$ relation is invariant under $\approx_{n+1}$, and even more so under $\approx\approx_{n+1}$, we have

$$\vec{\varphi}=\vec{\theta}^1,\ \vec{\psi}=\vec{\theta}^p, \text{ and } E(\vec{\theta}^i, \vec{\theta}^{i+1}) \text{ for every } i\in\{1, ..., p-1\}.$$

We have demonstrated the following refinement of (8):

(15) For $\vec{\varphi},\vec{\psi}\in N{\downarrow}(\ell,s)$, $\vec{\varphi}\approx_{n+1}\vec{\psi}$ iff there exists $p\in\mathbb{N}$, and $\vec{\theta}^2, ..., \vec{\theta}^{p-1}$ *in* $M(\ell, s)$ such that, with $\vec{\theta}^1=\vec{\varphi}$, $\vec{\theta}^p=\vec{\psi}$, we have $E(\vec{\theta}^i, \vec{\theta}^{i+1})$ for every $i=1, ..., p-1$.

Let $m(\ell, s)$ be the cardinality of $M(\ell, s)$. Then, obviously, by eliminating repetitions in the sequence $\vec{\theta}^2, ..., \vec{\theta}^{p-1}$, in (15) we can bound p by $p\le m(\ell,s)+2$, and get

(16) For $\vec{\varphi}, \vec{\psi}\in N{\downarrow}_{n+1}(\ell, s)$, $\vec{\varphi}\approx_{n+1}\vec{\psi}$ iff there exist $p\le m(\ell,s)+2$, and $\vec{\theta}^2, ..., \vec{\theta}^{p-1}$ *in* $M(\ell, s)$ such that, with $\vec{\theta}^1=\vec{\varphi}$, $\vec{\theta}^p=\vec{\psi}$, we have $E(\vec{\theta}^i, \vec{\theta}^{i+1})$ for every $i=1, ..., p-1$.

By what we already know, this shows that $\vec{\varphi}\approx_{n+1}\vec{\psi}$ is decidable for positive-length molecules $\vec{\varphi}, \vec{\psi}$. Removing the qualification "positive-length" is trivial.

This completes the proof of $(11)_{n+1}$.

It remains to show $(12)_{n+1}$.
Note that for a molecule $\vec{\varphi}=\varphi_1\cdot ...\cdot\varphi_\ell$, we have $\ell=\ell(\vec{\varphi})=\sum_{u\in \mathrm{supp}_{n+1}(\vec{\varphi})}[\vec{\varphi}](u)$.

Therefore, for any bound $s\in S_{n+1}$, $s\ge 0$,

(17) $\vec{\varphi}\in N{\downarrow}(s)_{n+1}$ implies that $\ell(\vec{\varphi})$ is bounded by the number

$$[\![s]\!]_{n+1} \overset{\mathrm{def}}{=} \sum_{\substack{u\in\mathrm{dom}(s)\\ \dim(u)=n+1}} s(u).$$

Recall the set $M(\ell, s)$ defined above. We put

$$f_{\leq n+1}(s) = f_{\leq n}(s) \cup \bigcup_{\ell \leq [\![s]\!]_{n+1}} M(\ell, s) .$$

By (14) and (17), we see that, for any $a \in N{\downarrow}_{n+1}(s)$, there is

$$b \in \bigcup_{\ell \leq [\![s]\!]_{n+1}} M(\ell, s) \subseteq f_{\leq n+1}(s)$$

such that $a \approx b$. Otherwise, if $a \in N{\downarrow}(s) - N{\downarrow}_{n+1}(s) = N{\downarrow}_{\leq n}(s)$, we have, by $(12)_n$, $b \in f_{\leq n}(s) \subseteq f_{\leq n+1}(s)$ with $a \approx b$.

This completes the proof of $(12)_{n+1}$ and that of Theorem 7.(3).

11. Proof of the existence of enough computopes.

We will prove theorem 6.(3). We are going to use section 5, especially the content function $[-]$, and sections 8 and 9 -- but not section 7, nor 10: we will not use "words".

The main tool will be the identity stated in 5.(12)(ix). Note that, as a special case of said identity, if $F:\boldsymbol{X}\to\boldsymbol{Y}$ is injective, then for any $a\in\|\boldsymbol{X}\|$ and $x\in|\boldsymbol{X}|$, $[Fa]_{\boldsymbol{Y}}(Fx)=[a]_{\boldsymbol{X}}(x)$.

From the content-function, we can deduce the *size*-functions: for any $m\in\mathbb{N}$, computad $\boldsymbol{X}$, and any pd $a\in\|\boldsymbol{X}\|$, we define

$$[\![a]\!]_m \overset{=}{\text{def}} [\![a]\!]_m^{\boldsymbol{X}} \overset{=}{\text{def}} \sum_{x\in|\boldsymbol{X}|_m} [a]_{\boldsymbol{X}}(x) = \sum_{x\in\mathrm{supp}_m(a)} [a]_{\boldsymbol{X}}(x) .$$

$[\![a]\!]_m$ is the total number of occurrences of m-indeterminates in a . Write $[\![a]\!]=\langle [\![a]\!]_m\rangle_m$, a vector of integers, only finitely many terms of which is non-zero.

Recall that for every $x\in\mathrm{supp}(a)$, $[a](x)\geq 1$. It follows that $\#\mathrm{supp}_m(a)\leq[\![a]\!]_m$: the number of distinct m-indets in a is bounded by $[\![a]\!]_m$.

By a *bound* I mean a vector $N=\langle N_m\rangle_{m\in\mathbb{N}}$ of integers N_m , only finitely many of which is non-zero. $\dim(N) \overset{=}{\text{def}} \max\{m:N_m\neq 0\}$. $N\upharpoonright n$ is the bound M for which $M_m=N_m$ for all $m\leq n$, and $M_m=0$ for all $m>n$.

Bounds are partially ordered by the pointwise order, denoted $\leq$. $a\in\|\boldsymbol{X}\|$ is *bounded* by N if $[\![a]\!]\leq N$.

(1) First Finiteness Lemma For any given finite computad $\boldsymbol{X}$, and any bound N , the number of pd's in $\boldsymbol{X}$ bounded by N is finite.

Proof This is a consequence of 8.(12).

The proof is by induction on $\mathrm{dim}(\boldsymbol{X})$. Let N be a bound, $\boldsymbol{X}$ a finite computad of dimension $n+1$, and assume $[\![\boldsymbol{X}]\!] \le N$. By *loc.cit.*, every $(n+1)$-pd a in $\boldsymbol{X}$ is (case 1) either 1_b for some $b \in \boldsymbol{X}_n$, or else (case 2) of the form

$$a = \varphi_1 \cdot \ldots \cdot \varphi_\ell ,$$

where

$$\varphi_i = b^i_n \cdot (b^i_{n-1} \cdot (\ldots (b^i_1 \cdot u^i \cdot e^i_1) \ldots) \cdot e^i_{n-1}) \cdot e^i_n ,$$

with b^i_m , $e^i_m \in \boldsymbol{X}_m$ and $u^i \in \boldsymbol{X}_{n+1}$. In case 1, a being bounded by N , b is also bounded by N , and the induction hypothesis (applied to $\boldsymbol{X}\restriction n$) says that there are only finitely many such b's , and therefore only finitely many such a's as well.

In case 2, a somewhat longer-stated, but similarly obvious, counting tells us that there are only a finite number of such a's that are bounded by N . Namely, if a in question is bounded by N , then:

first of all, since $\ell = [\![a]\!]_{n+1}$ (!), we have that ℓ is bounded by N_{n+1} ;

secondly, by 5.(12)(vii), each of the b^i_m , e^i_m is bounded by N , and hence, by the induction hypothesis, there are only a finite number of possibilities for these;

thirdly, since $\boldsymbol{X}$ is finite, there are a finite number ($= \#\boldsymbol{X}_{n+1}$) of possibilities for each u_i .

The last three facts clearly add up to what we want.

Let $\boldsymbol{X}$ be a finite computad. By the *size* of $\boldsymbol{X}$, $[\![\boldsymbol{X}]\!] = \langle [\![\boldsymbol{X}]\!]_m \rangle_{m \in \mathbb{N}}$, I mean the vector whose components are given by

$$[\![\boldsymbol{X}]\!]_m = \sum_{x \in |\boldsymbol{X}|_m} [\![x]\!] .$$

$\boldsymbol{X}$ is *bounded* by N if $[\![\boldsymbol{X}]\!] \le N$.

(2) Second Finiteness Lemma Given any bound N, there is a finite computad $\boldsymbol{Y}=\boldsymbol{Y}(N)$ such that: every time $\boldsymbol{X}$ is a finite computad bounded by N, there is a subcomputad $\boldsymbol{Z}$ of $\boldsymbol{Y}$ isomorphic to $\boldsymbol{X}$: $\boldsymbol{X}\cong\boldsymbol{Z}$.

Proof. By induction on $\mathrm{dim}(N)$. Suppose the assertion for n, to show it for $n+1$. Given N of dimension $n+1$, consider $N\upharpoonright n$. Let $\boldsymbol{W}=\boldsymbol{Y}(N\upharpoonright n)$. Let Φ be the set of all pairs (a, b) of pd's $a, b\in\boldsymbol{W}_n$ such that $a\|b$, and both a and b are bounded by N. By the first finiteness lemma, Φ is a finite set. For each pair $(a, b)\in F$, let $U_{(a, b)}$ be a set of cardinality equal to N_{n+1}, of $(n+1)$-indets u attached to $\boldsymbol{W}$ by $\mathrm{d}u=a$, $\mathrm{c}u=b$. Let $U=\dot{\bigcup}_{(a, b)\in F} U_{(a, b)}$. Define $\boldsymbol{Y}$ by $\boldsymbol{Y}=\boldsymbol{W}[U]$. I claim that $\boldsymbol{Y}$ works as $\boldsymbol{Y}(N)$.

Indeed, let $\boldsymbol{X}$ be any $(n+1)$-dimensional finite computad bounded by N. Consider $\boldsymbol{X}\upharpoonright n$, an n-dimensional finite computad. By the definitions, $[\![\boldsymbol{X}\upharpoonright n]\!] = [\![\boldsymbol{X}]\!]\upharpoonright n\leq N\upharpoonright n$. Therefore, there exists a subcomputad $\boldsymbol{V}$ of $\boldsymbol{W}$ such that $\boldsymbol{X}\upharpoonright n\cong\boldsymbol{V}$. Let $F:\boldsymbol{X}\upharpoonright n\xrightarrow{\cong}\boldsymbol{V}$ be an isomorphism.

Let $u\in|\boldsymbol{X}|_{n+1}$. Let $a=F(\mathrm{d}u)$, $b=F(\mathrm{c}u)$. We have that $a\|b$. Note (by a remark above) that $[\![a]\!]=[\![\mathrm{d}u]\!]$, $[\![b]\!]=[\![\mathrm{c}u]\!]$. Now, clearly, $[\![\mathrm{d}u]\!]$, $[\![\mathrm{c}u]\!]\leq[\![u]\!]\leq[\![\boldsymbol{X}]\!]\leq N$, hence, a and b are bounded by N. We conclude that the pair (a, b) belongs to the set Φ.

Let us set up an injection $G\colon|\boldsymbol{X}|_{n+1}\longrightarrow U=|\boldsymbol{Y}|_{n+1}$ such that, for every $u\in|\boldsymbol{X}|_{n+1}$, we have $G(u)\in U_{(a, b)}$, where $a=F(\mathrm{d}u)$, $b=F(\mathrm{c}u)$. This is possible since the cardinality of $|\boldsymbol{X}|_{n+1}$ is bounded by $[\![\boldsymbol{X}]\!]_{n+1}$ (since each $(n+1)$-indet contributes at least 1 to the sum which is $[\![\boldsymbol{X}]\!]_{n+1}$), hence, also by N_{n+1}, and each $U_{(a, b)}$ has cardinality N_{n+1}; thus there is enough room for the injection G.

The map F, construed as an injection $\boldsymbol{X}\upharpoonright n\longrightarrow\boldsymbol{W}=\boldsymbol{Y}\upharpoonright n$, together with the map $G\colon|\boldsymbol{X}|_{n+1}\longrightarrow U=|\boldsymbol{Y}|_{n+1}$, induces, by the universal property of $\boldsymbol{X}=(\boldsymbol{X}\upharpoonright n)[|\boldsymbol{X}|_{n+1}]$, a map $H:\boldsymbol{X}\to\boldsymbol{Y}$ of computads which is injective on indets. Its image denoted by $\mathrm{Im}(H)$, we have our required isomorphism $H:\boldsymbol{X}\xrightarrow{\cong}\mathrm{Im}(H)$ of $\boldsymbol{X}$ with a subcomputad of $\boldsymbol{Y}$.

Recall *principal* computads and, in particular, *computopes* from section 6.

Every time we have a computad $\boldsymbol{X}$ and an indeterminate $x\in|\boldsymbol{X}|$, we have the principal computad $A=\mathrm{Supp}_{\boldsymbol{X}}(x)$, a subcomputad of $\boldsymbol{X}$, such that $x=m_A$. For any $a\in\|A\|$, $[a]_A$ is the same as $[a]_{\boldsymbol{X}}$, as a consequence of our remark about "injective F's".

In what follows, A, B, ... denote principal computads.

Any principal computad A is, in particular, finite; thus, $[\![A]\!]$ is defined as above. However, in this case, another measure of size is more natural; we put $[\![A]\!]^{*} \overset{\mathrm{def}}{=} [\![m_A]\!]$. In fact, it does not really matter which one we use, because we have $[\![A]\!]^{*}_{m} \leq [\![A]\!]_{m} \leq ([\![A]\!]^{*}_{m})^{2}$. The first inequality is clear; for the second, note that $\#|A|_{m}=\#\mathrm{supp}_{m}(m_A)\leq[\![A]\!]^{*}_{m}$, so there are at most $[\![A]\!]^{*}_{m}$ summands in the sum that is $[\![A]\!]_{m}$; and each summand is at most $[\![m_A]\!]_{m} = [\![A]\!]^{*}_{m}$, by 5.(12)(x).

We will say that A is **-bounded* by N is $[\![A]\!]^{*}\leq N$. Thus, if A is *-bounded by N, then A is bounded by $N^{2}=\langle N^{2}_{m}\rangle_{m}$.

$[\![A]\!]^{*}$ is obviously invariant under isomorphism. However, something much stronger is true too.

If A and B are principal, $f:A\to B$ is a map of computads, then $|f|:|A|\to|B|$ is surjective iff $\dim(A)=\dim(B)$ iff $f(m_A)=m_B$ iff f is an epi (see 5.(9)). We have that if $f:A\to B$ is an epi, then $[\![A]\!]^{*}=[\![B]\!]^{*}$. This is clear from 5.(12)(ix):

$$[\![B]\!]^{*}_{m} = \sum_{y\in|B|_m}[m_B]_B(y) = \sum_{y\in|B|_m}[f(m_A)]_B(y) \overset{!}{=} \sum_{y\in|B|_m}\ \sum_{\substack{x\in|A|_m\\ fx=y}}[m_A]_A(x) = \sum_{x\in|A|_m}[m_A]_A(x) = [\![A]\!]^{*}_{m}.$$

where we used the quoted fact at $!$.

(3) Third Finiteness Lemma For any bound N, the set of isomorphism types of principal computads *-bounded by N is finite.

Proof. This is a direct consequence of the second finiteness lemma: by that lemma, every isomorphism type of finite computads bounded by N^2, hence every principal computad *-bounded by N, is represented by one of the finitely many principal subcomputads, each given as $\mathrm{Supp}(x)$ by a single one, x, of the finitely many indets of the finite computad $\boldsymbol{Y}(N^2)$.

(4) Theorem (=Theorem 6.(3)) Every principal computad is the specialization of a computope. In other words, for every principal computad B, there is at least one computope A, with an epimorphism $A \to B$ to B.

Proof Let B be a principal computad. Call a principal computad A for which there is an epimorphism $A \to B$ to B a *resolvent* of B. Being a resolvent of B is a property that is invariant under isomorphism. Each resolvent A of B has the same *-size as B: $[\![A]\!]^* = [\![B]\!]^*$. Therefore, by the third finiteness lemma, the isomorphism types of resolvents of B form a non-empty finite set. Let A be a resolvent such that $\#|A|$, the number of distinct indets of A, is maximal: $\#|A| \geq \#|C|$ for all resolvents C of B. Since $\#|A|$ is an isomorphism invariant of A, there are such A's. I claim that A is a computope.

Indeed, suppose that C is principal, and $C \xrightarrow{f} A$ is an epi. We have an epi $A \xrightarrow{g} B$; thus, we have an epi $C \xrightarrow{g \circ f} A$: C is a resolvent of B. But also, $|f| : |C| \to |A|$ is a surjective function, and thus, $\#|C| \geq \#|A|$. By the maximality of $\#|A|$, we must have that $\#|C| = \#|A|$. But then, the surjection $|f| : |C| \to |A|$ must be a bijection. A morphism $f: C \to A$ which becomes a bijection on indets is an isomorphism: f is an isomorphism. This is what is needed to show that A is a computope.

The proof of 6.(2) is clear from (3).

Appendix to section 1

Proof of 1.(1)

General reminders**:**

The Yoneda functor $\boldsymbol{E}\to\hat{\boldsymbol{E}}$ will be denoted by $\hat{(-)}:\boldsymbol{E}\to\hat{\boldsymbol{E}}$. $\hat{U} \underset{\mathrm{def}}{=} \boldsymbol{C}(-, U) \in \hat{\boldsymbol{C}}$; similarly for arrows.

For any category $\boldsymbol{E}$, $\hat{\boldsymbol{E}}=\mathrm{Set}^{\boldsymbol{E}^{\mathrm{op}}}$, $X\in\boldsymbol{E}$, $A\in\hat{\boldsymbol{E}}$, we have the (Yoneda) bijection

$$\frac{\hat{X}\xrightarrow{\ f\ }A}{x\in A(X)} \qquad \begin{array}{c} f \\ \downarrow \\ f_X(1_X) \end{array} \qquad \begin{array}{c} \ulcorner x \urcorner \\ \uparrow \\ x \end{array} \tag{1.1}$$

A *partial initial* (PI) object, in any category, is an object which is initial in a connected component (a full subcategory) of the category .

In $\mathrm{El}(\hat{\boldsymbol{E}})$, the objects $\tilde{X} \underset{\mathrm{def}}{=} (\hat{X}, (X, 1_X))$ ($X\in\mathrm{Ob}(\boldsymbol{E})$) are PI (the *standard* PIO's); up to iso, these are all the PIO's; every connected component of $\mathrm{El}(\hat{\boldsymbol{E}})$ has exactly one standard PIO in it.

Suppose:

$$\begin{array}{ccc} \hat{\boldsymbol{C}} & \xrightarrow[\simeq]{\ \Phi\ } & \hat{\boldsymbol{D}} \\ & \overset{\cong}{\underset{\varphi}{\Longrightarrow}} & \\ |-|_{\boldsymbol{C}} = |-| \searrow & & \swarrow |-| = |-|_{\boldsymbol{D}} \\ & \boldsymbol{S} & \end{array}$$

Want: $\overset{?}{F}: \boldsymbol{C}\xrightarrow{\ \cong\ }\boldsymbol{D}$ (*sic*!) . (1)

Reminders:

$$|A| = \coprod_{U\in\mathbf{C}} A(U) = \{(U,u): U\in\mathbf{C},\ u\in A(U)\}\ ,$$
$$|P| = \coprod_{X\in\mathbf{D}} P(X) = \{(X,x): X\in\mathbf{D},\ x\in P(X)\}\ ,$$

$$\mathrm{Ob}(\mathrm{El}(\hat{\mathbf{C}})) = \{(A\in\hat{\mathbf{C}}, a\in|A|\} = \{(A,U,u): A\in\hat{\mathbf{C}},\ U\in\mathbf{C},\ u\in A(U)\}$$
$$\mathrm{Ob}(\mathrm{El}(\hat{\mathbf{D}})) = \{(P\in\hat{\mathbf{C}}, p\in|P|\} = \{(P,X,x): P\in\hat{\mathbf{D}},\ X\in\mathbf{D},\ x\in P(X)\}$$

$$\varphi_A: |A| \xrightarrow{\cong} |\Phi A|\ ; \tag{2}$$

Definition of Ψ :

$$\begin{array}{ccc} \mathrm{El}(\hat{\mathbf{C}}) & \xrightarrow{\Psi} & \mathrm{El}(\hat{\mathbf{D}}) \\ (A,U,u) & \longmapsto & (\Phi A, \varphi_A((U,u))) \end{array} \tag{3}$$

Claim: Ψ is an equivalence. Easy.

It follows that Ψ sends a partial initial object to a partial initial object.

To define F (see (1)) on objects, let $U\in\mathbf{C}$. Take the standard PIO $\tilde{\hat{U}}$ in $\mathrm{El}(\hat{\mathbf{C}})$. $\Psi(\tilde{\hat{U}})$ is a PIO in $\mathrm{El}(\hat{\mathbf{D}})$. There is a unique standard PIO $\tilde{\hat{X}}$, $X\in\mathbf{D}$, such that $\tilde{\hat{X}} \cong \Psi(\tilde{\hat{U}})$; denote X by $F(U)$.

The mapping $F:\mathrm{Ob}(\mathbf{C}) \longrightarrow \mathrm{Ob}(\mathbf{D})$ is a bijection.

For every U , we have a unique isomorphism $(FU)^{\hat{}\,\tilde{}} \xrightarrow{\cong} \Psi(\tilde{\hat{U}})$ in $\mathrm{El}(\hat{\mathbf{D}})$, which is, in particular, an isomorphism in $\hat{\mathbf{D}}$:

$$\psi_U: (FU)^{\wedge} \xrightarrow{\cong} \Phi(\hat{U})\ .$$

We define the effect of F on arrows. Let $f:U\to V$. We have $\Phi(\hat{f}):\Phi\hat{U} \longrightarrow \Phi\hat{V}$. Using that

Yoneda is full and faithful, there is a unique arrow $Ff : FU \longrightarrow FV$ making this commute:

$$\begin{array}{ccc} F\hat{U} & \xrightarrow[\cong]{\psi_U} & \Phi\hat{U} \\ {\scriptstyle F\hat{f}}\downarrow & & \downarrow{\scriptstyle \Phi\hat{f}} \\ F\hat{V} & \xrightarrow[\psi_V]{\cong} & \Phi\hat{V} \end{array} .$$

Using Yoneda again, and that Φ is full and faithful, it follows that the induced mapping

$$F : \boldsymbol{C}(U, V) \longrightarrow \boldsymbol{D}(FU, FV)$$

is a bijection. It is clear that $F : \boldsymbol{C} \to \boldsymbol{D}$ is a functor, and thus an isomorphism of $\boldsymbol{C}$ and $\boldsymbol{D}$.

Proof of Proposition 1.(2)

Let us pick an initial object (U, u) in each connected component of $\mathrm{El}(\boldsymbol{A})$; let $\mathcal{U}$ be the set of all selected elements (U, u). Define the category $\boldsymbol{C}$ as follows. $\mathrm{Ob}(\boldsymbol{C}) \underset{\mathrm{def}}{=} \mathcal{U}$; an arrow $(U, u) \longrightarrow (V, v)$ is the same as an arrow $U \longrightarrow V$ in $\boldsymbol{A}$, with the obvious composition structure. We will show the assertion with $\boldsymbol{C}$ thus defined.

We have the canonical functor, a coproduct of representables,

$$F = \coprod_{(U, u) \in \mathcal{U}} \boldsymbol{A}(U, -) : \boldsymbol{A} \longrightarrow \mathrm{Set} . \tag{4}$$

The main assumption of the proposition, or rather, the choice of the set $\mathcal{U}$ made possible by that assumption, is precisely that F is isomorphic to $|-| = |-|_{\boldsymbol{A}}$, In fact, we have the natural transformation $\varphi : F \longrightarrow |-|$ for which $\varphi_A : \coprod_{(U, u) \in \mathcal{U}} \boldsymbol{A}(U, A) \longrightarrow |A|$ is defined by $\varphi_A((U, u), U \xrightarrow{f} A) = |f|(u) \in |A|$; said assumption is precisely the fact that φ_A is a bijection for every $A \in \mathrm{Ob}(\boldsymbol{A})$. We have the isomorphism

$$\varphi : F \xrightarrow{\cong} |-|_{\boldsymbol{A}} . \tag{5}$$

We have a full and faithful functor $\mathbf{C}\to\mathbf{A}$, which, however, is not an inclusion. Let $\mathbf{D}$ be the image of this functor; $\mathbf{D}$ is the full subcategory of $\mathbf{A}$ with objects all U such that $(U,u)\in\mathcal{U}$ for some u. Let's write $E:\mathbf{C}\to\mathbf{D}$ for the obvious equivalence $[(U,u)\mapsto U]$, and $\iota:\mathbf{D}\to\mathbf{A}$ for the inclusion. Accordingly, we have the functors

$$\mathbf{A}\xrightarrow{\ I\ }\hat{\mathbf{D}}\xrightarrow{\ E^{*}\ }\hat{\mathbf{C}}\,, \tag{6}$$

where I maps A to $\hat{A}\circ\iota$ (remember that $\hat{A}=\mathbf{A}(-,A)$), and $E^{*}(X)=X\circ E^{\mathrm{op}}$ ($X\in\hat{\mathbf{D}}$). Their composite is $G:\mathbf{A}\to\hat{\mathbf{C}}$. G maps A to the functor $[(U,u)\longmapsto\mathbf{A}(U,A)]$.

Consider the triangle

$$\begin{array}{ccc}\mathbf{A} & \xrightarrow{\quad G\quad} & \hat{\mathbf{C}}\\ & {\scriptstyle|-|_{\mathbf{A}}}\searrow\quad\swarrow{\scriptstyle|-|_{\hat{\mathbf{C}}}} & \\ & \mathrm{Set} & \end{array} \tag{7}$$

Inspection shows that the composite functor $|-|_{\hat{\mathbf{C}}}\circ G:\mathbf{A}\to\mathrm{Set}$ is identical to F (see (2)). Thus, (7) commutes up isomorphism (see (5)).

It remains to show that G is an equivalence, or equivalently, that I (see (6)) is an equivalence. The proof depends on two claims.

Claim 1 For each $(U,u)\in\mathcal{U}$, U is an *atom*, in the sense that $\mathbf{A}(U,-):\mathbf{A}\to\mathrm{Set}$ preserves (small) colimits.

As we know (see (5)), the functor $F=\coprod_{(U,u)\in\mathcal{U}}\mathbf{A}(U,-):\mathbf{A}\longrightarrow\mathrm{Set}$ is isomorphic to the forgetful functor $|-|_{\mathbf{A}}$, which is assumed to preserve colimits. Any coproduct of functors $\coprod_i F_i:\mathbf{A}\to\mathrm{Set}$ preserves colimits iff each component F_i does (as a consequence of the fact that a coproduct of a set of arrows f_i in Set is an isomorphism iff each f_i is an isomorphism). The claim follows.

Claim 2 The inclusion $\iota:\boldsymbol{D}\to\boldsymbol{A}$ is dense. That is: let $A\in\boldsymbol{A}$; we have the slice category $\boldsymbol{C}\downarrow A$ with the forgetful functor $\Phi=[(U\to A)\mapsto U]:\boldsymbol{C}\downarrow A\to\boldsymbol{A}$, and the cocone φ with vertex A on Φ for which $\varphi_{U\xrightarrow{f}A}\overset{\text{def}}{=}f:U\to A$. We claim that φ is a colimit cocone.

To ward against confusion, we will use square brackets, such as in $[V,f:V\to A]$, when denoting objects $\boldsymbol{C}\downarrow\boldsymbol{A}$, to distinguish them from elements (A,a) of $\mathrm{El}(\boldsymbol{A})$.

By our assumptions, it is enough to show that the assertion becomes true after applying the functor $|\text{-}|$. Accordingly, let $\langle\psi_{[V,f:V\to A]}:|V|\longrightarrow S\rangle_{[V,f]\in\boldsymbol{C}\downarrow A}$ be a cocone on $|\Phi|$, to show that there is unique $t:|A|\to S$ with

$$t\circ|f|=\psi_{[V,f]} \tag{?8}$$

for all $[V,f]\in\boldsymbol{C}\downarrow A$. To define t, let $a\in|A|$. Let $(U,u)\in\mathcal{U}$ be the initial object of the component of $\mathrm{El}(\boldsymbol{A})$ containing (A,a); we have an arrow $e:(U,u)\to(A,a)$ in $\mathrm{El}(\boldsymbol{A})$. Define

$$t(a)=\psi_{(U,e)}(u)\ . \tag{9}$$

To show (8), let $x\in|V|$ be any element; we want

$$(t\circ|f|)(x)=\psi_{[V,f]}(x)\ . \tag{10}$$

Put $a=|f|(x)\in|A|$. We have (9) with suitable e. Because of the presence of the arrow $f:(V,x)\to(A,a)$ in $\mathrm{El}(\boldsymbol{A})$, the element (V,x) is in the same component of $\mathrm{El}(\boldsymbol{A})$ as (A,a). (U,u) is initial in that component, so there is $g:(U,u)\to(V,x)$. But then $(U,u)\underset{f\circ g}{\overset{e}{\rightrightarrows}}(A,a)$; since (U,u) is initial, $e=f\circ g$. Since ψ is a cocone, $\psi_{[V,f]}\circ g=\psi_{[U,e]}$. Finally,

$$(t\circ|f|)(x)=t(a)=\psi_{[U,e]}(u)=\psi_{[V,f]}(|g|(u))=\psi_{[V,f]}(x)$$

as desired for (10).

The uniqueness of t is clear. We have proved Claim 2.

The fact that the functor $I = \mathbf{A}(-_{\mathbf{D}}, -_{\mathbf{A}}) : \mathbf{A} \to \hat{\mathbf{D}} = \mathrm{Set}^{\mathbf{D}^{\mathrm{op}}}$ is full and faithful is a direct consequence of the density of $\mathbf{D}$ in $\mathbf{A}$ (Claim 2). It remains to show that I is essentially surjective on objects.

Let $X \in \hat{\mathbf{D}}$. $\mathrm{el}(X)$ is, by definition, the category of elements of X, with objects the pairs (U, x) with $U \in \mathbf{D}$, $x \in X(U)$; an arrow $(U, x) \to (V, y)$ is an arrow $U \xrightarrow{f} V$ such that $(Ff)(y) = x$. We have the forgetful functor $\mathrm{el}(X) \longrightarrow \mathbf{D}$, which, composed with the inclusion $\mathbf{D} \xrightarrow{\iota} \mathbf{A}$, gives the diagram $\Phi : \mathrm{el}(X) \longrightarrow \mathbf{A}$. We define $A = \mathrm{colim}(\Phi)$, and prove that $X \cong I(A)$.

Calculating $I(A)$, we obtain

$$I(A) = \mathbf{A}(-_{\mathbf{D}}, \operatorname*{colim}_{(U, x) \in \mathrm{el}(X)} U) \cong \operatorname*{colim}_{(U, x) \in \mathrm{el}(X)} \mathbf{A}(-_{\mathbf{D}}, U) = \operatorname*{colim}_{(U, x) \in \mathrm{el}(X)} \mathbf{D}(-, U),$$

where the indicated isomorphism is a consequence of Claim 1. It is classic (a form of Yoneda's lemma) that the last colimit is isomorphic to X itself.

The proof of 1.(2) is complete.

Appendix to section 4.

Proof of 4.(2): A *magma* is defined like an ω-category except that we do not require the last four laws: unit laws, associativity and interchange, in the definition of ω-category. In particular, a magma has dimension, domain, codomain, identity and composition operations, the latter being defined under the usual domain/codomain conditions, and the operations are required to satisfy the domain/codomain laws.

First, we show that the free magma-extension $W =$ " $\mathbf{X}_{\mathrm{magma}}[U]$ " of $\mathbf{X}$ by U satisfies the following (we abbreviate W for $|W|$, $\mathbf{X}$ for $|\mathbf{X}|$):

(i) The canonical map $"\Gamma":\boldsymbol{X}\to W$ is monic; we take it to be an inclusion.

(ii) The canonical map $"\Lambda":U\to W$ factors through $W\text{-}\boldsymbol{X}$.

(iii) $1_a\in\boldsymbol{X}$ iff $a\in\boldsymbol{X}$.

(iv) For any $0\le k<n$, the partial composition $(w_1, w_2)\longmapsto w_1\#_k w_2$ maps the set $\boldsymbol{X}\times\boldsymbol{X}$ into $\boldsymbol{X}$ and the set $(W\times W)\text{-}(\boldsymbol{X}\times\boldsymbol{X})$ into $W\text{-}\boldsymbol{X}$. In other words, if $a\#_k b$ is well-defined, then $a\#_k b \in \boldsymbol{X}$ iff both a and b belong to $\boldsymbol{X}$.

The proof of this is straightforward; we can construct $|W|$ as a set of words satisfying (i) to (iv), without having to make any identifications.

Next, we observe that imposing on W the four identities mentioned above, in order to turn it into an ω-category (which will be $\boldsymbol{X}[U]$), we never have to identify an element of $\boldsymbol{X}$ with an element of $W\text{-}\boldsymbol{X}$. Looking at an instance of any one of the four identities, we see that if on either side, the expression is in $\boldsymbol{X}$, then the one on the other is also in $\boldsymbol{X}$. Take, for instance, the left unit law: $1_a^{(n)}\#_k b = b$, where $a=\mathrm{d}^{(k)}b$. If $b\in\boldsymbol{X}$, then $a\in\boldsymbol{X}$, and $1_a^{(n)}\in\boldsymbol{X}$, thus $1_a^{(n)}\#_k b\in\boldsymbol{X}$. This shows that if the right-hand side is in $\boldsymbol{X}$, so is the left-hand side. The converse is even more obvious. The same thing is true for the other laws, in a similarly more obvious manner.

Define the subclasses C_1 to C_4 of $(W\text{-}\boldsymbol{X})\times(W\text{-}\boldsymbol{X})$ as follows. They correspond to the four ω-category identities. Into C_i $(i\in\{1, 2, 3, 4\})$, we take up the pair $(w_1 . w_2)$ iff w_1 is an element of $W\text{-}\boldsymbol{X}$ on one side of an instance of identity number-i , and w_2 is the corresponding element, necessarily in $W\text{-}\boldsymbol{X}$, on the other side.

We can define an equivalence relation $\approx$ on $W\text{-}\boldsymbol{X}$ as the least equivalence on $W\text{-}\boldsymbol{X}$ that contains the classes C_1 to C_4 , and satisfies

$$a\approx b \implies 1_a\approx 1_b \, , \qquad (1)$$

$$a\#_k b\downarrow \;\&\; a\approx a' \;\&\; b\approx b' \;\&\; a'\#_k b'\downarrow \implies a\#_k b \approx a'\#_k b' \qquad (2)$$

The logical forms of the conditions tell us that such least $\approx$ exist.

Next, we show easily, "by induction", that we have

$$a\approx b \implies a\| b \qquad (3)$$

and as a consequence, we have

$$a\#_k b\downarrow \ \&\ a\approx a' \ \&\ b\approx b' \implies a'\#_k b'\downarrow \ \&\ a\#_k b\approx a'\#_k b' \tag{4}$$

which is a strengthening of (2). Finally, we extend $\approx$ to $W=(W-\mathbf{X})\cup\mathbf{X}$ by saying that, for $w_1, w_2\in W$, $w_1\approx w_2$ iff *either* $w_1, w_2\in W-\mathbf{X}$ and $w_1\approx w_2$ in the original sense, *or* $w_1=w_2\in\mathbf{X}$. It follows that $\approx$ is an equivalence on W that contains C_1 to C_4, and satisfies (1), (3) and (4). We can define the ω-category whose underlying set is $W/\approx$, the set of all equivalence classes of $\approx$ on W, by the method of representatives. It is easy to see that, with the *inclusion* $\mathbf{X}\longrightarrow W/\approx$, it has the universal property of the free ω-category extension $\Gamma:\mathbf{X}\to\mathbf{X}[U]$.

The assertions in 4.(2) are now clear.

Proof of 4.(8). Here is a complete construction of $\mathbf{X}[U]$ geared towards proving the lemma.

We define what we mean by a *word* w, and what are $\mathrm{d}w$, $\mathrm{c}w$; the latter two entities are elements of X_{n-1}, and they are parallel.

The words come in three disjoint sets, W_0, W_1 and W_2; the set W of all words is $W=W_0\dot\cup W_1\dot\cup W_2$.

W_0 consists of all the expressions of the form $\hat{\mathrm{id}}_x$ where $x\in X_{n-1}$ (identity words); $\mathrm{d}(\hat{\mathrm{id}}_x)\overset{=}{\mathrm{def}}\mathrm{c}(\hat{\mathrm{id}}_x)\overset{=}{\mathrm{def}}x$.

For $k<n$, a word $v\in W_0$ is a k-to-n *identity* if $v=\hat{\mathrm{id}}_x$ for a k-to-n-1 identity $x\in X_{n-1}$ in $\mathbf{X}$.

W_1 equals U, the given set of indeterminates; $\mathrm{d}u$ and $\mathrm{c}u$ are as given in the attachment of U to $\mathbf{X}$.

W_2 is defined inductively: for words v, w and $k<n$ such that $\mathrm{c}^{(k)}v=\mathrm{d}^{(k)}w$ [here, $\mathrm{d}^{(k)}v\overset{=}{\mathrm{def}}\mathrm{d}^{(k-1)}\mathrm{d}v$, and similarly for c],

the formal expression $v\hat{\#}_k w$ belongs to W_2 iff

either $v \in W_1 \cup W_2$, or $w \in W_1 \cup W_2$;

and

neither v nor w is a k-to-n identity.

Moreover, $d(v\hat{\#}_k w) \overset{=}{\text{def}} dv$ if $k=n-1$, and $d(v\hat{\#}_k w) \overset{=}{\text{def}} dv\#_k dw$ when $k<n-1$, and similarly for c . We do have that $d(v\hat{\#}_k w) \| c(v\hat{\#}_k w)$.

In other words, for words v and w , the formal expression $v\hat{\#}_k w$ is well-defined as a word, and belongs to W_2 provided v and w are " k-composable", and it is not the case that both v and w are identity words (in W_0), and, moreover, even if one of them is an identity word, it is not a k-to-n identity word.

We write $v\|w$ for $dv=dw$ & $cv=cw$.

It is understood, of course, that two words are equal iff they are formally identical; e.g., $v\hat{\#}_k w = v'\hat{\#}_\ell w'$ only if $k=\ell$, $v=v'$ and $w=w'$.

The words in W_2 are characterized among all words by the fact that they are of the form $v\hat{\#}_k w$ for words v and w .

We give a "fully defined" composition of words: a word $v\#_k w$ whenever $v, w \in W$, $k<n$ and $c^{(k)}v=d^{(k)}w$.

$v\#_k w \overset{=}{\text{def}} v\hat{\#}_k w$ whenever $v\hat{\#}_k w$ is well-defined.

If $v=\hat{\text{id}}_x$, $w=\hat{\text{id}}_y$, both from W_0 , then $v\#_k w \overset{=}{\text{def}} \hat{\text{id}}_x=\hat{\text{id}}_y=v=w$ when $k=n-1$ (and $x=y$), and $v\#_k w \overset{=}{\text{def}} \hat{\text{id}}_{x\#_k y}$ when $k<n-1$.

If v is a k-to-n identity, $v\#_k w \overset{=}{\text{def}} w$;

If w is a k-to-n identity, $v\#_k w \overset{=}{\text{def}} v$.

In the instances when two of the last three clauses apply, the definitions give the same result.

Notice that we always have that $d(v\#_k w)=dv$ when $k=n-1$, and $d(v\#_k w)=dv\#_k dw$

when $k<n-1$, and similarly for c .

Note that if either v or w is in W_2 , and $v\#_k w$ is well-defined, then $v\#_k w \in W_2$; and if either v or w is in W_1 , and $v\#_k w$ is well-defined, then $v\#_k w \in W_1 \cup W_2$.

Let us show the following version of (1.1):

(1.1)* Whenever $\ell<n$, $k<n$, $v, w\in W$, and $v\#_\ell w$ is well-defined, if $v\#_\ell w$ is a k-to-n identity, then both v and w are k-to-n identities.

Assume the hypotheses, including that $v\#_\ell w$ is a k-to-n identity. First of all, we must have that v and w are in W_0 , $v=\hat{\mathrm{id}}_x$ and $w=\hat{\mathrm{id}}_y$, $x, y\in X_{n-1}$. When $\ell=n-1$, we have $v=w=v\#_\ell w$, and the assertion is clear. When $\ell<n-1$, we have $v\#_\ell w = \hat{\mathrm{id}}_{x\#_\ell y}$; this being a k-to-n identity, $x\#_\ell y$ is a k-to-$n-1$ identity in $\boldsymbol{X}$; therefore, by (C_{n-1}) for $\boldsymbol{X}$, x and y are both k-to-$n-1$ identities in $\boldsymbol{X}$, and the assertion follows.

By $W_2\times_{\langle \mathrm{d},\mathrm{c}\rangle}W_2$, we mean the pullback $\{(v, w)\in W_2\times W_2 : v\|w\}$.

We define the relation $\approx$, an equivalence relation on W_2 , as the least relation $\approx\,\subset W_2\times_{\langle \mathrm{d},\mathrm{c}\rangle}W_2$ satisfying (i) to (iv) below; the variables v, v', w, w' range over W_2 , v_1, v_2, v_3, v_4 over W .

(i) $\approx$ is an equivalence.

(ii) $v\approx v'$ and $w\approx w'$ imply $v\#_k w \approx v'\#_k w'$ provided one of the composites, hence both, are well-defined.

(iii) $v\approx v'$ and $u\in W_0\cup W_1$ imply $v\#_k u \approx v'\#_k u$ provided one of the composite words is (hence both of them are) well-defined.

(iv) $v\approx v'$ and $u\in W_0\cup W_1$ imply $u\#_k v \approx u\#_k v'$ provided one of the composite words is (hence both of them are) well-defined.

(v) If v and w are in W_2 , and

either $v=(v_1\#_k v_2)\#_k v_3$ and $w=v_1\#_k(v_2\#_k v_3)$ for some $k<n$ and $v_1, v_2, v_3 \in W$,

or $v=(v_1\#_k v_2)\#_\ell(v_3\#_k v_4)$ and $w=(v_1\#_\ell v_3)\#_k(v_2\#_\ell v_4)$ for some $\ell<k<n$ and $v_1, v_2, v_3, v_4 \in W$,
then $v\approx w$.

One should note that each closure condition generates pairs in $\approx$ that are in the set $W_2 \times_{\langle d,c\rangle} W_2$.

We extend the equivalence $\approx$ to $\approx^*$ on the whole of W by declaring that for $v, w\in W$, $v\approx^* w$ iff either $v=w$, or $v, w\in W_2$ and $v\approx w$.

We **claim** that the conditions (i) to (iv) remain true for $\approx^*$. More precisely,

(i)* $\approx^*$ is an equivalence on W.

(ii)* $v, v', w, w'\in W$, $v\approx^* v'$ and $w\approx^* w'$ imply $v\#_k w \approx^* v'\#_k w'$ provided one of the composites, hence both, are well-defined.

(iv)* If $v, w\in W$, and

either **(a)** $v=(v_1\#_k v_2)\#_k v_3$ and $w=v_1\#_k(v_2\#_k v_3)$ for some $k<n$ and $v_1, v_2, v_3 \in W$,

or **(b)** $v=(v_1\#_k v_2)\#_\ell(v_3\#_k v_4)$ and $w=(v_1\#_\ell v_3)\#_k(v_2\#_\ell v_4)$ for some $\ell<k<n$ and $v_1, v_2, v_3, v_4 \in W$,

then $v\approx^* w$.

(What would be (iii)* is subsumed under (ii)*.)

(i)* is true.

For (ii)*, assume the hypotheses. We need to show that if $v\#_k w$ is in $W_0\cup W_1$, then $v'\#_k w' = v\#_k w$. By inspecting the definition of $v\#_k w$, we see that $v\#_k w\in W_0$ only if both v and w are in W_0, in which case $v'=v$ and $w'=w$, and the desired conclusion is reached; and similarly if $v\#_k w\in W_1$.

For (iv)*: the case (a) is similar and simpler than (b); we discuss (b) only; assume the hypotheses in (b).

If v or w is in W_0, then all of v_1, v_2, v_3, v_4 must be in W_0, and we have $v=w$.

If v or w is in W_1, that is, $v=u$ or $w=u$ for some $u\in U$, then, first of all, clearly, one of v_1, v_2, v_3, v_4 must be equal to u, and the others must be elements of W_0.

Assume $v=u\in U$. Suppose, e.g., $v_1=u$. $(u\#_k v_2)\#_\ell(v_3\#_k v_4) = u$ implies, on the one hand, that $u\#_k v_2=u$ and thus that v_2 is a k-to-n identity; on the other hand, that $v_3\#_k v_4$ is an ℓ-to-n identity. By (C_{n-1}) being true for $\boldsymbol{X}$, it follows that both v_3 and v_4 are ℓ-to-n identities. It follows that

$$w = (u\#_\ell v_3)\#_k(v_2\#_\ell v_4) = u\#_k v_2 = u\ .$$

The other cases: $v_i=u$ ($i\in\{2.3.4\}$) are similar. Also, the argument is similar when we start with $w=u\in U$.

The (equivalence) class of the word $w\in W$ under $\approx$ is denoted by $[w]$. Of course, the class $[v]$ for $v\in W_0\dot{\cup}W_1$ is $[v]=\{v\}$.

We define the n-category $\boldsymbol{Y}$ as follows. The $(n-1)$-truncation of $\boldsymbol{Y}$ is $\boldsymbol{X}$.

The set of n-cells of $\boldsymbol{Y}$ is $Y_n \overset{=}{\underset{\text{def}}{}} W/{\approx} = \{[w]:w\in W\}$.

We put, for $w\in W$, $\mathrm{d}[w]=\mathrm{d}w$, $\mathrm{c}[w]=\mathrm{c}w$; these are clearly well-defined.

The identity n-cells are given by the elements of W_1: for $x\in X_{n-1}$, $\mathrm{id}_x = [\hat{\mathrm{id}}_x]$.

For the composition of n-cells of $\boldsymbol{Y}$, we put

$$[v]\#_k[w] \underset{\text{def}}{=} [v\#_k w] \qquad (k<n\,,\ v, w\in W\,,\ \mathrm{c}^{(k)}v=\mathrm{d}^{(k)}w)$$

The well-definedness of composition is assured by (ii)*.

The domain/codomain laws were effectively pointed out above.

The identity laws, one for each $k<n$, concerning composition of n-cells, holds on the level of words already.

The remaining laws: associativity and middle exchange in dimension n are true as a consequence of (iv)*.

We have the obvious inclusion maps $\Gamma:\boldsymbol{X}\longrightarrow\boldsymbol{Y}$, $\Lambda:U\longrightarrow Y_n$; $(\boldsymbol{Y},\Gamma,\Lambda)$ is an object of $\boldsymbol{A}=\boldsymbol{A}[\boldsymbol{X},U,d,c]$. We claim that $(\boldsymbol{Y},\Gamma,\Lambda)$ is in fact an initial object of $\boldsymbol{A}$. This is verified by inspection; intuitively, we did not generate elements of $\boldsymbol{Y}$, and we did not make identifications between them, unless it was so dictated by the ω-category laws.

(1.1) holds by (1.1)*. The assertions concerning Γ and Λ are true directly by the construction. This completes the proof.

Appendix to section 5

Proof of 5.(3).

We have $n\mathtt{Comp}$, the category of n-computads, the n-truncated version of $\mathtt{Comp}$; its morphisms map indets to indets. $n\mathtt{Comp}$ has a non-full inclusion into $n\mathtt{Cat}$, and the forgetful functor $\boldsymbol{X} \mapsto |\boldsymbol{X}| = \coprod_{i=0}^{n} |\boldsymbol{X}|_i$ into $\mathtt{Set}$:

$$\mathtt{Set} \xleftarrow{|\text{-}|} n\mathtt{Comp} \rightarrowtail n\mathtt{Cat} \qquad (1)$$

By induction on n, we prove that $n\mathtt{Comp}$ has all (small) colimits, and the two functors in (1) preserve them. For $n{=}0$, this is right.

Assume the assertion true for n, to show it for $n{+}1$.

Let $(n{+}1)\mathcal{F}$ be the category of $(n{+}1)$-frames, the obvious $(n{+}1)$-truncated version of $\mathcal{F}$ of section 4 . We have, as before, the pair of adjoint functors:

$$(n{+}1)\mathcal{F} \underset{\mathcal{E}}{\overset{\tau}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} (n{+}1)\mathtt{Cat}\ .$$

Let $[n{+}1]\mathcal{F}$ be the non-full subcategory of $(n{+}1)\mathcal{F}$ with objects $(\boldsymbol{X};U)$ where $\boldsymbol{X}$ is an n-computad, and morphisms

$$(\boldsymbol{X} \xrightarrow{\Gamma} \boldsymbol{Y},\ U \xrightarrow{\Lambda} V) : (\boldsymbol{X};U) \longrightarrow (\boldsymbol{Y};V)\ .$$

in which Γ is a map of n-computads. The category $[n{+}1]\mathcal{F}$ is obtained by the same simple construction from the category $n\mathtt{Comp}$ as $(n{+}1)\mathcal{F}$ from $n\mathtt{Cat}$. We have the pair of forgetful functors

$$\begin{array}{ccccc} \mathtt{Set} & \longleftarrow & (n{+}1)\mathcal{F} & \longrightarrow & n\mathtt{Cat} \\ U & \longleftarrow\!\shortmid & (\boldsymbol{X},U) & \longmapsto & \boldsymbol{X} \end{array}$$

restricting to the ones in

$$\mathtt{Set} \longleftarrow [n{+}1]\mathcal{F} \longrightarrow n\mathtt{Comp}\ ,$$

and we have the combined diagram

$$\begin{array}{ccccc} \mathtt{Set} & \longleftarrow & [n{+}1]\mathcal{F} & \longrightarrow & n\mathtt{Comp} \\ \| & & \downarrow & & \downarrow \\ \mathtt{Set} & \longleftarrow & (n{+}1)\mathcal{F} & \longrightarrow & n\mathtt{Cat} \end{array} \qquad (2)$$

Both rows in (2) create colimits in their middle object, in the precise sense of "creation", and the middle vertical inclusion preserves those created colimits, by the induction assumption that the right vertical preserves colimits.

[A pair $\boldsymbol{A} \longleftarrow \boldsymbol{B} \longrightarrow \boldsymbol{C}$ of functors *creates colimits* if the following holds. Given a diagram Δ in $\boldsymbol{B}$, we take its images $\Delta_{\boldsymbol{A}}, \Delta_{\boldsymbol{C}}$ in $\boldsymbol{A}$ and $\boldsymbol{C}$. *Assume* we have found colimit diagrams $\Delta^*_{\boldsymbol{A}}$, $\Delta^*_{\boldsymbol{C}}$ in $\boldsymbol{A}$ and $\boldsymbol{C}$ extending $\Delta_{\boldsymbol{A}}, \Delta_{\boldsymbol{C}}$. Then, in $\boldsymbol{B}$, there is precisely one diagram, say Δ^*, consisting of Δ and a cocone on Δ, that maps to $\Delta^*_{\boldsymbol{A}}$ and $\Delta^*_{\boldsymbol{C}}$ by the two functors, and Δ^* is a colimit diagram. Note that saying that $\boldsymbol{A} \longleftarrow \boldsymbol{B} \longrightarrow \boldsymbol{C}$ creates colimit *does not* assume that $\boldsymbol{A}$ and $\boldsymbol{C}$ have all colimits; however, if they do, then so does $\boldsymbol{B}$, and the functors preserve them.]

Hence, by the induction assumption that $n\mathtt{Comp}$ has colimits, $[n{+}1]\mathcal{F}$ has colimits, and the three functors out of $[n{+}1]\mathcal{F}$ in (2) preserve them.

On the other hand, the functor $\mathcal{E}:\mathcal{F} \to \omega\mathtt{Cat}$ restricted to $[n{+}1]\mathcal{F}$ maps $(\boldsymbol{X}; U) \in [n{+}1]\mathcal{F}$ to $\boldsymbol{X}[U]$, a typical n-computad. Therefore, we have the commutative diagram of functors, with the horizontal arrows non-full inclusions:

$$\begin{array}{ccc} [n{+}1]\mathcal{F} & \rightarrowtail & (n{+}1)\mathcal{F} \\ \mathcal{E}\simeq\downarrow & \circ & \downarrow\mathcal{E} \\ (n{+}1)\mathtt{Comp} & \rightarrowtail & (n{+}1)\mathtt{Cat} \end{array} .$$

In fact, because of the definition of "morphism of $(n+1)$-computads", the induced functor $\mathcal{E}\simeq$ is an *equivalence* of categories.

The upper left corner has colimits; the upper horizontal preserves them; so does the right vertical, being a left adjoint; the composite from the upper left to the lower right preserves colimits. Since the left vertical is an equivalence, the lower left has colimits as well. By the commutativity of the diagram, the lower horizontal preserves colimits.

Looking at the diagram

$$\begin{array}{ccc} (n+1)\mathtt{Comp} & \rightarrowtail & (n+1)\mathtt{Cat} \\ {\scriptstyle (-)\restriction n}\Big\downarrow & & \Big\downarrow{\scriptstyle (-)\restriction n} \\ n\mathtt{Comp} & \rightarrowtail & n\mathtt{Cat} \end{array}$$

in which the verticals are truncations, and using that the horizontals and the right vertical preserve colimits, we conclude, also using that the lower horizontal reflects isomorphisms, that the left vertical preserves colimits.

We have the commutative diagram

$$\begin{array}{ccc} & \overset{\iota}{\longleftarrow} & [n+1]\mathcal{F} \\ \mathtt{Set} & \circ & \Big\downarrow{\scriptstyle \mathcal{E}\simeq} \\ & \underset{|\text{-}|_{n+1}}{\longleftarrow} & (n+1)\mathtt{Comp} \end{array}$$

where ι is the forgetful functor considered in (2) . It follows that $|\text{-}|_{n+1}$ preserves colimits.

From the preservation facts of the last two paragraphs, and the induction hypothesis that the left functor in (1) preserves colimits, we conclude that in

$$\mathtt{Set} \xleftarrow{\ |\text{-}|\ } (n+1)\mathtt{Comp} \rightarrowtail (n+1)\mathtt{Cat} \qquad (3)$$

the left functor preserves colimits. We have now shown all the properties of (3) that make up the induction statement for $(n+1)$.

Having constructed colimits in n-computads, now we have to pass to the level ω . This is done by the following simple abstract argument.

Suppose we have a limit diagram in CAT , with projections $\mathbf{C}\xrightarrow{\pi_\nu}\mathbf{C}_\nu$, $\nu\in N$, such that all the categories $\mathbf{C}_\nu$ have colimits, and all the connecting functors $F_{\mu\to\nu}:\mathbf{C}_\mu\longrightarrow\mathbf{C}_\nu$ preserve them. Then $\mathbf{C}$ has colimits, and the projections π_ν preserve them.

Note that $\mathtt{Comp}$, in a suitably large CAT , is the limit of the diagram consisting of the categories $n\mathtt{Comp}$, $n\in\mathbb{N}$, with connecting functors the truncations; the limit projections are truncations too. 5.(3) now follows from the corresponding fact for all finite n .

Proof of 5.(4): Suppose $\mathbf{X}\underset{G}{\overset{F}{\rightrightarrows}}\mathbf{Y}$ are such that $|F| = |G|$. We show by induction that $F\restriction n=G\restriction n$ $(n\in\mathbb{N})$; $F=G$ will follow. For $n=0$, the assertion is clear. Suppose $n\geq1$ and $F\restriction(n-1) = G\restriction(n-1)$. Then $F\restriction n=G\restriction n$ follows by the uniqueness of the universal property of $\mathbf{X}\restriction n = (\mathbf{X}\restriction(n-1))[|\mathbf{X}|_n]$.

Suppose $\mathbf{X}\xrightarrow{F}\mathbf{Y}$ is such that $|\mathbf{X}|\xrightarrow{|F|}|\mathbf{Y}|$ is a bijection. Recursively, we construct an inverse $(F\restriction n)^{-1}:\mathbf{Y}\restriction n\longrightarrow\mathbf{X}\restriction n$ For $n=0$, $(F\restriction n)^{-1}=(|F|_0)^{-1}$. Suppose we have $G=(F\restriction(n-1))^{-1}$ to construct $H=(F\restriction n)^{-1}$. The universal property of $\mathbf{Y}\restriction n = (\mathbf{Y}\restriction(n-1)([|\mathbf{Y}|_n]$ gives us $H:\mathbf{Y}\restriction n\longrightarrow\mathbf{X}\restriction n$ such that H on $\mathbf{Y}\restriction(n-1)$ is G , and for $y\in|\mathbf{Y}|_n$, $H(y)=g(y)$ with $g=(|F|_n)^{-1}$: indeed, the only precondition for this is that $\mathrm{d}(g(y))=G(\mathrm{d}y)$ and $\mathrm{c}(g(y))=G(\mathrm{c}y)$ hold, which is true since $F\restriction(n-1)$ applied to the two sides give the same result.

Proof of 5.(5)

Let $L=(L;\leq,\bot,\vee)$ be a join semilattice. We define an ω-category $\mathbf{L}$ from L . We also define a map $|\mathbf{L}|\longrightarrow L: a\mapsto[a]$.

Let $\mathbf{L}_{-1}=\{*\}$ and $[*]=\bot$. Let $\mathbf{L}_0 \overset{=}{\mathrm{def}} L$; $[-]$ on $\mathbf{L}_0$ the identity.

Recursively, assume $n\geq 0$ and the n-graph is

$$\mathbf{L}_{-1} \underset{c}{\overset{d}{\leftleftarrows}} \mathbf{L}_0 \underset{c}{\overset{d}{\leftleftarrows}} \cdots \underset{c}{\overset{d}{\leftleftarrows}} \mathbf{L}_{n-1} \underset{c}{\overset{d}{\leftleftarrows}} \mathbf{L}_n$$

defined; let

$$\mathbf{L}_{n+1} = \{(x, \delta, \gamma): x\in L,\ \delta, \gamma\in\mathbf{L}_n,\ [\delta], [\gamma]\leq x,\ \mathrm{d}\delta=\mathrm{d}\gamma,\ \mathrm{c}\delta=\mathrm{d}\gamma\};$$

for $a=(x, \delta, \gamma)\in\mathbf{L}_{n+1}$, put $[a]=x$, $\mathrm{d}a=\delta$, $\mathrm{c}a=\gamma$. Thus, $[\mathrm{d}a], [\mathrm{c}a]\leq[a]$.

For $a\in\mathbf{L}_n$, put $1_a \overset{=}{\mathrm{def}} ([a], a, a)\in\mathbf{L}_{n+1}$.

Assume $a, b\in\mathbf{L}_n$, $0\leq k<n$, $\mathrm{c}^{(k)}a=\mathrm{d}^{(k)}b$, to define $e = a\#_k b \in\mathbf{L}_n$. We put $[e]=[a]\vee[b]$, and $\mathrm{d}e=\mathrm{d}a$, $\mathrm{c}e=\mathrm{c}b$ if $k=n-1$, $\mathrm{d}e = \mathrm{d}a\#_k\mathrm{d}b$, $\mathrm{c}e = \mathrm{c}a\#_k\mathrm{c}b$ if $k<n-1$. The conditions $[\mathrm{d}e], [\mathrm{c}e]\leq[e]$ are satisfied since, by induction, $[\mathrm{d}e]=[\mathrm{d}a]\vee[\mathrm{d}b]$. The conditions $\mathrm{dd}e=\mathrm{dc}e$, $\mathrm{cd}e=\mathrm{cc}e$ are satisfied automatically (see the remark at Domain/Codomain laws in §2.).

The five identities are true as far as their $[-]$-values are concerned: for instance, in the interchange law, the left-hand side has $[-]$-value $([a]\vee[b])\vee([e]\vee[f])$, the right $([a]\vee[e])\vee([b]\vee[f])$, which are equal. The rest of the requisite equality are satisfied automatically again.

Let $\mathbf{X}$ be a computad. Let L be the poset $\mathcal{P}(|\mathbf{X}|)$, ordered by inclusion. We define the ω-category map $\varphi:\mathbf{X}\to\mathbf{L}$ recursively. On the set $\mathbf{X}_0$, $\varphi(x)=\{x\}$. Having defined φ on $\mathbf{X}\restriction n$, for $x\in|\mathbf{X}|_{n+1}$, we define $r=\varphi(x)\in\mathbf{L}_{n+1}$ by $[r]=\{x\}\cup[\varphi(dx)]\cup[\varphi(cx)]$, and, necessarily, $\mathrm{d}r=\varphi(\mathrm{d}x)$, $\mathrm{c}r=\varphi(\mathrm{c}x)$. By the universal property of $\mathbf{X}\restriction(n+1) = (\mathbf{X}\restriction n)[|\mathbf{X}|_{n+1}]$, this extends uniquely to $\varphi:\mathbf{X}\restriction(n+1)\longrightarrow\mathbf{L}$.

For $a\in\mathbf{X}$, let us write $\mathrm{supp}(a)$ for $[\varphi(a)]$. $\mathrm{supp}(-)$ satisfies the four identities by the definition of φ. The uniqueness of supp can be proved by "computad induction". Also, the "moreover" statements are seen easily by computad induction. Let's look at the third statement:

for fixed n, for all $a\in\boldsymbol{X}_n$, we have $\mathrm{supp}(a)\subseteq\boldsymbol{X}_{n-1} \Longrightarrow a=1_{da}$.

For $a=1_b$: $? \Rightarrow \top = \top$.
For $a=x\in|\boldsymbol{X}|_n$: $\bot \Rightarrow ? = \top$.
For $a=b\#_k e$: suppose $\mathrm{supp}(a)\subseteq\boldsymbol{X}_{n-1}$. Then $\mathrm{supp}(b)\subseteq\boldsymbol{X}_{n-1}$, $\mathrm{supp}(e)\subseteq\boldsymbol{X}_{n-1}$, hence, by induction, $b=1_{db}$, $e=1_{de}$; thus, $a=1_{db}\#_k 1_{de} = 1_{db\#_k de} = 1_{da}$; QED.

Proof of the computad induction principle

To show this, by induction on $n\in\mathbb{N}$, we first show that $|\boldsymbol{X}|_n\subseteq P$. For $n=-1$, true by (i). Suppose $n\geq 0$, $|\boldsymbol{X}|_{\leq n-1}\subseteq P$. Consider the set $Q = \dot{\bigcup}_{m\neq n}\boldsymbol{X}_m \cup (P\cap\boldsymbol{X}_n)$. First, Q contains $|\boldsymbol{X}|$. Indeed, let $x\in|X|$. Then if $\dim(x)\neq n$, $x\in Q$ clearly. If $\dim(x)=n$, $x\in P$ by (ii) and the induction hypothesis. Second, $b\in Q_{\geq 0} \Rightarrow 1_b\in Q$. True for $\dim(b)\neq n-1$ automatically, and by (iii) for $\dim(b)=n-1$. Thirdly, we have

for all b, e and k: $(b\#_k e\downarrow \;\&\; b\in Q \;\&\; e\in Q) \Longrightarrow b\#_k e\in Q$,

once again, with $m \overset{\mathrm{def}}{=} \dim(b)\neq n$ automatically, and for $m=n$ by (iv). The three statements, together with $*\in Q$, show that $\varnothing\langle|\boldsymbol{X}|\rangle\subseteq Q$, and, hence, $Q=|\boldsymbol{X}|$. The definition of Q then says that $\boldsymbol{X}_n\subseteq P$ as promised.

We have proved that $P=|\boldsymbol{X}|$.

Proof of 5.(6)

Temporarily, write $[-]$ for $\mathrm{supp}(-)$.

(i): True for $a=*$.

Let $a=x$; $[x]=\{x\}\cup[\mathrm{d}x]\cup[\mathrm{c}x]$. Let $b\in[x]$, to show $\mathrm{d}b,\mathrm{c}b\in\varnothing\langle[x]\rangle$. If $b=x$, this is true since, by induction hypothesis and $\dim(\mathrm{d}x)=\dim(x)-1$, $\mathrm{d}x\in\varnothing\langle[\mathrm{d}x]\rangle$. Similarly for $\mathrm{c}x$. If $b\in[\mathrm{d}x]$, then assertion is true, since we now assume that assertion is true for all a of dimension less than x . Similarly, for $b\in[\mathrm{c}x]$. $x\in\varnothing\langle[x]\rangle$ is obvious.

$[1_a]=[a]$ and $a\in\varnothing\langle[a]\rangle$ imply $1_a\in\varnothing\langle[a]\rangle$.

Suppose the assertion for a and b ; let $a\#_k b$ be well-defined. $[a\#_k b]=[a]\cup[b]$, and any union of down-closed sets is down-closed; hence, $[a\#_k b]$ is down-closed. Since $a,b\in\varnothing\langle[a\#_k b]\rangle$, it follows that $a\#_k b\in\varnothing\langle[a\#_k b]\rangle$.

(ii): For $a=*$: true. For $x\in|\boldsymbol{X}|_n$, assuming assertion for all b with $\dim(b)<n$: now, $Fx=y\in|\boldsymbol{Y}|$ (!) , thus $[Fx]=\{y\}\cup[\mathrm{d}y]\cup[\mathrm{c}y]$. Comparing this with $[x]$, using $F(\mathrm{d}x)=\mathrm{d}y$, $F(\mathrm{c}x)=\mathrm{c}y$, and all the induction hypotheses, the assertion is clear.
The rest is clear.

Proof of 5.(9)

(i): this is the same as 5.(7).

(ii): Since $\Phi:\mathrm{Comp}\to\omega\mathrm{Cat}$ and $|-|:\mathrm{Comp}\to\mathrm{Set}$ are faithful, the "if" parts are true.

Assume F is a mono in Comp , to show that $|F|:|\boldsymbol{X}|\to|\boldsymbol{Y}|$ is injective, and F is injective as a set-map $F:\|\boldsymbol{X}\|\to\|\boldsymbol{Y}\|$ (the latter being equivalent to saying that F is a mono in $\omega\mathrm{Cat}$). To do this, by induction on n , we prove that $|F|_n:|\boldsymbol{X}|_n\to|\boldsymbol{Y}|_n$ and $F_n:\boldsymbol{X}_n\to\boldsymbol{Y}_n$ are injective. For $n=-1$: true. Suppose *both* assertions are true for $\le n-1$, to show them for n .

Let $u,v\in|\boldsymbol{X}|_n$, $u\neq v$, and assume $Fu=Fv$, to deduce a contradiction to F being a mono in Comp . We have $F(\mathrm{d}u)=\mathrm{d}(Fu)=\mathrm{d}(Fv)=F(\mathrm{d}v)$, hence, by the induction hypothesis, $d \underset{\mathrm{def}}{=} \mathrm{d}u=\mathrm{d}v$. Similarly, $c \underset{\mathrm{def}}{=} \mathrm{c}u=\mathrm{c}v$. d and c are parallel elements of $\boldsymbol{X}_{n-1}$.

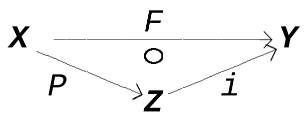
Write $\boldsymbol{Y}=\boldsymbol{X}\restriction(n-1)$, $U=|\boldsymbol{X}|_n$. Of course, $\boldsymbol{X}\restriction n=\boldsymbol{Y}[U]$. Consider $\boldsymbol{Z}\underset{\mathrm{def}}{=}\boldsymbol{Y}[U-\{v\}]$.

We can define maps of ω-cats $\boldsymbol{Z}\underset{H}{\overset{G}{\rightrightarrows}}\boldsymbol{X}$ such that G and H are identities on $\boldsymbol{Y}$, also identities on $U\text{-}\{u, v\}$, but $G(u)=u$ and $H(u)=v$; this is possible by the underlying universal property of $\mathbf{Z}$: note that $G(d)=H(d)=\mathrm{d}u=\mathrm{d}v$, and similarly for c . Now, clearly, $(F\circ G)(u)=Fu\neq Fv=(F\circ H)(u)$, thus $F\circ G\neq F\circ H$, contradicting F being a mono.

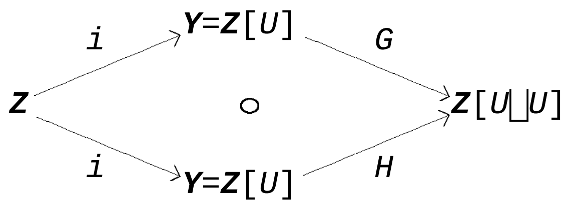
This shows that $|F|_n : |\mathbf{X}|_n \to |\mathbf{Y}|_n$ is injective. It remains to prove that $F_n:\mathbf{X}_n\to\mathbf{Y}_n$ is injective.

Consider the factorization $F=i\circ P$ according to (i), and take the truncations: $F\restriction n=(i\restriction n)\circ(P\restriction n)$. Since $|F\restriction n|$ is injective, so is $|P\restriction n|$. By construction, $|P\restriction n|$ is also surjective; thus, $|P\restriction n|$ is bijective. By (4), or rather, its obvious variant on n-computads, $P\restriction n$ is an isomorphism. It follows that $F\restriction n=(i\restriction n)\circ(P\restriction n)$ is injective as a set map $\|\mathbf{X}\restriction n\|\longrightarrow\|\mathbf{Y}\restriction n\|$. The assertion follows.

(iii): Assume $|F|$ is not surjective. Consider the factorization $F=i\circ P$ according to (i):



By 4.(7), we have $\mathbf{Y}=\mathbf{Z}[U]$, an iterated internal free extension, for a suitable set $U\subseteq|\mathbf{Y}|$. Since $|F|$ is not surjective, $U\neq\emptyset$. Consider the pushout



(see 5.(3)). Then $G\neq H$ since they send any $u\in U$ to the two different components of $U\bigsqcup U$. But also $G\circ F=H\circ F$. We have shown that F is not an epi.

Proof of 5.(12) .

The proof is similar to that of 5.(5) for the supp function.

Let A be any Abelian group. We define the ω-category $\mathbf{A}$; the definition is analogous to that of $\boldsymbol{L}$ from L in the proof of 5.(5).

For $n\geq 0$, we put $\boldsymbol{A}_n=\{(r,d,c):\ r\in A,\ d,c\in\boldsymbol{A}_{n-1},\ d\|c\}$. For $a=(r,d,c)\in\boldsymbol{A}_n$, we define $\mathrm{d}a=d$, $\mathrm{c}a=c$. We also define $[\![a]\!]=r$. Of course, $d\|c$ means $\mathrm{d}d=\mathrm{d}c$, $\mathrm{c}d=\mathrm{c}c$, which is a meaningful condition here, since $d,c\in\boldsymbol{A}_{n-1}$, and d and c are already defined on $\boldsymbol{A}_{n-1}$.

The definition so far specifies $\boldsymbol{A}$ as an ω-graph.

Define $1_a=([\![a]\!],a,a)$; that is, $[\![1_a]\!]=[\![a]\!]$, $\mathrm{d}(1_a)=\mathrm{c}(1_a)=a$.

The definition of the compositions $-\#_k-:\boldsymbol{X}_n\times_k\boldsymbol{X}_n\to\boldsymbol{X}_n$ is recursive in n.

Suppose $a,b\in\boldsymbol{A}_n$, $n>k\geq 0$, and $a\wedge_k b=\mathrm{c}^{(k)}a=\mathrm{d}^{(k)}b$. We define $e=a\#_k b\in\boldsymbol{A}_n$. We put $[\![e]\!]=[\![a]\!]+[\![b]\!]-[\![a\wedge_k b]\!]$. We are forced to define

$$\mathrm{d}e=\begin{cases}(\mathrm{d}a)\#_k(\mathrm{d}b) & \text{if } k<n-1\\ \mathrm{d}a & \text{if } k=n-1\end{cases}$$

$$\mathrm{c}e=\begin{cases}(\mathrm{c}a)\#_k(\mathrm{c}b) & \text{if } k<n-1\\ \mathrm{c}b & \text{if } k=n-1\end{cases}$$

The specification is correct since we will have $\mathrm{d}e\|\mathrm{c}e$ in $\boldsymbol{A}_{n-1}$.

To verify the interchange law, we use the notation of the statement of the law, including the notation introduced in the remarks after it; we assume $k<\ell$.

The $[\![-]\!]$-value of the LHS equals

$$\underbrace{([\![a]\!]+[\![b]\!]-[\![A]\!])}_{[\![a\#_k b]\!]}+\underbrace{([\![e]\!]+[\![f]\!]-[\![A]\!])}_{[\![e\#_k f]\!]}-\underbrace{([\![\varphi]\!]+[\![\psi]\!]-[\![A]\!])}_{[\![\varphi\#_k\psi]\!]\ =\ [\![a\#_k b\ \wedge\ e\#_k f]\!]}\ =$$

$$= \quad [\![a]\!] + [\![b]\!] + [\![e]\!] + [\![f]\!] - [\![\varphi]\!] - [\![\psi]\!] + [\![A]\!] \;;$$

that for the RHS:

$$\underbrace{(\,[\![a]\!] + [\![e]\!] - [\![\varphi]\!]\,)}_{[\![a\#_\ell e]\!]} + \underbrace{(\,[\![b]\!] + [\![f]\!] - [\![\psi]\!]\,)}_{[\![b\#_\ell f]\!]} - \underbrace{[\![A]\!]}_{[\![a\#_\ell e \;\wedge\; b\#_\ell f]\!]} = \;;$$

$$= \quad [\![a]\!] + [\![b]\!] + [\![e]\!] + [\![f]\!] - [\![\varphi]\!] - [\![\psi]\!] + [\![A]\!] \;;$$

we have $[\![\text{LHS}]\!] = [\![\text{RHS}]\!]$. By the remarks made for interchange at the statement of the law (section 3), this is enough for the interchange equality.

For associativity, we get, with $A=a\wedge_k b$, $B=b\wedge_k e$, that the $[\![-]\!]$-values of both sides are equal to $[\![a]\!] + [\![b]\!] + [\![e]\!] - [\![A]\!] - [\![B]\!]$.

In the case of the two-sided unit law, the common value, with $\varphi=a\wedge_k b$, is $[\![a\#_k b]\!] = [\![a]\!] + [\![b]\!] - [\![\varphi]\!]$, since $1_a\wedge_k 1_b = a\wedge_k b$.

For the right unit law, note that, for $\gamma=\mathrm{c}^{(k)}a$, we have $[\![1^{(n)}_\gamma]\!] = [\![\gamma]\!]$ and $a\wedge 1^{(n)}_\gamma = \gamma$. Thus

$$[\![a\#_k 1^{(n)}_\gamma]\!] = [\![a]\!] + [\![1^{(n)}_\gamma]\!] - [\![a\wedge 1^{(n)}_\gamma]\!] = [\![a]\!] + [\![\gamma]\!] - [\![\gamma]\!] = [\![a]\!] \;.$$

The left unit law is similar.

We have proved that $\boldsymbol{A}$ is an ω-category.

Let $\boldsymbol{X}$ be a computad. We let A be $|\boldsymbol{X}|\cdot\mathbb{Z}$, the Abelian group of finite multisets of indets of $\boldsymbol{X}$. We define the ω-category map $\varphi:\boldsymbol{X}\to\boldsymbol{A}$ recursively. On the set $\boldsymbol{X}_{-1}$, $\varphi(*)=0$. On $\boldsymbol{X}_0 = |\boldsymbol{X}|_0$, $\varphi(x)=\binom{x}{1}$. Having defined φ as $\boldsymbol{X}{\restriction}n\longrightarrow\boldsymbol{A}{\restriction}n$, for $x\in|\boldsymbol{X}|_{n+1}$, we define $r=\varphi(x)\in\boldsymbol{A}_{n+1}$ by $[\![r]\!]=\binom{x}{1}+[\![\varphi(\mathrm{d}x)]\!]+[\![\varphi(\mathrm{c}x)]\!]$, and, necessarily, $\mathrm{d}r=\varphi(\mathrm{d}x)$, $\mathrm{c}r=\varphi(\mathrm{c}x)$; since $\mathrm{d}x\|\mathrm{c}x$, we have $\mathrm{d}r\|\mathrm{c}r$, and the definition is legitimate. By the universal property of $\boldsymbol{X}{\restriction}(n+1)=(\boldsymbol{X}{\restriction}n)[\,|\boldsymbol{X}|_{n+1}]$, $\varphi:\boldsymbol{X}{\restriction}n\longrightarrow\boldsymbol{A}{\restriction}n$ and $\varphi:|\boldsymbol{X}|_{n+1}\longrightarrow\boldsymbol{A}_{n+1}$ extend uniquely to $\varphi:\boldsymbol{X}{\restriction}(n+1)\longrightarrow\boldsymbol{A}{\restriction}(n+1)$. We have defined

$\varphi:\mathbf{X}\to\mathbf{A}$.

For $a\in\mathbf{X}$, we write $[a]$ for $[\![\varphi(a)]\!]$. By construction, the equalities (i) to (iv) are satisfied.

Next, we prove (ix). Temporarily, let us denote the set $\{x\in|\mathbf{X}|:[a]_{\mathbf{X}}(x)\neq 0\}$ by $\|a\|_{\mathbf{X}}$. It is clear, by induction, that $\|a\|\subseteq\mathbf{X}_{\mathrm{dim}(a)}$.

We prove (ix) by computad induction.

First we take $a=u\in|\mathbf{X}|_n$; we assume that (ix) is true for arguments a of dimension less than n . Now, $Fa=Fu=v\in|\mathbf{Y}|_n$, and

$$[u]_{\mathbf{Y}}=\binom{u}{1}+[\mathrm{d}u]_{\mathbf{Y}}+[\mathrm{c}u]_{\mathbf{Y}} ,$$
$$[v]_{\mathbf{Y}}=\binom{v}{1}+[\mathrm{d}v]_{\mathbf{Y}}+[\mathrm{c}v]_{\mathbf{Y}} .$$

Let $y\in|\mathbf{Y}|$ and let us evaluate both sides of the equality in (ix) at y .

First, let $\mathrm{dim}(y)\geq n$. Since $\|dv\|_{\mathbf{Y}}$, $\|cv\|_{\mathbf{Y}}$ are subsets of $|\mathbf{Y}|_{n-1}$, we get

$$[v]_{\mathbf{Y}}(y)=\binom{v}{1}(y)=\begin{cases}1 & \text{if } y=v\\ 0 & \text{if } y\neq v\end{cases}$$

On the other hand, on the RHS we get

$$\sum_{\substack{x\in|\mathbf{X}|\\ Fx=y}}[u]_{\mathbf{X}}(x)=\sum_{\substack{x\in|\mathbf{X}|\\ Fx=y}}\binom{u}{1}(x)+\sum_{\substack{x\in|\mathbf{X}|\\ Fx=y}}[\mathrm{d}u]_{\mathbf{Y}}(x)+\sum_{\substack{x\in|\mathbf{X}|\\ Fx=y}}[\mathrm{c}u]_{\mathbf{Y}}(x) . \quad (1)$$

Since $\binom{u}{1}(x)=1$ unless $x=u$ when it is 1 , the first sum on the right equals 1 when $y=v$ (when $x=u$ is a possible x), otherwise it is 0 (when $x=u$ gives $Fx\neq y$). Since $Fx=y$ implies that $\mathrm{dim}(x)=\mathrm{dim}(y)\geq n$, the other two sums in (1) are 0 .

We have proved (ix) for arguments y of dimension $\geq n$.

For $y\in |\mathbf{Y}|_{<n}$, the value of the LHS in (ix) is that of $[\mathrm{d}v]_{\mathbf{Y}}+[\mathrm{c}v]_{\mathbf{Y}}$. But, by the induction hypothesis, and using (1),

$$
\begin{aligned}
([\mathrm{d}v]_{\mathbf{Y}}+[\mathrm{c}v]_{\mathbf{Y}})(y) &= [\mathrm{d}v]_{\mathbf{Y}}(y)+[\mathrm{c}v]_{\mathbf{Y}}(y)\\
&= [F(\mathrm{d}u)]_{\mathbf{Y}}(y)+[F(\mathrm{c}u)]_{\mathbf{Y}}(y)\\
&= \sum_{\substack{x\in |\mathbf{X}|\\ Fx=y}} [\mathrm{d}u]_{\mathbf{X}}(x) + \sum_{\substack{x\in |\mathbf{X}|\\ Fx=y}} [\mathrm{c}u]_{\mathbf{X}}(x)\\
&= \sum_{\substack{x\in |\mathbf{X}|\\ Fx=y}} [u]_{\mathbf{X}}(x)\ ,
\end{aligned}
$$

since $\sum_{\substack{x\in |\mathbf{X}|\\ Fx=y}} \binom{u}{1}(x)=0$ by $\dim(y)<\dim(u)$.

This proves the equality (ix) when $a=x$ is an indeterminate.

The rest of the cases are similar and easier. For instance, the "composition" clause only uses the "linearity" of the definition of $[a\#_k b]$.

Next, we prove that $[\mathrm{d}a]\le[a]$ and $0\le[a]$ simultaneously by computad induction on a.

For $a=*$: true.

Let $a=x$, an indeterminate. By the induction hypothesis, $[cx]\ge 0$, thus $[x]=\binom{x}{1}+[\mathrm{d}x]+[\mathrm{c}x]\ge[\mathrm{d}x]$. Since by the induction hypothesis, $[dx]\ge 0$, we also have $[x]\ge 0$.

$[1_a]=[a]\ge 0$, and $[\mathrm{d}(1_a)]=[a]\le[1_a]$; the identity clause is clear.

$[a\#_k b]=[a]+[b]-[a\wedge_k b]$, and $[\mathrm{d}(a\#_k b)]=[\mathrm{d}a]+[\mathrm{d}b]-[a\wedge_k b]$ when $k<n-1$ (since $\mathrm{d}a\wedge_k\mathrm{d}b=a\wedge_k b$ in this case), and $[\mathrm{d}(a\#_k b)]=[\mathrm{d}a]$ when $k=n-1$. $[a\#_k b]\ge[d(a\#_k b)]$ follows: in the first case, since $[a]\ge[\mathrm{d}a]$, $[b]\ge[\mathrm{d}b]$ by induction hypothesis, and in the second case since $[a]\ge[\mathrm{d}a]$ and $[b]\ge[a\wedge_k b]=[\mathrm{d}^{(k)}b]$ (and thus $[b]-[a\wedge_k b]\ge 0$) by the induction hypothesis. Of

course, $[a\#_k b] \geq 0$ follows.

The fact that $[ca]\leq[a]$ is similar.

Now, the fact that $[a]\leq[a\#_k b]=[a]+[b]-[a\wedge_k b]$ follows from the fact that $[b]\geq[a\wedge_k b]=[\mathrm{d}^{(k)}b]$. $[b]\leq[a\#_k b]$ is similar.

Finally, we show (viii). Let us apply (ix) to the inclusion $\mathrm{Supp}(a)\longrightarrow\boldsymbol{X}$ and to the element $a\in\mathrm{Supp}(a)$. We get, for any $y\in|\boldsymbol{X}|$, that

$$[a]_{\boldsymbol{X}}(y) = \sum_{\substack{x\in\mathrm{supp}(a)\\ x=y}} [a]_{\mathrm{Supp}(a)}(x) = \begin{cases} 0 & \text{if } y\notin\mathrm{supp}(a) \\ [a]_{\mathrm{Supp}(a)}(y) & \text{if } y\in\mathrm{supp}(a) \end{cases} .$$

This shows the left-to-right implication in (viii).

It remains to show that $y\in\mathrm{supp}(a) \Longrightarrow [a](y)\geq 1$ (we suppressed the subscript $\boldsymbol{X}$). Of course, we apply induction.

Let $a=x\in|\boldsymbol{X}|_n$; $[x]=\binom{x}{1}+[\mathrm{d}x]+[\mathrm{c}x]$. Assume $y\in\mathrm{supp}(x)=\{x\}\cup\mathrm{supp}(\mathrm{d}x)\cup\mathrm{supp}(\mathrm{c}x)$. If $y\in\mathrm{supp}(\mathrm{d}x)$, then $[\mathrm{d}x](y)\geq 1$, and $[x](y)\geq 1$ since $[x]\geq[\mathrm{d}x]$. Similarly for $y\in\mathrm{supp}(\mathrm{c}x)$. If (the remaining case) $y=x$, then $[x](x)\geq 1$ since $[\mathrm{d}x],[\mathrm{c}x]\geq 0$.

The remaining cases are omitted; they are similar to what we have seen.

Appendix to section 6

Proof of 6.(5)

To see (i), let A be small in $\mathcal{C}$. Consider the indets $x \in |A|$, and consider the following diagram $\Phi: G \to \mathtt{Comp}$. The graph G has two kinds of objects. The first is x, one for each $x \in |A|$; we put $\Phi(x) = \mathtt{Supp}_A(x)$. The second is the pair $\langle x, y \rangle$, one for each pair of distinct $x, y \in |A|$; we put $\Phi(\langle x, y \rangle) = \mathtt{Supp}_A(x) \cap \mathtt{Supp}_A(y)$, the intersection meant in the sense of the subobject lattice of A. The arrows of the graph G are $\langle x, y \rangle \to x$, $\langle x, y \rangle \to y$; $\Phi(\langle x, y \rangle \to x)$, $\Phi(\langle x, y \rangle \to y)$ are all inclusions. As coprojections from this diagram to A itself, take inclusions again. Since $\mathcal{C}$ is a sieve in $\mathtt{Comp}$, all objects and arrows in this diagram are in $\mathcal{C}$.

Note that by 4.(8)(ii), the forgetful functor $|{-}|: \mathtt{Comp} \to \mathtt{Set}$ takes the intersections $\Phi(\langle x, y \rangle)$ to corresponding intersections in $\mathtt{Set}$. If we apply the forgetful functor $|{-}|: \mathtt{Comp} \to \mathtt{Set}$ to all of the above, we get a colimit diagram in $\mathtt{Set}$, as a simple observation regarding the category of sets shows. Therefore, the original diagram is a colimit diagram in $\mathtt{Comp}$, hence, in $\mathcal{C}$ as well. Since A is $\mathcal{C}$-small, A is a retract of an object of this diagram, say $\Phi(x) = \mathtt{Supp}(x)$, in fact, A is necessarily isomorphic to $\mathtt{Supp}(x)$. This proves (i).

Turning to (ii), let A be a primitive object of $\mathcal{C}$. It is a colimit of a diagram of $\mathcal{C}$-small objects of $\mathcal{C}$, each of which is principal by (i). When the forgetful functor $|{-}|$ is applied to this diagram, it becomes a colimit diagram in $\mathtt{Set}$; therefore, the colimit coprojections are jointly surjective on indeterminates. There must be an object in the diagram, a $\mathcal{C}$-small principal object, say B, such that the colimit coprojection $B \to A$ has m_A in its image. Of course, the only element of $|B|$ that can be a preimage of m_A is m_B. We have an arrow $f: (B, m_B) \to (A, m_A)$ in $\mathtt{El}(\mathtt{Comp})$. By the definition of "primitive", f must be an isomorphism. Since B is $\mathcal{C}$-small, A is $\mathcal{C}$-small.

Concerning (iii): let $\boldsymbol{X}$ be an object in $\mathcal{C}$, and assume that it is $\mathcal{C}$-small, to show that it is $\mathtt{Comp}$-small. Let us abbreviate $\mathtt{Comp}(\boldsymbol{X}, -): \mathtt{Comp} \to \mathtt{Set}$ to $(\boldsymbol{X}, -)$.

By (i), we know that $(\boldsymbol{X}, -)$ commutes with filtered colimits. As $\boldsymbol{X}$ is non-empty (also by (i)), $(\boldsymbol{X}, -)$ commutes with empty colimits. To show that it commutes with all colimits, it suffices to show that it commutes with binary coproducts and coequalizers. The case of binary coproducts is similar to, but simpler than, the case of coequalizers; we deal with the latter only.

Suppose

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C \qquad (1)$$

is a coequalizer diagram in $\mathtt{Comp}$; our aim is to show that

$$(\boldsymbol{X}, A) \underset{g^*}{\overset{f^*}{\rightrightarrows}} (\boldsymbol{X}, B) \xrightarrow{h^*} (\boldsymbol{X}, C) \qquad (2)$$

is a coequalizer diagram in $\mathtt{Set}$.

Let $p:\boldsymbol{X}\to C$ be any element of $(\boldsymbol{X}, C)$. Let us factor p as $p=i\circ q$:

$$\boldsymbol{X} \xrightarrow{q} \boldsymbol{Y} \xrightarrow{i} C \ ,$$

where q is surjective, and i is injective. By condition (c) on $\mathcal{C}$, $\boldsymbol{Y}$ belongs to $\mathcal{C}$. Pull back (1) along i :

$$\begin{array}{ccccc} A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \xrightarrow{h} & C \\ {\scriptstyle a}\uparrow & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle i} \\ \bar{A} & \underset{\bar{g}}{\overset{\bar{f}}{\rightrightarrows}} & \bar{B} & \xrightarrow{\bar{h}} & \boldsymbol{Y} \end{array} \qquad (3)$$

By 5.(8)(iv), the lower part of (3) is a coequalizer in $\mathtt{Comp}$. Since $\mathcal{C}$ is a sieve and $\boldsymbol{Y}$ is in $\mathcal{C}$, that lower part is all in $\mathcal{C}$. Since $\mathcal{C}$ is assumed to satisfy (b) (colimits), that lower part is a coequalizer in $\mathcal{C}$. Since $\boldsymbol{X}$ is $\mathcal{C}$-small,

$$(\mathbf{X}, \bar{A}) \underset{\bar{g}^*}{\overset{\bar{f}^*}{\rightrightarrows}} (\mathbf{X}, \bar{B}) \xrightarrow{\bar{h}^*} (\mathbf{X}, \mathbf{Y}) \qquad (4)$$

is a coequalizer in $\mathtt{Set}$; $\bar{h}^*$ is surjective; there is $r:\mathbf{X}\to\bar{B}$ such that $\bar{h}\circ r=q$; for $s \overset{=}{\text{def}} b\circ r$, $h^*(s)=h\circ s=i\circ q=p$ (see (2) and (3)). Since $p\in(\mathbf{X}, C)$ was arbitrary, we have shown that h^* is surjective.

Next, let $\mathbf{X} \underset{s}{\overset{r}{\rightrightarrows}} B$ be a pair of arrows such that $h\circ r=h\circ s$. Let $\mathbf{Y}=\mathrm{Im}(h\circ r)=\mathrm{Im}(h\circ s)$, a subcomputad of C (since $\mathbf{X}=\mathrm{Supp}_{\mathbf{X}}(m_{\mathbf{X}})$, $\mathbf{Y}=\emptyset\langle \mathrm{supp}_B(hrm_{\mathbf{X}})\rangle$) . Since $\mathcal{C}$ satisfies (c), $\mathbf{Y}$ is in $\mathcal{C}$.

Let $i:\mathbf{Y}\to C$ be the inclusion. Taking pullbacks, we again have a diagram as in (3). As before, the lower part is a coequalizer in $\mathcal{C}$, and (4) is a coequalizer in $\mathtt{Set}$.

Because $\bar{B}$ was obtained as a pullback, and $i\circ q=h\circ r=h\circ s$, there are $\mathbf{X} \underset{\bar{s}}{\overset{\bar{r}}{\rightrightarrows}} \bar{B}$ such that $r=b\circ\bar{r}$, $s=b\circ\bar{s}$, $q = \bar{h}\circ r = \bar{h}\circ s$. We have $\bar{r}, \bar{s}\in(\mathbf{X}, \bar{B})$ identified by the map $\bar{h}^*:(\mathbf{X}, \bar{B}) \longrightarrow (\mathbf{X}, \bar{C})$. Since (4) is a coequalizer, there is a zig-zag

$$\begin{array}{ccccccccccc} & \bar{x}_1 & & & \bar{x}_2 & & & & \bar{x}_n & & \\ \bar{f}^*\swarrow & & \searrow\bar{g}^* & \bar{f}^*\swarrow & & \searrow\bar{g}^* & \cdots & \bar{f}^*\swarrow & & \searrow\bar{g}^* & \\ \bar{r}_1 & & & \bar{r}_2 & & & \bar{r}_3 \quad \bar{r}_n & & & & \bar{r}_{n+1} \end{array}$$

with $\bar{x}_i\in(\mathbf{X}, \bar{A})$, $\bar{r}_i\in(\mathbf{X}, \bar{A})$, $\bar{f}^*(\bar{x}_i)=\bar{r}_i$, $\bar{g}^*(\bar{x}_i)=\bar{r}_{i+1}$, and either $\bar{r}=\bar{r}_1$ and $\bar{s}=\bar{r}_n$, or vice versa. Let $x_i=a\circ\bar{x}_i$, $r_i=b\circ\bar{r}_i$. Then we have the zig-zag

$$\begin{array}{ccccccccccc} & x_1 & & & x_2 & & & & x_n & & \\ f^*\swarrow & & \searrow\bar{g}^* & f^*\swarrow & & \searrow g^* & \cdots & f^*\swarrow & & \searrow g^* & \\ r_1 & & & r_2 & & & r_3 \quad r_n & & & & r_{n+1} \end{array}$$

with $x_i\in(\mathbf{X},A)$, $r_i\in(\mathbf{X},A)$, $f^*(x_i)=r_i$, $g^*(x_i)=\bar{r}_{i+1}$, and either $r=r_1$ and $s=r_n$, or vice versa. This conclusion was reached for an arbitrary pair $\mathbf{X}\overset{r}{\underset{s}{\rightrightarrows}}B$ such that $h^*(r)=h^*(s)$. Together with the fact that h^* is surjective, this means that (2) is a coequalizer in Set .

Appendix to section 8

Proof of 8.(10)

We only give some details for part (i).

Thus, we are assuming that we have an ω-category in the new sense, and want to prove that, via the definitions in section 8, we have one in the old sense. Whenever we use the word "axiom", we mean a law that comes assumed with the new definition of ω-category.

Note that we defined $a\#_\ell b$ for a , b not necessarily of the same dimension although in the original definition of an ω-category, $a\#_\ell b$ is defined only when $\mathrm{dim}(a)=\mathrm{dim}(b)$. This is a minor difference in the definitions: if one has an ω-category in the original sense, one may put $a\#_\ell b \overset{=}{\mathrm{def}} 1_a^{(N)}\#_\ell 1_b^{(N)}$, with $N=\mathrm{max}(m,n)$.

Let us abbreviate $\mathrm{c}^{(\ell)}$, $\mathrm{d}^{(\ell)}$ by $\tilde{\mathrm{c}}$ and $\tilde{\mathrm{d}}$, and $\mathrm{c}^{(\ell+1)}$, $\mathrm{d}^{(\ell+1)}$ by $\bar{c}$ and $\bar{\mathrm{d}}$, respectively.

Let's show that the expressions used to define $a\#_\ell b$ are well-defined. The assumption is $\bar{\mathrm{c}}a=\bar{\mathrm{d}}b$.

For $a\cdot\bar{\mathrm{d}}b$, we need $\tilde{\mathrm{c}}a=\tilde{\mathrm{d}}\bar{\mathrm{d}}b$; this holds. Similarly for $\bar{\mathrm{c}}a\cdot b$, etc.

For $(a\cdot\bar{\mathrm{d}}b)\#_{\ell+1}(\bar{\mathrm{c}}a\cdot b)$ to be well-defined, we need $\bar{\mathrm{c}}(a\cdot\bar{\mathrm{d}}b)=\bar{\mathrm{d}}(\bar{\mathrm{c}}a\cdot b)$. But

$\bar{\mathrm{c}}(a \cdot \bar{\mathrm{d}}b) = \bar{\mathrm{c}}a \cdot \bar{\mathrm{d}}b$, and $\bar{\mathrm{d}}(\bar{\mathrm{c}}a \cdot b) = \bar{\mathrm{c}}a \cdot \bar{\mathrm{d}}b$, so this holds too.

The other expression for $a \#_{\ell} b$ is similarly seen to be well-defined.

We prove 8.(9). When $\ell = k = k(a, b)$, this is the commutative law. When $\ell < k$, we can use the induction hypothesis to the effect that the analogous law holds for $\ell + 1$.

The LHS of 8.(9) is rewritten by using the first expression for $(-)\#_{\ell}(-)$, for the RHS the second; in this way, we will get expressions in both of which a without d or c before it precedes b without d or c before it.

By using equalities like $\bar{\mathrm{d}}(\tilde{\mathrm{d}}a \cdot b) = \tilde{\mathrm{d}}a \cdot \bar{\mathrm{d}}b$, we obtain that the equality to be proved amounts to

$$((a \cdot \tilde{\mathrm{d}}b) \cdot (\tilde{\mathrm{c}}a \cdot \bar{\mathrm{d}}b))\#_{\ell+1}((\bar{\mathrm{c}}a \cdot \tilde{\mathrm{d}}b) \cdot ((\tilde{\mathrm{c}}a \cdot b) \stackrel{?}{=}$$
$$((\tilde{\mathrm{d}}a . \bar{\mathrm{d}}b) \cdot (a \cdot \tilde{\mathrm{c}}b))\#_{\ell+1}((\tilde{\mathrm{d}}a \cdot b) \cdot (\bar{\mathrm{c}}a \cdot \tilde{\mathrm{c}}b))$$

But the left factors of the two sides are equal by the ordinary commutative law, with a and $\tilde{\mathrm{d}}b$ as a and b . Similarly, the right factors are equal. 8.(9) is proved.

Next, as a lemma, we prove the following *generalized distributive law*:

$$(a\#_{\ell}b) \cdot e = (a \cdot e)\#_{\ell}(b \cdot e) \tag{1}$$

under the hypotheses that $p-1<\ell$ ($p=\mathrm{dim}(e)$), and $a\#_{\ell}b$, $a \cdot e$ and $b \cdot e$ are well-defined.

First of all, we note that $k[a, e]=k[b, e]=k[a\#_{\ell}b, e]=p-1$. $\tilde{\mathrm{c}}(a \cdot e)=\tilde{\mathrm{c}}a \cdot e$, and $\tilde{\mathrm{d}}(b \cdot e)=\tilde{\mathrm{d}}b \cdot e$, thus, since $\tilde{\mathrm{c}}a=\tilde{\mathrm{d}}b$, we have $\tilde{\mathrm{c}}(a \cdot e)=\tilde{\mathrm{d}}(b \cdot e)$, and the RHS in (1) is well-defined. Writing $\hat{\mathrm{c}}$ for $\mathrm{c}^{(p-1)}$, $\hat{\mathrm{d}}$ for $\mathrm{d}^{(p-1)}$, we have $\hat{\mathrm{c}}(a\#_{\ell}b)=\hat{\mathrm{c}}b=\hat{\mathrm{c}}a=\hat{\mathrm{d}}e$ (since $p-1<\ell$), and the LHS in (1) is well-defined.

The proof of (1) is by induction on $k-\ell$, with $k=k[a, b]$. When $k=\ell$, (1) is the distributivity axiom.

Using the definition for $a\#_\ell b$, we get

$$\text{LHS} = ((a\cdot\bar{\text{d}}b)\#_{\ell+1}(\bar{\text{c}}a\cdot b))\cdot e = ((a\cdot\bar{\text{d}}b)\cdot e)\#_{\ell+1}((\bar{\text{c}}a\cdot b)\cdot e),$$

where we used the induction hypothesis for $\ell+1$. Note that the application was legitimate, since $\hat{\text{c}}(a\cdot\bar{\text{d}}b)=\hat{\text{c}}a=\hat{\text{d}}e$, and $\hat{\text{c}}(\bar{\text{c}}a\cdot b)=\hat{\text{c}}b=\hat{\text{d}}e$. Since $p<m$, $p<n$ and $p<\ell+1$, we can apply the distributive axiom twice, to re-write the last as

$$((a\cdot e)\cdot(\bar{\text{d}}b\cdot e))\#_{\ell+1}((\bar{\text{c}}a\cdot e)\cdot(b\cdot e)). \qquad (2)$$

When we use the definition of $(-)\#_\ell(-)$ on the RHS in (1), we immediately see that we get the expression in (2). This proves (1).

We also have a dual form of (1):

$$a\cdot(b\#_\ell e) = (a\cdot b)\#_\ell(a\cdot e)$$

under the appropriate conditions.

We leave the proofs of the domain/codomain laws and the unit laws to the reader; they are straight-forward inductions on ℓ (in the formulation of those laws, we replace the original letter k by ℓ).

Next, we prove the associative law

$$(a\#_\ell b)\#_\ell e = a\#_\ell(b\#_\ell e) \qquad (3)$$

assuming that $a\#_\ell b$ and $b\#_\ell e$ are well-defined. For convenience, we assume that $\dim(a)=\dim(b)=\dim(e)=n$ (which is enough for our purposes, although the additional assumption is not necessary). The proof is by induction on $n-1-\ell$. When $\ell=n-1$, the law is a special case of the associative axiom.

Let $\ell<n-1$. We rewrite the LHS of (3) as

$$((a\cdot\bar{\mathrm{d}}b)\#_{\ell+1}(\bar{\mathrm{c}}a\cdot b))\cdot\bar{\mathrm{d}}e)\ \#_{\ell+1}\ ((\bar{\mathrm{c}}a\cdot\bar{\mathrm{c}}b)\cdot e)\ .$$

The generalized distributive law, (1), with $\ell+1$ in place of ℓ, applied on the first factor of the second $\#_{\ell+1}$, we get

$$(((a\cdot\bar{\mathrm{d}}b)\cdot\bar{\mathrm{d}}e)\#_{\ell+1}((\bar{\mathrm{c}}a\cdot b)\cdot\bar{\mathrm{d}}e))\ \#_{\ell+1}\ ((\bar{\mathrm{c}}a\cdot\bar{\mathrm{c}}b)\cdot e)\ .$$

As to the applicability of said law, note that $\mathrm{dim}(\bar{\mathrm{d}}e)=\ell+1$, thus $\mathrm{dim}(\bar{\mathrm{d}}e)-1<\ell+1$.

There are three places where we can use the associativity axiom, in fact in all three of its alternatives concerning the dimensions: we have

$\mathrm{dim}(a)=n>\mathrm{dim}(\bar{\mathrm{d}}b)=\mathrm{dim}(\bar{\mathrm{d}}e)=\ell+1$	for the first,
$\mathrm{dim}(b)=n>\mathrm{dim}(\bar{\mathrm{d}}\mathrm{f})=\mathrm{dim}(\bar{\mathrm{d}}e)=\ell+1$	for the second,
$\mathrm{dim}(e)=n>\mathrm{dim}(\bar{\mathrm{c}}a)=\mathrm{dim}(\bar{\mathrm{c}}b)=\ell+1$	for the third.

We obtain the expression

$$((a\cdot(\bar{\mathrm{d}}b\cdot\bar{\mathrm{d}}e))\#_{\ell+1}(\bar{\mathrm{c}}a\cdot(b\cdot\bar{\mathrm{d}}e)))\ \#_{\ell+1}\ (\bar{\mathrm{c}}a\cdot(\bar{\mathrm{c}}b\cdot e))\ .$$

Finally, an application of associativity for the operation $\#_{\ell+1}$, valid by the induction hypothesis, is used to obtain

$$(a\cdot(\bar{\mathrm{d}}b\cdot\bar{\mathrm{d}}e))\ \#_{\ell+1}\ ((\bar{\mathrm{c}}a\cdot(b\cdot\bar{\mathrm{d}}e))\#_{\ell+1}(\bar{\mathrm{c}}a\cdot(\bar{\mathrm{c}}b\cdot e)))\ . \qquad (4)$$

When we tackle the RHS in a similar manner, we get

$$(a\cdot(\bar{\mathrm{d}}b\cdot\bar{\mathrm{d}}e))\ \#_{\ell+1}\ (\bar{\mathrm{c}}a\cdot((b\cdot\bar{\mathrm{d}}e)\#_{\ell+1}(\bar{\mathrm{c}}b\cdot e)))\ .$$

The dual form of the generalized distributive law applied to the second factor of the first $\#_{\ell+1}$ will result in (4).

This completes the proof of the associative law for the "old" definition.

We turn to the interchange law.

We want to prove

$$(a\#_m b)\#_\ell(e\#_m f) \overset{?}{=} (a\#_\ell e)\#_m(b\#_\ell f) \qquad (5)$$

under the hypotheses that the dimensions of a, b, e, f are all equal to n, we have $0\le m<\ell<n$, and, with the (further) abbreviations $\dot{c} = c^{(m)}$, $\dot{d} = d^{(m)}$, $\hat{c} = c^{(m+1)}$, $\hat{d} = d^{(m+1)}$, the equalities

$$\dot{c}a = \dot{d}b\ ,\ \dot{c}e = \dot{d}f\ ,\ \tilde{c}a = \tilde{d}e\ ,\ \tilde{c}b = \tilde{d}f\ . \qquad (6)$$

Since $m+1\le\ell$,

$$\hat{c}(a\#_\ell e) = \hat{c}e\ , \qquad \hat{d}(b\#_\ell f) = \hat{d}b\ .$$

Using the definition of the operation $\#_m$ in terms of $\#_{m+1}$, we rewrite both the left and the right sides of (5), and obtain

$$((a\cdot\hat{d}b)\#_{m+1}(\hat{c}a\cdot b))\ \#_\ell\ ((e\cdot\hat{d}f)\#_{m+1}(\hat{c}e\cdot f)) \overset{?}{=}$$

$$((a\#_\ell e)\cdot\hat{d}b)\ \#_{m+1}\ (\hat{c}a\cdot(b\#_\ell f))\ . \qquad (7)$$

We proceed by induction on $\ell-m$. First, assume $\ell-m=1$ (the lowest value). Now, $\hat{c}=\tilde{c}$, $\hat{d}=\tilde{d}$.

The LHS of (7) becomes

$$((a\cdot\tilde{d}b)\#_\ell(\tilde{c}a\cdot b))\ \#_\ell\ ((e\cdot\tilde{d}f)\#_\ell(\tilde{c}e\cdot f))\ ,$$

to which associativity for $\#_\ell$ can be applied, to get

$$(a \cdot \tilde{\mathrm{d}} b) \#_\ell ((\tilde{\mathrm{c}} a \cdot b) \#_\ell (e \cdot \tilde{\mathrm{d}} f)) \#_\ell (\tilde{\mathrm{c}} e \cdot f) \; .$$

which, by (5), is the same as

$$(a \cdot \tilde{\mathrm{d}} b) \#_\ell ((\tilde{\mathrm{d}} e \cdot b) \#_\ell (e \cdot \tilde{\mathrm{c}} b)) \#_\ell (\tilde{\mathrm{c}} e \cdot f) \; .$$

To the middle term, generalized commutativity, 8.(9), already proved, can be applied, to get

$$(a \cdot \tilde{\mathrm{d}} b) \#_\ell ((e \cdot \tilde{\mathrm{d}} b) \#_\ell (\tilde{\mathrm{c}} e \cdot b)) \#_\ell (\tilde{\mathrm{c}} e \cdot f) \; ,$$

which, upon another use of the associative law for $\#_\ell$, and the generalized distributive law (1), becomes

$$((a \#_\ell e) \cdot \tilde{\mathrm{d}} b)) \#_\ell (\tilde{\mathrm{c}} e \cdot (b \#_\ell f)) \; ,$$

which is the RHS of (7).

To complete the induction. assume now $m+1<\ell$, and try to show (7). Now we have

$$\hat{\mathrm{c}} a = \hat{\mathrm{c}} e \underset{\mathrm{def}}{=} r \qquad \text{and} \qquad \hat{\mathrm{d}} b = \hat{\mathrm{d}} \mathrm{f} \underset{\mathrm{def}}{=} s \; .$$

and (7) becomes

$$((a \cdot s) \#_{m+1} (r \cdot b)) \#_\ell ((e \cdot s) \#_{m+1} (r \cdot f)) \overset{?}{=}$$

$$((a \#_\ell e) \cdot s) \#_{m+1} (r \cdot (b \#_\ell f)) \; . \tag{8}$$

By generalized distributivity, the RHS of (8) is

$$((a \cdot s) \#_\ell (e \cdot s)) \#_{m+1} ((r \cdot b) \#_\ell (r \cdot f)) \; ,$$

and (8) becomes an instance of the interchange law for $m+1$ and ℓ in place of m and ℓ, true by the induction hypothesis.

References

[A/R] J. Adamek and J. Rosicky, *Locally Presentable and Accessible Categories.* Cambridge University Press 1994.

[Bae/D] J. C. Baez and J. Dolan. Higher-dimensional algebra III. n-categories and the algebra of opetopes. *Advances in Mathematics* **135** (1998), 145-206.

[Bat] M. Batanin, Computads for Finitary Monads on Globular Sets. In: *Contemporary Mathematics* **230** (1998), 37-58; American Mathematical Society.

[Bat] M. Batanin, Computads and slices of operads. arXiv:math.CT/0209035 v1 4 Sep 2002.

[Bu] A. Burroni, Higher-dimensional word problems with applications to equational logic.

[Ha/M/Z] V. Harnik, M. Makkai and M. Zawadowski. Multitopic sets are the same as many-to-one computads. Preprint, June 2002. At: www.math.mcgill.ca/makkai.

[He/M/Po] C. Hermida, M. Makkai and A. J. Power. On weak higher-dimensional categories I. *J. Pure and Applied Algebra*; Part 1: **153** (2000), 221-246; Part 2: **157** (2001), 247-277; Part 3: **166** (2002), 83-104.

[J] M. Johnson, The combinatorial of n-categorical pasting. *J. Pure and Applied Algebra* **62** (1989), 211-225.

[L/Sc] J. Lambek and P. J. Scott, *Introduction to Higher Order Categorical Logic*. Cambridge University Press 1986.

[M1] M. Makkai, *First Order Logic with Dependent Sorts with Applications to Category Theory*. Preprint, 1995. Accepted for publication for Lecture Notes in Logic, Springer Verlag, 1995. At: www.math.mcgill.ca/makkai.

[M2] M. Makkai, Towards a Categorical Foundation of Mathematics. In: *Logic Colloquium '95*, Lecture Notes in Logic, Springer Verlag, **11** (1998); pp. 153-190.

[M3] M. Makkai, The multitopic ω-category of all multitopic ω-categories. September, 1999; corrected version: June, 2004. At: www.math.mcgill.ca/makkai.

[M4] M. Makkai, A 2-categorical pasting theorem: revisiting John Power's paper of the same title. In preparation.

[Pa] Thorsten Palm, Dendrotopic sets.

[Pe] J. Penon, Approche polygraphique des ∞-categories non strictes. *Cahiers de Topologie et Geometrie Differentielle Categoriques*, **XL–1** (1999), 31-80.

[Po1] A. J. Power, A 2-categorical pasting theorem. *J. Algebra* **129** (1990), 439-445.

[Po2] A. J. Power, An n-categorical pasting theorem. In: Category Theory, Proceedings, Como 1990, *Lecture Notes in Mathematics*, vol. **1488**, Springer 1991; pp. 326-358.

[Ste] R. Steiner, The algebra of directed complexes. *Applied Categorical Structures* **1** (1993), 247-284.

[Str1] R. Street, Limits indexed by category-valued 2-functors. *J. Pure and Applied Algebra* **8** (1976), 148-181.

[Str2] R. Street, The algebra of oriented simplexes. *J. Pure and Applied Algebra* **49** (1987), 283-335.

[Str3] R. Street, Categorical Structures. In: *Handbook of Algebra*, vol 1, M. Hazewinkel (ed.). Elsevier Science 1996; pp. 531-577.