MULTIPLECTY RESULTS FOR A SINGULAR AND QUASILINEAR EQUATION

J. Giacomoni and K. Sreeandh
MIP-Ceremath
Manufacture des Tabacs-Bat C
21, allée de Brienne
31000 Toulouse, France

Abstract. In this paper, we investigate the following quasilinear and singular problem:
\[
\begin{aligned}
-\Delta_p u &= \frac{\lambda}{u} + u^q \quad \text{in } \Omega \\
u|_{\partial \Omega} &= 0 , \quad u > 0 \quad \text{in } \Omega
\end{aligned}
\]
where \(\Omega\) is an open bounded domain with smooth boundary, \(1 < p, p-1 < q\) and \(\lambda, \delta > 0\). We first prove that there exist weak solutions for \(\lambda > 0\) small in \(W^{1,p}_0(\Omega) \cap C(\overline{\Omega})\) if and only if \(\delta < 2 + \frac{1}{p-1}\). Investigating the radial symmetric case (\(\Omega = B_R(0)\)), we prove by a shooting method the global multiplicity of solutions to (\(P\)) in \(C(\overline{\Omega})\) with \(0 < \delta, 1 < p\) and \(p-1 < q\).

1. Introduction. In this paper, we are interested in the following quasilinear and singular problem:
\[
(P) \begin{cases}
-\Delta_p u &= \frac{\lambda}{u} + u^q \quad \text{in } \Omega \\
u|_{\partial \Omega} &= 0 , \quad u > 0 \quad \text{in } \Omega
\end{cases}
\]
where \(\Omega\) is an open bounded domain with smooth boundary, \(1 < p, \lambda, \delta > 0\), \(\Delta_p = \nabla \cdot (|\nabla \cdot |^{p-2}\nabla \cdot )\) is the \(p\)-Laplace operator and \(p-1 < q\). Such equations arise in different models: filtration processes, pseudo-plastic flows. Concerning (\(P\)), we investigate the question of existence and multiplicity of solutions in respect to the parameters \(\lambda\) and \(\delta\). The main difficulties here are that the operator is degenerate (resp. singular) for \(p > 2\) (resp. \(p < 2\)) and that the nonlinearity is singular at 0. Singular problems have been already dealt in case of \(p = 2\) (i.e., the non degenerate case \(\Delta_2 = \Delta\)) in a significant number of papers. In particular, setting
\[
\begin{cases}
-\Delta u &= \frac{\lambda p(x)}{u} + \mu(x)u^q \quad \text{in } \Omega \\
u|_{\partial \Omega} &= 0 , \quad u > 0 \quad \text{in } \Omega
\end{cases}
\]
the case \(\mu = 0\) has been worked out in [9] where the authors prove the existence and uniqueness of the solution to (\(P\)) for any \(\lambda > 0\) and \(\delta > 0\). In [8], they prove the existence of one solution for small \(\mu > 0\). In [21], the authors use a variational method based upon the Nehari manifold to prove the existence of two solutions belonging to \(H^1(\Omega)\), to [3] in the subcritical case i.e. \(q < q^* - 1\) and for \(n \geq 3\). This result is extended to the critical case i.e. \(q = 2^* - 1\) in [15] and [10] using two different variational methods: the Perron’s method and the Nehari
manifold. These results concern the case $0 < \delta < 1$. In [4], the authors prove the multiplicity results when $n = 2$ and with critical nonlinearities of the type $h(u)e^{bu^2}$ with $b > 0$ and $0 < \delta < 3$ which is optimal for getting solutions in $H^1_0(\Omega)$. We stress that multiplicity is strongly related to the strong maximum principle which can be established in the singular case (see [7]).

When $p \neq 2$, few results are known concerning existence of solutions to $(P)$. In [3], using fixed point theorem, the authors prove for a class of problems like (4) in the unit ball of $\mathbb{R}^N$. Doing any weak solution to $(P)$ that the strong comparison principle is valid for singular problems like (P) and that any weak solution to $(P)$ is in $C^{1,\alpha}(\Omega)$ (for some $0 < \alpha < 1$) for any $\delta < 1$.

In the present work, we show that in the radial symmetry case (i.e. when $\Omega$ is a ball) multiplicity of solutions to $(P)$ holds in $C(\Omega)$ for any $\delta > 0$ and any $q$ such that $p - 1 < q$. The approach we use is based on O.D.E. techniques and shooting arguments.

Precisely, we are interested in
\[
\begin{cases}
-(r^{n-1}u'' + p-2u') = r^{n-1}(\frac{1}{w} + w^{p-1}) & \text{in } B_R(0) \\
u'(0) = 0 , u > 0 & \text{in } B_R(0).
\end{cases}
\] (4)

Note that by the change of variables $u(r) = \lambda^\mu w(\lambda^\theta r)$ with $\mu = \frac{-1}{\delta + q}$ and $\theta = \frac{q-p+1}{(\delta+q)(p-1)}$, (4) is equivalent to
\[
\begin{cases}
-(r^{n-1}w'' + p-2w') = r^{n-1}(\frac{1}{w} + w^{p-1}) & \text{in } B_{R\lambda^{-\theta}}(0) \\
w'(0) = 0 , w > 0 & \text{in } B_{R\lambda^{-\theta}}(0).
\end{cases}
\] (5)

Doing $R = \lambda^\theta$, the existence of solutions to (4) for small radius $R$ is equivalent to the existence of solutions to (5) in the unit ball of $\mathbb{R}^n$ for small $\lambda$. Concerning (5), using the Emden Fowler transformation $t = (\frac{r}{\lambda})^\mu$, $y(t) = u(r)$ (this technique has been already used in [6], [2], [17]), it is equivalent to study
\[
\begin{cases}
-(|y'|^{p-2}y')' = t^{-k}(\frac{1}{y} + y^{p-1}) & \text{in } (T, \infty) \\
y'(\infty) = 0 , y(T) = 0
\end{cases}
\] (6)

where $T = (\frac{1}{\lambda})^\mu$, $k = p\frac{n-1}{n-p}$. We use a shooting method to get solutions to (6).

Precisely, for $\gamma > 0$, we study
\[
(P_{\gamma}) \begin{cases}
-(|y'|^{p-2}y')' = t^{-k}(\frac{1}{y} + y^{p-1}) \\
y'(\infty) = 0 , y(\infty) = \gamma.
\end{cases}
\] (7)

The above “initial value” problem with the initial data prescribed at infinity admits a unique solution in $\{y > 0\}$ thanks to Proposition A4 in [12]. Let $y(t, \gamma)$ denote the solution to $(P_{\gamma})$, let $T_0(\gamma) := \inf\{t > 0 \mid y(\cdot, \gamma) > 0 \text{ in } (t, \infty)\}$, the first zero of $y(t, \gamma)$ as $t$ decreases from infinity. Since $\gamma \to T_0(\gamma)$ is continuous, for proving multiplicity results to $(P_{\gamma})$, it is enough to determine the asymptotics of $T_0(\gamma)$ when $\gamma \to 0^+$ and when $\gamma \to \infty$ given by the following lemma:

**Lemma 1.**
\[
\lim_{\gamma \to 0^+} T_0(\gamma) = \infty \text{ and } \lim_{\gamma \to \infty} T_0(\gamma) = \infty.
\] (8)
From Lemma 4, we get

**Theorem 1.** Let \(1 < p\) and \(\delta > 0\). Assume that \(\Omega = B_R(0)\). Then, there exists \(R^*\) such that for \(R < R^*\) there exists at least two radial and radially decreasing solutions in \(C(\Omega) \cap C^2(\Omega \setminus \{0\})\) and no solution for \(R > R^*\).

Finally, the outline of the paper is as follows: In the second section we prove that for a general domain with \(C^2\)-boundary, there exist weak solutions for \(\lambda > 0\) small enough if and only if \(p - 1 < q \leq 2 + \frac{1}{p - 1}\). By weak solution we mean a function \(u \in W^{1,p}_0(\Omega)\) satisfying \(\text{ess inf}_K u > 0\) over every compact set \(K \subset \Omega\), and

\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \lambda \int_\Omega u^{-\delta} \phi \, dx + \int_\Omega u^\theta \phi \, dx
\]

This result is well known for \(p = 2\) (see [9], [8]). We give here a simple proof based on sub-and-supersolutions technique for any \(1 < p\). In the third section, we deal with the radial symmetric case and prove Lemma 1 and Theorem 1.

2. **Existence of weak solutions to (2).**

In this section, we show the existence of weak solutions when \(0 < \delta < 2 + \frac{1}{p - 1}\) and for \(\lambda > 0\) small enough. Since the case \(0 < \delta < 1\) is completed in [13], we assume \(1 \leq \delta\).

Let \(\epsilon > 0, \alpha = \frac{p-1+\delta}{p} \geq 1\), \(\phi\) a positive eigenfunction associated to the first eigenvalue, \(\lambda_1\), to \(-\Delta_p\) in \(\Omega\) with Dirichlet boundary conditions and set:

\[
\begin{cases}
-\Delta_p u = \frac{1}{(u+\epsilon)^p} & \text{in } \Omega \\
u|_{\partial \Omega} = 0, \ u > 0 & \text{in } \Omega,
\end{cases}
\]

and

\[
\begin{cases}
-\Delta_p u = \frac{1}{(u+\epsilon)^p} + \lambda^0 u^\theta & \text{in } \Omega \\
u|_{\partial \Omega} = 0, \ u > 0 & \text{in } \Omega,
\end{cases}
\]

with \(\theta = \frac{q-(p-1)}{\delta+(p-1)}\). We have the following results:

**Lemma 2.** Assume \(0 < \delta, \lambda > 0\). Then, for all \(\epsilon > 0\), there exists a unique solution, \(u_\epsilon\), in \(W^{1,p}_0(\Omega)\) to (10) satisfying \(c((\phi + \epsilon^\alpha)^{\frac{p}{(p-1)+\delta}} - \epsilon) \leq u_\epsilon\) where \(c > 0\) does not depend on \(\epsilon\).

and

**Lemma 3.** Assume \(1 \leq \delta\). Then, there exists \(0 < \Lambda < \infty\) such that for all \(\epsilon > 0\) and \(0 < \lambda < \Lambda\), there exists a minimal solution, \(u_{\epsilon,\lambda}\), in \(W^{1,p}_0(\Omega)\) to (11) satisfying

\[
c((\phi + \epsilon^\alpha)^{\frac{p}{(p-1)+\delta}} - \epsilon) \leq u_{\epsilon,\lambda} \leq k(\phi + \epsilon^\alpha)^{\frac{p}{(p-1)+\delta}}
\]

where \(c < k\) do not depend on \(\epsilon\).

**Proof of Lemma 2** First, we have that:

\[
-\Delta_p(c((\phi + \epsilon^\alpha)^{\frac{p}{(p-1)+\delta}})} = \left(\frac{pc}{p-1+\delta}\right)^{p-1}\frac{1}{(\phi + \epsilon^\alpha)^{\frac{(p-1)(p-1+\delta)}{p-1}}} + \frac{1}{(p-1+\delta)^p} \int_{\Omega} |\nabla \phi|^p (\phi + \epsilon^\alpha)^{-\frac{p}{p-1+\delta}}
\]

Now, one has \(0 \leq (\phi + \epsilon^\alpha)^{\frac{p}{(p-1)+\delta}} - \epsilon\). Since \(\phi \in C^1(\Omega)\), we have that \(|\nabla \phi| \leq \mu\) in \(\Omega\). Therefore, for \(c > 0\) small enough we have

\[
\frac{1}{(p-1+\delta)^p} \int_{\Omega} |\nabla \phi|^p (\phi + \epsilon^\alpha)^{-\frac{p}{p-1+\delta}} \leq \frac{1}{(c(\phi + \epsilon^\alpha)^{\frac{p}{(p-1)+\delta}})}.
\]
From (13) and (14), we get that for \( c > 0 \) small enough
\[
-\Delta (c(\phi + \epsilon^\alpha) \frac{p}{p-\gamma}) \leq \frac{1}{(c(\phi + \epsilon^\alpha) \frac{p}{p-\gamma})^\delta}
\]
Therefore, \( c(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} - \epsilon \) is a subsolution to (10). Using similar arguments, we can show that for \( k > c \) large enough,
\[
-\Delta (k(\phi + \epsilon^\alpha) \frac{p}{p-\gamma}) > \frac{1}{(k(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} + \epsilon)^\delta}
\]
Note that from the strong maximum principle (see Theorem 5 in [20]), for \( \eta > 0 \) small enough,
\[
\phi \leq \eta \Rightarrow |\nabla \phi| \geq \nu > 0.
\]
which implies that in a small neighborhood of \( \partial \Omega \)
\[
\frac{(pk)^{p-1}(\delta - 1)(p-1)}{(p-1 + \delta)^p} |\nabla \phi|^p (\phi + \epsilon^\alpha)^{\frac{p}{p-\gamma}} \geq \frac{1}{(k(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} + \epsilon)^\delta}
\]
for \( k \) large enough. Therefore, \( k(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} \) is a strict supersolution to (10) such that \( c(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} - \epsilon \leq k(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} \). Then, the existence of \( u_0 \) follows. Finally, let us show that \( u = u_\epsilon \) is the unique weak solution in \( W^{1,p}(\Omega) \) to (10) : Assume that there is a second solution \( v \) to (10). Then,
\[
\begin{cases}
-\Delta_p v = \frac{\lambda}{(v+\epsilon)^\gamma} & \text{in } \Omega \\
v|_{\partial \Omega} = 0 & , v > 0 \text{ in } \Omega.
\end{cases}
\]
Multiplying (10) by \( u \) and (15) by \( \frac{u^p}{v_{p-\gamma}} \), subtracting and integrating on \( \{u > v\} \), we get
\[
0 \leq \int_{u > v} R(u, v) = \lambda \int_{u > v} (u^{1-p}(u + \epsilon)^{-\delta} - v^{1-p}(v + \epsilon)^{-\delta})u^p
\]
where
\[
R(u, v) = |\nabla u|^p - |\nabla v|^p - 2\nabla v \nabla \left( \frac{u^p}{v_{p-\gamma}} \right) \geq 0
\]
by Picone’s identity. Hence, since \( f(t) = t^{1-p}(t + \epsilon)^{-\delta} \) is non increasing on \( \mathbb{R}^+ \), we get from (16) \( u \leq v \) and reversing the roles of \( u \) and \( v \), we get \( u = v \). This completes the proof of Lemma 2.

**Proof of Lemma 3** We argue as in Lemma 2. First, we observe that \( u_\epsilon \) is a subsolution to (11). Moreover, since \( \phi \in C^1(\Omega) \), for \( \Lambda > 0 \) small enough depending on \( k \), \( k(\phi + \epsilon^\alpha) \frac{p}{p-\gamma} \) is a supersolution to (11) for any \( \lambda \in (0, \Lambda) \) and any \( \epsilon \leq \epsilon_0 \). Therefore, for any \( \lambda \in (0, \Lambda) \), there exists \( \hat{u}_{\epsilon, \lambda} \) a solution to (11) satisfying
\[
c((\phi + \epsilon^\alpha) \frac{p}{p-\gamma} - \epsilon) \leq \hat{u}_{\epsilon, \lambda} \leq k((\phi + \epsilon^\alpha) \frac{p}{p-\gamma}).
\]
To prove the existence of a minimal solution to (11), we use the weak comparison principle (see (10) or (19)) and the following monotone iterative scheme :
\[
\begin{cases}
-\Delta_p u_n - \frac{1}{(u_n+\epsilon)^\gamma} = \lambda \alpha u_{n-1}^\alpha & \text{in } \Omega \\
u_n|_{\partial \Omega} = 0
\end{cases}
\]
where \( u_0 = u_\epsilon \), the unique solution to (10). Note that \( u_0 \) is a weak subsolution to (2) and \( u_0 \leq \hat{u}_{\epsilon, \lambda} \). Then, using Proposition 2.3 in (10), we get easily that \( \{u_n\} \) is a nondecresasing sequence in \( W^{1,p}_0(\Omega) \) and \( u_n \leq \hat{u}_{\epsilon, \lambda} \). Hence, it is easy to prove that \( \{u_n\} \) is convergent to \( u_{\epsilon, \lambda} \) a solution to (11). Using again the weak comparison
principle and the uniqueness of the solution to (10), we can show that $u_{\epsilon, \lambda}$ is the minimal solution to (11) satisfying (12). This completes the proof of Lemma 3. \hfill $\square$

From Lemma 2 and 3, it follows

**Proposition 1.**

(i) Let $1 \leq \delta < 2 + \frac{1}{p-1}$. Then, $0 < \lambda \leq \Lambda$, there exists $u_{\lambda}$ a weak solution to (2).

(ii) If $\delta \geq 2 + \frac{1}{p-1}$, there is no weak solution to (2).

**Proof of Proposition 1.** Let us prove (i) : Let $u$ be a solution to (2). Setting $v = \lambda^\alpha u$, we observe that $v$ satisfies

$$\begin{cases}
-\Delta_p v = \frac{1}{\epsilon^p} + \lambda^\beta v^q \\
u|_{\partial \Omega} = 0, \; \epsilon > 0 \text{ in } \Omega,
\end{cases}$$

(18)

Letting $\epsilon \to 0^+$, it is easy to see that for $0 < \lambda \leq \Lambda$

$$u_{\epsilon, \lambda} \to v_{\lambda} \text{ pointwise a.e. in } \Omega.$$ 

From Lemma 2 and 3, $u_\epsilon \leq v_{\lambda}$ is positive (note that $u_\epsilon \uparrow$ as $\epsilon \downarrow 0$) and

$$\liminf_{\epsilon \to 0} \int_\Omega |\nabla u_\epsilon|^p \leq C \int_\Omega \phi^{-\frac{p}{p-1}} (\phi^{-\frac{p}{p-1}}) < \infty$$

if and only if $p < 2 + \frac{1}{p-1}$. Therefore, $v_{\lambda}$ is a weak solution if $p < 2 + \frac{1}{p-1}$.

(ii) follows from (2) and (19). \hfill $\square$

3. **The radial case.** Here we are dealing with the following problem :

$$\begin{cases}
-(r^{n-1}|u'|^{p-2}u')' = r^{n-1}(\frac{1}{p} + u^{p-1}) \text{ in } BR(0) \\
u'(0) = 0, \; u > 0 \text{ in } BR(0),
\end{cases}$$

(20)

Using O.D.E. techniques as the shooting method and the Emden-Fowler transformation, we get the existence of at least two solutions in $C^1(\Omega) \cap C(\Omega)$ for $p > 1$ and $\delta > 0$. Precisely as in [17], in prescribing the value $u(0) = \gamma$ and in evaluating the asymptotics of the first zero, $R_0(\gamma)$, of $u$ we prove Theorem 1. For this, using the Emden-Fowler Transformation : $t = (\frac{r}{\nu})^\nu$, $y(t) = u(r)$ with $\nu = \frac{n-1}{p-1}$, it is equivalent to deal with

$$\begin{cases}
-(r^{n-1}|y'|^{p-2}y')' = t^{-k}(y^{-\delta} + y^{p-1}) \text{ in } (T_0(\gamma), \infty) \\
y'(\infty) = \gamma
\end{cases}$$

(21)

where $T_0(\gamma) = \inf\{t > 0 | t \text{ is the first zero of } y(s)\} = (\frac{\nu}{R_0(\gamma)})^\nu$. We would like to point out that for any $\gamma > 0$, (21) has a unique solution in the set $\{y(t) > 0\}$ thanks to Proposition A4 in [12]. So let $y(\gamma)$ the solution of (21) associated to $\gamma$. Note that $y(\gamma)$ is concave in $(T_0(\gamma), \infty)$. Due to the continuity of $\gamma \to T_0(\gamma)$ we just need to evaluate the asymptotics of $T_0(\gamma)$ as $\gamma \to 0^+$ and as $\gamma \to \infty$. For this, we need some preliminar results. First, defining $f(s) = s^{p-1} + s^{-\delta}$ for $s > 0$, $k = p\frac{n-1}{n-p}$, $l = \frac{k-1}{p-1}$ and

$$z(t, \gamma) := \gamma t \left(t^{-l-1} + \gamma^{-1} \left(f(\gamma) \frac{k-1}{k-1} \right)^{-\frac{1}{p-1}} \right)^{-\frac{1}{k-1}},$$

we have

**Lemma 4.** Let $y(t, \gamma)$ be a solution of (21). Then $y(t, \gamma) < z(t, \gamma)$ for $T_0(\gamma) < t < \infty$.

**Proof of Lemma 4** The proof is similar as the proof of lemma 2.1 in [17]. \hfill $\square$
Lemma 5. \( T_0(\gamma) > 0, \forall \gamma > 0 \).

**Proof**: Very similar to the proof of lemma 2.2 in [17] (see p 882).

Now, let us evaluate the asymptotics of \( T_0(\gamma) \) as \( \gamma \to 0^+ \) and as \( \gamma \to \infty \):

**Proof of Lemma 5**. Let us deal with the case \( \gamma \to 0^+ \). From the equation satisfied by \( y \) and lemma 4, we have for \( \gamma > 0 \) and \( t \geq T_0(\gamma) \)

\[
(y'(t))^{p-1} - (y'(\eta t))^{p-1} = \int_{\eta t}^{t} s^{-k}(y(s)^q + y(s)^q)ds \geq \frac{1}{\gamma^\delta} \frac{t^{1-k}}{k-1}.
\]

Moreover from lemma 1 as \( \gamma \to 0^+ \)

\[
y(t) \leq z(t, \gamma) \leq o(\gamma)t.
\]

Using the concavity of \( y \), (24) and (25), we get \( T_0(\gamma) \to +\infty \) as \( \gamma \to 0^+ \). Now, let us deal with the asymptotics of \( T_0(\gamma) \) as \( \gamma \to \infty \). In this case, we have for \( t \geq T_0(\gamma) \):

\[
y(t) \leq z(t, \gamma) \leq o(\gamma^{-\frac{1}{\delta}})t \text{ as } \gamma \to \infty.
\]

Now, from (24), integrating the equation satisfied by \( y \) between \( T_1 > T_0(\gamma) \) and \( T_2 \geq T_1 \), we obtain:

\[
(y'(T_1))^{p-1} - (y'(T_2))^{p-1} = \int_{T_1}^{T_2} s^{-k}(y(s)^q + y(s)^q)ds \geq \frac{1}{y(T_2)^q} \int_{T_1}^{T_2} s^{-k}ds \geq C \frac{1}{y(T_2)^q} \left( \frac{T_2^{1-k}}{k-1} - \frac{T_1^{1-k}}{k-1} \right).
\]

Then, if there exists a sequence \( \{\gamma_k\} \) such that \( \sup_k T_0(\gamma_k) < \infty \), we have for some \( 0 < \theta \leq 1 \)

\[
y(y'(T_1))^{p-1} - (y'(T_2))^{p-1} \geq \frac{1}{y(T_2)^q} \left( \frac{T_2^{1-k}}{k-1} - \frac{T_1^{1-k}}{k-1} \right).
\]

This completes the proof of lemma 1.

**Proof of Theorem 1**. From lemma 1 it is sufficient to prove that \( \gamma \to T_0(\gamma) \), so equivalently \( \gamma \to R_0(\gamma) \) is continuous on \( \mathbb{R}^+ \). For this, let \( \{\gamma_k\}_{k \in \mathbb{N}} \) a sequence in \( \mathbb{R}^+ \) such that \( \gamma_k \to \gamma > 0 \) as \( k \to \infty \). Let \( u_k \) (resp. \( u \)) the corresponding solution to (4) with \( u_k(0) = \gamma_k \) (resp. \( u(0) = \gamma \)). Suppose for some \( \eta > 0 \) that

\[
R_0(\gamma) + 2\eta \leq \liminf_{k \to \infty} R_0(\gamma_k).
\]

From the uniqueness of the initial value problem (see [12]), for any \( \epsilon > 0 \), \( u_k \to u \) uniformly in \( \{x \mid u(x) \geq \epsilon\} \). Hence, for \( \epsilon > 0 \), there exists \( K(\epsilon) \in \mathbb{N}, R_k(\epsilon) \in \mathbb{R}^+ \) such that for \( k \geq K(\epsilon) \) we get

\[
R_k(\epsilon) \leq R_0(\gamma) + \eta \text{ and } u_k(R_k(\epsilon)) = \epsilon.
\]

Now, let \( 0 < \phi \in C_\infty^0([R_0(\gamma) + \eta, R_0(\gamma) + 2\eta]) \). Then, there exist \( c, K_1, K_2 \) independent of \( \epsilon \) such that

\[
\frac{c}{\epsilon^\eta} \leq \frac{K_1}{\epsilon^\eta} \int_{R_0(\gamma)}^{R_0(\gamma) + 2\eta} (\phi - u_k)^+ \leq K_2 \int_{R_0(\gamma)}^{R_0(\gamma) + 2\eta} (\phi - u_k)^+ (\frac{1}{u_k^\phi} + u_k^\phi) = \int_{R_0(\gamma)}^{R_0(\gamma) + 2\eta} (\phi - u_k)^+ (-\Delta u_k) \leq \int_{\phi \geq u_k} (-\Delta \phi) \phi
\]

(29)
from lemma 4.1 in [14]. Doing $\epsilon \to 0^+$ in (29), we get a contradiction. Similarly, we get a contradiction in other cases. This completes the proof of the continuity of $\gamma \to R_0(\gamma)$ in $\mathbb{R}^+$. Now, from lemma [1] we have

$$\lim_{\gamma \to 0^+} R_0(\gamma) = 0 \quad \text{and} \quad \lim_{\gamma \to \infty} R_0(\gamma) = 0.$$  

(30)

Setting $R^* = \sup_{\gamma \in \mathbb{R}^+} R_0(\gamma)$ ( $< \infty$ by lemma 5), Theorem 1 follows from (30) and the continuity of $\gamma \to R_0(\gamma)$.

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E-mail address: jgiacomo@univ-tlse1.fr
E-mail address: sreenadh@gmail.com