INTERVAL OSCILLATION OF A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH A DAMPING TERM

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Abstract. Using generalized Riccati transformations, we derive new interval oscillation criteria for a class of second order nonlinear differential equations with damping. Our theorems prove to be efficient in many cases where known results fail to apply.

1. Introduction. Oscillatory behavior of nonlinear second-order differential equations with damping has been attracting attention of researchers during the last few decades; see the papers [1]-[16] and the references cited there.

In this paper, we are concerned with a second order nonlinear differential equation

\[(r(t)\psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0,\]  

where \(t \in I = [t_0, +\infty), r \in C^1(I,(0, +\infty)), p, q \in C(I, \mathbb{R}), f \in C(\mathbb{R}, \mathbb{R}), \psi \in C^1(\mathbb{R}, (0, +\infty)),\) and \(xf(x) > 0\) for all \(x \neq 0.\) We assume that solutions of Eq. (1) exist for all \(t \geq t_0.\) A non-constant solution \(x(t)\) is called proper if \(\sup_{t \geq t_0} |x(t)| > 0.\) A proper solution \(x(t)\) is called oscillatory if it has no largest zero and non-oscillatory otherwise. A differential equation is called oscillatory if all its proper solutions are oscillatory.

Numerous results regarding oscillation of Eq. (1) and other classes of nonlinear differential equations with damping rely on so-called integral averaging technique which requires information on the behavior of coefficients of Eq. (1) on the entire positive semi-axis; see, for instance, the papers by Grace [2, 3], Grace and Lalli [4], Kirane and Rogovchenko [6], Mustafa et al. [7], Rogovchenko and Rogovchenko [10, 11], Tiryaki and Zafer [14]. Unfortunately, many theorems fail to apply to equations where \(p(t)\) is oscillatory or changes sign, cf. [6]. Criteria obtained for Eq.
demonstrate the efficiency of new theorems. Two illustrative examples are considered in the final part of the paper to show the applicability of the obtained results. The approach called for development of a new technique, referred to as an interval oscillation method, which uses information on the behavior of coefficients of a given differential equation only on an infinite sequence of intervals; see, for instance, El-Sayed [1], Kong [7], Li and Agarwal [8], Rogovchenko and Tuncay [12], Sun [13], Tiryaki and Zafer [14], Zheng [15].

In this paper, exploiting generalized Riccati-type transformations, we obtain for Eq. (1) efficient interval oscillation criteria which establish oscillatory nature of the class $W$ investigated by Tiryaki and Zafer [15], Zheng [16], Sayed [1], Kong [7], Li and Agarwal [8], Rogovchenko and Tuncay [12], Sun [13], Tiryaki and Zafer [14], Zheng [15].

Let $D = \{ (t, s) \mid -\infty < s \leq t < +\infty \}$. We say that a function $H(t, s)$ belongs to the class $W$ if

(i) $H \in C(D, [0, +\infty))$;

(ii) $H(t, t) = 0$ and $H(t, s) > 0$ for $-\infty < s < t < +\infty$;

(iii) $H$ has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ satisfying

$$
\frac{\partial H}{\partial t} = h_1(t, s)\sqrt{H(t, s)} \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t, s)\sqrt{H(t, s)},
$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

The following auxiliary result is a particular case of [15, Lemma 1.1].

**Lemma 1.** Suppose that a function $u \in C^1(D, \mathbb{R})$ satisfies the inequality

$$
u'(t) \leq -\alpha(t) - \beta(t)u^2(t),$$

for all $t \in (a, b) \subset I$, where the functions $\alpha \in C(I, \mathbb{R})$, $\beta \in C(I, (0, +\infty))$. Then, for any $c \in (a, b)$ and for any $H \in W$,

$$
\frac{1}{H(c, a)} \int_a^c \left[ \alpha(s)H(s, a) - \frac{1}{4\beta(s)}h_1^2(s, a) \right] ds + \frac{1}{H(b, c)} \int_c^b \left[ \alpha(s)H(b, s) - \frac{1}{4\beta(s)}h_2^2(b, s) \right] ds \leq 0.
$$

**2. Increasing nonlinearity.**

**Theorem 1.** Let $f$ be continuously differentiable and satisfy, for all $x \in \mathbb{R}$,

$$
f'(x) \geq \mu > 0.
$$

Assume that, for all $x \in \mathbb{R}$,

$$
0 < C \leq \psi(x) \leq C_1.
$$

Suppose also that there exists a function $g \in C^1(I, \mathbb{R})$ such that, for some $H \in W$ and $c \in (a, b)$,

$$
\frac{1}{H(c, a)} \int_a^c \left[ \varphi(s)H(s, a) - \frac{C_1}{4\mu}\nu(s)r(s)h_1^2(s, a) \right] ds + \frac{1}{H(b, c)} \int_c^b \left[ \varphi(s)H(b, s) - \frac{C_1}{4\mu}\nu(s)r(s)h_2^2(b, s) \right] ds > 0,
$$

where

$$
\varphi(t) = \nu(t) \left[ q(t) - g'(t) + \frac{g(t)(\mu g(t) - p(t))}{C_1 r(t)} + \left( \frac{1}{C_1} - \frac{1}{C} \right) \frac{p^2(t)}{4\mu r(t)} \right]
$$

and

$$
\frac{\partial H}{\partial t} = \alpha(t) - \beta(t)u^2(t),
$$

for all $t \in (a, b) \subset I$, where the functions $\alpha \in C(I, \mathbb{R})$, $\beta \in C(I, (0, +\infty))$. Then, for any $c \in (a, b)$ and for any $H \in W$,
and
\[ v(t) = \exp \left[ \int^t p(s) - 2\mu g(s) ds \right]. \quad (6) \]

Then every solution of Eq. (1) has at least one zero in \((a, b)\).

**Proof.** For the sake of contradiction, assume that there exists a solution \(x(t)\) of Eq. (1) that has no zeros in \((a, b)\), and thus has constant sign in this interval. Without loss of generality, suppose that \(x(t) > 0\) for all \(t \in (a, b)\). Let \(g \in C^1(I, \mathbb{R})\) and let \(v\) be defined by (6). Introduce a new function \(u\) by
\[ u(t) = v(t) \left( \frac{r(t)\psi(x(t))x'(t)}{f(x(t))} + g(t) \right). \quad (7) \]

Note that \(u\) is well-defined because \(f(x(t)) > 0\) for \(t \in (a, b)\). Differentiating (7) and using (1)-(3), (6), and (7), we obtain
\[ u'(t) = \frac{v'(t)}{v(t)} u(t) - v(t) \left\{ p(t) \frac{x'(t)}{f(x(t))} + q(t) - g'(t) \right\} \]
\[ + r(t) \psi(x(t)) \left( \frac{x'(t)}{f(x(t))} \right)^2 f'(x(t)) \leq \frac{v'(t)}{v(t)} u(t) - v(t) (q(t) - g'(t)) \]
\[ - \frac{1}{r(t) \psi(x(t))} \left\{ \mu \frac{u^2(t)}{v(t)} - 2\mu g(t) u(t) + p(t) u(t) + \mu g^2(t) v(t) - p(t) v(t) g(t) \right\} \]
\[ = \frac{v'(t)}{v(t)} u(t) - v(t) (q(t) - g'(t)) \]
\[ - \frac{1}{r(t) \psi(x(t))} \left\{ \left( u(t) \sqrt{\frac{\mu}{v(t)}} + \frac{1}{2} \left( p(t) - 2\mu g(t) \right) \sqrt{\frac{v(t)}{\mu}} \right)^2 - \frac{1}{4\mu} p^2(t) v(t) \right\} \]
\[ \leq \frac{v'(t)}{v(t)} u(t) - v(t) (q(t) - g'(t)) + \frac{1}{4\mu Cr(t)} p^2(t) v(t) \]
\[ - \frac{1}{C_1 r(t)} \left( u(t) \sqrt{\frac{\mu}{v(t)}} + \frac{1}{2} \left( p(t) - 2\mu g(t) \right) \sqrt{\frac{v(t)}{\mu}} \right)^2, \]

which finally yields
\[ u'(t) \leq -\varphi(t) - \frac{\mu}{C_1 v(t)r(t)} u^2(t), \quad (8) \]

where \(\varphi\) is defined by (5). Applying to (8) Lemma 1 we conclude that, for any \(c \in (a, b)\) and any \(H \in W\),
\[ \frac{1}{H(c, a)} \int_a^c \left[ \varphi(s) H(s, a) - \frac{C_1}{4\mu} v(s) r(s) h_1^2(s, a) \right] ds \]
\[ + \frac{1}{H(b, c)} \int_c^b \left[ \varphi(s) H(b, s) - \frac{C_1}{4\mu} v(s) r(s) h_2^2(b, s) \right] ds \leq 0, \]

which contradicts (4). Thus, \(x(t)\) has at least one zero in \((a, b)\). \(\square\)

The following result is an immediate consequence of Theorem 1.

**Theorem 2.** Assume that conditions (2), (3) are satisfied, and suppose that, for any \(\tau \geq t_0\), there exist a function \(g \in C^1(I, \mathbb{R})\) and real numbers \(a, b, \tau \leq a < b\) such that, for some \(H \in W\) and \(c \in (a, b)\), (4) holds with \(\varphi\) and \(v\) defined by (5) and (6) respectively. Then Eq. (1) is oscillatory.
Proof. Pick a sequence of real numbers, \( t_0 \leq \tau_1 \leq \tau_2 \leq \cdots \), satisfying \( \tau_n \to +\infty \) as \( n \to +\infty \). By the assumptions of the theorem, for any \( \tau_n \), there exist a function \( g \in C^1(I, \mathbb{R}) \) and a pair of real numbers \( a_n, b_n, \tau_n \leq a_n < b_n \) such that, for some \( H \in \mathcal{W} \) and some \( c_n \in (a_n, b_n) \),

\[
\frac{1}{H(c_n, a_n)} \int_{a_n}^{c_n} \left[ \varphi(s)H(s, a_n) - \frac{C_1}{4\mu} v(s)r(s)h_2^2(s, a_n) \right] ds \\
+ \frac{1}{H(b_n, c_n)} \int_{c_n}^{b_n} \left[ \varphi(s)H(b_n, s) - \frac{C_1}{4\mu} v(s)r(s)h_2^2(b_n, s) \right] ds > 0.
\]

Then, by virtue of Theorem 1, any solution \( x(t) \) of Eq. (1) has at least one zero in \((a_n, b_n)\). Taking into account that \( a_n \to +\infty \) and \( b_n \to +\infty \) as \( n \to +\infty \), we conclude that the sequence of zeros of \( x(t) \) diverges to +\( \infty \), that is, \( x(t) \) oscillates. Therefore, Eq. (1) is oscillatory.

In applications, a class of difference kernels \( \mathcal{W}_0 \subset \mathcal{W} \) plays an important role. It contains all functions \( H \in \mathcal{W} \) that satisfy \( H(t, s) = H(t - s) \), in which case \( h_1(t - s) = h_2(t - s) \) if \( t \to +\infty \). For the class \( \mathcal{W}_0 \), a consequence of Theorem 2 is the following result.

**Theorem 3.** Assume that conditions \( [2], [3] \) are satisfied. Suppose further that, for any \( \tau \geq t_0 \), there exist a function \( g \in C^1(I, \mathbb{R}) \) and real numbers \( a, c, \tau \leq a < c \) such that, for some \( H \in \mathcal{W}_0 \),

\[
\int_a^c H(s - a) [\varphi(s) + \varphi(2c - s)] ds \\
> \frac{C_1}{4\mu} \int_a^c [v(s)r(s) + v(2c - s)r(2c - s)] h^2(s - a) ds,
\]

(9)

where \( v \) and \( \varphi \) are as in Theorem 2. Then Eq. (1) is oscillatory.

Proof. In Theorem 2 let \( c = 2^{-1}(a + b) \). Then,

\[
H(b - c) = H(c - a) = H \left( \frac{b - a}{2} \right).
\]

Since, for any integrable on \([a, b]\) function \( f \), one has

\[
\int_{(a+b)/2}^b f(t) dt = \int_a^{(a+b)/2} f(a + b - s) ds,
\]

we conclude that

\[
\frac{1}{H(c - a)} \int_a^c H(s - a) \varphi(s) ds + \frac{1}{H(b - c)} \int_c^b H(b - s) \varphi(s) ds \\
= \frac{1}{H(c - a)} \int_a^c H(s - a) [\varphi(s) + \varphi(2c - s)] ds
\]

and

\[
\frac{1}{H(c - a)} \int_a^c [v(s)r(s)h^2(s - a)] ds + \frac{1}{H(b - c)} \int_c^b [v(s)r(s)h^2(b - s)] ds \\
= \frac{1}{H(c - a)} \int_a^c [v(s)r(s) + v(2c - s)r(2c - s)] h^2(s - a) ds.
\]

Thus, (9) yields (4), and Eq. (1) is oscillatory by Theorem 2. \( \Box \)
Choosing $H(t,s) = (t-s)^n$, $n > 1$, in Theorem 3, we obtain the following useful oscillation criterion.

**Corollary 1.** Assume that (2), (3) hold and, for any $\tau \geq t_0$, there exist a function $g \in C^1(I, \mathbb{R})$ and real numbers $a, c, \tau \leq a < c$ such that

$$
\int_a^c (s-a)^n [\varphi(s) + \varphi(2c-s)] ds \\
> \frac{C_1n^2}{4\mu} \int_a^c [v(s)r(s) + v(2c-s)r(2c-s)] (s-a)^{n-2} ds,
$$

where $\varphi$ and $\psi$ are as in Theorem 1. Then Eq. (1) is oscillatory.


**Theorem 4.** Let $f(x)$ be continuous on $\mathbb{R}$ and satisfy, for all $x \in \mathbb{R}$,

$$
\frac{f(x)}{x} \geq \mu > 0. \tag{10}
$$

Assume that $[3]$ holds and, for all $t \in (a,b) \subset I$,

$$
q(t) \geq 0, \tag{11}
$$

where $q(t)$ is not identical zero on $(a,b)$. Suppose also that there exists a function $g \in C^1(I, \mathbb{R})$ such that, for some $H \in \mathbb{W}$ and some $c \in (a,b)$,

$$
\frac{1}{H(c,a)} \int_a^c \left[ \varphi(s)H(s,a) - \frac{C_1}{4} v(s)r(s)h_1^2(s,a) \right] ds \\
+ \frac{1}{H(b,c)} \int_c^b \left[ \varphi(s)H(b,s) - \frac{C_1}{4} v(s)r(s)h_2^2(b,s) \right] ds > 0, \tag{12}
$$

where

$$
\varphi(t) = v(t) \left[ \mu q(t) - g'(t) + \frac{g(t)(g(t) - p(t))}{C_1r(t)} + \left( \frac{1}{C_1} - \frac{1}{C} \right) \frac{p^2(t)}{4r(t)} \right]; \tag{13}
$$

and

$$
v(t) = \exp \left[ \int_t^b \frac{p(s) - 2g(s)}{C_1r(s)} ds \right]. \tag{14}
$$

Then every solution of Eq. (1) has at least one zero in $(a,b)$.

**Proof.** As in Theorem 1 assume, without loss of generality, that there exists a solution $x(t)$ of Eq. (1) such that $x(t) > 0$ for $t \in (a,b)$. Let $g \in C^1(I, \mathbb{R})$ and $v$ be defined by (14). Introduce $u$ by

$$
u(t) = v(t) \left[ \frac{r(t)\psi(x(t))x'(t)}{x(t)} + g(t) \right]. \tag{15}$$
Differentiating (15) and using (1), we obtain
\[ u'(t) = \frac{v'(t)}{v(t)}u(t) - v(t) \left( q(t) \frac{f(x(t))}{x(t)} - g'(t) \right) \]
\[ - \frac{1}{r(t)\psi(x(t))} \left\{ \frac{u^2(t)}{v(t)} - 2g(t)u(t) + p(t)u(t) + g^2(t)v(t) - p(t)v(t)g(t) \right\} \]
\[ = \frac{v'(t)}{v(t)}u(t) - v(t) \left( q(t) \frac{f(x(t))}{x(t)} - g'(t) \right) \]
\[ - \frac{1}{r(t)\psi(x(t))} \left\{ \frac{u(t)}{\sqrt{v(t)}} + (p(t) - 2g(t)) \frac{\sqrt{v(t)}}{2} \right\}^2 - \frac{1}{4}p^2(t)v(t) \].

Using (3), (10), and (11), one has
\[ u'(t) \leq \frac{v'(t)}{v(t)}u(t) - \mu q(t)v(t) + v(t)g'(t) \]
\[ - \frac{1}{C_1r(t)} \left( \frac{u(t)}{\sqrt{v(t)}} + (p(t) - 2g(t)) \frac{\sqrt{v(t)}}{2} \right)^2 + \frac{1}{4C_1r(t)p^2(t)v(t)}, \]
which yields
\[ u'(t) \leq -\varphi(t) - \frac{1}{C_1r(t)r(t)}u^2(t), \tag{16} \]
where \( \varphi \) is defined by (13). Applying to (16) Lemma 1 with \( \alpha(t) = \varphi(t) \) and \( \beta(t) = (C_1v(t)r(t))^{-1} \), we conclude that, for any \( c \in (a, b) \) and any \( H \in \mathcal{W} \),
\[ \frac{1}{H(c, a)} \int_a^c \left[ \varphi(s)H(s, a) - \frac{C_1}{4}v(s)r(s)h_1^2(s, a) \right] ds \]
\[ + \frac{1}{H(b, c)} \int_b^c \left[ \varphi(s)H(b, s) - \frac{C_1}{4}v(s)r(s)h_2^2(b, s) \right] ds \leq 0, \]
which contradicts (12). Hence, \( x(t) \) has at least one zero in \( (a, b) \).

The following result is an immediate consequence of Theorem 4 and can be established with a minor modification of the proof of Theorem 2.

**Theorem 5.** Assume that (3) and (10) are satisfied, and, for any \( \tau \geq t_0 \), there exist a function \( g \in C^1(I, \mathbb{R}) \) and real numbers \( a, b, \tau \leq a < b \) such that condition (11) is met and, for some \( H \in \mathcal{W} \) and some \( c \in (a, b) \), (12) holds with \( v \) and \( \varphi \) defined as in Theorem 4. Then Eq. (1) is oscillatory.

Next result is similar to Corollary 1 and is deduced analogously by proving first a counterpart of Theorem 3.

**Corollary 2.** Assume that (3) and (10) hold and, for any \( \tau \geq t_0 \), there exist a function \( g \in C^1(I, \mathbb{R}) \) and real numbers \( a, c, \tau \leq a < c \) such that (11) is satisfied and
\[ \int_a^c (s-a)^n \left[ \varphi(s) + \varphi(2c-s) \right] ds \]
\[ \geq \frac{C_1n^2}{4} \int_a^c \left[ v(s)r(s) + v(2c-s)r(2c-s) \right] (s-a)^{n-2}ds, \]
where \( v \) and \( \varphi \) are as in Theorem 4. Then Eq. (1) is oscillatory.
Remark 1. Note that replacing the assumption \( [2] \) of monotonicity of \( f \) with the condition \([10]\), we do not practically change the appearance of the oscillation criteria, although different classes of equations are treated. Using instead of \([15]\) transformations

\[
u(t) = v(t) \left[ \frac{r(t)\psi(x(t))x'(t)}{x(t)} + \beta g(t) \right]
\]

or

\[
u(t) = v(t)r(t) \left[ \frac{\psi(x(t))x'(t)}{x(t)} + \beta g(t) \right],
\]

where \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), one can select

\[
v(t) = \exp \left[ \int_{t_0}^{t} \frac{\alpha p(s) - 2\beta g(s)}{\alpha C_1 r(s)} ds \right]
\]

and adjust \( \varphi \) accordingly to derive sufficient conditions for oscillation that look a bit different but are essentially the same. This can be verified by applying various modifications to the given differential equation with the same choice of \( \nu(t) \). For instance, using in Theorem \([4]\) transformation \([17]\) with \( \alpha = 1 \) and \( \beta = \mu \) instead of \([15]\), one obtains condition \([4]\) with \( \varphi \) and \( v \) defined as in Theorem \([1]\). A similar reasoning applies to generalized Riccati substitution \([7]\).

Interval oscillation criteria are especially efficient when applied to differential equations with a “weird” behavior of coefficients on the entire interval \( I \). In Example \([1]\) one has

\[
\int_{t_0}^{\infty} q(t) dt = -\infty,
\]

and standard oscillation criteria derived in \([2, 3, 6, 9, 11]\) and \([13]\) by using integral averaging method cannot be used to prove its oscillatory character.

Example 1. Consider Eq. \([1]\) on \([3, +\infty)\) with \( r(t) = t^{-1}, p(t) = t^{-2}, \)

\[
q(t) = \begin{cases} 
\gamma (t - 3n), & 3n \leq t \leq 3n + 1, \\
-\gamma (t - 3n - 2), & 3n + 1 < t \leq 3n + 2, \\
e^x (t - 3n - 2)(t - 3n - 3), & 3n + 2 < t \leq 3n + 3,
\end{cases}
\]

\( \psi(x) = (x^2 + 1)(x^2 + 2)^{-1} \) and \( f(x) = 4x + x^3 \). Applying Corollary \([1]\) with \( a = 3n, c = 3n + 1, H(t-s) = (t-s)^2, g(t) = 0, v(t) = t \), we conclude that the given equation is oscillatory for

\[
\gamma > \max_{n \geq 1} \left( \frac{15}{2\mu(30n + 11)} \right) \left( 6 + 3n \ln \left( \frac{3n \ln (3n + 2)}{3n + 2} \right) + 2\ln \left( \frac{3n + 1}{3n + 2} \right) \right)
\]

\[
= 15 \left( 6 + \ln \left( \frac{432}{3125} \right) \right) \approx 0.18390.
\]

Example 2. Consider Eq. \([1]\) on \([2, +\infty)\) with \( f(x) = x \left( 1/8 + (1 + x^2)^{-1} \right) \),

\[
q(t) = \begin{cases} 
\gamma (t - 2n), & 2n \leq t \leq 2n + 1, \\
-\gamma (t - 2n)(t - 2n - 2), & 2n + 1 < t \leq 2n + 2,
\end{cases}
\]

and all other functions as above. Then \( f'(x) = (8(1 + x^2))^{-2}(x^2 - 3)^2 \), and \( f \) does not satisfy \([2]\). An application of Corollary \([2]\) with \( a = 2n, c = 2n + 1, H, g \)
and $v$ as in Example 1 yields oscillation of the given equation for
\[ \gamma > \max_{n \geq 1} \left( \frac{120}{2n+11} \left( 3 - \ln \frac{2(n+1)}{2n+1} - n \ln \frac{n+1}{n} \right) \right) \]
\[ = \frac{120}{31} \left( 3 - \ln \frac{8}{3} \right) \approx 7.8161. \]

**Remark 2.** Note that to apply general oscillation criteria due to Tiryaki and Zafer [15], one has to require that, for some $\alpha_2, \alpha_3 > 0$ and all $u, v \in \mathbb{R}$, either
\[ vf(u) \geq \alpha_2 \psi^2(u)v^2 \]

or
\[ uv \geq \alpha_3 \psi^2(u)v^2, \]

and both conditions fail to hold in our examples.

**Remark 3.** Oscillation criteria reported in this paper are also new in the case where $\psi(x) = 1$ and do not reduce to theorems established for the differential equation
\[ (r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0 \]

by Li and Agarwal [8], Rogovchenko and Tuncay [12], Sun [13] or Zheng [16].

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