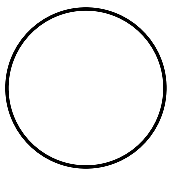
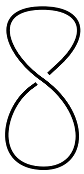
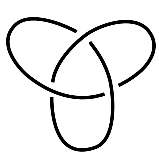
Knot Theory's Odd Origins

The modern study of knots grew out an attempt by three 19th-century Scottish physicists to apply knot theory to fundamental questions about the universe

Daniel S. Silver







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Take a length of rope, loop and weave it around itself and connect its ends. The result, of course, is a knot.

Creating a knot seems simple, yet knot theory is one of the most active fields in mathematics today, yielding thousands of peer-reviewed articles over the past decade. There's even an academic journal devoted entirely to knot theory. One reason for its popularity is the conviction that knots have profound implications for fields other than mathematics. Already knot theory has been applied to subjects as diverse as quantum mechanics and genetics. But these are only the latest of a number of attempts to use knots to untangle some of science's mysteries.

In fact, knot theory grew out of a bold (and wildly misguided) effort to develop an early "theory of everything." Two 19th-century Scottish physicists, William Thomson and Peter Guthrie Tait, believed that chemical elements were knotted tubes of ether, which, given the contemporary state of physics, might have explained the nature of atoms and how they acted. A third Scottish physicist, James Clerk Maxwell, famous for his work on electromagnetism, encouraged Thomson and Tait in the development of this "vortex-atom theory."

No solid evidence for Thomson's vortex-atom theory was ever found, and today it is long forgotten. But the attempt to apply knot theory to fundamental questions about science, the ultimate nature of matter and even the existence of an afterlife is an inspiring tale. Today, scientists are often accused of being excessively narrow in their research. This colorful triumvirate of Scottish physicists would have been free of any such charges.

This is the story of a magnificent failure.

Knot History

The most fundamental question in knot theory is determining whether two knots are the same. Mathematicians usually regard knots as identical if one can be deformed—stretched or twisted, but never broken—so that it looks exactly like the other. It is relatively easy to prove that two knots are the same—simply deform one until it appears identical to the other—but showing that two knots are different is difficult because the possible contortions are infinite.

Knot theory includes the study of links, which consist of any number of knots (called components) intertwined in any manner. A popular motif in ancient Rome, links were often added to mosaics adorning homes and temples. Celtic knot and link patterns, the best examples of which can be found in the Book of Kells, appeared in Ireland in the 7th century and spread from there to Scotland.

In the 19th century, knots became a subject of mathematical, and not just artistic, interest. Johann Carl Friedrich Gauss (1777–1855), the son of a bricklayer and the greatest mathematician of his day, was the first to discover a nontrivial fact about links. In 1833, he showed that the number of "intertwinings," what we call today the linking number of two knots, can be computed by an integral. Gauss discussed knots with his doctoral student, Johann Benedict Listing (1808–1882), who later coined the word *topology*, a combination of the Greek words *topos* (place) and *logos* (reason), to describe the new geometry of position.

Gauss and Listing were both curious about knots, but further progress required someone obsessed with the topic.

Lord Kelvin

William Thomson was a brilliant mathematician and physicist. He was also unreasonably confident. As a Cambridge undergraduate he was so sure that he would be senior wrangler, the student who scored highest on the grueling mathematical tripos examination, that after taking the test he told his servant to find out who was second wrangler. "You, sir," was the servant's devastating reply.

Thomson's interests were diverse. From the late 1850s to the mid-1860s, Thomson contributed to the efforts to lay the first trans-Atlantic telegraph cable, and, in 1866, he was knighted for his work. (He became Lord Kelvin, or more precisely, Baron Kelvin of Largs, 26 years later.) In 1858, he invented the mirror galvanometer

Juerg Alean

Figure 1. A smoke ring rises above Mount Etna, an active volcano on the island of Sicily. Rings such as this one are caused by small emissions of smoke from narrow vents on the volcano; they can reach more than 600 feet in diameter and remain stable for more than 10 minutes. In 1867, Scottish physicist Peter Guthrie Tait performed experiments with much smaller but fundamentally similar smoke rings to learn more about vortex motion. After watching one of these experiments, William Thomson, later Lord Kelvin, concluded that perhaps the most basic forms of matter were stable, permanent knots of ether endowed with vortex motion. The efforts of Thomson and Tait gave rise to knot theory, today one of the most active branches of mathematics. On the facing page are a circle (also called the unknot), and a circle twisted over itself, which is mathematically identical to the circle above it. The trefoil knot (*bottom*) is the simplest nontrivial knot.

while working on the trans-Atlantic cable. This device, which could detect the faint electric currents running through the cable more easily than previous equipment, made Thomson a wealthy man.

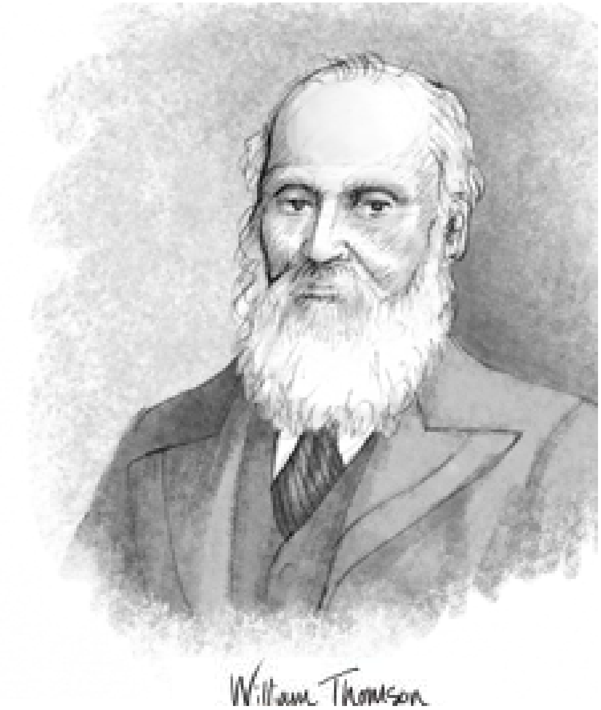
But it is for the absolute temperature scale that bears his name that Thomson is best known. The scale was a by-product of his most important research, which was in the field of thermodynamics. (The Kelvinator refrigerator was also named in his honor—certainly a lesser tribute.) After reading French scientist Sadi Carnot's paper, "Reflections on the Motive Power of Heat," Thomson proposed two laws of thermodynamics establishing the indestructibility of energy. All physics, he concluded, should derive from energy principles.

Such sweeping conclusions were characteristic of Thomson, a trait that caused him some trouble. In 1862, before the discovery of radioactivity, he used principles of thermodynamics to estimate the age of the Earth, concluding that it was far younger than geologists believed. Despite all the evidence, including Darwin's recently published theory of natural selection, Thomson refused to accept that the Earth could be more than 100 million years old.

As Thomson grew older, he continued to resist new scientific developments. David Lindley, in his engaging biography of Thomson, contends that Thomson became a crank in his later years, "a living fossil, a holdover from a forgotten era." Thomson rejected Maxwell's universally embraced theory of electricity and magnetism, he rejected radioactivity, insisting that the Earth was only 100 million years old, and he rejected the rapidly developing atomic theory. Until the end of his long life, however, Thomson was one of Britain's most brilliant scientific minds.

P. G. Tait

Unlike Thomson, Peter Guthrie Tait did make senior wrangler of his class at Cambridge. Tait, born in 1831, was aggressive, argumentative and fiercely loyal to Thomson. "We never agreed to differ," Thomson wrote in a 1901 obituary for his lifelong friend and collaborator, "[we] always fought it out. But it was almost as great a pleasure to fight with Tait as to agree with him."



In 1860, the curators of the University of Edinburgh offered the vacant Chair of Natural Philosophy to Tait, passing over the more scientifically accomplished Maxwell because of Tait's stronger teaching ability. (J. M. Barrie, the author of *Peter Pan* and a student of Tait, once said that "Never, I think, can there have been a more superb demonstrator.")

Tait joined the fray over the age of the Earth on Thomson's side. In a public lecture in 1885 he summarized his position, arguing that the Earth could not be more than 10 or 15 million years old. If this upset geologists, he said, then "so much the worse for geology."

The two friends were perfectly matched: energetic, confident and profoundly playful.

Smoke-ring Science

Victorian scientists, including Thomson and Tait, believed in the existence of an invisible, perfect fluid called the ether, a notion that originated with Aristotle. Such a medium seemed essential if, say, the Sun were to exert its gravitational pull on the Earth. Thomson sought to come up with a mechanical model of the ether that would explain how it interacted with physical phenomena.

Thomson was also grappling with another fundamental question of his time: What, exactly, were atoms? The existence of atoms was widely accepted, but the details remained elusive. Thomson could not accept the idea of atoms as small, hard bodies, which was one popular explanation. How could atoms, so conceived, account for the great variety of chemical elements? How could they vibrate and emit visible light? An experiment performed by Tait led to another possible explanation.

In 1867, Tait showed Thomson how to make smoke rings do tricks. Tait had learned earlier from a paper by German scientist Hermann von Helmholtz that a vortex ring in an ideal fluid would be stable and permanent. Air is not an ideal fluid, but Tait was content with an approximate model. He put a large hole in one end of a wooden box and replaced the other end with a tightly stretched towel. Inside the box he sprinkled a strong solution of ammonia and placed a dish containing sulfuric acid poured over common salt. As Tait explained during a lecture seven years later:

> These two gases combine, and form solid sal-ammoniac, so that anything visible which escapes from the box is simply particles of sal-ammoniac, which are so very small that they remain suspended by fluid-friction, like smoke in the air. Now notice the effect of a sudden blow applied to the end of the box opposite the hole.

The air in the room would have been pungent. As Tait whacked the towel, vortex rings emerged, vibrating violently, "just as if they were solid rings of India-rubber." Tait marveled at their stability. If an elliptical or square hole was used instead of a round hole, the vortex shape would shake and vibrate until it assumed a circular shape, which Tait regarded as a "position of stable equilibrium."

Tait worked hard to perfect his smoke-ring experiment. It appears that Thomson did as well. An 1876 letter from Tait to Thomson proposed some deadly recipes:

> Have you ever tried plain air in one of your boxes? The effect is very surprising. But eschew NO_5 and Zn. The true thing is

SO_3 + NaCl. Have NH_3 rather in excess—and the fumes are very dense + not unpleasant. NO_5 is DANGEROUS. Put your head into a ring and feel the draught.

More than three decades later, the great French mathematician Henri Poincaré would maintain that useful combinations of ideas are the most beautiful. A subconscious "aesthetic sieve," he believed, seeks out those beautiful associations. As Thomson watched Tait's smoke rings gliding gently across the room, he had just such an inspiration: Perhaps vortex rings of ether were the basic blocks of matter. Vortex motion imparted by a divine creator had broken the otherwise homogeneous ether into its chemical parts—chemical elements were knotted tubes of ether. Simplicity was part of this theory's appeal. No cumbersome hypotheses would be needed to explain chemical properties; they were a result of topology. It was simple and beautiful—it had to be true.

Elements exhibit characteristic colors, or spectra, when brought to a sufficiently high temperature. Sodium exhibits two spectral lines. Thomson explained sodium's "D-lines" by concluding that sodium consists of two vortex rings linked in the simplest fashion. In 1867, Thomson wrote to Helmholtz about his theory:

> ... every variety of combinations might exist. Thus a long chain of vortex rings, or three rings, each running through each other, would give each very characteristic reactions upon other such kinetic atoms.

Tait described Thomson's notion of vortex rings in a series of lectures published in 1874. He compared the interactions of vortex rings with the effects of drawing a teaspoon across the surface of a cup of tea. Lifting the spoon from the surface will create "a couple of little eddies or whirlpools going round in the tea rotating in opposite directions." "These two little eddies," he continued, "are simply the ends of a half vortex-ring."

> There can be ends in such a case as that, because these two ends are in the free surface of the liquid. A vortex-ring, then, cannot have ends ... and if we adopt Thomson's notion of a perfect fluid filling infinite space, of course there can then be no ends. All vortex-rings—and therefore, according to Sir William Thomson, all atoms of matter—must necessarily be endless, that is to say, must have their ends finally united together after any number of convolutions or knottings.

The details of Thomson's theory remained rather vague, but the general idea fit wonderfully into a dynamical vision of the universe. Victorian science was imbued with Newton's mechanical philosophy. French mathematician Pierre-Simon Laplace even asserted that if one could somehow know the forces at every point of the universe at some instant, then one could predict the future and describe the past. God, it seemed, was a billiards player resting after a particularly good shot.

Tait's published lectures contain a detailed description of his smoke-ring experiments, including measurements of the boxes and descriptions of the chemicals. But the text and accompanying woodcut illustration can give only a limited idea of what Tait and Thomson experienced. Determined to learn the feel—and

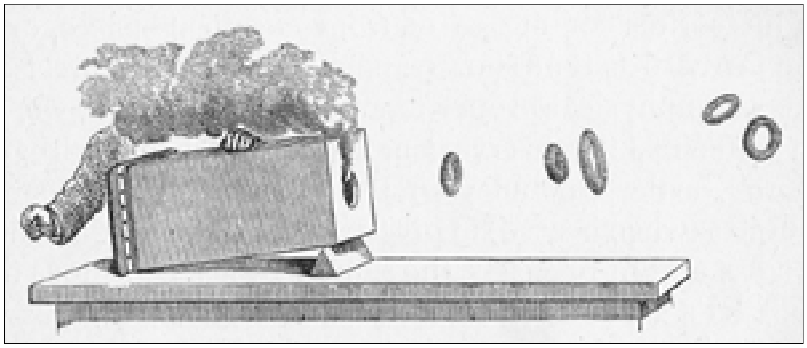


Figure 2. In 1867, Peter Guthrie Tait constructed a box to study smoke rings. He placed a towel on one end, carved a circular hole on the other end and placed ammonia and sulfuric acid inside the box. Striking the towel would cause smoke rings to emerge through the hole. As smoke reached the opening, it would move outward and then backward in a circular motion as air outside the box pushed against it. This circular pattern would create a vortex, stabilizing the smoke ring. If a shape other than a circle was used, the rings would gradually take on a circular shape after passing through the hole. This sketch of the smoke-ring box comes from Tait's book *Recent Advances in Physical Science*.

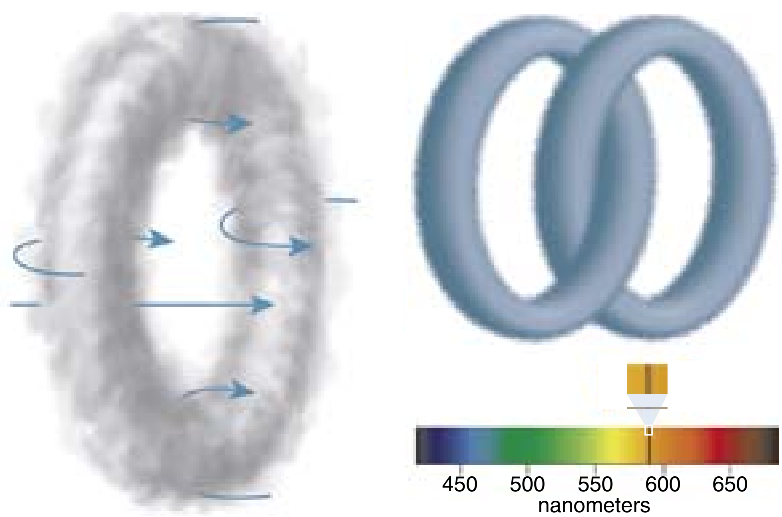


Figure 3. William Thomson's vortex-atom theory arose out of his observations of Tait's experiments with smoke rings. The arrows around the smoke ring at left indicate the direction of the vortex motion. Air pushes against the greater surface area of the outer part of the ring, moving the ring in the direction of the air passing through the inside of the ring. In a perfect fluid, such as the ether, this vortex motion would be permanent. Thomson posited that the behavior of atoms could be explained if they were knots of ether imparted with vortex motion. He used sodium as an example, thinking that the element's distinct "D-lines," or doublet of spectral emissions, might be caused by two linked rings of ether.

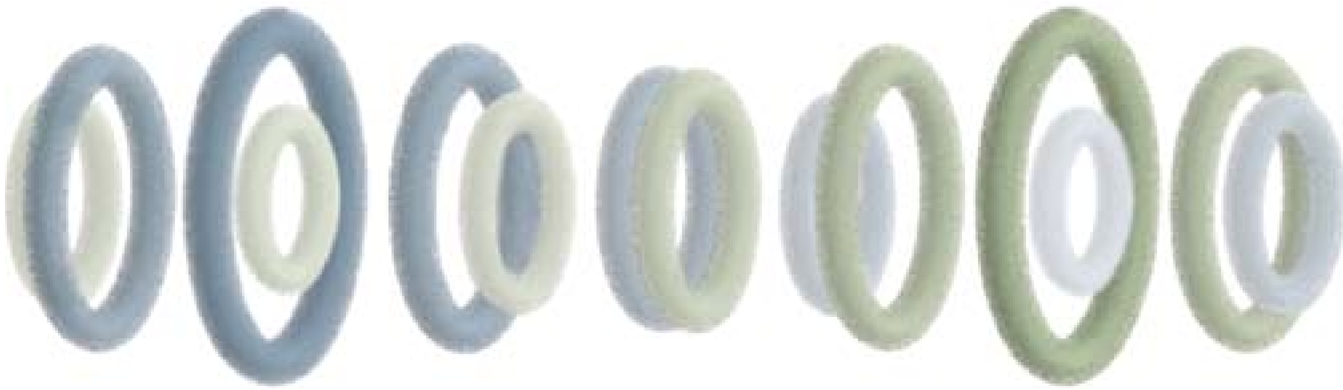


Figure 4. In a paper published in 1858, Hermann von Helmholtz described the motion of vortex rings in a perfect fluid. He concluded that if two rings (here, one blue and one green) moved in the same direction on the same plane, the first ring would enlarge and slow down, and the second ring would shrink and pass through the first ring. This process would repeat indefinitely. Tait used his smoke-ring box to illustrate Helmholtz's conclusions and found that he was able to reproduce this interaction.

the smell—of the experiment, I made my own smoke-ring boxes and spent a bizarre afternoon poisoning myself. Colleagues Andrzej Wierzbicki and Susan Williams and I whacked boxes filled with the combined fumes of acid and ammonia. The rings wobbled and shook as they emerged, but as they grew in diameter, often as much as two feet across, they became stable. The sight of the rings sailing gracefully across the room must have been deeply satisfying to Tait and Thomson. The corrosive fumes probably less so.

dp/dt

James Clerk Maxwell was cheerful, humorous, athletic and brilliant—his mind raced ahead of conversations, leaving others baffled in its wake. He and Tait first met as students at Edinburgh Academy. Later, as colleagues, the two friends corresponded by the new halfpenny postcards almost daily. Mail delivery was prompt, and it was possible to send a card and receive a reply in a single day. Fortunately, many of the cards that Maxwell sent have been preserved. He often used cryptic shorthand: Thomson was T, Tait was T′ and John Tyndall, a successful popularizer of science whom Maxwell and Tait regarded as mediocre, was T″, a private joke suggesting that Tyndall was a "second-order quantity." Maxwell signed himself dp/dt, an abbreviation of an equation from thermodynamics that produced his initials: JCM = dp/dt.

Maxwell's deep interest in knots and topology likely grew out of Thomson's vortex-atom theory and the influence of Tait. A card sent by Maxwell to Tait in November 1867 suggests that he had been thinking about Helmholtz's papers on vortex motion and knots, most likely because of their implications for electricity and magnetism.

In a letter written the next month, Maxwell commented that "I have amused myself with knotted curves for a day or two." He then explained that the linking number of two knots has physical significance. Running electrical current through one knot produces a magnetic field. The linking number is essentially the work done by a charged particle moving along the path of the second knot. Maxwell expressed the linking number as a double integral that Gauss had discovered earlier.

Maxwell's deep interest in knots and links is revealed in recently published letters and notes. Included among these are stereoscopic slides of knots intended to be viewed with a stereoscope of Maxwell's own improved design. There is also a photograph of a zoetrope, or wheel of life. A 19th-century biography of Maxwell describes the design:

> In the ordinary instrument, on looking through the slits in the revolving cylinder the figures are seen moving on the oppo-

site side of the cylinder. Maxwell inserted concave lenses in place of the slits, the lenses being of such focal length that the virtual image of the object at the opposite extremity of the diameter of the cylinder was formed on the axis of the cylinder, and consequently appeared stationary as the cylinder revolved.

A close look at the hand-drawn figures in the photograph of Maxwell's zoetrope reveals that they consist of three simple rings, representing smoke rings. In his 1858 paper, Helmholtz had described how two vortex rings traveling in the same direction would interact:

> If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till, finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same goes on in the opposite order, so that the rings pass through each other alternatively.

Having appreciated the *pas de deux*, Maxwell wished to understand the three-ring dance. In a letter to Thomson on October 6, 1868, he announced that Helmholtz's conclusions held true for three rings as well as two.

Despite his racing mind and brilliant wit, Maxwell maintained a careful and objective voice in all his writings. The same cannot be said of his friend Tait.

The Unseen Universe

The success of Newton's mechanical vision of the universe was a source of national pride for Victorian-era Britain, but the materialistic philosophy it spawned caused some unease. If all phenomena could be scientifically explained, then so could miracles, including religious miracles. The certainty of revelation would be at risk.

One popular response was to turn to spiritualism, which became widely popular in Britain by 1870. The Society for Psychical Research, founded in 1886, included prominent Victorians such as William Gladstone, Lewis Carroll, John Ruskin and Alfred Lord Tennyson. At the core of their efforts was the desire to buttress religious belief with science. Some even hoped to prove the existence of life after death.

A few scientists, such as mathematician William Kingdon Clifford, proclaimed their skepticism—even atheism—openly. John Tyndall, a champion of materialism, provoked many listeners with his address before the British Association meeting at Belfast in 1874 by arguing that science must be divorced from religious doctrine. Religious arguments about the nature of the universe, he said, must accept their inferiority to scientific explanations.

This outraged Tait, who responded by publishing, with Belfast physicist Balfour Stewart, *The Unseen Universe*, a hastily written, rambling attempt to justify miracles, spirits and, above all, an afterlife. Despite its literary and scientific shortcomings, *The Unseen Universe* was a commercial success. A sequel, *Paradoxical Philosophy*, followed soon thereafter.

In short, Tait and Stewart argued that, although the ether around us is imperfect, there are other, parallel universes with ever more perfect ether. They included a diagram of a series of concentric circles in which the innermost ring denoted an "evanescent smoke-ring," the next circle represented our world, or the "visible universe," and each successively larger circle denoted an ever more perfect "invisible universe." "Just as the smoke-ring was developed out of ordinary molecules," they explained, "so let us imagine ordinary molecules to be developed as vortex rings out of something much finer and more subtle than themselves, which we have agreed to call the invisible universe." If we go infinitely far back, the authors contended, we reach "a universe possessing infinite energy, and of which the intelligent developing agency possesses infinite energy."

Kelvin Fagan, the Cavendish Laboratory

Figure 5. James Clerk Maxwell, a friend of Thomson and Tait, became interested in their work on vortices and used this zoetrope to examine the interactions of three rings. As the zoetrope spun, it would appear to viewers watching through the openings on the outside that the rings drawn on the interior band were passing through each other in the same manner described by Helmholtz for two rings.

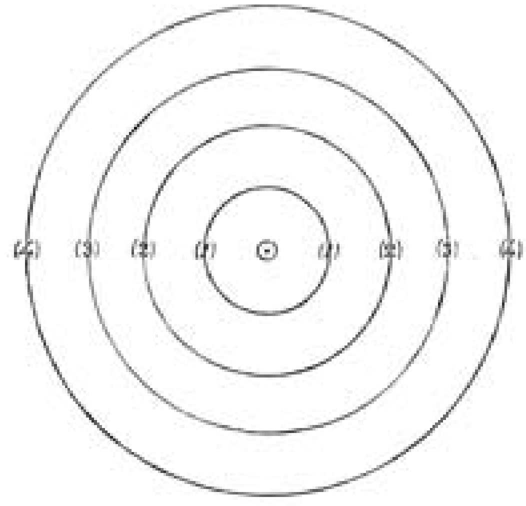


Figure 6. In *The Unseen Universe*, authors Tait and Balfour Stewart hypothesized about the nature of the universe. They postulated that just as a smoke ring is composed of ordinary molecules, so too those molecules might be vortex rings of "something much finer and more subtle than themselves." In this sketch, the innermost circle represents a smoke ring within our visible universe (1). In the same way, our universe is an impermanent part of a more perfect universe (2), which is itself part of an even more perfect universe (3), and so on, until reaching a universe with "infinite energy" created by a divine agent.

Tait and Stewart believed that thoughts cause molecular disturbances that rippled into the next world. Vibrations in our world dissipate and fade but transmit into the perfect ethereal realm a motion that is eternal. They mingled technical arguments about thermodynamics with the words of Saint Paul. Not surprisingly, a trefoil knot adorns the spine and title page of the book.

In *The Unseen Universe* we see that Tait was not interested merely in mathematical questions about knots; he was hoping to answer the most intractable questions about consciousness, the soul and the afterlife: "No doubt religion informs us … that there are other beings above man, but these do not live in the visible universe, but in that which is unseen and eternal." Thus, the authors concluded, "We have now reached the stage from which we can very easily dispose of any scientific difficulty regarding miracles." But Clifford's review in the *Fortnightly Review* was predictably sharp, commenting on "the reposeful picture of the universal divan, where these intelligent beings while away the tedium of eternity by blowing smoke rings from sixty-three kinds of mouths."

Like Tait, Maxwell was deeply religious, but his beliefs were personal. Although he admired the vortex-atom theory, he was skeptical of Tait's attempts to use it in defense of religious arguments. In a letter to Tait in 1878, he commented sarcastically that "If you think of extending the collection of hymns given in the original work [*Unseen Universe*], do not forget to insert 'How happy could I be with Ether.'"

A few months later, Maxwell reviewed *Paradoxical Philosophy* in *Nature*, again questioning the link between scientific and religious arguments:

> To exercise the mind in speculations on [ether] may be a most delightful employment for those who are intellectually fitted to indulge in it, though we cannot see why they should on that account appropriate the words of St Paul ... No new discoveries can make the argument against the personal existence of man after death any stronger than it has appeared to be ever since men began to die.

Maxwell took a more playful poke at Tait's speculations in a poem, included in a letter to Tait, which began "My soul is an entangled knot, Upon a liquid vortex wrought." Maxwell may not have been able to restrain his friend's spiritual impulses, but he did inform and inspire Tait's explorations of the mathematics of knots and links.

Tait's Program

In 1878, Tait began an ambitious program of cataloging knots, which, if the vortex-atom theory was correct, would actually be a sort of table of elements. Tait did not have rigorous techniques for showing that his pictures represented different knots, but he did have sound geometric intuition and courage. (By the 1920s, techniques of Henri Poincaré and James Alexander could verify that Tait's tables were essentially correct.)

After building tables for knots that can be drawn with six or fewer crossings, Tait was still dissatisfied. The problem was that the large number of spectral lines in some elements

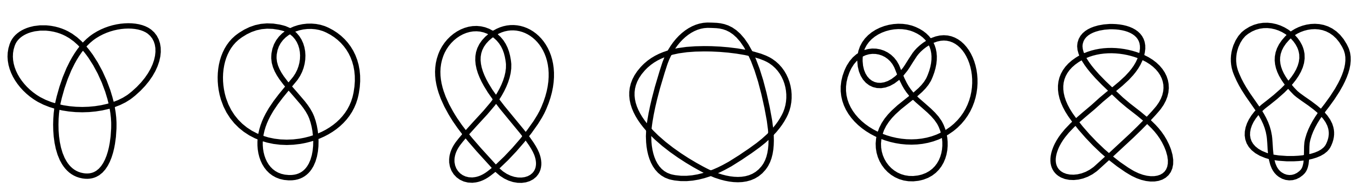


Figure 7. In an effort to develop a table of elements based on the vortex-atom theory, Tait began to classify knots according to the number of crossings. The seven simplest knots, above, extend through six crossings. Tait eventually built tables through ten-crossing knots. Current tables extend through 16 crossings and contain more than 1.7 million entries.

meant that the corresponding vortex atoms would be very complicated. Tait then produced a table of "the first seven orders of knottiness" (knots with up to seven crossings), but even that was insufficient.

Knots with eight or more crossings, wrote Tait, "are not likely to be attacked by a rigorous process until the methods are immensely simplified." It would take someone with "the requisite leisure," as he put it, to extend the list further. Inspired by Tait, others did become involved. The Reverend Thomas P. Kirkman sent Tait a redundant list of 10-crossing knots. Tait worked hard to weed out duplications. When Charles Little, a mathematician and civil engineer at Nebraska State University, sent Tait his own list, there was some slight disagreement, but Tait successfully located his single error before publication. Kirkman also sent Tait a list of 1,581 knots with 11 crossings, but Tait decided he could spare no more time for the project.

Others would take up his labor. Tabulation, although no longer the sole focus of knot theory, continues. Current tables extend through 16 crossings and contain about 1.7 million entries. Morwen Thistlethwaite of the University of Tennessee is in the process of completing a table of 17-crossing knots.

The Meaning of Knots

Thomson's confidence and aesthetic judgment combined with Tait's enthusiasm and philosophical nature to promote the first sustained research program in knot theory. Maxwell contributed as well by providing Tait with a steady stream of ideas and encouragement.

Research into knots continues, but the vortex-atom theory faded like Tait's smoke rings. Thomson, however, remained hopeful in his 1889 presidential address to the Institution of Electrical Engineers. One can well imagine him thinking, "Too bad, it would have been beautiful":

> And here, I am afraid I must end by saying that the difficulties are so great in the way of forming anything like a comprehensive theory that we cannot even imagine a finger-post pointing to a way that leads us towards the explanation. That is not putting it too strongly. I can only say we cannot now imagine it. But this time next year,—this time ten years,—this time one hundred years,—probably it will be just as easy as we think it is to understand that glass of water, which seems now so plain and simple. I cannot doubt but that these things, which now seem to us so mysterious, will be no mysteries at all; that the scales will fall from our eyes; that we shall learn to look on things in a different way—when that which is now a difficulty will be the only common-sense and intelligible way of looking at the subject.

The vortex-atom theory represented what we might call the first attempt at a physical application of knot theory. Since then, knot theory techniques have made their way into fluid dynamics, solar physics, DNA research and quantum computation. But the physical significance of knots remains elusive.

It is possible that a knot, like its simpler geometric cousin, the circle, represents a fundamental relation of quantities. Two thousand years of reflection on Euclid's geometry have produced many abstract applications. It is understandable if a few more years of thought are needed before we can break free from an overly literal view of knots and links. When we finally understand their deepest nature, profound physical applications will blossom. And it will be beautiful.

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Bibliography

Campell, L., and W. Garnet. 1882. *The Life of James Clerk Maxwell*. London: Macmillan and Co.

Epple, M. 1998. Topology, matter, and space, I: Topological notions in 19th-century natural philosophy. *Archive for History of Exact Sciences* 52:297–392.

Epple, M. 1999. Geometric aspects in the development of knot theory. In *History of Topology*, ed. I. M. James. Amsterdam: North-Holland, 301–358.

Helmholtz, H. 1867. On the integrals of the hydrodynamical equations, which express vortex motion. *Philosophical Magazine* ser. 4, 33:485–512.

Hoste, J., M. Thistlethwaite and J. Weeks. 1998. The first 1,701,936 knots. *The Mathematical Intelligencer* 20:33–48.

Knott, C. G. 1911. *Life and Scientific Work of P. G. Tait*. Cambridge: Cambridge University Press.

Laplace, P. 1820. *Essai philosophique sur les probabilités*. Trans. F. L. Emory. New York: Dover, 1951.

Lindley, D. 2004. *Degrees Kelvin: A Tale of Genius, Invention, and Tragedy*. Washington, D.C.: Joseph Henry Press.

Maxwell, J. C. 1965. *The Scientific Papers of James Clerk Maxwell*, ed. W. D. Niven. Mineola: Dover Publications.

Maxwell, J. C. 1990, 1995, 2002. *The Scientific Letters and Papers of James Clerk Maxwell*. 3 vols., ed. P. M. Harman. Cambridge: Cambridge University Press.

Poincaré, H. 1956. Mathematical creation. In *The World of Mathematics*, ed. J. R. Newman. New York: Simon and Schuster, 1956. Vol. IV, 2041–2050.

Stewart, B., and P. G. Tait. 1881. *The Unseen Universe or Physical Speculations on a Future State*. London: MacMillan and Co.

Thomson, W. 1890. *Mathematical and Physical Papers, III*. Cambridge: Cambridge University Press.

Tyndall, J. 1874. *Address Delivered Before the British Association Assembled at Belfast, with Additions*. London: Longmans, Green, and Co.

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