Linear Complementarity and Oriented Matroids

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Abstract

A combinatorial abstraction of the linear complementarity theory in the setting of oriented matroids was first considered by M.J. Todd. In this paper, we take a fresh look at this abstraction, and attempt to give a simple treatment of the combinatorial theory of linear complementarity. We obtain new theorems, proofs and algorithms in oriented matroids whose specializations to the linear case are also new. For this, the notion of sufficiency of square matrices, introduced by Cottle, Pang and Venkateswaran, is extended to oriented matroids. Then, we prove a sort of duality theorem for oriented matroids, which roughly states: exactly one of the primal and the dual system has a complementary solution if the associated oriented matroid satisfies "weak" sufficiency. We give two different proofs for this theorem, an elementary inductive proof and an algorithmic proof using the criss-cross method which solves one of the primal or dual problem by using surprisingly simple pivot rules (without any perturbation of the original problem).

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1 Introduction

The linear complementarity problem (LCP) is to find vectors w and z in R^n satisfying

$$\begin{aligned} &w = Az + b, z \geq 0, w \geq 0 \\ &z^T w = 0 \end{aligned} \tag{1.1}$$

where b is a given rational n-vector, and A is a given rational $n \times n$ matrix.

The theory of linear complementarity, introduced by Cottle [5], Cottle and Dantzig [6] and Lemke [17], has been extensively studied for the last two decades. One main importance of this theory lies in the fact that it provides a unified framework for the two important optimization problems with wide applications, linear programming and convex quadratic programming.

Solving the LCP (1.1) in general, however, appears to be difficult. One simple reason is that there is no known good characterization of the nonexistence of a solution to the system for any given matrix A and a vector b. In other words, the decision problem of LCP is *NP* (i.e. if there is a solution, there is a solution with size polynomially bounded by the input size) but may not be *co-NP*. Several different classes of matrices A for which the associated decision problem is *co-NP* are known. The P-matrices and the positive semidefinite (PSD-) matrices constitute such classes. The LCP's arising from linear programming problems or convex quadratic programming problems have PSD-matrices. Recently, Cottle, Pang and Venkateswaran [7, 8] have introduced the class of sufficient matrices properly containing the two classes, for which the decision problem is *co-NP*.

For solving such "good" classes of LCP's, there are two major approaches, the classical pivoting approach and the recent approach using interior-point algorithms [16, 26]. While polynomial-time algorithms for LCP's have been obtained only by the latter approach, a satisfactory foundation of linear complementarity theory depends much on the combinatorial nature of the former approach.

Todd [23, 24] was the first to explain the essence of combinatorics used in the classical pivoting algorithms for LCPs and extend many results on LCP to the notion of oriented matroids. In order to achieve these he obtained elegant characterizations of symmetric matrices, P-matrices and PSD-matrices in terms of sign conditions on the linear subspaces associated with the extended matrix $[-A\ I]$. Another difficulty he had to overcome was to generalize perturbation techniques to oriented matroids, since the usual pivoting algorithms for LCP rely heavily on nondegeneracy.

On the other hand, the recent approach by Klafszky and Terlaky [14, 15], Fukuda and Terlaky [13] and den Hertog, Roos and Terlaky [9] showed that the nondegeneracy assumption is not important for the finiteness of pivoting algorithm if the criss-cross type of "minimum-index" pivot strategy is employed.

In this paper, we attempt to show two things. The first thing is to obtain a general theorem (the duality theorem) of complementarity in the setting of oriented matroids,

which implies a good characterization of the nonexistence of a solution in the special setting of sufficient LCP case. For this, a certain combinatorial abstraction of sufficient matrices is useful for the simplicity of the statement and also for a short elementary proof. The second thing is to study essential conditions on oriented matroids under which the criss-cross method [15], a natural extension of the same algorithm for linear programming [21, 22, 25] "solves" the associated complementarity problem. For this we extract the combinatorial ideas used in [9]. It turns out that if an oriented matroid satisfies essentially the same assumption as for the duality theorem, the criss-cross method solves the complementarity problem, and hence gives a constructive proof of the duality theorem. It is noteworthy that we do not depend on the nondegeneracy assumption. And therefore, we not only generalize some theorems of Cottle et al [7] and Todd [23, 24], but simplify their proofs, and obtain a new algorithm which is extremely simple.

2 LCP Duality Theorem

In order to write the LCP (1.1) in a form that will be convenient for extending it to the setting of oriented matroids, we set

$$V(A,b) = \{x \in R^{2n+1} \ \mid \ [-b \ \ -A \ I] \ x = 0\}. \tag{2.2}$$

We denote the components of a vector $x \in R^{2n+1}$ by $x = (x_0, x_1, x_2, \ldots, x_{2n})$ instead of the usual $x = (x_1, x_2, \ldots, x_{2n+1})$, while we denote a vector $x \in R^{2n}$ by $x = (x_1, \ldots, x_{2n})$. Then, one can rewrite the LCP (1.1) as

$$\text{find } x \in V(A,b) \text{ satisfying} \tag{2.3}$$

$$x \geq 0, \ x_0 = 1 \tag{2.4}$$

$$x_i \cdot x_{n+i} = 0 \text{ for all } i = 1, \ldots, n. \tag{2.5}$$

A vector x in R^{2n} or R^{2n+1} is called *complementary* if the complementarity condition (2.5) is satisfied. A solution to the LCP is a nonnegative complementary vector x in $V(A,b)$ with $x_0 = 1$.

If the LCP has a solution, then it is easy to see that there is a basic solution, i.e., one having a minimal support among all nonzero vectors in $V(A,b)$, and hence the LCP is in NP (see [20], Corollary 3.2b). In general, there is no known good characterization of the nonexistence of a solution to the LCP. However we can write conditions on the linear subspace under which such characterization exists.

A vector x in R^{2n} or in R^{2n+1} is called *strictly sign-preserving* (s.s.p.) if

$$x_i \cdot x_{n+i} \geq 0 \text{ for all } i = 1, \ldots, n \text{ and} \tag{2.6}$$

$$x_j \cdot x_{n+j} > 0 \text{ for some } j, \tag{2.7}$$

and called *strictly sign-reversing* (s.s.r.) if

$$x_i \cdot x_{n+i} \leq 0 \text{ for all } i = 1, \ldots, n \text{ and} \tag{2.8}$$

$$x_j \cdot x_{n+j} < 0 \text{ for some } j. \tag{2.9}$$

Now we have the following general theorem, which we call the duality theorem of linear complementarity.

Theorem 2.1 (LCP Duality Theorem) : *Let V be a linear subspace of R^{2n+1} satisfying the following condition:*

$$\begin{aligned} \textit{either} \quad & V \textit{ contains no s.s.r. vector } x \textit{ with } x_0 = 0 \\ \textit{or} \quad & V \textit{ contains no s.s.p. vector } x \textit{ with } x_0 \neq 0, \end{aligned} \tag{2.10}$$

and suppose the orthogonal complement $V^\perp$ also satisfies the same condition, i.e.,

$$\begin{aligned} \textit{either} \quad & V^\perp \textit{ contains no s.s.r. vector } y \textit{ with } y_0 = 0 \\ \textit{or} \quad & V^\perp \textit{ contains no s.s.p. vector } y \textit{ with } y_0 \neq 0. \end{aligned} \tag{2.11}$$

Then exactly one of the following statements holds:

(a) *there exists a nonnegative complementary vector x in V with $x_0 = 1$;*

(b) *there exists a nonnegative complementary vector y in $V^\perp$ with $y_0 = 1$.*

One can easily see that this theorem is self-dual, i.e., replacing V by $V^\perp$ yields the same theorem. This self-duality makes the statement look simple, and quite importantly it makes a short elementary proof possible. Also, this theorem suggests us to consider the *dual* problem of the LCP (2.3)∼(2.5):

$$\text{find } y \in V(A, b)^\perp \text{ satisfying} \tag{2.12}$$

$$y \geq 0, \; y_0 = 1 \tag{2.13}$$

$$y_i \cdot y_{n+i} = 0 \text{ for all } i = 1, \ldots, n, \tag{2.14}$$

which is solvable if and only if the *primal* problem is not, under the assumptions of theorem.

Remark that if V in the duality theorem has an explicit representation as $V(A, b)$, then one can write the statements (a) and (b) more explicitly as

(a') there exist vectors w, z in R^n satisfying $w = Az + b, z \geq 0, w \geq 0,$ and $z^T w = 0$;

(b') there exists a vector u in R^n satisfying $b^T u = -1, u \geq 0, A^T u \leq 0,$ and $u^T A^T u = 0$.

The assumptions (2.10) and (2.11) for $V(A,b)$ are satisfied by several well-known classes of square matrices A, for any vector b. For example, the class of P-matrices and the class of PSD-matrices satisfy the conditions because, for any b,

$$V(A,b) \text{ contains no s.s.r. vector } x \text{ with } x_0 = 0 \text{ and}$$
$$V(A,b)^{\perp} \text{ contains no s.s.p. vector } w \text{ with } w_0 \neq 0.$$

In fact, the sufficient matrices are exactly those matrices with these properties. More explicitely, an $n \times n$ matrix A is called *column sufficient* if

$$w = Az \text{ implies } (w,z) \text{ is not s.s.r.,} \tag{2.15}$$

called *row sufficient* if

$$v = -A^T u \text{ implies } (u,v) \text{ is not s.s.p.,} \tag{2.16}$$

and called *sufficient* if it is both column and row sufficient. Clearly, the row-sufficiency of A is equivalent to the column-sufficiency of A^T. Theorem 2.1 implies that for any sufficient matrix A and any vector b, exactly one of statements (a') or (b') holds.

As an example of linear subspaces V satisfying the conditions (2.10) and (2.11), we can set $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and $V = V(A,b)$. In this example, the primal problem has a complementary solution $x = (x_0, x_1, x_2, x_3, x_4) = (1,0,1,0,0)$. Remark that if we set $b = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$ and use the same A, the conditions of the theorem are not fulfilled because both the s.s.r vector $(0,0,1,1,-2)$ and the s.s.p. vector $(2,0,-1,3,-2)$ are in $V^{\perp}$. In fact, in this case, neither of V nor $V^{\perp}$ have any nonnegative complementary solutions, although V contains the nonnegative vector $(1,0,1,0,\frac{1}{2})$. Both of the matrices above are column sufficient (and not row sufficient), and these examples explain the difficulty of solving the LCP with a column sufficient matrix in general.

It should be noted that Theorem 2.1 can be considered as a generalization of the well-known LP duality theorem and Cottle's QP duality theorem since they correspond to certain subclasses of PSD-matrices.

In the following section, we will see how the duality theorem, Theorem 2.1, can be generalized and proved in purely combinatorial fashion using the notion of oriented matroids.

3 Oriented Matroids and Complementarity

Let E be a finite set. A *signed vector on* E is a vector in $\{+,0,-\}^E$. The opposite $-X$ of a signed vector X is defined in the obvious way: $-(-0-++) = (+0+--)$. Let X and Y be signed vectors on E. An element $j \in E$ is said to *separate* X and Y if X_j and

Y_j are nonzero and opposite in signs. The set of elements separating X and Y is denoted by $D(X,Y)$. The composition of X and Y, denoted by $X \circ Y$, is the signed vector on E defined by

$$(X \circ Y)_j = \begin{cases} X_j & \text{if } X_j \neq 0 \\ Y_j & \text{otherwise} \end{cases} \qquad (j \in E).$$

The *support* $\underline{X}$ of X is the set $\{j \in E : X_j \neq 0\}$.

An *oriented matroid on* E is a pair $M = (E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a set of signed vectors on E satisfying the following axioms:

(OM1) $\mathbf{0} \in \mathcal{F}$; (where $\mathbf{0}$ denotes the zero vector)
(OM2) $X \in \mathcal{F} \Longrightarrow -X \in \mathcal{F}$;
(OM3) $X, X' \in \mathcal{F} \Longrightarrow X \circ X' \in \mathcal{F}$;
(OM4) $X, X' \in \mathcal{F}$ and $f \in D(X, X') \Longrightarrow$ there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z_j = (X \circ X')_j$ for all $j \notin D(X, X')$.

We call each member of $\mathcal{F}$ a *vector* of the oriented matroid. Each vector with minimal nonempty support is called a *circuit*. One can define oriented matroids in terms of circuits, see [2, 3].

Note that the notion of oriented matoids was introduced independently by Bland-Las Vergnas [3] and Folkman-Lawrence [10]. The definition of oriented matroids above is due to Edmonds, Fukuda and Mandel [11, 18].

For a vector $x \in R^E$, we denote the signed incidence vector of x as $\delta(x)$, i.e., $\delta(x)$ is the signed vector on E with $\delta(x)_j = sign(x_j)$ for each $j \in E$. One can easily check that for any linear subspace V of R^E, the set $\mathcal{F} = \delta(V)$ of all signed vectors coming from V satisfies the axioms (OM1)$\sim$(OM4). Such an oriented matroid is called *representable* or *linear*. It is well-known that there are infinitely many non-linear oriented matroids. For constructions, see [11, 18, 4, 1].

We shall give below some basic properties of oriented matoids without proofs. For those unfamilier with oriented matroids, good references are [1, 3, 10, 11, 18].

It will be convenient to define the *product* $\alpha \cdot \beta$ (or simply $\alpha\ \beta$) of two signs $\alpha, \beta \in \{+, 0, -\}$ in the natural way: $+$ if both are nonzero and of same sign, $-$ if both are nonzero and of opposite sign, and 0 otherwise. Two signed vectors X and Y are called *orthogonal* (denoted by $X * Y$) if the existence of an element $f \in E$ with $X_f \cdot Y_f \neq 0$ implies the existence of an element $g \in E$ with $X_g \cdot Y_g = -X_f \cdot Y_f$. The *dual* $M^* = (E, \mathcal{F}^*)$ of an oriented matroid M is defined by

$$\mathcal{F}^* = \{Y \in \{+, -, 0\}^E \mid X * Y \text{ for all } X \in \mathcal{F}\}.$$

The dual M^* is also an oriented matroid, and

$$\mathcal{F}^{**} = \mathcal{F} \tag{3.17}$$

Furthermore, if $F = \delta(V)$ for a linear subspace V of R^E then $F^* = \delta(V^\perp)$. A vector, circuit, ... of M^* may be called *covector*, *cocircuit* ... of M.

There are two basic operations on oriented matroids which will be used frequently in this paper. For a subset S of E, let

$$\mathcal{F}\backslash S = \{X\backslash S \mid X \in \mathcal{F} \text{ and } X_j = 0 \text{ for all } j \in S\} \tag{3.18}$$

$$\mathcal{F}/S = \{X\backslash S \mid X \in \mathcal{F}\}, \tag{3.19}$$

where $X\backslash S$ denotes the subvector $(X_j : j \in E\backslash S)$ of X. It is easy to see from the definition, both $(E, \mathcal{F}\backslash S)$ and $(E, \mathcal{F}/S)$ are oriented matroids, which are denoted by $M\backslash S$ and M/S, respectively. When S is a singleton set $\{e\}$, $M\backslash\{e\}$ and $M/\{e\}$ will be denoted by $M\backslash e$ and M/e, respectively. The following relations hold for any oriented matroid $M = (E, \mathcal{F})$:

$$(\mathcal{F}\backslash S)^* = \mathcal{F}^*/S \tag{3.20}$$

$$(\mathcal{F}/S)^* = \mathcal{F}^*\backslash S \tag{3.21}$$

For a nonnegative integer n, let E_{2n} denote a set of $2n$ elements partitioned into prescribed n pairs; for each $e \in E_{2n}$ there is a unique element $\overline{e}$ in E_{2n}, called the complement of e such that $\overline{\overline{e}} = e$. For a subset S of E_{2n} let $\overline{S} = \{\overline{e} \mid e \in S\}$. A subset S is called *complementary* if $S \cap \overline{S} = \emptyset$. We use $\hat{E}_{2n}$ to denote $E_{2n} \cup \{g\}$ for a given E_{2n} and a distinguished element g which is assumed to be not in E_{2n}.

One canonical choice of E_{2n} and $\hat{E}_{2n}$ is $\{1, 2, \ldots, 2n\}$ and $\{g = 0, 1, 2, \ldots, 2n\}$, respectively, where $\overline{i} = i + n$ if $i \leq n$, and $\overline{i} = i - n$ if $i > n$.

For a given oriented matroid $M = (\hat{E}_{2n}, \mathcal{F})$ with $n \geq 0$, the *complementarity problem* (OMCP) is to

$$\text{find } X \in \mathcal{F} \text{ satisfying} \tag{3.22}$$

$$X \geq 0,\ X_g = + \tag{3.23}$$

$$X_e \cdot X_{\overline{e}} = 0 \text{ for all } e \in E_{2n} \tag{3.24}$$

Here, we use the inequality $\geq$ in the obvious way and we shall do so in the sequel. The conditions (3.22), (3.23) and (3.24) are straightforward abstractions of (2.3), (2.4) and (2.5), respectively.

Again, we simply make the "automatic" abstraction of definitions in the LCP case. A signed vector X on E_{2n} or on $\hat{E}_{2n}$ is called *complementary* if the condition (3.24) is satisfied, called *strictly sign-preserving* (s.s.p.) if

$$X_e \cdot X_{\overline{e}} \geq 0 \text{ for all } e \in E_{2n} \text{ and} \tag{3.25}$$

$$X_f \cdot X_{\overline{f}} > 0 \text{ for some } f, \tag{3.26}$$

and called *strictly sign-reversing* (s.s.r.) if

$$X_e \cdot X_{\overline{e}} \leq 0 \text{ for all } e \in E_{2n} \text{ and} \tag{3.27}$$

$$X_f \cdot X_{\overline{f}} < 0 \text{ for some } f. \tag{3.28}$$

The following theorem is a combinatorial generalization of the duality theorem, Theorem 2.1.

Theorem 3.1 (OMCP Duality Theorem) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the following conditions:*

$$\begin{aligned} \textit{either} \quad & \mathcal{F} \textit{ contains no s.s.r. vector } X \textit{ with } X_g = 0 \\ \textit{or} \quad & \mathcal{F} \textit{ contains no s.s.p. vector } X \textit{ with } X_g \neq 0 \textit{ and,} \qquad (3.29) \\ \textit{either} \quad & \mathcal{F}^* \textit{ contains no s.s.r. vector } Y \textit{ with } Y_g = 0 \\ \textit{or} \quad & \mathcal{F}^* \textit{ contains no s.s.p. vector } Y \textit{ with } Y_g \neq 0. \qquad (3.30) \end{aligned}$$

Then exactly one of the following statements holds:

(a) *there exists a nonnegative complementary vector X in $\mathcal{F}$ with $X_g = +$;*

(b) *there exists a nonnegative complementary vector Y in $\mathcal{F}^*$ with $Y_g = +$.*

We call an oriented matroid $M' = (E_{2n}, \mathcal{F}')$ *sufficient* if $\mathcal{F}'$ contains no s.s.r. vector and its dual $(\mathcal{F}')^*$ contains no s.s.p. vector. If $M = (\hat{E}_{2n}, \mathcal{F})$ is an single-element extension of a sufficient oriented matroid $M' = (E_{2n}, \mathcal{F}')$ by g, i.e., $M \backslash g = M'$, then clearly M satisfies the conditions (3.29) and (3.30) of Theorem 3.1. Therefore we have the following corollary of the duality theorem.

Corollary 3.2. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be any single-element extension of a sufficient oriented matroid M' by g. Then exactly one of the following statements holds:*

(a) *there exists a nonnegative complementary vector X in $\mathcal{F}$ with $X_g = +$;*

(b) *there exists a nonnegative complementary vector Y in $\mathcal{F}^*$ with $Y_g = +$.*

In this section, we prove the duality theorem in a most elementary way. It will be interesing to compare this proof with an algorithmic proof for (a slightly different version of) this theorem, since they are closely related.

For the elementary proof, what we need is the notion of minors of OMCP, which was introduced by Todd [23]. Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid with $n \geq 0$. For each complementary subset S of E_{2n}, let $M(S) = (\hat{E}_{2n} \backslash (S \cup \overline{S}), \mathcal{F}(S))$ denote the oriented matroid $M \backslash S / \overline{S}$, which is called a *complementary minor* of M. For $e \in E$, we denote $M(\{e\})$ by $M(e)$. Of course, the set $\hat{E}_{2n} \backslash (S \cup \overline{S})$ can be considered as a set of form $\hat{E}_{2(n-s)}$, for any complementary subset S with $\mid S \mid = s$. From the definitions, we have the following relations:

$$X \in \mathcal{F} \text{ and } X_e = 0 \text{ for all } e \in S \Rightarrow X \backslash (S \cup \overline{S}) \in \mathcal{F}(S); \qquad (3.31)$$

$$X' \in \mathcal{F}(S) \Rightarrow \exists X \in \mathcal{F} \text{ such that } X \backslash (S \cup \overline{S}) = X' \text{ and } X_e = 0 \ \forall\, e \in S; \qquad (3.32)$$

$$Y \in \mathcal{F}^* \text{ and } Y_{\overline{e}} = 0 \text{ for all } e \in S \Rightarrow Y \backslash (S \cup \overline{S}) \in \mathcal{F}(S)^*; \qquad (3.33)$$

$$Y' \in \mathcal{F}(S)^* \Rightarrow \exists Y \in \mathcal{F}^* \text{ such that } Y \backslash (S \cup \overline{S}) = Y' \text{ and } Y_{\overline{e}} = 0 \ \forall\, e \in S. \qquad (3.34)$$

These immediately imply the following:

Lemma 3.3. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (3.29) and (3.30) of Theorem 3.1. Then the complementary minor $M(S)$ of M satisfies the same conditions for any complementary subset S of E_{2n}.*

Lemma 3.4. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (3.29) and (3.30) of Theorem 3.1. Then for each $e \in E_{2n}$, at most one of the following four statements holds:*

(a1) *$\exists$ a complementary vector X^1 in $\mathcal{F}$ with $X^1 \backslash \{e, \overline{e}\} \geq 0, X^1_{\overline{e}} = -$ and $X^1_g = +$;*

(a2) *$\exists$ a complementary vector X^2 in $\mathcal{F}$ with $X^2 \backslash \{e, \overline{e}\} \geq 0, X^2_e = -$ and $X^2_g = +$;*

(b1) *$\exists$ a complementary vector Y^1 in $\mathcal{F}^*$ with $Y^1 \backslash \{e, \overline{e}\} \geq 0, Y^1_e = -$ and $Y^1_g = +$;*

(b2) *$\exists$ a complementary vector Y^2 in $\mathcal{F}^*$ with $Y^2 \backslash \{e, \overline{e}\} \geq 0, Y^2_{\overline{e}} = -$ and $Y^2_g = +$.*

Proof. We prove by contradiction. Suppose the conclusion is false, i.e., two of the four statements hold. We have six cases to consider but because of the duality (3.17), (3.20) and (3.21), we have essentially three cases:

Case 1. (a1) and (a2) hold. (This covers the case (b1) and (b2), also.)
Let $Z' = X^1 \circ X^2$. By the axiom (OM3), $Z' \in \mathcal{F}$. It is easy to see from (a1) and (a2) that Z' is s.s.p. and $Z'_g \neq 0$. On the other hand, by using the axiom (OM4) with $X = X^1$, $X' = -X^2$ and $f = g$, we have $Z'' \in \mathcal{F}$ such that $Z''_g = 0$ and $Z''_j = (X^1 \circ -X^2)_j$ for all $j \notin D(X^1, -X^2)$. Again using (a1) and (a2), we have $Z''_j \cdot Z''_{\overline{j}} \leq 0$ for all $j \in E_{2n}$ and $Z''_e \cdot Z''_{\overline{e}} < 0$. This means that Z'' is s.s.r. and $Z''_g = 0$. The existence of Z' and Z'' violates the condition (3.29). Thus (a1) and (a2) cannot hold simultaneously.

Case 2. (a1) and (b1) hold. (This covers the case (a2) and (b2), also.)
Clearly, the vectors X^1 and Y^1 are not orthogonal, a contradiction. Thus (a1) and (b1) cannot hold simultaneously.

Case 3. (a1) and (b2) hold. (This covers the case (a2) and (b1), also.)
Clearly, the vectors X^1 and Y^2 are not orthogonal, a contradiction. Thus (a1) and (b2) cannot hold simultaneously.

This completes the proof. ∎

Now we are ready to prove the duality theorem.

Proof of Theorem 3.1. We use induction on n. If $n = 0$ then the theorem is obvious. Let $n > 0$, and suppose that the theorem is true for all smaller values of n. Assume the assumptions (3.29) and (3.30) are satisfied. Let e be an element of E_{2n}. By Lemma 3.3, the assumptions of the theorem are satisfied by the minors $M(e)$ and $M(\overline{e})$. By inductive hypothesis, exactly one of

(a1') there exists a nonnegative complementary vector X^1 in $\mathcal{F}(e)$ with $X^1_g = +$;

(b1') there exists a nonnegative complementary vector Y^1 in $\mathcal{F}(e)^*$ with $Y^1_g = +$

holds, and exactly one of

(a2') there exists a nonnegative complementary vector X^2 in $\mathcal{F}(\overline{e})$ with $X^2_g = +$;

(b2') there exists a nonnegative complementary vector Y^2 in $\mathcal{F}(\overline{e})^*$ with $Y^2_g = +$

holds. Now assume that the conclusion of the theorem is false, i.e., neither (a) nor (b) of the theorem holds. Then, using the relations (3.32) and (3.34), each of the statements (a1'), (a2'), (b1') and (b2') implies

(a1) $\exists$ a complementary vector X^1 in $\mathcal{F}$ with $X^1 \backslash \{e, \overline{e}\} \geq 0, X^1_{\overline{e}} = -$ and $X^1_g = +$;

(a2) $\exists$ a complementary vector X^2 in $\mathcal{F}$ with $X^2 \backslash \{e, \overline{e}\} \geq 0, X^2_e = -$ and $X^2_g = +$;

(b1) $\exists$ a complementary vector Y^1 in $\mathcal{F}^*$ with $Y^1 \backslash \{e, \overline{e}\} \geq 0, Y^1_e = -$ and $Y^1_g = +$;

(b2) $\exists$ a complementary vector Y^2 in $\mathcal{F}^*$ with $Y^2 \backslash \{e, \overline{e}\} \geq 0, Y^2_{\overline{e}} = -$ and $Y^2_g = +$,

respectively. The induction hypothesis forces that two of the statements (a1), (a2), (b1) and (b2) must hold. But this is impossible from Lemma 3.4. Thus the one of two statements (a) and (b) must hold. It is clear that (a) and (b) cannot hold simultaneously, and the theorem follows. ∎

4 Basis, Tableau and the Duality Theorem

In the previous section, we have proved the duality theorem by an elementary inductive argument. The proof was very simple and, in fact, it gives some general idea of how the OMCP can be solved; once you solve two subproblems $M(e)$ and $M(\overline{e})$, then one of the two solutions will be a solution of the original OMCP.

In this section, we will see how this simple idea can be extended to a finite algorithm. For this, we need the notion of basis, as this is essential in pivoting algorithms for the linear complementarity problem.

Let $M = (E, \mathcal{F})$ be an oriented matroid. A *basis* is a maximal subset B of E such that there is no nonzero vector X in $\mathcal{F}$ with $\underline{X} \subseteq B$. The following properties of basis are well-known:

Property 4.1. *Let $M = (E, \mathcal{F})$ be an oriented matroid.*

(1) *Every basis has the same cardinality, called the* rank $r(M)$ *of M;*

(2) *For each basis B and for each $j \in E \backslash B$, there exist a unique circuit, called the* fundamental *circuit $X(B, j) = X_M(B, j)$, such that $X(B, j)_j = +$ and $\underline{X(B, j)} \subseteq B \cup \{j\}$;*

(3) *A subset B of E is a basis of M if and only if $E \backslash B$ is a cobasis, i.e., a basis of the dual M^*.*

We will usually denote by N the cobasis $E \backslash B$, and denote by $Y(N, i)$ the fundamental cocircuit $X_{M^*}(N, i)$ for each $i \in B$

For each basis B and the cobasis $N = E \backslash B$, the *tableau* $T(B) = (t_{ij} : i \in B, j \in N)$ is the $\{+, 0, -\}$-matrix defined by

$$t_{ij} \quad = \quad X(B, j)_i \ \text{ for } i \in B, j \in N. \tag{4.35}$$

Clearly, the tableau $T(B)$ contains not only all information about the fundamental circuits $X(B, j), j \in N$, but also contains that of the fundamental cocircuits $Y(N, i), i \in B$ through the following well-known relationship:

$$t_{ij} \quad = \quad -Y(N, i)_j \ \text{ for } i \in B, j \in N. \tag{4.36}$$

Therefore, the row vectors $(t_{ij} : j \in N), i \in B$ represent the essential part of the fundamental cocircuits with opposite sign, while the column vectors $(t_{ij} : i \in B), j \in N$ represent that of the fundamental circuits with same sign.

The following property is well-known and can be easily verified from the oriented matroid axioms (OM1)$\sim$(OM4) and the definitions.

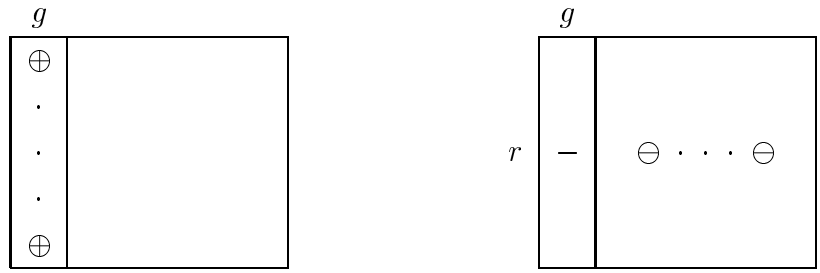

Figure 4.1: feasible and cofeasible tableaux

Lemma 4.2. *Let $M = (E, \mathcal{F})$ be an oriented matroid. Let B be a basis, and let $t_{rs}, r \in B, s \in N$ be any nonzero entry. Then the set $B' = B \backslash \{r\} \cup \{s\}$ is a basis, and the tableaux $T(B) = (t_{ij})$ and $T(B') = (t'_{ij})$ are related as follows: for each $i \in B \backslash \{r\}$ and $j \in N \backslash \{s\}$,*

$$\begin{aligned} t'_{sr} &= t_{rs} \ , \\ t'_{sj} &= -t_{rs} \cdot t_{rj} \ , \\ t'_{ir} &= t_{rs} \cdot t_{is} \ , \ \textit{and} \\ t'_{ij} &= t_{ij} \circ (-t_{rs} \cdot t_{is} \cdot t_{rj}) \quad \textit{unless} \ \ t_{ij} = t_{rs} \cdot t_{is} \cdot t_{rj} \neq 0. \end{aligned}$$

The replacement of a tableau $T(B)$ by an "adjacent" tableau $T(B')$ is called a *pivot operation on the position* (r, s).

Now we come back to the complementarity problem. Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid with $n \geq 0$. For the complementarity problems for which we give an algorithm, we assume that

Assumption 4.3. *$M = (\hat{E}_{2n}, \mathcal{F})$ contains a basis B with $\mid B \mid = n$, $g \notin B$ and complementary, i.e., $B \cap \overline{B} = \emptyset$.*

This assumption is always satisfied for the case of LCP (1.1) where $\mathcal{F}$ arises from the linear subspace $V(A, b)$ in (2.2) the set $\{n+1, n+2, \ldots, 2n\}$ is a complementary basis.

We define here two types of basis. A basis B or tableau $T(B)$ is called *feasible* if $g \notin B$ and $X(B, g) \geq 0$, and called *cofeasible* if $g \notin B$, there exists $r \in B$ such that $t_{rg} < 0$, and $t_{rj} \leq 0$ for all $j \in N$, where $N = \hat{E}_{2n} \backslash B$. (See Figure 4.1 where the symbols $\oplus$ and $\ominus$ indicate a nonnegative entry and a nonpositive entry.)

Property 4.4. *Let B be a basis with $g \notin B$, and let N be the cobasis $\hat{E}_{2n} \backslash B$.*

(1) *If B is feasible, then $X(B, g)$ is feasible, and if in addition complementary then $X(B, g)$ is complementary as well and solves the OMCP.*

(2) *If B is cofeasible, then $Y(N, r)$ is cofeasible, and if in addition B is complementary and $Y(N, r)_{\overline{r}} = t_{r\overline{r}} = 0$, then $Y(N, r)$ solves the dual OMCP, where r is a basic element for which $t_{rg} < 0$, and $t_{rj} \leq 0$ for all $j \in N$.*

Theorem 4.5 (Basis-Form Duality Theorem) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying Assumption 4.3 and the assumptions (3.29) and (3.30) of Theorem 3.1. Then exactly one of the following statements holds:*

(a) *there exists a complementary feasible basis;*

(b) *there exists a complementary cofeasible basis.*

A basis B or tableau $T(B)$ is called *terminal* if B is complementary and either feasible or cofeasible. We shall prove the theorem above using the following algorithm, called the criss-cross method, which will be shown to find a terminal tableau after a finite number of steps if the assumptions are satisfied.

In order to describe the algorithm, the following lemma is important.

Lemma 4.6. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the assumptions of Theorem 4.5. Let B be a complementary basis with $g \notin B$. Then, for each $r, s \in B$ the following statements hold:*

(1) *$t_{rg} \neq 0$ implies $t_{r\overline{r}} \geq 0$;*

(2) *$t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{r}} = 0$ and $t_{r\overline{s}} = +$ imply $t_{sg} = -$ and $t_{s\overline{s}} = +$;*

(3) *$t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{s}} = 0$ and $t_{r\overline{s}} = +$ imply $t_{s\overline{r}} = -$.*

Proof. For the proof, we set $X^0 = X(B, g)$, $X^1 = X(B, \overline{r})$, $X^2 = X(B, \overline{s})$, $Y^1 = Y(N, r)$, and $Y^2 = Y(N, s)$.

(1) Assume $t_{rg} \neq 0$. Let $t_{rg} \neq 0$ for some $r \in B$, and suppose $t_{r\overline{r}} = -$. Then X^1 is s.s.r. and $X^1_g = 0$. On the other hand, the vector $Z = (t_{rg}X^0) \circ X^1 \in \mathcal{F}$ is s.s.p. and $Z_g \neq 0$. This contradicts to the assumption (3.29).

(2) Assume $t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{r}} = 0$ and $t_{r\overline{s}} = +$. Let $\gamma = t_{sg}$ and $\beta = t_{s\overline{s}}$. First we show $\beta = +$. Suppose $\beta \leq 0$. Then the vector $Z = X^2 \circ (-X^1) \in \mathcal{F}$ is s.s.r. with $Z_g = 0$, and the vector $W = X^0 \circ (-X^1) \in \mathcal{F}$ is s.s.p with $W_g \neq 0$. This contradicts to the assumption (3.29). Therefore $\beta = +$.

Now, we show $\gamma = -$. Suppose $\gamma \geq 0$. Clearly, the vector $W = Y^1 \circ (-Y^2) \in \mathcal{F}^*$ is s.s.p. with $W_g \neq 0$. It suffices to show that there is an s.s.r. vector, say Z in $\mathcal{F}^*$ with $Z_g = 0$, since this contradicts to the assumption (3.30). There are two cases to consider.

Case: $\gamma = 0$**.** In this case, the vector $Z = Y^2$ is an s.s.r. vector with $Z_g = 0$, and hence a contradiction.

Case: $\gamma = +$**.** In this case, by eliminating the g-component from the vectors Y^1 and Y^2 using the axiom (OM4), we have a vector $Z \in \mathcal{F}^*$, which is s.s.r. with $Z_g = 0$.

This completes the proof.

$$
\begin{array}{c|c|c}
 & & \overline{r} \\
\hline
r & \bullet & \alpha \\
\hline
\end{array}
\qquad \Longrightarrow \alpha = \oplus
$$

($\bullet$ indicates a nonzero entry)

$$
\begin{array}{c|c|cc}
 & & \overline{r} & \overline{s} \\
\hline
r & - & 0 & + \\
s & \gamma & 0 & \beta \\
\hline
\end{array}
\qquad \Longrightarrow \gamma = -, \beta = +
$$

$$
\begin{array}{c|c|cc}
 & & \overline{r} & \overline{s} \\
\hline
r & - & 0 & + \\
s & & \delta & 0 \\
\hline
\end{array}
\qquad \Longrightarrow \delta = -
$$

Figure 4.2: Sign Pattern of Complementary Tableaux

(3) Assume $t_{rg} = -$, $t_{r\overline{r}} = t_{s\overline{s}} = 0$ and $t_{r\overline{s}} = +$. Let $\delta = t_{s\overline{r}}$. We have to show $\delta = -$. Suppose $\delta \geq 0$. Then the vector $W = X^0 \circ (-X^1) \in \mathcal{F}$ is s.s.p. with $W_g \neq 0$ and the vector $Z = X^2 \circ (-X^1)$ is s.s.r. with $Z_g = 0$. A contradiction to the assumption (3.29).

∎

Figure 4.2 illustrates the meaning of the lemma.

Criss-Cross method

The criss-cross method starts with any complementary basis B such that $g \notin B$. The algorithm assumes that the n complementary pairs $\{\{e, \overline{e}\} : e \in E_{2n}\}$ are linearly ordered, and this order induces a linear order on any complementary subset of E_{2n}. We denote by $\min S$ the smallest element in S, for any complementary subset S.

If it terminates then it outputs a pair (B, e) where B is a terminal basis, and $e = g$ if B is feasible, $e = r$ if B is detected cofeasible because of the r-row of $T(B)$.

```
procedure Criss-Cross(B:complementary basis);
repeat
    if B is feasible then
        STOP: return the pair (B; g)                {feas. compl. basis found }
    else
        r := min{i ∈ B | t_ig < 0};
        if t_rī ≤ 0 for all i ∈ B then
            STOP: return the pair (B; r)            {cofeas. compl. basis found}
        else
            s := min{i ∈ B | t_rī > 0};
            p := max{r, s};
            if t_pp̄ ≠ 0 then
(*)             DIAGONAL PIVOT: B := B\{p} ∪ {p̄}
                                                    {B is agian a basis, by Lemma 4.2}
            else
(**)            EXCHANGE PIVOTS: B := B\{r, s} ∪ {r̄, s̄}
                                                    {B is agian a basis, by Lemma 4.7 }
            endif
        endif
    endif
until true=false.
```

Lemma 4.7. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (3.29) and (3.30) of Theorem 3.1. When the criss-cross method is applied to M, the exchange pivots (**) always produce a complementary basis.*

Proof. Let B be a complementary basis produced by the criss-cross method for which the exchange pivots (**) are executed. Let r, s, p be indices chosen for B by the algorithm. First consider the case $p = r$. Since (**) is executed, $r > s$, $t_{rg} = -$, $t_{r\overline{r}} = 0$ and $t_{r\overline{s}} = +$. Also, we have $t_{sg} \geq 0$. By Lemma 4.6 (2), $t_{s\overline{r}} \neq 0$. This together with Lemma 4.2 implies $B\backslash\{r, s\} \cup \{\overline{r}, \overline{s}\}$ is a basis again.

Now consider the case $p = s$. Since (**) is executed, $s > r$, $t_{rg} = -$, $t_{r\overline{s}} = +$ and $t_{s\overline{s}} = 0$. Also, by the description of the algorithm and Lemma 4.6 (1), $t_{r\overline{r}} = 0$. By Lemma 4.6 (3), we have $t_{s\overline{r}} = -$. This implies $B\backslash\{r, s\} \cup \{\overline{r}, \overline{s}\}$ is a basis again. ∎

Lemma 4.8. *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying the conditions (3.29) and (3.30) of Theorem 3.1. Then for each $e \in E_{2n}$, at most one of the following four statements holds:*

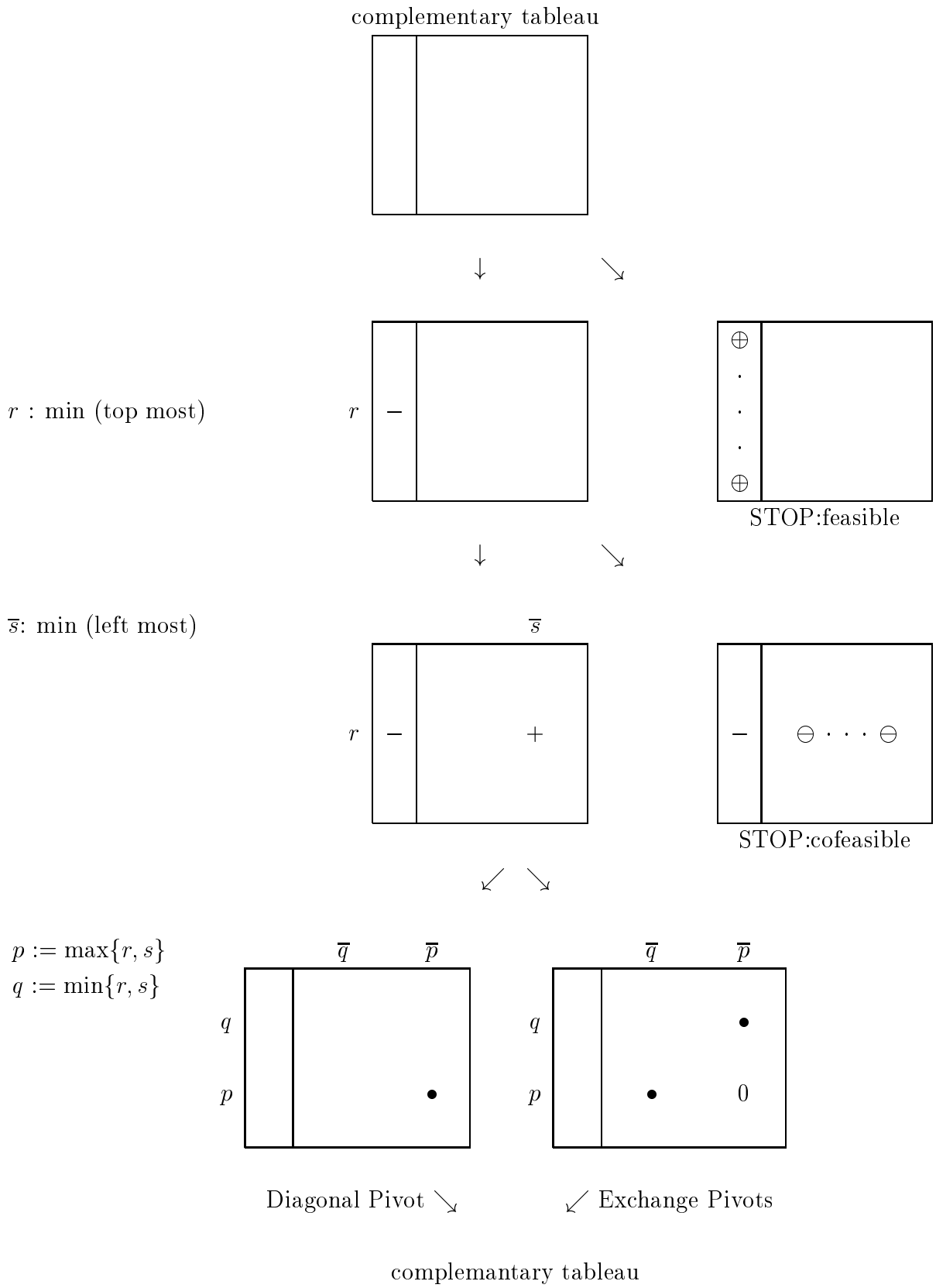

Figure 4.3: Criss-Cross Algorithm for the Complementarity Problem

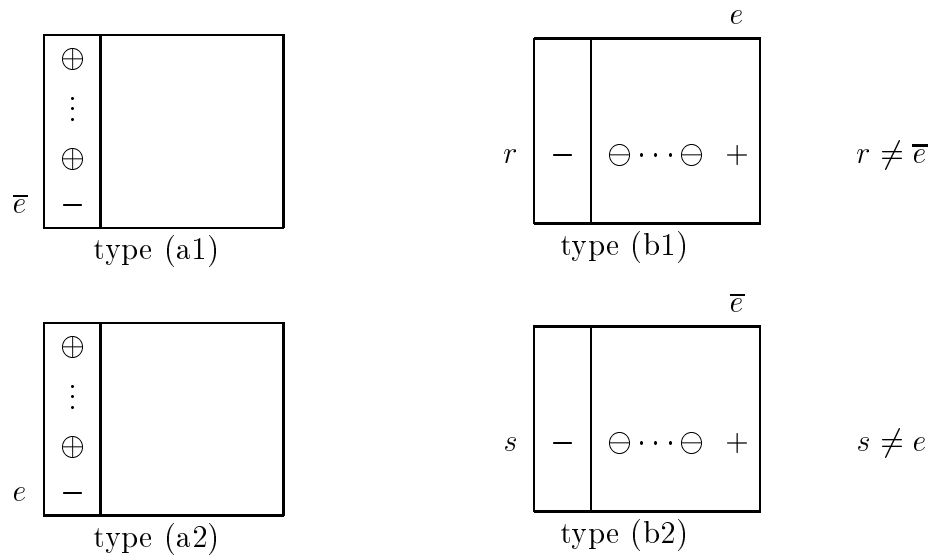

Figure 4.4: Four types of tableaux – Only one type can exist!

(a1) $\exists$ *a complementary basis* B^1 *not containing* g *such that* $\overline{e} \in B^1, t^1_{ig} \geq 0$ *for all* $i \in B^1 \backslash \{\overline{e}\}$ *and* $t^1_{\overline{e}g} = -$*;*

(a2) $\exists$ *a complementary basis* B^2 *not containing* g *such that* $e \in B^2, t^2_{ig} \geq 0$ *for all* $i \in B^2 \backslash \{e\}$ *and* $t^2_{eg} = -$*;*

(b1) $\exists$ *a complementary basis* B^3 *not containing* g *such that* $\overline{e} \in B^3, t^3_{rg} < 0, t^3_{r\overline{i}} \leq 0$ *for all* $i \in B^3 \backslash \{\overline{e}\}$ *and* $t^3_{re} = +$*, for some* $r \in B^3$ *with* $r \neq \overline{e}$*;*

(b2) $\exists$ *a complementary basis* B^4 *not containing* g *such that* $e \in B^4, t^4_{sg} < 0, t^4_{s\overline{i}} \leq 0$ *for all* $i \in B^4 \backslash \{e\}$ *and* $t^4_{s\overline{e}} = +$*, for some* $s \in B^4$ *with* $s \neq e$*,*

where $T(B^k) = (t^k_{ij})$ *for* $k = 1, 2, 3, 4$*.*

Proof. Let $X^i = X(B^i, g)$ for $i = 1, 2$, $Y^1 = Y(N^3, r)$ and $Y^2 = Y(N^4, s)$, where $N^j = \hat{E}_{2n} \backslash B^j$. By Lemma 4.6 (1), $t^3_{r\overline{r}} = t^4_{s\overline{s}} = 0$, and therefore the dual vectors Y^1 and Y^2 are both complementary. It is easy to see that the vectors X^1, X^2, Y^1 and Y^2 satisfy the conditions in (a1), (a2), (b1) and (b2) of Lemma 3.4, respectively. By Lemma 3.4, at most one of the four vectors exists. Hence, at most one of the four bases B^1, B^2, B^3, B^4 exists. ∎

The four types of tableaux described in Lemma 4.8 are illustrated in Figure 4.4.

Theorem 4.9 (Finiteness of the Criss-Cross Method) : *Let* $M = (\hat{E}_{2n}, \mathcal{F})$ *be an oriented matroid satisfying Assumption 4.3 and the assumptions (3.29) and (3.30) of Theorem 3.1. Then the criss-cross method solves the OMCP associeted with* M*, that is, it terminates after a finite number of pivots, and obtains either a complementary feasible basis or a complementary cofeasible basis.*

Proof. Suppose the criss-cross method does not terminate for some OMCP problem satisfying the conditions of the theorem, and let $M = (\hat{E}_{2n}, \mathcal{F})$ be such an example with minimal number of elements. Since there are finitely many bases in M, it must produce a cycle of adjacent bases: $B^0, B^1, \ldots, B^k = B^0$. Because of the minimality of number of elements, within this cycle every element in E_{2n} must leave and enter some basis. Let $\{e, \overline{e}\}$ be the largest complementary pair in E_{2n}. Consider the situation when e enters a basis. By the description of the algorithm, there are two possible cases:

Case E1: $r = \overline{e}$ and $t_{\overline{e}g}$ is the only negative entry in the g-column of the tableau;

Case E2: $r \neq \overline{e}$, $s = e$ and t_{re} is the only positive entry in the r-row of the tableau.

When e leaves a basis, there are also two possible cases:

Case L1: $r = e$ and t_{eg} is the only negative entry in the g-column of the tableau;

Case L2: $r \neq e$, $s = e$ and $t_{r\overline{e}}$ is the only positive entry in the r-row of the tableau.

These four cases (E1), (E2), (L1), (L2) fit exactly the cases (a1), (b1), (a2), (b2) of Lemma 4.8, respectively. By the same lemma, no two of the four statements can occur, and no two of (E1), (E2), (L1) and (L2) can occur, a contradiction. Therefore, the criss-cross method must terminate. Clearly, when it terminates, the final basis is either a complementary feasible basis or a complementary cofeasible basis. ∎

The duality theorem, Theorem 4.5, is a straightforward consequence of Theorem 4.9.

5 Concluding Remarks

The criss-cross method we described in the previous section is merely one of finite algorithms for the OMCP (satisfying the conditions of Theorem 4.9). As for the special case of the Oriented Matroid Programming case, one can design different finite pivoting algorithms. In some sense, the criss-cross method is a most rigid algorithm among them, since each pivot operation is uniquely determined and a sequence of complementary bases generated is completely determined once a linear order of complementary pairs $\{e, \overline{e}\}$ is fixed.

One way to see the possibility of relaxing the algorithm is to prove the finiteness of the criss-cross method more constructively as in [12]. In fact, we can show that some binary number associated with each iteration of the algorithm increases monotonically. For each $e \in E_{2n}$, we denote by $[e]$ the complementary pair $\{e, \overline{e}\}$. Clearly $[e] = [\overline{e}]$. Let $[E_{2n}] = \{[e] \mid e \in E_{2n}\}$. Thus $[E_{2n}]$ is the set E_{2n} with each complementary pairs e, $\overline{e}$ identified. As it was assumed for the criss-cross method, we assume that the n complementary pairs $\{\{e, \overline{e}\} : e \in E_{2n}\}$ are linearly ordered, and this order is just a linear order on $[E_{2n}]$. For a $\{0,1\}$-vector L in $\{0,1\}^{[E_{2n}]}$, and for $j \in [E_{2n}]$, let $L + j$ denote the

vector in $\{0,1\}^{[E_{2n}]}$

$$(L+j)_i = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ L_i & \text{if } i > j \end{cases} \qquad (i \in [E_{2n}]).$$

We say L' is *lex greater* than L if $L \neq L'$ and L' is larger than L in the last subscript j for which they differ. Obviously, if $L_j = 0$ for some j, then $L + j$ is lex greater than L.

Let us consider the following updating of L at each iteration of the criss-cross method:

$$\begin{array}{l} \text{set L=0 initially, and} \\ \text{update L by L+p} \end{array} \tag{5.37}$$

where p is the larger element $\max\{r, s\}$ between the two basic elements r and s chosen at each iteration.

Theorem 5.1 (Monotonic Improvements) : *Let $M = (\hat{E}_{2n}, \mathcal{F})$ be an oriented matroid satisfying Assumption 4.3 and the assumptions (3.29) and (3.30) of Theorem 3.1. If the criss-cross method is applied to M with the vector L updated by (5.37), then L strictly increases at each iteration, and therefore the algorithm must terminate in $2^n - 1$ iterations.*

We omit the proof of this theorem. One just have to show $L_p = 0$ holds before being updated by $L + p$, and the fundamental lemma Lemma 4.7 is essential again as it was for the proof of Theorem 4.9.

One implication of the theorem is that as long as L increases at each iteration, the algorithm is finite. Consider L at a general stage of the criss-cross method. A *zero interval* of L is a set of consecutively ordered indices: $j_1, j_2, \ldots, j_k \in [E_{2n}]$ such that $L_{j_1} = L_{j_2} = \ldots = L_{j_k} = 0$. One can easily see that the linear order among the indices (complementary pairs) is not essential at this stage. So one can pretend that each zero intervals are not ordered, and one can just set any linear ordering on a zero interval whenever necessary.

For the case of linear programming, interesting experimental results supporting the practicality of some flexible version of the criss-cross method have been reported in [19].

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