A unifying investigation of interior-point methods for convex programming

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Abstract

In the recent past a number of papers were written that present low complexity interior-point methods for different classes of convex programs. Goal of this article is to show that the logarithmic barrier function associated with these programs is self-concordant, and that the analyses of interior-point methods for these programs can thus be reduced to the analysis of interior-point methods with self-concordant barrier functions.

Key words: interior-point method, barrier function, dual geometric programming, (extended) entropy programming, primal and dual l_p-programming, relative Lipschitz condition, scaled Lipschitz condition, self-concordance.

1 Introduction

The efficiency of a barrier method for solving convex programs strongly depends on the properties of the barrier function used. A key property that is sufficient to prove fast convergence for barrier methods is the property of self-concordance introduced in [17]. This condition not only allows a proof of polynomial convergence, but numerical experiments in [1, 11, 14] and others further indicate that numerical algorithms based on self-concordant barrier functions are of practical interest and effectively exploit the structure of the underlying problem.

A well-known barrier function for solving convex programs is the logarithmic barrier function, introduced by Frisch [4] and Fiacco and McCormick [3]. To describe the logarithmic barrier function more precisely, we will first give a general form for the classes of problems considered in this paper:

$$(\mathcal{CP}) \quad \begin{cases} \min \ f_0(x) \\ f_i(x) \leq 0, \ i = 1, \cdots, m, \\ Ax = b, \end{cases}$$

where A is $p \times n$ matrix and b an p-dimensional vector. The logarithmic barrier function for this program is given by

$$\phi(x, \mu) = \frac{f_0(x)}{\mu} - \sum_{i=1}^{m} \ln(-f_i(x)),$$

where $\mu > 0$ is the barrier parameter. We show that for several classes of convex problems for which interior-point methods were presented in the literature the logarithmic barrier function is self-concordant. These classes are: dual geometric programming, (extended) entropy programming, primal and dual l_p-programming. Since for dual geometric programming and dual l_p-programming no complexity results are known in the literature, these self-concordance proofs enlarge the class of problems for which polynomiality can be proved. (In [12] only a convergence analysis is given.) Moreover, we show that some other smoothness conditions used in the literature (relative Lipschitz condition [9, 7], scaled Lipschitz condition [25, 13], Monteiro and Adler's condition [16]) are also covered by this self-concordance condition. These observations allow a unification of the analyses of interior-point methods for a number of convex problems.

The article is divided in three parts. In Section 2 we give the definition of self-concordance and state some basic lemmas about self-concordant functions. In Sections 3 - 6 we prove self-concordance for the classes of problems treated in [5, 12, 23], and in Section 7 we show that the smoothness conditions used in [7, 9, 13, 16, 25] imply self-concordance of the barrier function.

2 Some general composition rules

Let us first give the precise definition of self-concordance as given by Nesterov and Nemirovsky [17]:

Definition of self-concordance: Let $\mathcal{F}^0$ be an open convex subset of $\mathbb{R}^n$. A function $\varphi : \mathcal{F}^0 \to \mathbb{R}$ is called κ-self-concordant on $\mathcal{F}^0$, $\kappa \geq 0$, if φ is three times continuously differentiable in $\mathcal{F}^0$ and if for all $x \in \mathcal{F}^0$ and $h \in \mathbb{R}^n$ the following inequality holds:

$$\nabla^3 \varphi(x)[h, h, h] \leq 2\kappa \left(h^T \nabla^2 \varphi(x) h\right)^{\frac{3}{2}},$$

where $\nabla^3\varphi(x)[h,h,h]$ denotes the third differential of φ at x and h.

Intuitively, since $\nabla^3\varphi$ describes the change in $\nabla^2\varphi$, and since $\nabla^3\varphi$ is bounded by a suitable power of $\nabla^2\varphi$, this condition implies that the *relative* change of $\nabla^2\varphi$ is bounded by 2κ. The associated norm to measure the relative change is given by $\nabla^2\varphi(x)$, i.e. for $h \in I\!R^n$ the norm associated with the point x is $\|h\|_{\nabla^2\varphi(x)} := (h^T\nabla^2\varphi(x)h)^{1/2}$. (See [10] and [6] for example, where also a brief analysis is given, showing that the property of self-concordance of the barrier function of a convex program is sufficient to prove polynomial convergence. A more detailed analysis that includes certain nonconvex programs and that uses an additional condition relating the first and second derivatives of φ is given in [17].)

The following lemma gives some helpful composition rules for self-concordant functions. The proof follows immediately from the definition of self-concordance.

Lemma 1 *(Nesterov and Nemirovsky [17])*

- *(addition and scaling) Let φ_i be κ_i-self-concordant on $\mathcal{F}_i^0$, $i = 1,2$, and $\rho_1, \rho_2 \in I\!R_+$ then $\rho_1\varphi_1 + \rho_2\varphi_2$ is κ-self-concordant on $\mathcal{F}_1^0 \cap \mathcal{F}_2^0$, where $\kappa = max\{\frac{\kappa_1}{\sqrt{\rho_1}}, \frac{\kappa_2}{\sqrt{\rho_2}}\}$.*
- *(affine invariance) Let φ be κ-self-concordant on $\mathcal{F}^0$ and let $\mathcal{B}(x) = Bx + b : I\!R^k \to I\!R^n$ be an affine mapping such that $\mathcal{B}(I\!R^k) \cap \mathcal{F}^0 \neq \emptyset$. Then $\varphi(\mathcal{B}(.))$ is κ-self-concordant on $\{x \,:\, \mathcal{B}(x) \in \mathcal{F}^0\}$.*

The next lemma gives a sufficient condition for an objective function f to guarantee that f "combined" with the logarithmic barrier function for the positive orthant of $I\!R^n$ is self-concordant. This lemma will help to simplify self-concordance proofs in the sequel.

Lemma 2 *Let $f(x) \in C^3(\mathcal{F}^0)$ be convex. If there exists a β such that*

$$|\nabla^3 f(x)[h,h,h]| \leq \beta h^T\nabla^2 f(x)h\sqrt{\sum_{i=1}^n \frac{h_i^2}{x_i^2}}, \tag{1}$$

$\forall x \in \mathcal{F}^0$ and $\forall h \in I\!R^n$, then

$$\varphi(x) := f(x) - \sum_{i=1}^n \ln x_i$$

is $(1+\frac{1}{3}\beta)$-self-concordant on $\mathcal{F}^0$, and

$$\psi(\nu, x) := -\ln(\nu - f(x)) - \sum_{i=1}^n \ln x_i$$

is $(1+\frac{1}{3}\beta)$-self-concordant on $\tilde{\mathcal{F}}^0$. Here, $\tilde{\mathcal{F}}^0 \subset I\!R \times \mathcal{F}^0$ is the set $\{(\nu, x) \mid x \in \mathcal{F}^0,\ \nu > f(x)\}$.

At a first glance, condition (1) may look somewhat arbitrary. We give a brief motivation right after the following proof, and we will see that the lemma has indeed useful applications.

Proof:

We start by proving the first part of the lemma. Straightforward calculations yield

$$\nabla\varphi(x)^T h = \nabla f(x)^T h - \sum_{i=1}^n \frac{h_i}{x_i} \tag{2}$$

$$h^T\nabla^2\varphi(x)h = h^T\nabla^2 f(x)h + \sum_{i=1}^n \frac{h_i^2}{x_i^2} \tag{3}$$

$$\nabla^3\varphi(x)[h,h,h] = \nabla^3 f(x)[h,h,h] - 2\sum_{i=1}^n \frac{h_i^3}{x_i^3}. \tag{4}$$

We show that

$$(\nabla^3\varphi(x)[h,h,h])^2 \le 4(1+\frac{1}{3}\beta)^2(h^T\nabla^2\varphi(x)h)^3, \tag{5}$$

from which the lemma follows. Since f is convex, the two terms on the right-hand side of (3) are nonnegative, i.e. the right-hand side can be abbreviated by

$$h^T\nabla^2\varphi(x)h = a^2+b^2, \tag{6}$$

with $a,b \ge 0$. Because of (1) we have that

$$|\nabla^3 f(x)[h,h,h]| \le \beta a^2 b.$$

Obviously

$$\left|\sum_{i=1}^n \frac{h_i^3}{x_i^3}\right| \le \sum_{i=1}^n \frac{h_i^2}{x_i^2}\sqrt{\sum_{i=1}^n \frac{h_i^2}{x_i^2}} = b^3.$$

So we can bound the right-hand side of (4) by

$$|\nabla^3\varphi(x)[h,h,h]| \le \beta a^2 b + 2b^3. \tag{7}$$

It is straightforward to verify that

$$(\beta a^2 b + 2b^3)^2 \le 4(1+\frac{1}{3}\beta)^2(a^2+b^2)^3.$$

Together with (6) and (7) our claim (5) follows and hence the first part of the lemma.

Now we prove the second part of the lemma. Let

$$\tilde{x} = \begin{pmatrix} \nu \\ x \end{pmatrix}, \quad h = \begin{pmatrix} h_0 \\ \vdots \\ h_n \end{pmatrix} \quad \text{and} \quad g(\tilde{x}) = \nu - f(x) > 0, \tag{8}$$

then

$$\psi(\tilde{x}) = -\ln g(\tilde{x}) - \sum_{i=1}^n \ln x_i \tag{9}$$

$$\nabla\psi(\tilde{x})^T h = -\frac{\nabla g(\tilde{x})^T h}{g(\tilde{x})} - \sum_{i=1}^n \frac{h_i}{x_i} \tag{10}$$

$$h^T\nabla^2\psi(\tilde{x})h = -\frac{h^T\nabla^2 g(\tilde{x})h}{g(\tilde{x})} + \frac{(\nabla g(\tilde{x})^T h)^2}{g(\tilde{x})^2} + \sum_{i=1}^n \frac{h_i^2}{x_i^2} \tag{11}$$

$$\begin{aligned}\nabla^3\psi(\tilde{x})[h,h,h] &= -\frac{\nabla^3 g(\tilde{x})[h,h,h]}{g(\tilde{x})} + 3\frac{(h^T\nabla^2 g(\tilde{x})h)\nabla g(\tilde{x})^T h}{g(\tilde{x})^2} \\ &\quad -2\frac{(\nabla g(\tilde{x})^T h)^3}{g(\tilde{x})^3} - 2\sum_{i=1}^n \frac{h_i^3}{x_i^3}.\end{aligned} \tag{12}$$

We show that

$$(\nabla^3\psi(\tilde{x})[h,h,h])^2 \le 4(1+\frac{1}{3}\beta)^2(h^T\nabla^2\psi(\tilde{x})h)^3, \tag{13}$$

which will prove the lemma. Since g is concave, all three terms on the right-hand side of (11) are nonnegative, i.e. the right-hand side can be abbreviated by

$$h^T\nabla^2\psi(\tilde{x})h = a^2+b^2+c^2, \tag{14}$$

with $a, b, c \geq 0$. Due to (1) we have

$$\left| \frac{\nabla^3 g(\tilde{x})[h,h,h]}{g(\tilde{x})} \right| \leq \beta a^2 c,$$

so that we can bound the right-hand side of (12) by

$$|\nabla^3 \psi(\tilde{x})[h,h,h]| \leq \beta a^2 c + 3a^2 b + 2b^3 + 2c^3. \tag{15}$$

It is straightforward to verify that

$$(\beta a^2 c + 3a^2 b + 2b^3 + 2c^3)^2 \leq 4(1 + \frac{1}{3}\beta)^2 (a^2 + b^2 + c^2)^3,$$

by eliminating all odd powers in the left-hand side via inequalities of the type $2ab \leq a^2 + b^2$. Together with (14) and (15) our claim (13) follows. This proves the lemma. *Q.E.D.*

We now explain property (1) in more detail. Let $\phi(x) = -\sum_{i=1}^n \ln x_i$ be the logarithmic barrier for $I\!R^n_+$. Observe that

$$\sqrt{\sum_{i=1}^n \frac{h_i^2}{x_i^2}} = \sqrt{h^T \nabla^2 \phi(x) h} = \|h\|_{\nabla^2 \phi(x)}.$$

We recall that (as mentioned above) the canonical norm associated with some barrier function ϕ at a point x is given by $\nabla^2 \phi(x)$. Loosely speaking, property (1) tells us that for $\|h\|_{\nabla^2 \phi(x)} = 1$, the spectral norm of the third derivative $\nabla^3 f$ is bounded by a multiple β of the spectral norm of the second derivative $\nabla^2 f$. This property is defined in [17] as f being compatible with ϕ, and, as we have seen, it implies self-concordance of the combined barrier functions φ and ψ.

Clearly, if f satisfies (1), then so does $\frac{1}{\mu} f$ for any (fixed) parameter $\mu > 0$. In particular, this implies that also the function $f(x)/\mu - \sum \ln x_i$ is $(1 + \frac{1}{3}\beta)$-self-concordant. Finally we note that for any parameter $q \geq 1$, the above proof also holds true for $-q \ln(\nu - f(x)) - \sum_{i=1}^n \ln x_i$. This observation can be used to prove that for the classes of problems considered in this paper not only the logarithmic barrier function but also the center function of Huard [8] (also used in e.g. [21, 9, 10, 6]) is self-concordant.

3 The dual geometric programming problem

Let $\{I_k\}_{k=1,\cdots,r}$ be a partition of $\{1, \cdots, n\}$ (i.e. $\cup_{k=1}^r I_k = \{1, \cdots, n\}$ and $I_k \cap I_l = \emptyset$ for $k \neq l$). The dual geometric programming problem [2] is then given by

$$(\mathcal{DGP}) \quad \begin{cases} \min \ c^T x + \sum_{k=1}^r \left[\sum_{i \in I_k} x_i \ln x_i - \left(\sum_{i \in I_k} x_i \right) \ln \left(\sum_{i \in I_k} x_i \right) \right] \\ Ax = b \\ x \geq 0, \end{cases}$$

where A is an $m \times n$ matrix and c and b are n- and m-dimensional vectors, respectively. For this problem we have the following lemma.

Lemma 3 *The logarithmic barrier function of the dual geometric programming problem* $(\mathcal{DGP})$ *is* 2-*self-concordant*[1].

[1] This corrects a remark in [12], in which it is claimed that the self-concordance property does not hold for this problem.

Proof: Because of Lemma 1, it suffices to verify 2-self-concordance for the following logarithmic barrier function

$$\varphi(x) = \sum_{i\in I_k} x_i \ln x_i - \Big(\sum_{i\in I_k} x_i\Big) \ln\Big(\sum_{i\in I_k} x_i\Big) - \sum_{i\in I_k} \ln x_i, \tag{16}$$

for some fixed k. For simplicity, we will drop the subscript $i \in I_k$. Now we can use Lemma 2, so that we only have to verify that (1) holds for

$$f(x) := \sum x_i \ln x_i - \Big(\sum x_i\Big) \ln\Big(\sum x_i\Big),$$

and $\beta = 3$, which is equivalent to the following inequality:

$$\left|\sum \frac{h_i^3}{x_i^2} - \frac{(\sum h_i)^3}{(\sum x_i)^2}\right| \le 3\left(\sum \frac{h_i^2}{x_i} - \frac{(\sum h_i)^2}{\sum x_i}\right)\sqrt{\sum \frac{h_i^2}{x_i^2}}. \tag{17}$$

Here $x_i > 0$ and h_i arbitrary. Dividing the whole inequality by $\sum x_i$ and then substituting first $h_i = y_i x_i$ and thereafter $t_i = x_i / \sum x_j$ we get the equivalent inequality

$$y^{3^T} t - (y^T t)^3 \le 3(y^{2^T} t - (y^T t)^2)\sqrt{y^T y},$$

where y_i are arbitrary, t_i positive and $\sum t_i = 1$. (Here y^3, e.g., is the vector with entries y_i^3.) Since $y^T t = E(y)$ can be interpreted as the expected value of some random variable y, the last inequality is equivalently rewritten as

$$E(y^3) - E(y)^3 \le 3(E(y^2) - E(y)^2)\sqrt{\sum y_i^2},$$

relating the variance of y to some third moment. By adding

$$\begin{aligned}(E(y^2) - E(y)^2)\sqrt{\sum y_i^2} &\ge (E(y^2) - E(y)^2)\max y_i = E((y - E(y))^2 \max y_i) \\ &\ge E((y - E(y))^2 y) = E(y^3) - 2E(y)E(y^2) + E(y)^3\end{aligned}$$

and

$$2(E(y^2) - E(y)^2)\sqrt{\sum y_i^2} \ge 2(E(y^2) - E(y)^2)E(y) = 2E(y)E(y^2) - 2E(y)^3$$

we get

$$3(E(y^2) - E(y)^2)\sqrt{\sum y_i^2} \ge E(y^3) - E(y)^3,$$

i.e. inequality (17) follows. *Q.E.D.*

4 The extended entropy programming problem

The extended entropy programming problem is defined as

$$(\mathcal{EEP}) \quad \begin{cases} \min\ c^T x + \sum_{i=1}^n g_i(x_i) \\ Ax = b \\ x \ge 0, \end{cases}$$

where A is an $m \times n$ matrix and c and b are n- and m-dimensional vectors, respectively. Moreover, it is assumed that the scalar functions $g_i \in C^3$ satisfy $|g_i'''(x_i)| \le \kappa_i \frac{g_i''(x_i)}{x_i}$, $i = 1, \cdots, n$. This class of problems is studied in Ye and Potra [23] and Han et al[2] [5]. In the case of entropy programming we have $g_i(x_i) = x_i \ln x_i$, for all i, and $\kappa_i = 1$. Self-concordance for the logarithmic barrier function of this problem simply follows from the following lemma.

[2] In [5] it is conjectured that these problems do not satisfy the self-concordance condition. The lemma shows that this conjecture is not true.

Lemma 4 *Suppose that* $|g_i'''(x_i)| \leq \kappa_i \frac{g_i''(x_i)}{x_i}$, $i = 1, \cdots, n$, *then the logarithmic barrier function for the extended entropy programming problem* $(\mathcal{EEP})$ *is* $(1 + \frac{1}{3}\max_i \kappa_i)$*-self-concordant.*

Proof: Using Lemma 1 it suffices to show that

$$g_i(x_i) - \ln x_i$$

is $(1 + \frac{1}{3}\kappa_i)$-self-concordant. Since (1) reduces in the present case to

$$|g_i'''(x_i)| \leq \kappa_i g_i''(x_i)\frac{1}{x_i},$$

this immediately follows from Lemma 2. *Q.E.D.*

5 The primal l_p-programming problem

Let $\{I_k\}_{k=1,\cdots,r}$ be a partition of $\{1, \cdots, m\}$ (i.e. $\cup_{k=1}^r I_k = \{1, \cdots, m\}$ and $I_k \cap I_l = \emptyset$ for $k \neq l$). Let $p_i \geq 1$, $i = 1, \cdots, m$. Then the primal l_p-programming problem [18, 22] can be formulated as

$$(\mathcal{PL}_p) \quad \begin{cases} \max \ \ \eta^T x \\ \sum_{i \in I_k} \frac{1}{p_i}|a_i^T x - c_i|^{p_i} + b_k^T x - d_k \leq 0, \ k = 1, \cdots, r, \end{cases}$$

where (for all i and k) a_i, b_k, and η are n-dimensional vectors, and c_i and d_k are real numbers. Nesterov and Nemirovsky [17] treated a special case of this problem, namely the so-called l_p-approximation problem. We will reformulate $(\mathcal{PL}_p)$ such that all problem functions remain convex, contrary to Nesterov and Nemirovsky's reformulation.

In a first step, the primal l_p-programming problem can be reformulated as:

$$\begin{cases} \max \ \ \eta^T x \\ \sum_{i \in I_k} \frac{1}{p_i} t_i + b_k^T x - d_k \leq 0, \ k = 1, \cdots, r \\ \left.\begin{array}{l} s_i^{p_i} \leq t_i \\ a_i^T x - c_i \leq s_i \\ -a_i^T x + c_i \leq s_i \end{array}\right\} i = 1, \cdots, m \\ s \geq 0. \end{cases} \tag{18}$$

In the same way as we will prove Lemma 5, it can be proved that the logarithmic barrier function for this reformulated l_p-programming problem is $(1 + \frac{1}{3}\max_i |p_i - 2|)$-self-concordant, i.e. the concordance parameter depends on p_i. We can eliminate this dependence as follows. Replace the constraints $s_i^{p_i} \leq t_i$ by the equivalent constraints $s_i \leq t_i^{\pi_i}$, where $0 < \pi_i := \frac{1}{p_i} \leq 1$, and replace the (redundant) constraints $s \geq 0$ by $t \geq 0$. So, we obtain the following reformulated l_p-programming problem:

$$(\mathcal{PL}_p') \quad \begin{cases} \max \ \ \eta^T x \\ \sum_{i \in I_k} \frac{1}{p_i} t_i + b_k^T x - d_k \leq 0, \ k = 1, \cdots, r \\ \left.\begin{array}{l} s_i \leq t_i^{\pi_i} \\ a_i^T x - c_i \leq s_i \\ -a_i^T x + c_i \leq s_i \end{array}\right\} i = 1, \cdots, m \\ t \geq 0. \end{cases} \tag{19}$$

Observe that the transformed problem has $4m + r$ constraints, compared with r in the original problem $(\mathcal{PL}_p)$. Now we have the following lemma.

Lemma 5 *The logarithmic barrier function for the reformulated l_p-programming problem $(\mathcal{PL}'_p)$ is $\frac{5}{3}$-self-concordant.*

Proof:
First note that the logarithmic barrier function for the linear constraints is 1-self-concordant. Moreover, since $f_i(t_i) := -t_i^{\pi_i}$, $0 < \pi_i \leq 1$, satisfies (1) with $\beta = 2 - \pi_i$, we have from Lemma 2 that

$$-\ln(t_i^{\pi_i} - s_i) - \ln t_i$$

is $(1 + \frac{1}{3}(2 - \pi_i))$-self-concordant, where $0 < \pi_i \leq 1$. From Lemma 1 it follows that the logarithmic barrier function is $\frac{5}{3}$-self-concordant. *Q.E.D.*

6 The dual l_p-programming problem

Let q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $1 \leq i \leq m$, and let the rows of a matrix A be a_i, $i = 1, \cdots, m$, and the rows of a matrix B be b_k, $k = 1, \cdots, r$. Then, the dual of the l_p-programming problem $(\mathcal{PL}_p)$ is (see [18]–[22])

$$(\mathcal{DL}_p) \quad \begin{cases} \min\ c^T y + d^T z + \sum_{k=1}^{r} z_k \sum_{i \in I_k} \frac{1}{q_i} \left| \frac{y_i}{z_k} \right|^{q_i} \\ A^T y + B^T z = \eta \\ z \geq 0. \end{cases}$$

(If $y_i \neq 0$ and $z_k = 0$, then $z_k |y_i / z_k|^{q_i}$ is defined as ∞.) The above problem is equivalent to

$$\begin{cases} \min\ c^T y + d^T z + \sum_{i=1}^{n} \frac{1}{q_i} t_i \\ s_i^{q_i} z_k^{-q_i+1} \leq t_i, \quad i \in I_k, \quad k = 1, \cdots, r \\ y \leq s \\ -y \leq s \\ A^T y + B^T z = \eta \\ z \geq 0 \\ s \geq 0. \end{cases} \tag{20}$$

Similarly as in the proof of the next lemma, it can be proved that the logarithmic barrier function of this reformulated dual l_p-programming problem is $(1 + \frac{\sqrt{2}}{3} \max_i (q_i + 1))$-self-concordant. Again, the dependence on q_i can be eliminated: the constraints $s_i^{q_i} z_k^{-q_i+1} \leq t_i$ are replaced by the equivalent constraints $t_i^{\rho_i} z_k^{-\rho_i+1} \geq s_i$, where $0 < \rho_i := \frac{1}{q_i} \leq 1$, and the redundant constraints $s \geq 0$ are

replaced by $t \geq 0$. The new reformulated dual l_p-programming problem becomes:

$$(\mathcal{DL}'_p) \quad \begin{cases} \min\ c^T y + d^T z + \sum_{i=1}^n \frac{1}{q_i} t_i \\ s_i \leq t_i^{\rho_i} z_k^{-\rho_i+1}, \quad i \in I_k, \quad k = 1, \cdots, r \\ y \leq s \\ -y \leq s \\ A^T y + B^T z = \eta \\ z \geq 0 \\ t \geq 0. \end{cases} \tag{21}$$

Note that the original problem $(\mathcal{DL}_p)$ has r inequalities, and the reformulated problem $(\mathcal{DL}'_p)$ $4m + r$. We now have the following lemma.

Lemma 6 *The logarithmic barrier function of the reformulated dual l_p-programming problem $(\mathcal{DL}'_p)$ is 2-self-concordant.*

Proof:
By Lemma 1 it suffices to show that

$$-\ln(t_i^{\rho_i} z_k^{-\rho_i+1} - s_i) - \ln z_k - \ln t_i$$

is $(1 + \frac{\sqrt{2}}{3}(\rho_i + 1))$-self-concordant, or equivalently by Lemma 2, that (we will omit the subscript i and k in the sequel of this proof) $f(t, z) := -t^\rho z^{-\rho+1}$ with $0 < \rho < 1$ satisfies (1) for $\beta = \sqrt{2}(\rho+1)$, i.e. that

$$|\nabla^3 f(t, z)[h, h, h]| \leq \sqrt{2}(\rho + 1) h^T \nabla^2 f(t, z) h \sqrt{\frac{|h_1|^2}{t^2} + \frac{|h_2|^2}{z^2}}, \tag{22}$$

where $h^T = (h_1, h_2)$. After doing some straightforward calculations we obtain for the second order term

$$\begin{aligned} h^T \nabla^2 f(t, z) h &= \rho(1-\rho) t^{\rho-3} z^{-\rho-2} (t z^3 h_1^2 + t^3 z h_2^2 - 2t^2 z^2 h_1 h_2) \\ &= \rho(1-\rho) t^{\rho-3} z^{-\rho-2} (z h_1 - t h_2)^2 t z, \end{aligned}$$

and for the third order term

$$\begin{aligned} |\nabla^3 f(t, z)[h, h, h]| &= \rho(1-\rho) s^{\rho-3} z^{-\rho-2} |(\rho-2) z^3 h_1^3 - (\rho+1) t^3 h_2^3 - 3(\rho-1) t z^2 h_1^2 h_2 + 3\rho t^2 z h_1 h_2^2| \\ &= \rho(1-\rho) t^{\rho-3} z^{-\rho-2} (z h_1 - t h_2)^2 |(\rho-2) z h_1 - (\rho+1) t h_2| \\ &\leq \rho(1-\rho)(\rho+1) t^{\rho-3} z^{-\rho-2} (z h_1 - t h_2)^2 (z|h_1| + t|h_2|). \end{aligned}$$

Now we obtain

$$\frac{|\nabla^3 f(t, z)[h, h, h]|}{h^T \nabla^2 f(t, z) h} \leq (\rho + 1) \left(\frac{|h_1|}{t} + \frac{|h_2|}{z} \right) \leq \sqrt{2}(\rho + 1) \sqrt{\frac{|h_1|^2}{t^2} + \frac{|h_2|^2}{z^2}}.$$

This proves (22). The lemma follows since $\rho_i \leq 1$. *Q.E.D.*

7 Other smoothness conditions

Relative Lipschitz condition

Jarre [9] introduced the following relative Lipschitz condition (also used in e.g. [7]) for the Hessian matrix of the problem functions $f_i(x)$, $0 \leq i \leq m$, of (CP):

$$\exists M > 0 : \forall v \in I\!R^n \ \ \forall x, x+h \in \mathcal{F}^0 :$$

$$|v^T(\nabla^2 f_i(x+h) - \nabla^2 f_i(x))v| \leq M\|h\|_H v^T \nabla^2 f_i(x) v, \tag{23}$$

where H is the Hessian matrix of the corresponding logarithmic barrier function. As shown in Jarre [10], if the Hessians of the problem functions f_i of $(\mathcal{CP})$ fulfil this relative Lipschitz condition with parameter M, and if $f_i \in C^3$, then the associated logarithmic barrier function is $(1+M)$-self-concordant. (The converse is not true.) Moreover, in [10] it is shown that the relative Lipschitz condition for the logarithmic barrier function is equivalent to self-concordance if the underlying function is three times continously differentiable.

Monteiro and Adler's condition

Monteiro and Adler [16] considered minimization problems with linear equality constraints and a separable convex objective function on the positive orthant of $I\!R^n$. The objective function $f(x) = \sum_i g_i(x_i)$ must satisfy the following condition:

There exist positive numbers T and p such that for all reals $x > 0$ and $y > 0$ and all $i = 1, \cdots, n$, we have

$$y|g_i'''(y)| \leq T \max\left\{(\frac{x}{y})^p, (\frac{y}{x})^p\right\} g_i''(x).$$

Using Lemma 2 and substituting $y = x$ in the above condition, it is easy to see that g_i satisfies (1) with $\beta = T$, i.e. that the logarithmic barrier function for such a problem is $(1+\frac{1}{3}T)$-self-concordant. Using Lemma 2 we may simplify the condition of [16] to the (weaker) condition that there exists a positive number T such that for all reals $y > 0$ and all $i = 1, \cdots, n$, we have

$$y|g_i'''(y)| \leq T g_i''(x).$$

This condition is not only simpler, also the dependence on some extra parameter p is eliminated.

Scaled Lipschitz condition

In [25] and [13] interior-point methods are given and analyzed for problems with linear equality constraints and convex objective function $f(x)$ on the positive orthant of $I\!R^n$. The objective function has to satisfy the following scaled Lipschitz condition:

There exists $M > 0$, such that for any γ, $0 < \gamma < 1$,

$$\|X(\nabla f(x+\Delta x) - \nabla f(x) - \nabla^2 f(x)\Delta x)\| \leq M \Delta x^T \nabla^2 f(x) \Delta x, \tag{24}$$

whenever $x > 0$ and $\|X^{-1}\Delta x\| \leq \gamma$. (Here, $\|.\|$ is the Euclidean norm.)

This condition is also covered by the self-concordance condition if f is three times continuously differentiable in the interior of the feasible domain. More precisely we will show in the next lemma that the corresponding logarithmic barrier function is $(1+\frac{2}{3}M)$-self-concordant.

Lemma 7 *Suppose $f(x) \in C^3$ fulfils the scaled Lipschitz condition with parameter M. Then the logarithmic barrier functions φ and ψ from Lemma 2 are $(1+\frac{2}{3}M)$-self-concordant.*

Proof:
It suffices to prove (1). Set $h = \Delta x$ as in definition (24). First note that

$$\sqrt{\sum_{i=1}^{n} \frac{h_i^2}{x_i^2}} = \|X^{-1}\Delta x\|.$$

Since $f \in C^3$ we may expand ∇f as follows:

$$\nabla f(x + \Delta x) = \nabla f(x) + \nabla^2 f(x)\Delta x + \frac{1}{2}\nabla^3 f(x)[\Delta x, \Delta x, .] + o(\|\Delta x\|^2),$$

where $\nabla^3 f(x)[\Delta x, \Delta x, .]$ is a vector whose i-th component is equal to

$$\sum_{j,k} \frac{\partial^3 f(x)}{\partial x_i \, \partial x_j \, \partial x_k} \Delta x_j \, \Delta x_k.$$

Replacing Δx by $\lambda \Delta x$ in definition (24), inserting the above expansion, dividing by λ^2, and taking the limit as λ tends to zero we obtain

$$\|X\nabla^3 f(x)[\Delta x, \Delta x, .]\| \leq 2M\Delta x^T \nabla^2 f(x)\Delta x. \tag{25}$$

Considering $X\nabla^3 f(x)[\Delta x, \Delta x, .]$ as a column vector we may continue

$$\|X\nabla^3 f(x)[\Delta x, \Delta x, .]\| \geq \frac{(X^{-1}\Delta x)^T}{\|X^{-1}\Delta x\|} X\nabla^3 f(x)[\Delta x, \Delta x, .] = \frac{\nabla^3 f(x)[\Delta x, \Delta x, \Delta x]}{\|X^{-1}\Delta x\|},$$

and obtain that

$$\nabla^3 f(x)[\Delta x, \Delta x, \Delta x] \leq 2M\|X^{-1}\Delta x\|\Delta x^T \nabla^2 f(x)\Delta x,$$

which is exactly relation (1). *Q.E.D.*

Before we conclude this work we would like to briefly point out a class of problems considered by Mehrotra and Sun [15] (and also by Zhang [24]) which does not have a self-concordant logarithmic barrier function. Mehrotra and Sun introduced a curvature constraint of the following form: There exists a number $\kappa \geq 1$ such that for all x, y and h in $\mathbb{R}^n$

$$h^T \nabla^2 f_i(x) h \leq \kappa h^T \nabla^2 f_i(y) h.$$

For constraint functions f_i satisfying this condition, they present a polynomial time interior-point algorithm (that needs at most $O(\kappa^5 \sqrt{m} \ln \epsilon)$ Newton iterations to reduce the error by a factor of ϵ). Clearly, there are constraints with self-concordant barriers that do not satisfy this condition, and conversely, this condition covers some constraint functions that do not have a self-concordant barrier function. For most applications however, we believe that the self-concordance condition is more practical.

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