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A Strongly Polynomial Rounding Procedure Yielding
A Maximally Complementary Solution for
$P_*(\kappa)$ Linear Complementarity Problems

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Abstract

We deal with Linear Complementarity Problems (LCPs) with $P_*(\kappa)$ matrices. First we establish the convergence rate of the complementary variables along the central path. The central path is parameterized by the barrier parameter μ, as usual. Our elementary proof reproduces the known result that the variables on, or close to the central path fall apart in three classes in which these variables are $\mathcal{O}(1)$, $\mathcal{O}(\mu)$ and $\mathcal{O}(\sqrt{\mu})$, respectively. The constants hidden in these bounds are expressed in, or bounded by, the input data. All this is preparation for our main result: a strongly polynomial rounding procedure. Given a point with sufficiently small complementarity gap and close enough to the central path, the rounding procedure produces a maximally complementary solution in at most $\mathcal{O}(n^3)$ arithmetic operations.

The result implies that Interior Point Methods (IPMs) not only converge to a complementary solution of $P_*(\kappa)$ LCPs but, when furnished with our rounding procedure, they can produce a maximally complementary (exact) solution in polynomial time.

Key words: *linear complementarity problems, $P_*(\kappa)$ matrices, error bounds on the size of the variables, optimal partition, maximally complementary solution, rounding procedure.*

AMS Subject Classification : **90C33**

Contents

1 Introduction

In this paper we deal with a class of Linear Complementarity Problems (LCPs):

$$(LCP) \qquad s(x) := Mx + q \geq 0, \quad x \geq 0, \quad xs(x) = 0, \tag{1}$$

where M is an $n \times n$ real matrix, $q \in \mathbb{R}^n$ and $xs(x)$ denotes the coordinatewise product of the vectors x and $s(x)$. We say that an algorithm solves (LCP) if either it produces a vector x satisfying the constraints of (LCP) or it provides a certificate that no such vector exist. In the first case we say that x solves (LCP).

The vector x is a *strictly complementary solution* of (LCP) if it solves (LCP) and $x + s(x) > 0$. Contrary to Linear Optimization (LO) [25], in general no strictly complementary solution exists for (LCP): there might exist pairs of complementary variables x_i and $s_i(x)$ that are both zero in all solutions of the (LCP). Complementary solutions with the maximal number of nonzero coordinates will be referred to as *maximally complementary solutions*. The existence of maximally complementary solutions follows from the convexity of the solution set, proved by Cottle et al. in [4]. Kojima et al. [17] under some additional assumptions showed that solutions on the central path converge to a maximally complementary solution of (LCP).

All known algorithms for solving (LCP) need some assumption on the matrix M. So do Interior Point Methods (IPMs) as well. IPMs for solving (LCP) are widely studied in the last decade. A survey on recent results is written by Yoshise [33]. Kojima, Mizuno and Yoshise [14] presented a polynomial time algorithm that produces an exact solution for LCPs where M is positive semidefinite. The same authors [15] established an $\mathcal{O}(\sqrt{n}L)$[1] iteration bound for a *potential reduction* algorithm. Ji, Potra and Huang [10] developed a polynomial, $\mathcal{O}(\sqrt{n}L)$ *predictor-corrector* method for positive semidefinite LCPs under the assumption that the sequence of iterates generated by their interior-point algorithm converges to a strictly complementary solution. Later, Ye and Anstreicher [31] proved the same iteration bound, $\mathcal{O}(\sqrt{n}L)$ for predictor-corrector methods, removing the assumption given in [10]. In 1991, Kojima et al. [17] extended all the previously known results to the wider class of so called $P_*(\kappa)$ LCPs and unified the theory of LCPs from the view point of interior point methods. Jansen, Roos and Terlaky [9] introduced a family of *primal-dual affine-scaling* algorithms for positive semidefinite LCPs. These results were recently extended to LCPs with $P_*(\kappa)$ matrices by Illés, Roos and Terlaky [8]. The iteration bound of those algorithms are $\mathcal{O}((1 + 4\kappa)n \log x_0^T s(x_0)/\epsilon)$, where x_0 is the initial iterate and ϵ is the *complementarity gap* $x^T s(x)$ at termination.

Interior-point methods need an interior feasible point to start with. Among others, Ji, Potra and Sheng [11] studied the initialization problem and proposed a predictor-corrector method for solving the $P_*(\kappa)$ LCPs from infeasible starting points. Kojima et al. [16, 17] gave a big-M construction that allows to solve the problem in one phase.

The aim of this paper is twofold. First we derive some bounds on the magnitude of the variables in the vicinity of the central path[2], when the complementarity gap is small enough. Second, a strongly polynomial rounding procedure is presented that provides a maximally complementary (exact) solution from any interior point solution that is in a certain neighborhood of the central path and for which the complementarity gap is sufficiently small.

For deriving results on the magnitude of the variables in a given neighborhood of the central path we use some known results from the theory of error bounds for systems of linear inequalities [20]. The theory of error bounds goes back to the early fifties [7]; for recent developments we refer to the survey paper [23] and the references therein. For LCPs, a well-known local error bound is given by Robinson

[1] L is the binary input length of the problem [17].
[2] The central path is defined in the usual way. See Section 2.

[24] which says that there exists a constant $\epsilon > 0$ and $\tau > 0$ such that

$$\mathrm{dist}(x, ?^*) \leq \tau \| \min(x, s(x)) \|, \qquad (2)$$

for all x satisfying $\| \min(x, s(x)) \| \leq \epsilon$, where $?^*$ denotes the solution set of (LCP) in $\mathbb{R}^n_+$, $\mathrm{dist}(x, ?^*) = \min_{y \in \Gamma} \|y - x\|$ and the minimum $\min(x, s(x))$ is taken coordinatewise. By using the properties of the central path and some results on error bounds of Cook et al. [3] and Mangasarian and Shiau [3, 20], we derive some bounds on these constants in terms of the input data if x is on or close to the central path. To the best of our knowledge, this is the first result yielding easy to calculate bounds for these constants in the study of LCPs.

The bounds on the magnitude of the variables along the central path depend on the dimension n of the problem, on the parameter κ and the barrier parameter μ that parameterizes the central path, and on two condition numbers σ_{LCP} and ν_{LCP} of (LCP). The condition number σ_{LCP} is closely related to that defined by Ye [32] and studied by Vavasis and Ye [28] for polyhedra with real number data and slightly modified by Roos, Vial and Terlaky [25] for the case of LO problems. The second condition number, ν_{LCP}, will be introduced later. Other condition numbers for LCP are defined in [18, 29]. It will be shown in a quite elementary way that in a given neighborhood of the central path the variables fall apart in three classes and their magnitudes are $\mathcal{O}(1), \mathcal{O}(\mu)$ and $\mathcal{O}(\sqrt{\mu})$ respectively, provided the parameter μ is sufficiently small.

The rounding procedure we describe for (LCP) resembles the one presented in the papers [30, 21] and in the book [25]. We show that IPMs with a rounding procedure terminate in a finite (polynomial) number of iterations and yield a maximally complementary solution.

There are some other methods [14, 18] in the literature that generate an exact solution to (LCP) in $\mathcal{O}(n^3 L)$ iterations, but those are different from ours and do not generate a maximally complementary solution. Kojima, Mizuno and Yoshise [14] in Appendix B of their paper, presented a method which leads to a basic solution of the LCPs, thus not providing a maximally complementary solution. They compute a solution, $\bar{z} = (\bar{x}, \bar{s})$ such that the variables can be split into the complementary sets $I_1 = \{k : \bar{z}_k < 2^{-L}\}$ and $I_2 = \{k : \bar{z}_k \geq 2^{-L}\}$ in such a way that the submatrix of $(E, -M)$ corresponding to I_2 contains only linearly independent columns and by setting the variables for I_1 equal to zero a complementary solution of 1 is obtained. Kojima et al. ([17], page 16) pointed out that: "Practically, however, it might be too complicated to compute with the number 2^{-2L} because it is too small."[3] The required complementarity gap for our rounding procedure is bigger in general (see Section 5), however, we do not claim that the accuracy theoretically needed to start our rounding procedure is practically reachable.

Our rounding procedure generates a maximally complementary (exact) solution, while all the previously known rounding procedures produce a complementary basic solution, which in general, is not maximally complementary. Having a maximally complementary solution, a complementary basic solution can be computed in strongly polynomial time by using the basis identification procedure described in Berkelaar et al. [2]. However, no strongly polynomial algorithm is known to generate maximally complementary solution from a complementary basic solution.

The paper is organized as follows. Some preliminary results are discussed in Section 2. In Section 3 the optimal partition is defined and the related concept of maximally complementary solutions. We introduce two condition numbers for (LCP) and derive local bounds on the magnitude of the variables on the central path. The main result in this section describes how the optimal partition can be determined if the barrier parameter μ is small enough. In Section 4 we generalize the results of Section 3 to points that are close to the central path, so-called approximate centers, and we show that the optimal partition can be identified from x if x belongs to a certain neighborhood of the central path and if $x^T s(x)$ is small

[3] The value 2^{-2L} is an upper bound on the required complementarity gap that is sufficient to identify a complementary basic solution.

enough; such a vector x can be obtained in polynomial time by any interior point method. Section 5 presents a strongly polynomial rounding procedure that yields a maximally complementary solution. Some concluding remarks close the paper in Section 6.

Throughout, we shall use $\|\cdot\|_p$ ($p \in [1, \infty]$) to denote the p-norm on $\mathbb{R}^n$, with $\|\cdot\|$ denoting the Euclidean norm $\|\cdot\|_2$. E will denote the identity matrix, e will be used to denote the vector which has all its components equal to one. Given an n-dimensional vector x, we denote by X the $n \times n$ diagonal matrix whose diagonal entries are the coordinates x_j of x. If x, $s \in \mathbb{R}^n$ then $x^T s$ denotes the dot product of the two vectors. Further, xs and x^α for $\alpha \in \mathbb{R}$ will denote the vectors resulting from coordinatewise operations. For any matrix $A \in \mathbb{R}^{m \times n}$, A_i, $A_{.j}$ are the i-th row and the j-th column of A, respectively. Furthermore,

$$\pi(A) := \prod_{j=1}^{n} \|A_{.j}\|.$$

For any index set $J \subset \{1, 2, \ldots, m\}$, $|J|$ denotes the cardinality J and $A_J \in \mathbb{R}^{|J| \times n}$ the submatrix of A whose rows are indexed by elements in J. Moreover, if $K \subset \{1, 2, \ldots, n\}$, $A_{JK} \in \mathbb{R}^{|J| \times |K|}$ denotes the submatrix of A_J whose columns are indexed by elements in K.

2 Preliminaries

For further use we first recall some well-known results and definitions. The reader may consult the papers [17] and [33] for proofs and details.

A matrix $M \in \mathbb{R}^{n \times n}$ is a $P_*(\kappa)$ matrix if

$$(1 + 4\kappa) \sum_{i \in I_+(x)} x_i [Mx]_i + \sum_{i \in I_-(x)} x_i [Mx]_i \geq 0, \quad \forall x \in \mathbb{R}^n, \tag{3}$$

where

$$I_+(x) := \{1 \leq i \leq n \ : \ x_i [Mx]_i > 0\}, \quad I_-(x) := \{1 \leq i \leq n \ : \ x_i [Mx]_i < 0\},$$

and κ is a nonnegative real number. Note that the index sets $I_+(x)$ and $I_-(x)$ depend not only on x but on the matrix M as well. The matrix M is a P_* matrix if it is a $P_*(\kappa)$ matrix for some nonnegative κ:

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa).$$

One easily verifies that M is a $P_*(0)$ matrix if and only if M is positive semidefinite. Furthermore, if M is $P_*(\overline{\kappa})$ for some $\overline{\kappa} \geq 0$ then M is $P_*(\kappa)$ for all $\kappa \geq \overline{\kappa}$.

The class of *sufficient* matrices (SU) was introduced by Cottle et al. [4]. A matrix $M \in \mathbb{R}^{n \times n}$ is *column sufficient* if for all $x \in \mathbb{R}^n$,

$$X(Mx) \leq 0 \Rightarrow X(Mx) = 0$$

and *row sufficient* if M^T is column sufficient. The matrix M is *sufficient* if it is both row and column sufficient. Recently, Väliaho [27] proved that $P_* = SU$.

The sets of feasible and positive feasible vectors are denoted respectively by

$$? = \{x \ : \ x \geq 0, \, s(x) \geq 0\},$$
$$?^0 = \{x \ : \ x > 0, \, s(x) > 0\},$$

and the set of solutions of (LCP) by

$$?^* = \{x \ : \ x \geq 0, \ s(x) \geq 0, \ xs(x) = 0\}.$$

It is known (cf. [17], Theorem 4.6.) that if $M \in P_*$ and $? \neq \emptyset$ then $?^* \neq \emptyset$. Further, if $?^0 \neq \emptyset$, then $?^*$ is compact,[4] moreover for every $\mu > 0$ there exists a unique $x \in ?^0$ such that

$$xs(x) = \mu e.$$

In other words, assuming that $?^0$ is nonempty the *central path*

$$\mathcal{C} := \{x \in ?^0 \ : \ xs(x) = \mu e \text{ for some } \mu > 0\}$$

exists. Kojima et al. [17] showed that the assumption $?^0 \neq \emptyset$ can be made without loss of generality. Hence we may assume that the central path $\mathcal{C}$ exists. The central path $\mathcal{C}$ is a one-dimensional smooth curve that leads to a solution of (LCP) when μ approaches 0.[5]

We insert here the following technical lemma that will be used at several places below.

Lemma 2.1 *Let x be a solution of the equation $Dx = d$, where D is an integral and nonzero $m \times n$ matrix and d an integral vector. If J denotes the support of x and the columns of $D_{.J}$ are linearly independent, then*

$$\frac{1}{\Delta(D)} \leq |x_j| \leq \Delta(D) \|d\|_1, \quad j \in J.$$

Here $\Delta(D)$ denotes the largest absolute value of the determinants of the square submatrices of D. The right inequality holds also if d is not integral.

Proof: For completeness we include a proof here. Let x be as in the lemma and let the index set K be such that D_{KJ} is a nonsingular square submatrix of D; such K exists because the columns in $D_{.J}$ are linearly independent. Now we have $D_{KJ}x_J = d_K$, and hence, by Cramer's rule,[6]

$$x_j = \frac{det\left(D_{KJ}^{(j)}\right)}{det\left(D_{KJ}\right)}, \quad \forall j \in J, \tag{4}$$

where $D_{KJ}^{(j)}$ denotes the matrix arising when the j-th column in D_{KJ} is replaced by d. Since the denominator in the above quotient is at least 1 we obtain

$$|x_j| \leq \det\left(D_{KJ}^{(j)}\right), \quad j \in J. \tag{5}$$

By evaluating the last determinant to its j-th column, while using that each square submatrix is also a square submatrix of D, the right hand side inequality follows. For the left inequality we use that d is integral; since $x_j \neq 0$ this implies that the numerator in (4) is at least one. □

Corollary 2.1 *If the columns of D are all nonzero then, under the assumptions of Lemma 2.1,*

$$\frac{1}{\pi(D)} \leq |x_j| \leq \pi(D) \|d\|, \quad j \in J.$$

The right inequality holds also if d is not integral.

[4] The key observation for this is Lemma 4.5 of Kojima et al. [17].
[5] For further details we refer to Chapters 2 and 4 in [17].
[6] The idea of using Cramer's rule in this way was applied first by Khachiyan in [12].

Proof: If the columns of D are all nonzero, then the left inequality in Lemma 2.1 remains valid if we replace $\Delta(D)$ by $\pi(D)$. This is immediate from the well-known Hadamard inequality for determinants and because D and d are integral. The inequality at the right follows by applying Hadamard's inequality to (5).[7] □

3 The optimal partition and two condition numbers

In the rest of the paper we assume that $M \in P_*(\kappa)$ for some $\kappa \geq 0$. This implies that the matrix M is sufficient.

3.1 Optimal partition

Let us denote the index set $\{1, 2, \ldots, n\}$ by I and define the sets

$$\begin{aligned} B &:= \{i \in I \, : \, x_i > 0 \text{ for some } x \in ?^*\}, \\ N &:= \{i \in I \, : \, s_i(x) > 0 \text{ for some } x \in ?^*\}, \\ T &:= \{i \in I \, : \, x_i = s_i(x) = 0 \text{ for all } x \in ?^*\}. \end{aligned}$$

We show that these index sets are disjoint and $B \cup N \cup T = I$, i.e. they form the so-called *optimal partition* of the index set I with respect to (LCP).

Lemma 3.1 ([17]) *The index sets B, N and T form a partition of the index set I.*

Proof: ¿From the definition of the sets B, N and T it is obvious that $B \cap T = \emptyset$, $N \cap T = \emptyset$ and $I = B \cup N \cup T$. Let us assume that $B \cap N \neq \emptyset$. Then there exist $x', x'' \in ?^*$ such that $x'_j > 0, s_j(x') = 0, x''_j = 0$, and $s_j(x'') > 0$, for some $j \in I$. Let us denote $x := x' - x'', s' = s(x'), s'' = s(x''), s := s' - s''$ and $X := \mathrm{diag}(x)$. It is easy to see that

$$Xs = XMx \leq 0$$

and

$$x_j s_j = x_j (Mx)_j = (x'_j - x''_j)(s'_j - s''_j) = -x'_j s''_j < 0,$$

what contradicts the column sufficiency of the matrix M. □

Corollary 3.1 *Let x' and x'' solve (LCP), so $x', x'' \in ?^*$. Then $x's(x'') = 0$ and $x''s(x') = 0$.*

Proof: The definition of the classes B and N implies $\{i \in I \, : \, x'_i > 0\} \subset B$ and $\{i \in I \, : \, s_i(x'') > 0\} \subset N$. Since $B \cap N = \emptyset$, it follows that x' and $s(x'')$ are complementary. The proof for x'' and $s(x')$ is analogous. □

Corollary 3.2 *The solution set $?^*$ is convex.*

[7] The idea for deriving bounds from Hadamard's inequality is due to Klafszky and Terlaky [13] (in Hungarian).

Proof: Let $x', x'' \in ?^*$ and $\lambda \in [0,1]$. If $x := \lambda x' + (1-\lambda)x''$ then $x \geq 0$ and $s(x) = \lambda s(x') + (1-\lambda)s(x'') \geq 0$. Thus $x \in ?$. Further, Corollary 3.1 gives that $xs(x) = 0$, whence $x \in ?^*$. □

A solution $x \in ?^*$ is called *maximally complementary*, if $x_B > 0$ and $s_N(x) > 0$. Since $?^*$ is convex (and polyhedral) a maximally complementary solution exists.[8]

¿From now on we assume that $?^0 \neq \emptyset$. If the i^{th} column of M is zero then the P_* property implies that the i^{th} row is zero as well. Therefore $s_i(x) = q_i$ in that case, for every x. Hence, if $q_i < 0$, then (LCP) is infeasible. If $q_i \geq 0$, then the constraint $s_i(x)$ is always satisfied and we may reduce the problem by removing the i^{th} row and column of M. Thus we will assume that all columns of M are nonzero. When $q = 0$ then the (LCP) has a trivial solution ($x = 0$). Therefore, without loss of generality we further assume that $q \neq 0$.[9]

Our goal is to find the optimal partition of the index set and, finally, to round off to a maximally complementary solution. In fact, we will show that given $x(\mu)$ we can find the optimal partition provided μ is small enough. To this end we need to give bounds for the size of the variables along the central path. In the next two sections we obtain such bounds in terms of two condition numbers for (LCP).

3.2 The first condition number for (LCP).

In this section we introduce our first *condition number* of (LCP). This is done in a similar way as in Roos et al. [25] for LO problems. Since $?^0 \neq \emptyset$, $?^*$ is nonempty and compact (see Section 2), so the following two numbers are well defined.

$$\sigma^x_{LCP} := \min_{i \in B} \max_{x \in \Gamma^*} \{x_i\}, \qquad \sigma^s_{LCP} := \min_{i \in N} \max_{x \in \Gamma^*} \{s_i(x)\}.$$

By convention we take $\sigma^x_{LCP} = \infty$ if B is empty and $\sigma^s_{LCP} = \infty$ if N is empty, thus both σ^x_{LCP} and σ^s_{LCP} are positive. If B is nonempty then σ^x_{LCP} is finite and if N is nonempty then σ^s_{LCP} is finite. Since $q \neq 0$ it cannot happen that both B and N are empty, thus under the *interior point condition* ($?^0 \neq \emptyset$), at least one of the two numbers is finite. As a consequence, the number

$$\sigma_{LCP} := \min\{\sigma^x_{LCP}, \sigma^s_{LCP}\}$$

is positive and finite. One can easily verify that σ_{LCP} can also be written as

$$\sigma_{LCP} := \min_{i \in B \cup N} \max_{x \in \Gamma^*} \{x_i + s_i(x)\}.$$

In general, we have to solve a problem without knowing its condition number σ_{LCP}. In such cases there is a cheap way to get a lower bound for σ_{LCP} if the problem data (M, q) are integer. We proceed by deriving such a lower bound.

Lemma 3.2 *If M and q are integral then $\sigma_{LCP} \geq \frac{1}{\pi(M)}$.* □

[8] The convexity of Γ^* is proved in another way in [4], (see Theorem 5-6, pages 240-241). Furthermore it is shown that Γ^* is a polyhedron.

[9] It may be noted that in this paper we find a strictly complementary solution of (LCP) under the assumptions $\Gamma^0 \neq \emptyset$ and $q \neq 0$. If $q = 0$ we have the trivial solution $x = 0$, but this solution will in general not be maximally complementary. The case $q = 0$ is interesting in itself. E.g., if M is skew-symmetric it covers LO and there exists a strictly complementary solution [25]; the other extreme occurs if M is a positive definite matrix (e.g., if M is the identity matrix): then $B = N = \emptyset$ and $T = I$.

Proof: For any vector $x \in ?$ we have, with $s = s(x)$,

$$\begin{pmatrix} s_B \\ s_N \\ s_T \end{pmatrix} = \begin{pmatrix} M_{BB} & M_{BN} & M_{BT} \\ M_{NB} & M_{NN} & M_{NT} \\ M_{TB} & M_{TN} & M_{TT} \end{pmatrix} \begin{pmatrix} x_B \\ x_N \\ x_T \end{pmatrix} + \begin{pmatrix} q_B \\ q_N \\ q_T \end{pmatrix}. \tag{6}$$

Further, $x \in ?^*$ holds if and only if $x_N = 0$, $x_T = s_T = 0$, $s_B = 0$. This is equivalent to

$$\begin{pmatrix} M_{BB} & 0_{BN} \\ M_{NB} & -E_{NN} \\ M_{TB} & 0_{TN} \end{pmatrix} \begin{pmatrix} x_B \\ s_N \end{pmatrix} = \begin{pmatrix} -q_B \\ -q_N \\ -q_T \end{pmatrix}, \quad x_B \geq 0, s_N \geq 0. \tag{7}$$

Any maximally complementary solution x yields a positive solution of this system. In order to get a lower bound on σ_{LCP} we need to derive a lower bound on the maximal value of each coordinate of the vector $z := (x_B, s_N)$ when this vector runs through all possible solutions of (7). For each i we know that there exists a solution z with $z_i > 0$. Hence there exists a basic solution z of (7) with $z_i > 0$. Therefore, Corollary 2.1 yields the following lower bound on the biggest coordinate of z.

$$\max_{x \in \Gamma^*} z_i \geq \frac{1}{\pi\left(M_{.B}\right)}.$$

Since $\pi(M) \geq \pi\left(M_{.B}\right)$, the lemma follows. □

Now we are ready to estimate the size of the variables $x_i, s_i(x)$ when x lies on the central path, i.e. $xs(x) = \mu e$, and $i \in B$ or $i \in N$. We denote $s(\mu) := s\left(x(\mu)\right)$.

Theorem 3.1 *For any positive μ one has*

$$\begin{array}{llll} x_i(\mu) \geq \frac{\sigma_{LCP}}{n(1+4\kappa)}, & i \in B, & x_i(\mu) \leq \frac{n\mu(1+4\kappa)}{\sigma_{LCP}}, & i \in N, \\ s_i(\mu) \leq \frac{n\mu(1+4\kappa)}{\sigma_{LCP}}, & i \in B, & s_i(\mu) \geq \frac{\sigma_{LCP}}{n(1+4\kappa)}, & i \in N. \end{array}$$

Proof: We first consider the case $i \in N$. Let us assume that $\overline{x} \in ?^*$ and $\overline{s} := s(\overline{x})$. Taking into consideration that $M \in P_*(\kappa)$, and $x(\mu), s(\mu), \overline{x}, \overline{s} \geq 0$ we get

$$\begin{aligned} (x(\mu) - \overline{x})^T (s(\mu) - \overline{s}) &= (x(\mu) - \overline{x})^T M (x(\mu) - \overline{x}) \\ &\geq -4\kappa \sum_{i \in I_+(x(\mu) - \overline{x})} (x(\mu) - \overline{x})_i [M(x(\mu) - \overline{x})]_i \\ &= -4\kappa \sum_{i \in I_+(x(\mu) - \overline{x})} (x(\mu) - \overline{x})_i (s(\mu) - \overline{s})_i \\ &= -4\kappa \sum_{i \in I_+(x(\mu) - \overline{x})} ((x(\mu)s(\mu))_i - (x(\mu)\overline{s})_i - (\overline{x}s(\mu))_i + (\overline{x}\overline{s})_i) \\ &\geq -4\kappa \sum_{i \in I_+(x(\mu) - \overline{x})} (x(\mu)s(\mu))_i \\ &\geq -4\kappa n \mu. \end{aligned} \tag{8}$$

The last inequality holds because $x(\mu)s(\mu) = \mu e$ and $\overline{x} \in ?^*$. On the other hand

$$(x(\mu) - \overline{x})^T (s(\mu) - \overline{s}) = n\mu - x(\mu)^T \overline{s} - \overline{x}^T s(\mu).$$

Combining this with (8) we have

$$x(\mu)^T \overline{s} + s(\mu)^T \overline{x} \leq n\mu(1 + 4\kappa)$$

which implies

$$x_i(\mu)\overline{s}_i \leq x(\mu)^T\overline{s} \leq n\mu(1+4\kappa) \qquad \forall\, i \in I. \tag{9}$$

Now if $i \in N$ and $\overline{s}$ such that $\overline{s}_i$ is maximal, then by definition $\overline{s}_i \geq \sigma_{LCP}$. Dividing by $\overline{s}_i$ in (9) we obtain

$$x_i(\mu) \leq \frac{n\mu(1+4\kappa)}{\overline{s}_i} \leq \frac{n\mu(1+4\kappa)}{\sigma_{LCP}}. \tag{10}$$

Since $x_i(\mu)s_i(\mu) = \mu$, it also follows that

$$s_i(\mu) \geq \frac{\sigma_{LCP}}{n(1+4\kappa)}, \text{ for all, } i \in N.$$

This proves the second and fourth inequality in the lemma. The first and third inequalities for $i \in B$ are obtained from (9) analogously. □

3.3 The second condition number for (LCP).

In this section we derive bounds that will help us to get control on the variables $x_i(\mu)$ and $s_i(\mu)$ if $i \in T$. Before dealing with the main theorem in this section we review some results about systems of linear inequalities and equalities.

Let $A \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{k\times n}$ be two real matrices. For given $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^k$, consider the following system of linear inequalities

$$Ax \leq b, \quad Cx = d. \tag{11}$$

Cook et al. [3] and Mangasarian and Shiau [20] studied the Lipschitz continuity of solutions of (11) with respect to right-hand side perturbations of (11). We will use a variant of those results. For completeness, we give a simple proof, similar to that presented in [3].

Lemma 3.3 *Let the system (11) have nonempty feasible sets $?^1$ and $?^2$ for the right-hand side (b^1, d^1) and (b^2, d^2), respectively. For each $x^1 \in ?^1$ there exists an $x^2 \in ?^2$ such that*

$$\|x^1 - x^2\|_\infty \leq \nu(A;C)\left\|\begin{pmatrix} b^1 - b^2 \\ d^1 - d^2 \end{pmatrix}\right\|_\infty, \tag{12}$$

where[10]

$$\nu(A;C) := \max_{u,v}\left\{ \left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_1 \;\middle|\; \begin{array}{c} A^Tu + C^Tv = z - y,\ e^T(z+y) = 1,\ u,y,z \geq 0, \\ \textit{the columns of } (A^T, C^T) \textit{ corresponding to nonzero} \\ \textit{elements of } (u,v) \textit{ are linearly independent} \end{array} \right\}.$$

Proof: We are interested in finding t such that $t = \left\|x - x^1\right\|_\infty$, with $x \in ?^2$, is minimal. This amounts to solving the linear minimization problem

$$\min_x \left\{ t \ :\ Ax \leq b^2,\ Cx = d^2,\ te + x \geq x^1,\ te - x \geq -x^1 \right\}. \tag{13}$$

Note that this problem is feasible, since $?^2 \neq \emptyset$, and bounded. Hence, the optimal value t^* is equal to the optimal value of the dual problem of (13). This gives

$$t^* = \max\left\{ u^Tb^2 + v^Td^2 + y^Tx^1 - z^Tx^1 \ :\ A^Tu + C^Tv + y - z = 0,\ e^T(z+y) = 1,\ u,y,z \geq 0 \right\}. \tag{14}$$

[10] This definition is a slight modification of the one given by Mangasarian and Shiau [20]. They simply require $\left\|A^Tu + C^Tv\right\|_1 = 1$, not using the variables y and z. Our definition has the advantage that the feasible region of the optimization problem defining $\nu(A;C)$ are the vertices of a polyhedral set.

Let (u, v, y, z) be an optimal solution of this problem. Then we may write

$$\begin{aligned}
t^* &= u^T b^2 + v^T d^2 + (y - z)^T x^1 \\
&= u^T b^2 + v^T d^2 - \left(A^T u + C^T v\right)^T x^1 \\
&= u^T b^2 + v^T d^2 - u^T A x^1 - v^T C x^1 \\
&\le u^T b^2 + v^T d^2 - u^T b^1 - v^T d^1 \\
&\le u^T \left(b^2 - b^1\right) + v^T \left(d^2 - d^1\right) \\
&\le \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_1 \left\| \begin{pmatrix} b^2 - b^1 \\ d^2 - d^1 \end{pmatrix} \right\|_\infty .
\end{aligned}$$

Hence, the proof will be complete if we show that (14) has an optimal solution (u, v, y, z) such that the columns of (A^T, C^T) corresponding to the nonzero components of (u, v) are linearly independent. This can be shown as follows. If, to the contrary, the columns corresponding to the nonzero coordinates of (u, v) are dependent then there exist vectors $\bar{u}$ and $\bar{v}$, not both zero, such that $A^T \bar{u} + C^T \bar{v} = 0$ and $\bar{u}_i = 0$ if $u_i = 0$ and $\bar{v}_i = 0$ if $v_i = 0$. Note that $z(\lambda) := (u, v, y, z) + \lambda(\bar{u}, \bar{v}, 0, 0)$ will be feasible for (14) if λ is such that $u + \lambda \bar{u} \ge 0$. Due to the definition of $\bar{u}$ and $\bar{v}$ the set of λ for which this happens is a closed interval $[\alpha, \beta]$ with $\alpha < 0 < \beta$ (here we allow $\alpha = -\infty$ and $\beta = \infty$). Hence we necessarily have $\bar{u}^T b^2 + \bar{v}^T d^2 = 0$, otherwise a contradiction with the optimality of $z(0)$ would arise. As a consequence, $z(\lambda)$ is optimal for all $\lambda \in [\alpha, \beta]$. Clearly, by choosing λ appropriately, we can obtain a solution of (14) with fewer nonzero coordinates. By repeating this procedure we obtain a solution (u, v, y, z) of (14) for which the columns of (A^T, C^T) corresponding to the nonzero components of (u, v) are linearly independent. For such a solution we have

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_1 \le \nu(A; C),$$

by the definition of $\nu(A; C)$. This completes the proof. □

We proceed by deriving a lower bound for $\nu(A; C)$.

Lemma 3.4 *One has*

$$\nu(A; C) \ge \frac{1}{\min_{i,j} \left(\|a_i\|_1, \|c_j\|_1\right)},$$

where a_i runs through the rows of A and c_j through the rows of C.

Proof: Let a denote the i-th row of A. Then, if e_i denotes the i-th unit vector, one has $\left\|A^T e_i\right\|_1 = \|a\|_1$. Hence, assuming $a \ne 0$, taking $u = e_i / \|a\|_1, v = 0, z_j = a_j / \|a\|_1$ if $a_j \ge 0, y_j = -a_j / \|a\|_1$ if $a_j < 0$, and all remaining entries of y and z equal to zero, the quadruple (u, v, y, z) is feasible for the maximization problem defining $\nu(A; C)$. Therefore, $\nu(A; C) \ge \|u\|_1 = 1 / \|a\|_1$. A similar argument yields that $\nu(A; C) \ge 1 / \|c\|_1$ for each row c of C, and hence the lemma follows. □

An upper bound for $\nu(A; C)$ can be derived if all the entries of A and C are integral.

Lemma 3.5 *For integer A, C one has*

$$\nu(A; C) \le n \Delta(A^T; C^T) \le n \pi(A^T, C^T). \tag{15}$$

Proof: Let (u, v, y, z) be a feasible solution for the maximization problem in the definition of $\nu(A; C)$. Let $w^T = (u^T, v^T)$, $\bar{A} = (A^T, C^T)$. Then $\bar{A} w = z - y$. Since the columns of $\bar{A}$ corresponding to nonzero elements of w are linearly independent, we may apply Lemma 2.1, which yields

$$\|w\|_\infty \leq \Delta\left(\bar{A}\right)\|z-y\|_1 \leq \Delta\left(\bar{A}\right).$$

The last inequality follows since $\|z-y\|_1 \leq \|z+y\|_1 = 1$. Since $\|w\|_1 \leq n\|w\|_\infty$ the first inequality in the lemma follows from this. The rest of the lemma follows from the Hadamard inequality for determinants. Hence the proof is complete. □

We now are going to apply Lemma 3.5 to a second condition number for (LCP) which enables us to bound the variables along the central path. This second condition number, denoted as ν_{LCP}, depends on the input matrix M and the optimal partition (B, N, T). It is defined as follows.

Definition 3.1 *Let I_1, I_2 be a partition of the index set I such that $B \subseteq I_1$ and $N \subseteq I_2$. Let us define:*

$$\nu_{LCP} := \max_{I_1+I_2=I} \nu\left[\begin{pmatrix} M & -E \\ E_{I_2} & 0 \\ 0 & E_{I_1} \end{pmatrix}; \begin{pmatrix} -E_{I_1} & 0 \\ 0 & -E_{I_2} \end{pmatrix}\right].$$

If the matrix M is integral, then we can give a lower bound and an easily computable upper bound for ν_{LCP}.

Lemma 3.6 *If M is integral, then*

$$1 \leq \nu_{LCP} \leq \max_{I_1+I_2=I} n\Delta\left[\begin{pmatrix} M & -E \\ E_{I_2} & 0 \\ 0 & E_{I_1} \end{pmatrix}^T; \begin{pmatrix} -E_{I_1} & 0 \\ 0 & -E_{I_2} \end{pmatrix}^T\right] = n\Delta(M) \leq n\pi(M).$$

Proof: The first inequality is immediate from Lemma 3.4, the second inequality follows from Lemma 3.5, the equality is obvious and the last inequality is Hadamard's inequality. □

Now we are ready to state our main theorem in this section.

Theorem 3.2 *If*

$$\mu < \frac{\sigma_{LCP}^2}{n^2(1+4\kappa)^2}, \tag{16}$$

then

$$\frac{\sqrt{\mu}}{\nu_{LCP}} \leq x_i(\mu),\ s_i(\mu) \leq \nu_{LCP}\sqrt{\mu}, \quad i \in T.$$

Proof: When (16) holds, one can easily verify that

$$x_i(\mu) \geq \frac{\sigma_{LCP}}{n(1+4\kappa)} > \frac{n\mu(1+4\kappa)}{\sigma_{LCP}} \geq s_i(\mu),\ \forall i \in B,$$

and

$$s_i(\mu) \geq \frac{\sigma_{LCP}}{n(1+4\kappa)} > \frac{n\mu(1+4\kappa)}{\sigma_{LCP}} \geq x_i(\mu),\ \forall i \in N.$$

Letting

$$I_1 = \{i \,:\, x_i(\mu) \geq s_i(\mu)\}, \quad I_2 = \{i \,:\, x_i(\mu) < s_i(\mu)\},$$

we have $B \subset I_1$ and $N \subset I_2$ if (16) holds. Hence, defining

$$H(x) := \min(x, s(x)),$$

we have

$$H_i(x(\mu)) = \begin{cases} s_i(\mu) & \text{if} \quad i \in I_1, \\ x_i(\mu) & \text{if} \quad i \in I_2. \end{cases}$$

¿From the fact that $H(x(\mu)) = \min(x(\mu), s(\mu))$ and $x(\mu)_i s_i(\mu) = \mu$ we conclude that $H_i(x(\mu)) \leq \sqrt{\mu}$.

Consider the following linear system:

$$\begin{array}{rcl} Mx - s &=& -q \\ x_{I_2} &=& 0 \qquad\qquad -x_{I_1} \leq 0 \\ s_{I_1} &=& 0 \qquad\qquad -s_{I_2} \leq 0. \end{array} \tag{17}$$

It is easy to see that the feasible set of the system (17) is the solution set ?* of (LCP). Let this set play the role of ?[2] in the Lemma 3.3. Further, let the solution set of the following linear system play the role of ?[1]:

$$\begin{array}{rcl} Mx - s &=& -q \\ x_{I_2} &=& H_{I_2}(x(\mu)) \qquad\qquad -x_{I_1} \leq 0 \\ s_{I_1} &=& H_{I_1}(x(\mu)) \qquad\qquad -s_{I_2} \leq 0. \end{array} \tag{18}$$

Clearly ?[1] is not empty, because $x(\mu)$ satisfies (18). Now it follows from Lemma 3.3 that there exists a solution x^* of (17), i.e. $x^* \in$?*, such that

$$\left\| \begin{pmatrix} x^* - x(\mu) \\ s^* - s(\mu) \end{pmatrix} \right\|_\infty \leq \nu \left[\begin{pmatrix} M & -E \\ E_{I_2} & 0 \\ 0 & E_{I_1} \end{pmatrix} ; \begin{pmatrix} -E_{I_1} & 0 \\ 0 & -E_{I_2} \end{pmatrix} \right] \|H(x(\mu))\|_\infty.$$

Using the definition of ν_{LCP} and $H_i(x(\mu)) \leq \sqrt{\mu}$ it follows that

$$\left\| \begin{pmatrix} x^* - x(\mu) \\ s^* - s(\mu) \end{pmatrix} \right\|_\infty \leq \nu_{LCP}\sqrt{\mu}.$$

Since $x_i^* = 0$, for $i \in T$, we conclude that for all $i \in T \cap I_1$ one has

$$\frac{\sqrt{\mu}}{\nu_{LCP}} \leq s_i(\mu) \leq \sqrt{\mu} \leq x_i(\mu) \leq \nu_{LCP}\sqrt{\mu}.$$

Similarly for all $i \in T \cap I_2$, it holds

$$\frac{\sqrt{\mu}}{\nu_{LCP}} \leq x_i(\mu) \leq \sqrt{\mu} \leq s_i(\mu) \leq \nu_{LCP}\sqrt{\mu}.$$

This proves the theorem. □

3.4 Finding the optimal partition

In Table 1 we collected the results of the last two theorems (Theorem 3.1 and Theorem 3.2).

These results have an important consequence. If μ is so small that

$$\frac{n\mu(1 + 4\kappa)}{\sigma_{LCP}} < \frac{\sqrt{\mu}}{\nu_{LCP}}$$

	$i \in B$	$i \in N$	$i \in T$
$x_i(\mu)$	$\geq \frac{\sigma_{LCP}}{n(1+4\kappa)}$	$\leq \frac{n\mu(1+4\kappa)}{\sigma_{LCP}}$	$\frac{\sqrt{\mu}}{\nu_{LCP}} \leq x_i(\mu) \leq \nu_{LCP}\sqrt{\mu}$
$s_i(\mu)$	$\leq \frac{n\mu(1+4\kappa)}{\sigma_{LCP}}$	$\geq \frac{\sigma_{LCP}}{n(1+4\kappa)}$	$\frac{\sqrt{\mu}}{\nu_{LCP}} \leq s_i(\mu) \leq \nu_{LCP}\sqrt{\mu}$

Table 1: Local bounds for the variables on the central path.

and

$$\nu_{LCP}\sqrt{\mu} < \frac{\sigma_{LCP}}{n(1+4\kappa)}$$

then we have a complete separation of the variables. Both inequalities give the same bound on μ, namely

$$\mu < \frac{{\sigma_{LCP}}^2}{\nu_{LCP}^2 n^2 (1+4\kappa)^2}. \tag{19}$$

This means that if a point on the central path is given such that (19) holds, then we can determine the optimal partition (B, N, T) of (LCP).

Unfortunately, in practice we may not assume that we can calculate points on the central path exactly. Practical algorithms generate points in the vicinity of the central path. Therefore, in the next section we deal with the situation that a point x is given in an appropriate neighborhood of the central path. We will show that if x is close enough to $x(\mu)$, with μ small enough, we also have a complete separation of the variables into the three different classes B, N and T. This will imply that all path-following IPMs eventually produce iterates that are suitable to identify the optimal partition of (LCP).

4 Optimal partition identification from approximate centers

In this section we generalize the results of the previous section to the case where a point x is given in a specific neighborhood of the central path. On the central path all the coordinates of the vector $xs(x)$ are equal. This suggests that a good measure of centrality could be the ratio of the smallest and largest coordinate. If we bound this ratio, then a neighborhood of the central path is obtained. We therefore use the following *measure of centrality*[11]

$$\delta_c(x) := \frac{\max(xs(x))}{\min(xs(x))},$$

where $\max(xs(x))$ denotes the largest coordinate of $xs(x)$ and $\min(xs(x))$ denotes the smallest one.

4.1 Finding the optimal partition from approximate centers

We first generalize the results of Theorem 3.1 and Theorem 3.2 to the case where x is not on the central path $\mathcal{C}$.

[11] This measure of centrality is introduced by Ling [19] and used in [8, 9]. The same measure of centrality is used throughout the book of Roos et al. [25].

Lemma 4.1 *Let $x \in ?^0$ and $s = s(x)$. If $\delta_c(x) \leq \tau$, for some $\tau > 1$, and $\mu := \frac{x^T s(x)}{n}$ then one has*

$$x_i \geq \frac{\sigma_{LCP}}{\tau n(1+4\kappa)},\ i \in B, \qquad x_i \leq \frac{(1+4\kappa)n\mu}{\sigma_{LCP}},\ i \in N,$$

$$s_i \leq \frac{(1+4\kappa)n\mu}{\sigma_{LCP}},\ i \in B, \qquad s_i \geq \frac{\sigma_{LCP}}{\tau n(1+4\kappa)},\ i \in N.$$

If, further

$$\mu \leq \frac{\sigma_{LCP}^2}{\tau n^2(1+4\kappa)^2}, \tag{20}$$

then

$$\frac{\sqrt{\mu}}{\tau\sqrt{\tau}\nu_{LCP}} \leq x_i,\ s_i \leq \sqrt{\tau}\nu_{LCP}\sqrt{\mu},\ i \in T.$$

Proof: The proof uses essentially the same arguments as the proofs of Theorem 3.1 and Theorem 3.2. The arguments leading to (10) in the proof of Theorem 3.1 are still valid, so

$$x_i \leq \frac{(1+4\kappa)x^T s}{\sigma_{LCP}} = \frac{(1+4\kappa)n\mu}{\sigma_{LCP}} \quad \text{for } i \in N. \tag{21}$$

The rest of the proof is a little complicated by the fact that x is not on, but only in a certain neighborhood of the central path. If $\delta_c(x) \leq \tau$ then there are $\alpha, \beta \in (0, \infty)$ such that

$$\alpha e \leq xs \leq \beta e, \quad \text{with} \quad \frac{\beta}{\alpha} = \tau. \tag{22}$$

These inequalities replace the identity $x_i(\mu)s_i(\mu) = \mu$ used in the proof of Theorem 3.1. Due to the left inequality in (22) we also have $x_i s_i \geq \alpha$ for all i. Hence using (21) we must have

$$s_i \geq \frac{\alpha\sigma_{LCP}}{(1+4\kappa)x^T s}.$$

The right inequality in (22) gives $x^T s \leq n\beta$, thus

$$s_i \geq \frac{\alpha\sigma_{LCP}}{n\beta(1+4\kappa)} = \frac{\sigma_{LCP}}{\tau n(1+4\kappa)}.$$

This proves the second and fourth inequality in the lemma. The proof of the first and third inequalities can be obtained in the same way, therefore their proof is left to the reader.

To prove the last statement of the lemma, we notice that for the current point (x, s), it obviously holds

$$\frac{x_i s_i}{\mu} = \frac{n x_i s_i}{x^T s} \geq \frac{n \min(xs)}{n \max(xs)} \geq \frac{1}{\tau}, \quad \forall i = 1, 2, \ldots, n,$$

and

$$\frac{x_i s_i}{\mu} = \frac{n x_i s_i}{x^T s} \leq \frac{n \max(xs)}{n \min(xs)} \leq \tau, \quad \forall i = 1, 2, \ldots, n.$$

Let $H(x) = \min(x, s)$, the above two inequalities give

$$[H(x)]_i \leq \sqrt{\tau}\sqrt{\mu}, \quad \forall i = 1, 2, \ldots, n.$$

Following similar arguments as in the proof of Theorem 3.2, one can easily derive the conclusion. □

$\mu = \frac{x^T s(x)}{n}$	$i \in B$	$i \in N$	$i \in T$
x_i	$\geq \frac{\sigma_{LCP}}{\tau n(1+4\kappa)}$	$\leq \frac{(1+4\kappa)n\mu}{\sigma_{LCP}}$	$\frac{\sqrt{\mu}}{\tau\sqrt{\tau}\nu_{LCP}} \leq x_i \leq \sqrt{\tau}\nu_{LCP}\sqrt{\mu}$
$s_i(x)$	$\leq \frac{(1+4\kappa)\mu}{\sigma_{LCP}}$	$\geq \frac{\sigma_{LCP}}{\tau n(1+4\kappa)}$	$\frac{\sqrt{\mu}}{\tau\sqrt{\tau}\nu_{LCP}} \leq s_i \leq \sqrt{\tau}\nu_{LCP}\sqrt{\mu}$

Table 2: Local estimates for variables belonging to index sets B, N and T if $\delta_c(x) \leq \tau$.

In Table 2 we collected the results of the above lemma.

We conclude that the partition (B, N, T) can be identified if $x^T s(x)$ is so small that

$$\frac{(1+4\kappa)n\mu}{\sigma_{LCP}} < \frac{\sqrt{\mu}}{\tau\sqrt{\tau}\nu_{LCP}},$$

and

$$\sqrt{\tau}\nu_{LCP}\sqrt{\mu} < \frac{\sigma_{LCP}}{\tau n(1+4\kappa)}.$$

It is easy to verify that both inequalities give the same bound for μ thus for complete separation of the variables we need

$$\mu < \frac{\sigma_{LCP}^2}{n^2\tau^3\nu_{LCP}^2(1+4\kappa)^2}. \tag{23}$$

Therefore we may state without further proof our main result.

Theorem 4.1 *Let $x \in ?^0$ be such that $\delta_c(x) \leq \tau$, for some $\tau > 1$, and $\mu = \frac{x^T s(x)}{n}$. If (23) is true, then, with $s = s(x)$, the optimal partition of (LCP) follows from*

$$T = \{i \;:\; \frac{\sqrt{\mu}}{\tau\sqrt{\tau}\nu_{LCP}} \leq x_i,\, s_i \leq \sqrt{\tau}\nu_{LCP}\sqrt{\mu}\},$$

$$B = \{i \notin T \;:\; x_i > s_i\} \quad \textit{and} \quad N = \{i \notin T \;:\; x_i < s_i\}.$$

□

4.2 Complexity of finding the optimal partition

In this section we assume that we have given a point $x^{(0)} \in ?^0$ close to the central path (i.e. $\delta_c(x^{(0)}) \leq \tau$ for some $\tau > 1$). We define μ^0 by $n\mu^0 = \left(x^{(0)}\right)^T s^{(0)}$. Starting at x^0 interior point methods for solving (LCP) need $\mathcal{O}(\sqrt{n}\log(n\mu^0/\epsilon))$ iterations (see, e.g., [11, 15, 17, 31]), or $\mathcal{O}(n\log(n\mu^0/\epsilon))$ (see, e.g., [8]) iterations to generate a point x such that $\delta_c(x) \leq \tau$ and $x^T s(x) \leq \epsilon$. The first bound holds for methods with small updates of the barrier parameter whereas the second bound is typical for methods using large updates, and also for methods using a Dikin-type affine-scaling direction. Hence, by substituting the value of ϵ according to Theorem 4.1, we can get iteration bounds to identify the optimal partition.

The above will be illustrated below for the Dikin affine-scaling algorithm presented in [8]. If $n \geq 4$, this algorithm, with $\tau = 2$, requires at most

$$3(1+4\kappa)n\,\log\frac{n\mu^0}{\epsilon} \tag{24}$$

iterations to generate a point x such that $\delta_c(x) \leq 2$ and $x^T s(x) \leq \epsilon$.

Theorem 4.2 *Starting at a point $x^{(0)} \in ?^0$ with $\delta_c\left(x^{(0)}\right) \le 2$, and $n \ge 4$, the Dikin affine-scaling algorithm reveals the optimal partition after at most*

$$3(1+4\kappa)n \log \frac{8n^2(1+4\kappa)^2\nu_{LCP}^2\mu^0}{\sigma_{LCP}^2} \le 3(1+4\kappa)n \log\left(8n^4(1+4\kappa)^2\pi(M)^4\mu^0\right)$$

iterations.

Proof: The expression (24) gives the number of iterations to reach an ϵ-solution. With μ as in Theorem 4.1 and $\epsilon = n\mu$ we obtain the first bound. The inequality follows by using the upper bound for ν_{LCP} in Lemma 3.6 and the lower bound for σ_{LCP} in Lemma 3.2. □

Similar results can be derived for other polynomial IPMs.

5 Rounding to a strictly complementary solution

We just established that the optimal partition of (LCP) can be found after a polynomial number of iterations with any known path-following IPMs for $P_*(\kappa)$ LCPs. The required number of iterations depends on the starting point $x^{(0)}$, the parameter κ, and on the condition numbers ν_{LCP} and σ_{LCP}. Our ultimate goal is not only to find the optimal partition but to find an exact and maximally complementary solution of (LCP). Assuming that the optimal partition (B,N,T) has been determined, with B nonempty,[12] we describe a rounding procedure that can be applied to any sufficiently centered positive vector x with $x^T s(x)$ small enough, and the rounding procedure yields a vector $\tilde{x}$ such that (7) is satisfied and $\tilde{x}_B > 0, s_N(\tilde{x}) > 0$. As might be expected, the accuracy that was sufficient to find the optimal partition is not enough to perform the rounding procedure. In Theorem 5.1 we will give a bound on the complementary gap that provides sufficient accuracy for our rounding procedure. The rounding procedure yields a maximally complementary solution in strongly polynomial time. Finally, the number of iterations, required to reach the necessarily small complementarity gap, is bounded by Theorem 5.2.

5.1 Rounding procedure

Let $x \in ?^0$, $s = s(x)$ be given and assume that the optimal partition (B,N,T) is known. Now we want to compute $\Delta x_B, \Delta s_N$ such that

$$x_B + \Delta x_B > 0 \quad \text{and} \quad s_N + \Delta s_N > 0, \tag{25}$$

and

$$\begin{pmatrix} 0 \\ s_N + \Delta s_N \\ 0 \end{pmatrix} = \begin{pmatrix} M_{BB} & M_{BN} & M_{BT} \\ M_{NB} & M_{NN} & M_{NT} \\ M_{TB} & M_{TN} & M_{TT} \end{pmatrix} \begin{pmatrix} x_B + \Delta x_B \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} q_B \\ q_N \\ q_T \end{pmatrix}, \tag{26}$$

because then $\tilde{x} := (x_B + \Delta x_B, 0, 0) \in ?^*$, and this solution is maximally complementary. Since x and s satisfy (6), we may subtract (6) from (26), leading to the following system

$$\begin{pmatrix} -s_B \\ \Delta s_N \\ -s_T \end{pmatrix} = \begin{pmatrix} M_{BB} & M_{BN} & M_{BT} \\ M_{NB} & M_{NN} & M_{NT} \\ M_{TB} & M_{TN} & M_{TT} \end{pmatrix} \begin{pmatrix} \Delta x_B \\ -x_N \\ -x_T \end{pmatrix}, \tag{27}$$

[12] If $B = \emptyset$, then $x = 0$ and $s = q$ is the only possible solution. The vector $(0, q)$ solves the problem if and only if $q \ge 0$.

which is thus equivalent to (26). We can rewrite this as

$$\begin{pmatrix} M_{BB} & 0_{BN} \\ M_{NB} & -E_{NN} \\ M_{TB} & 0_{TN} \end{pmatrix} \begin{pmatrix} \Delta x_B \\ \Delta s_N \end{pmatrix} = \begin{pmatrix} M_{BN}x_N + M_{BT}x_T - s_B \\ M_{NN}x_N + M_{NT}x_T \\ M_{TN}x_N + M_{TT}x_T - s_T \end{pmatrix}. \tag{28}$$

We conclude that we can round x to a maximally complementary solution $\tilde{x}$ if we can find a solution $(\Delta x_B, \Delta s_N)$ of (28) that satisfies (25). We show below that if $x^T s(x) = n\mu$ is small enough and x is close enough to the central path, then such $(\Delta x_B, \Delta s_N)$ can be found by Gaussian elimination.

It may be useful to point out that the analysis below works out well because the variables x_T, x_N, s_B and s_T that occur in the right hand side of (28) are 'small' if μ is small enough and x is close enough to the central path. These variables are bounded above by Lemma 4.1. Since x_B and s_N are 'large', by the same lemma, it is therefore not surprising that (28) admits a solution such that (25) holds.

Theorem 5.1 *Let $x \in ?^0$ be such that $\delta_c(x) \le \tau = 2$. If*

$$\mu < \frac{\sigma_{LCP}^2}{8n^3(1+4\kappa)^2\nu_{LCP}^2 \|M\|_\infty^2 \pi(M)^2} \tag{29}$$

then the rounding procedure yields a maximally complementary solution in at most $\mathcal{O}(n^3)$ arithmetic operations.

Proof: To keep the expressions simple we introduce the following notations:

$$A := \begin{pmatrix} M_{BB} & 0_{BN} \\ M_{NB} & -E_{NN} \\ M_{TB} & 0_{TN} \end{pmatrix}, \ \Delta z := \begin{pmatrix} \Delta x_B \\ \Delta s_N \end{pmatrix} \text{ and } r := \begin{pmatrix} M_{BN}x_N + M_{BT}x_T - s_B \\ M_{NN}x_N + M_{NT}x_T \\ M_{TN}x_N + M_{TT}x_T - s_T \end{pmatrix}.$$

Then equation (28) becomes

$$A\Delta z = r. \tag{30}$$

When solving (30) by Gaussian elimination, which needs $\mathcal{O}(n^3)$ arithmetic operations, we obtain a solution such that the columns of A corresponding to its support are linearly independent. Hence, using Corrollary 2.1,

$$\|\Delta z\|_\infty \le \pi(A)\|r\| = \pi(M_{.B})\|r\| \le \pi(M)\|r\|. \tag{31}$$

We proceed by estimating $\|r\|$. We use the trivial inequality $\|r\| \le \sqrt{n}\|r\|_\infty$ and

$$\|r\|_\infty \le \left\| \begin{pmatrix} M_{BN} & M_{BT} & -E_B & 0 \\ M_{NN} & M_{NT} & 0 & 0 \\ M_{TN} & M_{TT} & 0 & -E_T \end{pmatrix} \right\|_\infty \left\| \begin{pmatrix} x_N \\ x_T \\ s_B \\ s_T \end{pmatrix} \right\|_\infty. \tag{32}$$

Observe that the value of μ given by (29) satisfies the hypothesis of Theorem 4.1. Therefore, we have a complete separation of the variables. As a consequence, all entries in the vectors x_N, x_T, s_B and s_T are bounded above by $\sqrt{\tau}\nu_{LCP}\sqrt{\mu}$. Hence, the infinity norm of the concatenation of these vectors, that appears at the right in (32), is bounded above by this number. Obviously the infinity norm of the matrix in (32) is bounded above by the infinity norm of M. Thus we find

$$\|r\| \le 2n\sqrt{n}\nu_{LCP}(1+4\kappa)\sqrt{\mu}\,\|M\|_\infty.$$

Substitution in (31) yields

$$\|\Delta z\|_\infty \le \sqrt{n}\nu_{LCP}\sqrt{\tau}\sqrt{\mu}\,\|M\|_\infty\,\pi(M). \tag{33}$$

Using the lower bound of Lemma 4.1 (with $\tau = 2$) for the entries of x_B and s_N, we conclude that the rounding procedure certainly yields a maximally complementary solution if

$$\sqrt{2n}\nu_{LCP}\sqrt{\mu}\,\|M\|_\infty\,\pi(M) < \frac{\sigma_{LCP}}{2n(1+4\kappa)}.$$

This inequality is equivalent to

$$\sqrt{\mu} < \frac{\sigma_{LCP}}{2\sqrt{2}n\sqrt{n}\nu_{LCP}(1+4\kappa)\,\|M\|_\infty\,\pi(M)}.$$

which yields the bound for μ in the theorem. This completes the proof. □

5.2 Complexity of finding an exact solution

We apply the results of the previous section to estimate the number of iterations required by the Dikin affine-scaling algorithm to reach the state where the rounding procedure yields a maximally complementary solution. Without further proof we may state our final result.

Theorem 5.2 *Starting at a point $x^{(0)} \in ?^0$ with $\delta_c\left(x^{(0)}\right) \leq 2$, and $n \geq 4$, the Dikin affine-scaling algorithm requires at most*

$$3(1+4\kappa)n\,\log\frac{8n^3(1+4\kappa)^2\nu_{LCP}^2\,\|M\|_\infty^2\,\pi(M)^2\,\mu^0}{\sigma_{LCP}^2}$$

iterations to generate a point x at which the rounding procedure produces a maximally complementary solution. □

6 Concluding remarks

The aim of this paper was to show that one can determine a maximally complementary solution of (LCP) in polynomial time, thus extending a well-known result for LO (cf. Roos et al. [25]). We assumed that $?^0 \neq \emptyset$, $q \neq 0$ and that a starting point $x^{(0)} \in ?^0$ is given. Under these assumption we could derive the desired result.

A crucial point in the analysis is the convergence rate along the central path of the variables in the index set T, which is $\mathcal{O}(\sqrt{\mu})$. All known proofs of this result use a corollary of Robinson [24] related to the theory of polyhedral multifunctions. In Section 3 we presented a new and relatively simple proof.

In the analysis we need two condition numbers for $P_*(\kappa)$ LCPs, both of which appear in the achieved iterations bound. Both numbers were bounded by expressions in the input data. Using Theorem 3.2 and Theorem 4.1 we showed that if $x \in ?^0$ is sufficiently close to the central path and $x^Ts(x)$ sufficiently small then we can identify the optimal partition and compute a maximally complementary solution by using Gaussian elimination (Theorem 5.1). Similar bounds were presented by Kojima et al. [17, 14] to generate a complementary basic solution of (LCP).

The number of iterations to obtain the accuracy necessary to run the rounding procedure is computed for Dikin affine-scaling algorithm [8] in Theorem 5.2. Similar results for other known IPMs can be obtained, as well.

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