Designing Incentives in Online Collaborative Environments[1]

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Abstract – We consider collaborative systems where users make contributions across multiple available projects and are rewarded for their contributions in individual projects according to a local rule of sharing the value produced by contributors to the given project. This serves as a model of online social computing systems such as online Q&A forums where users invest their efforts in producing online content and receive in return credits in terms of attention or social status, or momentary rewards. The central question that we study is how the value produced should be split among contributors so that individual incentives are approximately aligned with that of the system as a whole. We formulate a game-theoretic framework that allows for quite some generality with respect to the nature of contributions (e.g. supplements vs. complements) and the production costs. We find that optimal social welfare can be well approximated by simple local sharing rules where users are approximately rewarded in proportion to their marginal contributions. The framework allows us to identify the main factors that contribute to social inefficiency, including the nature of contributions and production costs, and we quantify the social inefficiency for a number of interesting cases. Our work provides insights and guidelines for the design of incentives when eliciting contributions in online services.

Keywords: Social computing, Online Q&A forums, User generated content (UGC), Crowdsourcing, Implementation, Game Theory

1 Introduction

Many economic domains are ones where self-interested individuals participate in joint ventures by investing time, effort, money or other personal resources, to produce some value which is then shared among the participants. Examples include traditional surplus sharing games [29, 12], co-authorship settings where the wealth produced is in the form of credit in scientific projects that is implicitly split among the authors of a paper [24], CEO labor markets where the wealth is shareholder value that is split among high-rank employers in the form of the end-of-year bonuses [32], but most importantly online services contexts where users collaborate on various projects (e.g. Q&A Forums, Wikipedia, TopCoder). One important such web service that is becoming increasingly widespread, is that of online social computing systems where web users generate content in exchange for virtual or monetary rewards (e.g. Yahoo! Answers, Quora, Stack Overflow) and has also been given recent attention from the game theoretic standpoint [14, 15, 16, 9, 4, 22].

In the context of social computing each venture represents a specific topic on a user-generated website such as Yahoo! Answers or Stack Overflow. Each web user can be modeled as having a budget of time that he spends on such a web service, which he chooses how to split among different topics/questions that arise. The attention produced at each topic can be thought of as a value produced locally at that topic. Responses by different web users affect the attention produced in significantly different ways; a unit of time spent by a more specialized user produces a higher quality response. The attention produced is implicitly split among the responders of the topic in a non-uniform manner, since the higher the slot that the response is placed in the feed the higher the attention it gets. Hence, the website designer has the power to implicitly choose the attention-sharing mechanism locally at each topic, by strategically ordering the responses according to some quality factor and potentially randomizing. A natural goal of the website designer is to devise the attention-sharing mechanism such that the global attention (number of visitors) produced on his website is maximized.

Our goal is to study what are the properties of the collaborative setting that allow the existence of good local sharing rules and what are natural sharing rules that achieve good efficiency under different assumptions on the value functions. The main question that we examine can be formulated as follows:

> *How should the value produced be split among the contributors so that the individual incentives are approximately aligned with those of the system as a whole and are there fundamental limits on this approximation that cannot be surpassed?*

The design of simple and local mechanisms is of great importance for online applications where many mechanisms are going to be ran in an automated way and thereby cooperative game theoretic solution concepts, such as the core or bargaining, that require ad-hoc negotiations and global redistribution of value are not applicable.

The main conceptual message of our work is that high efficiency *globally*, is achieved if each player is awarded a significant fraction of his marginal contribution to the value *locally* at each project. The ability to satisfy the marginal contribution property, and thereby the ability to achieve good efficiency guarantees by using local sharing mechanisms, is guided by the degree of complementarity among the players' efforts. The more complementary the efforts are, the higher the inefficiency is. If each player is quintessential in producing any value at a given project then the inefficiency can grow linearly with the number of participants. On the other hand, if the players' efforts are substitutes, then there exist local sharing rules that achieve at least half of the optimal social welfare. Substitute effort settings are a good fit for online user-generated content websites, since the marginal increase in attention produced by a submission is decreasing in the amount and quality of existing submissions. On the other hand, studying complementary efforts allows us to capture scenarios where projects require diverse set of skills, e.g. classical collaborative environments such as scientific co-authorship settings.

1.1 Our Results

We formulate a framework to study the effects of various rewarding schemes in online collaborative environments where the value produced is a function of invested efforts and is shared among contributors using a local sharing rule. The framework allows to accommodate for quite some generality with respect to the utility generated to the players and the underlying production costs.

Game Theoretic Model. We consider the following model of collaboration: the system consists of set of players and a set of projects. We make the assumption that each player has some arbitrary feasibility constraint on the efforts that he can invest across projects. Each project produces some value that is a function of the efforts invested by the different participants. This common value produced by each project is then shared among the participants of the project according to some pre-specified sharing rule. We are interested in understanding the crucial parameters of such settings that determine the overall efficiency, as measured by the total value produced, when the players act selfishly trying to maximize their own share, rather than the overall wealth produced.

Solution Concepts. Since we model collaborative settings we study game theoretic solution concepts that also take into account the fact that players can coalitionally deviate. Discussing with your collaborators to agree on changing a strategy is quite natural. We capture such an effect by studying the efficiency at *strong Nash equilibria* of the game, as well as more relaxed notions of where only players that participate in a common project can coalitionally deviate. We study worst-case inefficiency by quantifying the efficiency achieved at the worst such solution concept.

For the case where players' efforts are substitutes, which could capture online collaborative environments with higher degree of anonymity, we also examine efficiency in non coalitionally robust solution concepts such that Nash Equilibria and outcomes of no-regret learning behavior, as well as settings with incomplete information about parameters of the game, such as abilities of players.

Marginal Contribution. We show that if each player is awarded at least his marginal contribution to the value of a project, locally, then individual incentives are globally well-aligned with social ones and the overall efficiency is a good approximation to the optimal. Specifically, every strong Nash equilibrium achieves at least half of the optimal social welfare. If the project value functions satisfy the diminishing marginal returns property then the latter guarantee is satisfied at every Nash equilibrium too, and even when players employ no-regret learning strategies. The efficiency guarantee degrades smoothly with the fraction of the marginal contribution that a player is awarded. These results provide a theoretical support that efficient outcomes can be well approximated by local value sharing rules.

Efficiency Limits. We introduce a measure of complementarity for the value functions, which we denote as *the degree of substitutability of efforts* and we show that it is the main driving force of efficiency in our setting, imposing a limit on the welfare approximation achievable by local sharing rules.

Simple and Efficient Sharing Rules. We show that sharing locally the value proportionally to the marginal contribution of each player always achieves an efficiency that is tight under the latter measure. When project value functions satisfy the diminishing marginal property then splitting locally according to the Shapley value or according to a large class of sharing methods that have been well studied by the axiomatic surplus sharing literature, achieves at least $1/2$ of the optimal social welfare at every Nash as well as coarse correlated equilibrium. If the value is a function of the weighted sum of efforts (e.g. each effort is weighted by an ability factor of the player) then sharing proportional to the weighted effort achieves the same guarantee. The latter guarantee holds even if the abilities of the players are not common knowledge but are rather drawn from commonly known distributions.

Sources of Inefficiency. Even if intuitive sharing rules are used and each player is awarded his marginal contribution, inefficiency (though limited) may arise in the system. Intuitively, the sources of inefficiency in our model is either over-collaboration or under-collaboration. We observe that in settings where players' efforts are substitutes, then at equilibrium the players tend to concentrate at specific projects, more than the in optimal solution, rather than distribute their efforts (Example 9). On the contrary, when players' efforts are complementary, then players tend to isolate themselves more than the optimal and work on atomic projects (Example 4).

Effort Costs and a Threshold Phenomenon. In the last part of the paper we examine the case where apart from the feasibility constraints the players also incur a cost that is some function of their efforts. We focus on games where both the produced value at any project and the cost of a player are functions of total effort. We characterize the inefficiency as a function of the convexity of the cost functions, i.e. how fast the marginal cost, per extra unit of effort, increases. We measure the convexity of a function using the measure of elasticity which is a well-studied concept in economics. We show that as long as the elasticity of the cost functions is strictly greater than 1 (i.e. the marginal cost is strictly increasing) and value functions are concave, then the inefficiency both in terms of produced value and in terms of social welfare (including player effort costs) is a constant independent of the number of players. This stands in a stark contrast with the case when cost functions are linear, where the inefficiency can be in the order of the number of participants.

Applicability Outside the Online Setting. Another interesting application of our work is in the context of sharing scientific credit in paper co-authorship scenarios. One could think of players as researchers splitting their time among different scientific projects. Given the efforts of the authors at each project there is some scientific credit produced. This credit heavily depends on the amount of effort of the players and could potentially depend asymmetrically in the efforts of different authors. Local sharing rules in this scenario translate to scientific credit-sharing rules among the authors of a paper. In any paper there is an implicit sharing rule which is the order of authors in the paper. Hence, different ordering conventions in different communities correspond to different sharing mechanisms, with the alphabetical ordering corresponding to equal sharing of the credit while the contribution ordering corresponding to splitting proportionally to the marginal contribution of each author. Our results imply that the equal sharing rule works well only when the number of authors per paper is small. The former is reassured by the fact that scientific communities (e.g. theory community) that endorse the alphabetical ordering of authors in a paper, generally produce papers with a small number of co-authors. On the contrary, communities like physics and biology where papers can consist of many as 30 authors, generally order authors according to contribution.

Techniques. To prove our results we derive novel efficiency bounds for general utility maximization games that might be of independent interest. We show that in any utility maximization game where the social welfare is monotone (can only decrease if a player is removed from the game), if a player receives at least his marginal contribution to the social welfare then the social welfare at any strong Nash equilibrium (Nash equilibrium that is stable under coalitional deviations) is at least half of the optimal social welfare (Theorem 1). We refer to the latter condition as the Vickrey condition, since it is closely related with the intuition behind the well-known VCG auction. More generally, we show that if a player receives a k fraction of his marginal contribution, then the social welfare at any strong Nash equilibrium is at least a $1/(k+1)$ fraction of the optimal. If the game satisfies the stronger monotonicity property that a player by participating has only non-negative externalities in the utility of the rest of the players, then the latter guarantee becomes $1/k$. In addition, we show when the strategy space of the players is some subset of Euclidean space and the social welfare satisfies a diminishing marginal property, closely related to submodularity, we give an adaptation and an improvement of the results of Vetta [43] that shows that the latter guarantees are satisfied at any Nash equilibrium too (Theorem 3). When the social welfare doesn't satisfy any monotonicity property then we give efficiency guarantees if the game admits a potential and the potential is closely related to the social welfare (Theorem 2). Our result also portrays a very strong connection between analysis of efficiency of the best Nash equilibrium with that of the worse strong Nash equilibrium in potential games, implying that for utility maximization potential games the latter too are close to each other. When the value produced is a function of total effort we give much tighter efficiency guarantees by adapting the recently introduced *local smoothness* framework [37] to the case of utility maximization rather than cost-minimization.

1.2 Structure of the Paper

We start with introducing the assumptions and notation in Section 3. Section 4 shows our main results on the efficiency for the class of utility sharing games that satisfy a monotonicity and a Vickrey condition. In Section 5 we present our results on the efficiency for our model of collaboration under various assumptions on the project value functions and sharing rules. In Section 6 we derive our results for the case where effort is costly. Most of the proofs and some extensions of our work are deferred to Appendix.

2 Related Work

The design of incentives for online collaborative environments has been studied from both empirical, e.g. [44, 30], and theoretical standpoint, e.g. [22, 9, 16]. We first discuss the work that is most closely related to ours in this context. In a recent line of work, Ghosh and Hummel [14, 15] studied similar problem as in our paper, motivated by online social computing systems, but focusing on the case of a single project. Our work differs in that we consider a multiplicity of projects across which a contributor can strategically invest his effort and we allow for more general class of project value functions. The multiplicity of projects creates endogenous outside options that can significantly affect equilibrium outcomes. The effect of mechanisms to incentivize high quality contributions in crowdsourcing environments has been studied by Ghosh and McAfee [16], Chawla, Hartline and Sivan [7], and Archak and Sundararajan [4]. In the specific context of incentivizing contributions in online Q&A forums, Jain, Chen, and Parkes [22] studied mechanisms to solicit high-quality contributions and incentivize early submissions under assumption that individual contributions are either supplements or complements. All these prior work can be seen as focusing on the case of a single project. In [9], the authors studied a model of crowdsourcing systems where users can choose from a set of multiple projects. That work focused on the case where each user selects at most one project and each project offers a fixed prize that is shared according to the winner-take-all sharing rule among the contributors of a project. Our work is different in that we allow for the case of the produced value sharing among the contributors to a project according to a local sharing rule, and we allow individual contributors to invest their efforts across multiple projects.

There have been several works on the efficiency of Nash equilibria of utility sharing games [43, 17], also relating efficiency with the marginal contribution property. Here we extend that literature by mainly studying strong Nash equilibria and by studying stronger monotonicity properties. We are able to drop the assumption of submodularity that is prevalent in the literature. The study of the efficiency of the strong Nash equilibrium was introduced in [11, 2]. However, the literature focused mainly on specific models of cost minimization games like network design games (see e.g. [1, 11]). Here, we analyze efficiency of strong Nash equilibrium in general utility sharing games. Our model of effort market games is very similar to the contribution games model of [3]. However, in [3], the authors assume that all players get the same value from a project. This corresponds to the special case of equal sharing rule in our model. Moreover, they mainly focus on network games where each project is restricted to two participants. For that case, all of our results give a constant factor efficiency.

Splitting scientific credit among collaborators was recently studied by Kleinberg and Oren [24] where they examine the case where players choose a single project to work on. They also consider the equal sharing rule when the game is symmetric and a sharing rule that is related to a player's marginal contribution for asymmetric ability players. However, their main focus is (in our terminology) how to change globally the value functions of the projects so that optimality is achieved at some equilibrium of the resulting perturbed game.

Common value sharing games also have a strong connection with the bargaining literature [20, 25, 6]. The main question in that literature is similar to what we ask here: how should a commonly produced value be split among the participants. Our approach though is very different than the bargaining literature since we focus on simple mechanisms that use only local information of a project and not global properties of the game. In many of those works it is shown that the right splitting of value achieves optimality. However, that "right splitting" is a complex function of the whole game instance. Here, we show that in many cases, simple local utility sharing rules are able to give approximate optimality.

The existence of strong Nash equilibria has generated a long line of research in both the economic and game theory community (see e.g. [31, 19, 11] for some latest work). When feasibility constraints of each player are such that a player can only choose one project to put all his effort, everyone has the same budget of effort and value produced is a function of total effort then the marginal contribution sharing scheme and the proportional sharing scheme collapse to just splitting the value produced among the players that choose a project. Hence, in this case, our model is a singleton utility congestion game where the utility associated with each resource is $v(n)/n$. The existence of strong Nash equilibria in such games has been studied in [38, 21]. Rozenfeld and Tennenholtz [38] show that when $v(n)/n$ is increasing (i.e. the value function is convex) then there always exists a strong nash equilibrium and can be computed in polynomial time by a greedy algorithm. Holzman et al. [21] shows that when $v(n)/n$ is decreasing (i.e. value function is concave) then the set of pure Nash equilibria which is non-empty, coincides with the set of strong Nash equilibria. These two results provide strong evidence of the existence of strong Nash equilibria in our collaboration model and we view the problem of examining the existence of strong nash equilibria in our more general framework as a very interesting future direction. In this work we focus on proving efficiency characteristics that hold for all strong Nash equilibria whenever they exist.

3 Assumptions and Notation

We consider a normal form game with a set of players $N = \{1, 2, \ldots, n\}$ on the strategy space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_n$, where $\mathcal{X}_i$ is the strategy set of player i, and $u_i : \mathcal{X} \to \mathbb{R}$ is the payoff function of player $i \in N$. We define $x_i^{out} \in \mathcal{X}$ to be an out strategy for player $i \in N$, which intuitively corresponds to player i not entering the game. Given a strategy profile $x \in \mathcal{X}$, we define the social welfare to be the total utility, i.e. $SW(x) = \sum_{i \in N} u_i(x)$.

We will refer to this class of games as *utility sharing games* as most of our analysis would be concerned with how the relation between the individual payoffs $u_i(x)$ and the total utility $SW(x)$ affects the social efficiency.

We will consider a special class of utility sharing games that are monotone, which intuitively means that at any strategy profile, the social welfare cannot be increased by excluding some of the players from participating in the game.

Definition 1 (Monotone games) *A utility sharing game is said to be monotone if for every player $i \in N$ and for every strategy profile $x \in \mathcal{X}$: $SW(x) \geq SW(x_i^{out}, x_{-i})$. If, in addition, $u_j(x) \geq u_j(x_i^{out}, x_{-i})$, for every $j \in M$, then the game is said to be strongly monotone.*

We will focus on games where if all players play their out strategy then the social welfare is 0. In some of our results, we will assume a class of utility sharing games where each player is guaranteed a payoff that is at least his own marginal contribution to the social welfare. We refer to this as a Vickrey condition, which is defined as follows.

Definition 2 (Vickrey condition) *A utility sharing game satisfies the Vickrey condition if for every player $i \in N$ and strategy profile $x \in \mathcal{X}$: $u_i(x) \geq SW(x) - SW(x_i^{out}, x_{-i})$. We also say that a game satisfies the k-approximate Vickrey condition if each player gets at least $1/k$ of his marginal contribution to the social welfare.*

It is interesting to examine what the Vickrey and Monotonicity condition imply on the externalities in a game. The externality of a player i from playing a strategy x_i is the change in the utility of the rest of the players when the player plays this strategy rather than staying out: $\sum_{j \neq i} u_j(x) - \sum_{j \neq i} u_j(x_i^{out}, x_{-i})$. If $u_i(x_i^{out}, x_{-i}) = 0$ then a game satisfies the k-approximate Vickrey and the monotonicity condition if the externality of any strategy of a player is closely related with his utility. Specifically it is at least $-u_i(x)$ and at most $(1 - \frac{1}{k})u_i(x)$. Thus the Vickrey and monotonicity condition imply that the externality of any strategy of a player is neither very positive nor very negative when compared to his utility. Interestingly for a game to satisfy the exact Vickrey condition it needs to have only negative externalities.

Common Value Sharing Games We then consider a special class of utility sharing games where the strategies of players correspond to investments across a set of economic activities $M = \{1, \ldots, m\}$. Each player i participates in a subset of the activities $M_i \subseteq M$. The strategy profile of player i is specified by the amount of investment $x_{i,j} \in \mathbb{R}_+$ in activity $j \in M_i$, thus $x = (x_{i,j}, i \in N, j \in M_i)$.

We assume that each player has a feasibility constraint on his investments across activities and thereby his strategy space $\mathcal{X}_i$ is some subset of $\mathbb{R}^{|M_i|}$. For instance, it could be the polytope defined by a weighted budget constraint $\sum_{j \in M_i} t_{i,j} x_{i,j} \leq B_i$ or it could be a discrete set of points of $\mathbb{R}^m$ that could capture integral versions of effort market games where players can only put all of their budget in only one project. Most of our efficiency theorems hold for arbitrary such constraints.[1]

Each activity $j \in M$ is associated with a set of participating players N_j and with a common value function $v_j : \mathbb{R}^{|N_j|} \to \mathbb{R}_+$. We denote with $\phi_{i,j}(x_j)$ the share of project value $v_j(x_j)$ assigned to player i at profile of efforts x_j. We will consider sharing rules where the value of a project is fully distributed to players, i.e. $\sum_{i \in N} \phi_{i,j}(x_j) = 1$, for every $j \in M$ and any profile of efforts $x \in X$. Therefore, the payoff to player i for efforts x is given by

$$u_i(x) = \sum_{j \in M_i} u_{i,j}(x_j) = \sum_{j \in M_i} \phi_{i,j}(x_j) v_j(x_j). \tag{1}$$

We restrict the set of sharing rules to *local sharing rules* where the value of a project is distributed across the set of players that participate in this project. Hence, if the set of players that participate to project j is N_j, then the sharing rule is restricted to distribute the value of the project across players in N_j.

For general value functions a major role in defining good redistribution mechanisms will be the *local marginal contribution* of a player to the value at the given effort levels:

$$\partial_i v_j(x) = v_j(x) - v_j(0, x_{-i}). \tag{2}$$

We will derive some of our results for special classes of project value functions. In particular, we will consider a subclass of concave utility functions that satisfy a decreasing marginal property, which we define as follows.

Definition 3 (diminishing marginal property) *A function f is said to satisfy the diminishing marginal property if $f(y + z) - f(y) \geq f(x + z) - f(x)$, for every x, y, z such that $x \geq y$. Where $x \geq y$ denotes a coordinate-wise comparison.*

Several natural classes of functions satisfy the above property. For example, if $v_j(x) = f(g(x))$, where f is a concave function and g an additive one. This class of functions could accommodate the case when each player has project specific abilities that are modeled as coefficients in his effort. Then the value of a project is a concave

[1] In Theorem 12 we require that the constraint set is of the form of a simple budget $\sum_{j \in M_i} x_{i,j} \leq B_i$ and in Theorem 8 we require that it is some convex set.

function of the weighted sum of efforts: $v_j(x) = f(\sum_i w_{i,j} x_{i,j})$. Another, case is when $v_j(x) = \sum_i f_i(x_i)$ and f_i are concave functions. If the function f is continuous and twice differentiable the diminishing marginal property coincides with the property that the cross partial derivative for any two inputs must be non-positive: $\partial f / \partial x_i \partial x_j \leq 0$. The diminishing marginal property and it's counterpart of increasing differences has appeared in several works in economics mainly under the scope of comparative statics and in studying structural properties of games (e.g. supermodular games).

Production Costs In the last part of the paper we consider the case where the players apart from their feasibility constraints on their investment levels they also incur a cost that is a function of the feasible investment vector they chose. We assume that the utility of a player is quasi-linear in allocation and costs:

$$u_i(x) = \sum_{j \in M_i} \phi_{i,j}(x_j) v_j(x_j) - c_i(x_i). \tag{3}$$

Solution Concepts We will consider equilibria of utility sharing games for several solution concepts. The most classical solution that we consider is the *pure-strategy Nash equilibrium*, defined as a strategy profile $x \in \mathcal{X}$ such that for every player $i \in N$,

$$u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i}), \text{ for every } x_i' \in \mathcal{X}_i.$$

The unilateral stability of the Nash Equilibrium can be quite restrictive in several settings. Specifically, in settings where cooperation and communication among the players is natural, it is reasonable to also consider strategy profiles that are resilient to coalitional deviations. The *strong Nash equilibrium* [5] proposed by Aumann is one such concept. A strategy profile x is a strong Nash equilibrium (SNE) if and only if there is no coalitional deviation such that *each* player in the coalition receives strictly larger utility after the deviation. Strong Nash equilibria are a more reasonable outcome in cooperative settings but they don't always exist. However, whenever they exist they are the most natural prediction and therefore the study of their efficiency properties is of great importance.

4 Utility Sharing Games and the Vickrey Condition

In this section, we show that the monotonicity and the Vickrey condition are sufficient to give very high efficiency at any strong Nash equilibrium of a utility sharing game, if one exists.

Theorem 1 (SNE in Monotone Games) *Any strong Nash equilibrium of a monotone utility sharing game that satisfies the Vickrey condition achieves at least* $1/2$ *of the optimal social welfare. If the game satisfies the* k*-approximate Vickrey condition, then any strong Nash equilibrium achieves at least* $1/(k+1)$ *fraction of the optimal social welfare. If the game is strongly monotone and satisfies the* k*-approximate Vickrey condition then any strong Nash equilibrium achieves at least* $1/k$ *fraction of the optimal social welfare.*

In an orthogonal direction, one could ask what efficiency guarantees could be given for utility sharing games that don't satisfy any monotonicity property. Without any additional assumption it would be hard to imagine that any general result could be given. However, we are able to give results for non-monotone games in the case when they have a potential function and the potential is closely related with social welfare. A utility sharing game is a potential game if it admits a potential function $\Phi : X \to \mathbb{R}$ such that: $u_i(x_i', x_{-i}) - u_i(x) = \Phi(x_i', x_{-i}) - \Phi(x)$, for every $x, x' \in X$.

Theorem 2 (SNE in Potential Games) *If a utility sharing game is a potential game with non-negative utilities such that* $\alpha SW(x) \leq \Phi(x) \leq \beta SW(x)$*, for every* $x \in X$ *and* $\alpha, \beta > 0$*, then it has strong price of anarchy at most* $\frac{\beta+1}{\alpha}$.

The above theorem is reminiscent of the potential method [33] for bounding the inefficiency of the *best Nash equilibrium* of a potential game. Specifically, using the potential method we would get that for games that satisfy the premises of Theorem 2 the best Nash Equilibrium achieves a β/α fraction of the optimal efficiency (the price of stability is at most β/α). Hence, in games where the potential method produces tight results, we see that there is a very interesting connection between the quality of the strong Nash equilibrium and that of the best Nash equilibrium.

There is a strong connection of the above results and the literature on *monotone valid utility games* [43]. A monotone valid utility game is a utility maximization game that apart from the monotonicity and the Vickrey condition also satisfies that the social welfare function is submodular (if we view social welfare as a function on sets of player strategies). Vetta [43] proved that any *Nash equilibrium* of a monotone valid utility game achieves at least half the optimal social welfare (or a $k+1$ fraction if Vickrey is replaced by the approximate version). Later, this result was extended to coarse correlated equilibria by Roughgarden [35].

Theorem 1 complements Vetta's result by stating that if we remove the assumption of submodularity then we can recover the good efficiency by moving to the strong Nash equilibrium solution concept. In addition, we show a tighter efficiency guarantee for *Nash equilibria* if we replace the social welfare monotonicity assumption of Vetta with strong monotonicity. We also adapt Vetta's result to games where the social welfare function is defined on a vector space, where the submodularity assumption is replaced by the *diminishing marginal property*.

Theorem 3 (NE in Monotone Games) *Consider a monotone utility sharing game that satisfies the k-approximate Vickrey condition. If the strategy space of each player is a subset of some vector space and the social welfare function satisfies the diminishing marginal property then any Nash equilibrium achieves at least a $k+1$ fraction of the optimal social welfare. If the game is strongly monotone then any Nash equilibrium achieves at least a k fraction of the optimal social welfare.*

In the appendix we also give a novel efficiency guarantee for *Nash equilibria* of potential games where the potential is a monotone submodular function. We show that in congestion games where resources are associated with decreasing and positive utility functions of the congestion, then the potential at any Nash Equilibrium is at least half of the maximum potential and if the potential is at most β the social welfare then the social welfare at any Nash equilibrium is at least $1/(\beta+1)$ of the maximum social welfare.

In the sections that follow we will use the above general theorems as quintessential tools for the analysis of the efficiency of redistribution mechanisms in our context.

5 Common Value Sharing Games

In this section we focus on a subclass of utility sharing games where the utility of the players is of the form of a sum of shares of local common values produced at different activities that they participate in as defined in Equation (1). We will try to understand what are the properties of the value functions $v_j(\cdot)$ and of the sharing scheme $\phi_{i,j}(\cdot)$ such that the induced game satisfies the monotonicity, Vickrey and diminishing marginal condition and thereby would lead to efficient outcomes either at Nash or strong Nash equilibria based on our general theorems in the previous section.

Observe that in such settings, since we assumed the common value is completely shared among the participants of an activity, we get that the social welfare is equal to the total value produced:

$$SW(x) = \sum_{i\in N} u_i(x) = \sum_{j\in M} v_j(x_j) = V(x). \tag{4}$$

Since the utility of a player is linearly separable across activities we can easily conclude that the game will satisfy the Vickrey and monotonicity condition if it satisfies those conditions locally at each venture. The local monotonicity condition would simply imply that the value functions $v_j(x_j)$ are monotone in every input. The local

k-approximate Vickrey condition means that

$$\phi_{i,j}(x_j)v_j(x_j) \geq \tfrac{1}{k}\partial_i v_j(x_j). \tag{5}$$

Adding over all players we get that for a game to satisfy the k-approximate Vickrey condition locally, the value functions must satisfy:

$$\mathcal{S}_{v_j} = \sup_{x_j \text{ feasible}} \frac{\sum_{i\in N_j} \partial_i v_j(x_j)}{v_j(x_j)} \leq k. \tag{6}$$

We refer to the above quantity as the *degree of substitutability of efforts*. If we assume that our value functions come from a class of functions $\mathcal{C}$ then we will refer to the degree of substitutability of efforts in this class as $\mathcal{S}_\mathcal{C} = \sup_{v_j\in\mathcal{C}} \mathcal{S}_{v_j}$.

Observe that $\mathcal{S}_{v_j} \in [0, |N_j|]$. If $\mathcal{S}_{v_j}$ is close to 0 then this implies that the function is almost constant. If the value functions satisfy the diminishing marginal property then $\mathcal{S}_{v_j} \leq 1$. If $\mathcal{S}_{v_j} = |N_j|$ then every players effort is quintessential for producing any value, i.e. $v_j(0, x_j^{-i}) = 0$.

In fact there always exist a mechanism that will induce a game that satisfies the $\mathcal{S}_{v_j}$-approximate Vickrey condition. Namely the *marginal contribution mechanism*, under which the share of a player is proportional to his marginal contribution:

$$\phi_{i,j}(x_j)v_j(x_j) = \frac{\partial_i v_j(x_j)}{\sum_l \partial_l v_j(x_j)} v_j(x_j) \geq \frac{1}{\mathcal{S}_{v_j}}\partial_i v_j(x_j). \tag{7}$$

In addition, if we allow for $\mathcal{S}_{v_j} \geq k$ then for any local sharing scheme there exists an instance such that the social welfare achieved at the unique strong Nash equilibrium is only a k fraction of the optimal social welfare. Therefore, our analysis above is tight, showing that the degree of substitutability is exactly the quantity that captures what worst-case efficiency guarantees we can achieve by using local sharing rules.

Example 4 *There are n players each with an effort budget of 1. They all participate in a shared project C that has the following value function $v_C(x_C) = n$ if $x_C = (1, \ldots, 1)$ and 0 otherwise. Suppose that all the players put their effort on C. The mechanism $\mathcal{M}$ must share the surplus among the participants. Hence, no matter what the mechanism is, there will be at least on player $i_\mathcal{M}$ that receives a share ≤ 1. Hence, in the instance $I_\mathcal{M}$, where only player $i_\mathcal{M}$ also has an outside private project P on which only he participates and with value function $v_P(1) = 1 + \epsilon$ and 0 otherwise, then the unique strong Nash equilibrium is for player $i_\mathcal{M}$ to put all his effort on P. This equilibrium will then have social welfare $1 + \epsilon$, while the optimal would be all players putting their effort on C leading to social welfare n.*

Corollary 5 *If the class of functions is defined by $\mathcal{C} = \{v_j : \mathcal{S}_{v_j} \leq k\}$. then no local sharing rule can achieve an efficiency guarantee of better than $\lfloor k \rfloor$ fraction of the optimal at every strong Nash equilibrium and for every instance in this class.*

Corollary 6 *If we restrict to a class of value functions $\mathcal{C}$ with $\mathcal{S}_\mathcal{C} \leq k$ then there exists a local sharing rule (e.g. marginal contribution rule) that always achieves a $k + 1$ fraction of the optimal social welfare at any strong Nash Equilibrium. If the value functions satisfy the diminishing marginal property then the latter efficiency is holds for the bigger set of all Nash equilibria and Coarse Correlated Equilibria (no-regret learning).*

In the following sections we examine natural mechanisms other than the marginal contribution sharing rule, that satisfy the Vickrey condition for subclasses of value functions.

Diminishing Marginal Value Functions Many mechanisms proposed in the axiomatic surplus sharing literature satisfy the exact Vickrey condition in the case when value functions satisfy the diminishing marginal property. An important and broad class of mechanisms are the fixed-path generated methods and convex combinations of them (see [27] for a complete analysis and Appendix for a brief description and the proof of why they satisfy the Vickrey condition). One of the most commonly used convex combination of fixed path routing methods is the Shapley-Shubik method [40], which corresponds to taking the average over all fixed path methods where players effort units are not mixed (i.e. ordering of player's arrivals), also called incremental paths. Another interesting property of the Shapley-Shubik sharing method is that when the value functions are also concave, then the utility of a player from a project as a function of his effort is a sum of concave functions and, therefore, also concave. Hence, by [34] we also get that the game induced by the Shapley-Shubik mechanism always possesses a pure Nash equilibrium.

Concave Functions of Total Effort Another interesting class of common value sharing games is when the value of a project is an increasing concave function of the total effort invested into it: $v_j(x_j) = \tilde{v}_j(X_j)$, where $X_j = \sum_{i \in N} x_{i,j}$ and $\tilde{v}_j : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing concave function. For such a class of functions a sharing rule is the *proportional sharing rule*:

$$\phi_{i,j}(x_j) v_j(x_j) = \frac{x_{i,j}}{X_j} v_j(x_j). \tag{8}$$

This sharing rule was considered in various economic systems, including rent-seeking contests [42] and resource allocation games [23]. It is noteworthy that our framework of utility sharing games under proportional sharing rule accommodates traditional Cournot oligopoly [8, 13] as a special case by interpreting the function $\tilde{v}_j(x)/x$ as the inverse demand function.

Here we show that for the above class of functions the proportional sharing rule satisfies the exact Vickrey condition and thereby every Nash equilibrium of the game that it induces achieves at least half of the optimal social welfare.

Lemma 7 *The proportional sharing rule satisfies the Vickrey condition when the value functions are increasing and concave in the total effort and $\tilde{v}_j(0) = 0$.*

It is noteworthy that total effort games have a strong connection with splittable selfish routing games: the resulting game could be viewed as a utility version of a splittable selfish routing game where players choose only singleton strategies and the utility function of each resource is $\bar{v}_j(x) = v_j(x)/x$. We note that the above result remains to hold even when players choose sets and not singleton strategies. When $v_j(x)$ is increasing and concave, then $\bar{v}_j(x)$ is decreasing and quasi-concave. Thus, our analysis implies that the price of anarchy of the utility version of splittable selfish routing is at most 2, when the project value functions $v_j(x_j)$ are decreasing and $xv_j(x)$ is increasing concave.

When the strategy space of each player is a convex subset of $\mathbb{R}^{|M_i|}$ (e.g. weighted budget constraint) and the value functions are continuous and differentiable then our game falls into the games captured by the local smoothness framework of Roughgarden and Schoppmann [37]. This framework enables us to use the latter structure of the strategy spaces and the value functions to get tighter bounds. Since the local smoothness framework was initially defined for cost minimization games, in Appendix we give a brief description of how it adapts to utility maximization games. Using the local smoothness framework we are able to produce much tighter results for a natural class of value functions.

Theorem 8 *Suppose that the project value functions are of the form $v_j(x_j) = q_j X_j^{\alpha}$, for $0 \leq \alpha \leq 1$ and $q_j > 0$. Then, the proportional sharing rule has the price of anarchy at most $\frac{2-\alpha}{2^{1-\alpha}}$. Furthermore, the worst-case price of anarchy is $1/[\log(2) 2^{\frac{1}{\log(2)}-1}] \approx 1.0615$.*

It is interesting to notice that the above bound coincides with the optimality result for the case of linear functions $\alpha = 1$. The other interesting special case is that as $\alpha \to 0$ then the price of anarchy also converges to 1. In other words if value functions are almost constant, then any pure-strategy Nash equilibrium is almost optimal.

For another natural class of project value functions, namely the functions of the form $v_j(x) = q_j(1 - e^{-r_j x})$, for $q_j, r_j > 0$, we show that the best price of anarchy bound that is provable using the local smoothness framework is 2 (see Appendix), thus coinciding with the more general result in Theorem 7. In fact the bound of 2 is tight for this class of functions as is shown by the following example.

Example 9 *Consider the following instance: there are n players and n projects. Every player participates in every project. Each player has a budget of effort of 1 that he can split arbitrarily among his projects. Project 1 has value function $v_1(x) = 1 - e^{-\alpha x}$. The rest of the $n-1$ projects have value function $q(1 - e^{-\beta x})$. We still assume that value is shared proportional to the effort. We will define α, β and q in such a way that the unique equilibrium will be for all players to invest their whole budget on project 1, while the optimal will be for each of the players' efforts to be spread out among all the projects. We show that if we let $\alpha \to \infty$, $q = \frac{n-1}{\beta n^2}$ and $\beta \to 0$, then the unique equilibrium is for all players to put their whole budget on project 1. The optimal on the other hand is for players efforts to be spread out among all the projects. A good approximation to the optimal is for each player to pick a different project and devote his whole effort on it. The ratio of the welfare of the above solution over the equilibrium welfare in the limit of the above values will be: $1 + (1 - 1/n)^2$. Hence, as the number of players grows the latter ratio converges from below to our upper bound of 2. A more detailed exposition of the example is given in Appendix.*

Intuitively what causes inefficiency in the above example is that players prefer to congest a low value project with a very high rate of success (i.e. produces very high value for a very small effort) rather than trying their own luck alone on a hard project that would yield very high value but would require a lot of effort to produce it.

Total Weighted Effort and Incomplete Information Similar to the previous section, we can prove that if value functions are increasing concave functions of weighted total effort

$$v_j(x_j) = \tilde{v}_j \left(\sum_{i \in N} w_{i,j} x_{i,j} \right),$$

then sharing proportional to the weighted effort, $w_{i,j} x_{i,j}$, also satisfies the Vickrey condition.

The weights $w_{i,j}$ in the context of a collaborative environment correspond to an ability factor of a player i for a project j. For instance, this could be how knowledgeable a player is on a topic in Q&A online service. In such settings, if we interpret the effort $x_{i,j}$ as units of time spend on a response and $w_{i,j}$ as an ability factor, then $q_{i,j} = w_{i,j} x_{i,j}$ corresponds to a quality factor associated with the response of player i. Hence, the proportional to the weighted effort mechanisms, corresponds to splitting attention proportional to the quality of the response. Such a mechanism has been studied under a different perspective in recent work on Q&A web services [14, 15].

In such online environments it is interesting to try to extend our results even when players don't have complete knowledge of the abilities of the participants. We show here that in fact such an extension holds. Specifically, using the recent smoothness framework introduced by Roughgarden [36] and Syrgkanis [41] we manage to show that the game defined by the proportional to the weighted effort sharing rule satisfies the stronger smoothness property (universal smoothness in [41] required for an efficiency guarantee to carry over to settings where players have incomplete information about parameters of the game.

Lemma 10 *If weights $w_{i,j}$ are drawn from commonly known distributions, then any Bayes-Nash equilibrium of the game defined by the proportional to the weighted effort sharing rule achieves at least $1/2$ of the expected optimal social welfare.*

Projects that Require Specific Skills In this section we examine the case where players can be categorized in groups $\mathcal{G}$ where each group has a specific skill. We denote with $\mathcal{G}_j \subseteq \mathcal{G}$ the subset of skills that contribute to a specific project j.

We assume that the value function of a project satisfies the decreasing marginal contribution property skill-wise. That is, keeping the effort levels x_{-g} of players outside of a given skill group $g \in \mathcal{G}_j$ then the group restricted value functions $v_j(x_g, x_{-g})$ satisfies the decreasing marginal contribution property. For instance, the value could be of the form of a product of skill-specific functions: $v_j(x_j) = \prod_{g \in G_j} v_j^g(x_g)$, such that each $v_j^g(x_g)$ is a function that satisfies the decreasing marginal property. This way we could capture settings where a positive effort from each skill is needed to produce any value. The class of functions that satisfy the decreasing marginal property are a special case of the latter general class where $|\mathcal{G}_j| = 1$.

We show that the degree of substitutability of such a value function is at most $|\mathcal{G}_j|$. Hence, by Theorem 6 the marginal contribution sharing rule will achieve a $\max_j |\mathcal{G}_j| + 1$ fraction of the optimal social welfare at any strong Nash equilibrium.

Lemma 11 *If the players participating in a project can be categorized in skill groups such that the value function $v_j(x_j)$ satisfies the decreasing marginal contribution property within each group, then the degree of substitutability of the value function $\mathcal{S}_{v_j}$ is at most $|\mathcal{G}_j|$.*

Another sharing rule that yields the same guarantee as the marginal contribution rule, is the following alteration of the Shapley Value: take the average of the marginal contribution of a player over all permutations that put the players of the same skill group consecutively. Observe that the Shapley value takes the average over all possible permutations while the above method restricts the feasible permutations.

Local Public Goods and Equal Sharing An interesting special case of our sharing games is when the value produced is a non-divisible local public good that can be used by all the participating players. Assuming that all players will make equal use of the good and that they value it the same, this setting corresponds to our sharing games when the equal sharing rule is applied, i.e. everyone gets an equal share of the publicly produced value:

$$\phi_{i,j}(x_j) v_j(x_j) = \frac{1}{|N_j|} v_j(x_j) \tag{9}$$

Therefore studying the properties of the equal sharing rule is interesting in it's own right.

Additionally, in a scientific credit setting the equal sharing rule corresponds to sorting the authors of a paper in alphabetical order, rather than according to their contribution. The latter is a commonly used practice in many communities and it is interesting to compare its efficiency guarantee's with those of other mechanisms studied so far.

First of all we observe that when the value functions are monotone the equal sharing rule induces a strongly monotone game that satisfies the $\max_j |N_j|$-approximate Vickrey condition. Therefore by Theorems 1 and 3 we get that it achieves an $\max_j |N_j|$ fraction of the optimal social welfare at any Nash equilibrium when the values satisfy the diminishing marginal property and at any strong Nash equilibrium for arbitrary value functions.

In addition, it is easy to see that the equal sharing rule induces a potential game with potential $\Phi(x) = \sum_j \frac{v_j(x_j)}{|N_j|}$. The latter allows us to claim efficiency results even when *value functions are not monotone* by using Theorem 2. Specifically, we observe that the potential satisfies that

$$\frac{1}{\max_j |N_j|} SW(x) \leq \Phi(x) \leq SW(x)$$

and, therefore, Theorem 2 implies that any strong Nash equilibrium achieves at least a $2 \max_j |N_j|$ fraction of the optimal social welfare.

5.1 Relaxing Coalitional Stability

In the context of contribution games Anshellevich et al [3] defined the notion of a setwise Nash equilibrium in which only players that share a project can coalitionally deviate.

We study here to what extend our previous results hold for this relaxed notion of stability. First observe that for the case of diminishing marginal value functions our results don't need any coalitional stability since they hold even for the set of Nash equilibria. If the value functions don't satisfy the latter property then we show that setwise stability is enough for our main results to hold when the value functions are convex and the feasibility constraints of the players have the form of simple budget constraints $\sum_{j\in M_i} x_{i,j} \leq B_i$.

Theorem 12 *If the project value functions are convex, the feasibility constraints have the form of a budget constraint and the sharing rule satisfies the k-approximate Vickrey condition, then any setwise Nash equilibrium achieves a $k+1$ fraction of the optimal social welfare. If the sharing rule induces a strongly monotone game, then it achieves a k fraction of the optimal social welfare.*

Anshellevich et al. [3] showed that the equal sharing rule achieves at every setwise Nash equilibrium at least at $\max_j |N_j|$-fraction of the optimal social welfare. They mainly focused on social network settings where projects are bilateral relationships and in which case the latter bound implies that every setwise Nash equilibrium achieves half of the optimal social welfare. This bound is a special case of the above theorem, since the equal sharing rule satisfies the $\max_j |N_j|$-approximate Vickrey condition and induces a strongly monotone utility game.

6 Common Value Sharing Games with Production Costs

In this section we study the case when apart from the feasibility constraints each player incurs an extra cost that is a function of the effort vector he exerts. We analyze efficiency with respect to the value produced and with respect to the optimal social welfare (including players' effort costs). We exhibit an interesting threshold phenomenon in the inefficiency of the setting: if marginal cost is constant (i.e. linear effort cost) then the inefficiency grows almost linearly with the number of participants. However, when effort cost is strictly convex, then the inefficiency can be at most a constant independent of the number of participants. This extends the previously studied case of players endowed with effort budgets to a much general case where players incur production costs from a broad set of functions that satisfy the increasing returns property.

In this setting a player's utility is:

$$u_i(x) = \sum_{j\in M_i} \phi_{i,j}(x_j) v_j(x_j) - c_i(x_i). \tag{10}$$

Observe that unlike in the previous section, the social welfare is not equal to the value produced, but instead

$$SW(x) = \sum_{i\in N} u_i(x) = \sum_{j\in M} v_j(x_j) - \sum_{i\in N} c_i(x_i) = V(x) - C(x).$$

We will refer to $V(x)$ as the production of an outcome x and to $C(x)$ as the production cost.

We show that the production plus the social welfare at equilibrium is at least the value of the optimal social welfare.

Theorem 13 *Consider a common value sharing game with production costs and a sharing scheme that satisfies locally the Vickrey condition. Let x be a strong Nash equilibrium and x^* be the outcome that maximizes social welfare. Then $SW(x) + V(x) \geq SW(x^*)$. For diminishing marginal value functions, the latter holds even if x is a pure Nash equilibrium.*

Later in this section, we will use the last theorem to derive efficiency bounds for both production and social welfare in the case of total effort games and as a function of the elasticity of the value and cost functions.

Linear Effort Costs Lead to Inefficiency. As a first step we show that costs can significantly affect the efficiency of equilibria. Specifically, if we allow the costs to be linear in the total effort of a player then the efficiency at a Nash equilibrium can be as high as $\Theta(n)$ even when the value functions are concave functions of total effort and the value is shared proportional to the effort. Remember that for this exact setting, if instead of linear costs the players had a budget of effort, our results in the previous sections gave an inefficiency of at most 2. In Appendix, we provide a simple example with a single project whose value is a function of total effort and where players incur linear costs, such that the inefficiency at the unique Nash equilibrium is $\Theta(n)$. The example is reminiscent of the Tragedy of the Commons [18].

The above result is interesting as it stands in sharp contrast to the model with budget constraints where the price of anarchy is upper bounded by a constant, independent of the number of players. Hence, there must be some phase transition as the convexity of the cost functions changes from linear effort costs to budget constraints. This is exactly the topic of the next section where we basically show that *linear effort costs are a corner case and as long as the cost functions are strictly convex then the inefficiency is some constant independent of the number of players*.

6.1 Efficiency and the Elasticity of Costs

In this section we consider the case where both the value and cost are functions of the total effort rather than of the vector of efforts. We will also assume that they are continuous and differentiable functions and that the players have no other feasibility constraints, i.e. $\mathcal{X}_i = \mathbb{R}_+^{|M_i|}$.

We will show that the more convex the cost functions are and the more concave the value functions are, the better the efficiency both with respect to production and to social welfare. The budget case that we studied can be seen as a limit of a family of convex functions and for this reason it leads to efficient outcomes.

We need to use some measure of convexity to quantify the inefficiency of our game. If cost functions are convex and assuming $c_i(0) = 0$ it follows that for any $x > 0 : xc_i'(x) \geq c(x)$. In addition, since value functions are concave and $v_j(0) = 0$ it follows that for any $x > 0 : xv_j'(x) \leq v_j(x)$. We will use a measure of convexity/concavity that will correspond to strict versions of the above inequalities. Our inefficiency bounds will be parameterized exactly by the slackness of the above two inequalities. In fact, the latter measure is the well-studied concept of elasticity in economics.

Definition 4 (Elasticity) *The x-elasticity of a function $f(x)$ is defined as:* $\epsilon_f(x) = \left|\frac{f'(x)x}{f(x)}\right|$.

An increasing convex function has an elasticity of at least 1 and a concave function has an elasticity of at most 1. We will quantify the inefficiency in our game as a function of how far from 1 the elasticities of the value and cost functions are. For instance if $c(x) = x^{1+a}$, then $c(x)$ has elasticity of $1 + a$. If $v(x) = x^{1-\beta}$ then $v(x)$ has elasticity of $1 - \beta$.

The key element of our proof is the inequality in Theorem 13. Specifically, if we manage to show that for the social welfare outcome x^*, $SW(x^*) \geq \rho V(x^*)$, then Theorem 13 will imply that $V(x) \geq \frac{\rho}{2}V(x^*)$ and thereby, there will be at most a constant inefficiency in the production level. In addition, if we manage to show that for any Nash Equilibrium x, $SW(x) \geq \rho V(x)$, then Theorem 13 will imply that $SW(x) \geq \frac{1}{\rho+1}SW(x^*)$ and thereby there will be at most a constant inefficiency in social welfare as well. Strict convexity and concavity of cost and value functions, respectively, will enable us to prove such properties.

Theorem 14 *If the elasticity of the cost functions is at least $1 + \mu$ and the elasticity of value functions is at most $1 - \lambda$, then the social welfare at any pure Nash equilibrium is at least $\frac{\mu+\lambda/n}{1+2\mu+\lambda/n}$ of the optimal. In addition production at equilibrium is at least $\frac{1}{2}\frac{\lambda+\mu}{1+\mu}$ of the production at the social welfare maximizing outcome.*

Our result above has implications about the efficiency of Cournot equilibria as a function of the elasticity of the inverse demand curve and of the production cost curve of the firms. Specifically, is $p(x)$ is the inverse demand curve, then $v(x) = xp(x)$. Then, our assumption that $v(x)$ has elasticity at most $1 - \lambda$ translates to the property that the inverse demand curve must have elasticity at least λ. The latter implies that the *price elasticity of demand* should be at most $\frac{1}{\lambda}$, and therefore demand should not be very elastic.

Observe that from our theorem we get that as long as $\mu > 0$ the social welfare inefficiency at any equilibrium, even if $\lambda = 0$ is at most $\frac{\mu}{1+2\mu}$, which is a constant independent of the number of players. Thereby, the $\Omega(n)$ inefficiency that we presented in the previous section for linear effort costs, was only a corner case of linear effort costs. The moment the effort costs become strictly convex, the inefficiency becomes a constant independent of the number of players.

In addition, the budget constraint case that we studied in previous sections can be seen as a limit of a family of convex functions that converge to a limit function of the form $c_i(x_i) = 0$ if $x_i < B_i$, and ∞ otherwise. Such limit function can be thought as a convex function with infinite elasticity. The reason is that at any equilibrium and at the optimal outcome players will be exhausting their budget. Therefore, at that point the cost function has infinite elasticity since a very small percentage change in the input effort will lead to an infinite percentage change in the output. Observe that if we take the limit as $\mu \to \infty$ in the theorem above then we get that the social welfare at equilibrium is at least half the optimal social welfare. This bound matches exactly the bound we gave for the case of budgets in previous sections.

7 Discussion and Conclusions

We examined how the relation between social and individual utility affects efficiency in a natural setting where collaborating agents choose their engagement level, focusing on online joint ventures such as Yahoo! Answers, Quora or Topcoder where the agents split their budget of effort across various projects. We examined the effect different *local* sharing rules have on the overall *global* efficiency achieved in equilibrium, and showed that by awarding each agent a significant fraction of her marginal contribution for each local project, we can achieve a high degree of *global* efficiency.

We used a game theoretic model where agents distribute their bounded resources across various projects, and where the value produced at each project is shared among its participants according to a pre-decided local sharing rule, and examined the strong Nash equilibrium and Nash Equilibrium of the game. We showed that when the wealth is redistributed so that each agent obtains at least his marginal contribution locally at each project, the incentives are well-aligned so that the global social welfare in equilibrium is at least half of the optimal social welfare. Our analysis was based on novel efficiency guarantees for general utility maximization games, of independent interest, while at the same time applying recent techniques from the game theoretic community to the context of collaborative environments.

We showed that the ability to award each player at least his marginal contribution depends on the degree of complementarity among the players; the more complementary the worse. If players can be categorized in skill groups and the efforts of people within a group are substitutable, then the number of different skill groups is a limit of the approximation achievable. For settings with no complementarities, such as when the value produced is a concave function of the weighted sum of player's efforts, where the weight represents a project specific ability, we showed that good efficiency globally is achieved by redistributing value locally proportionally to the weighted effort, which corresponds to the quality of the submission, and that this result extends to settings where abilities of the players are private and drawn from commonly known distributions.

When the players incur a cost for investing effort (instead of having a budget of effort/time) we showed that the rate of increase of the marginal cost per-unit of effort for each player (i.e. convexity of the cost function) plays an important role in efficiency. We portrayed a surprising threshold phenomenon, where if the marginal cost is constant (i.e. linear cost functions) then the inefficiency can grow with the number of participants, while the moment it becomes strictly increasing the inefficiency is a constant independent of the number of participants.

Our results have interesting implications in many domains such as online collaborative tasks, question answering services and crowdsourcing markets. They indicate that in order to to improve the efficiency of such services, one has to use a compensation scheme where agents are rewarded based on their marginal contribution to the achieved welfare. In cases where such a scheme is not used, our results indicate that the strategic behavior of the selfish agents results in inherent inefficiencies. However, when a marginal contribution based scheme is used by such services, a very high efficiency can be achieved despite the self-interested nature of the agents. What's more important our results portray that in settings with small degree of complementarities among participants, local sharing rules are sufficient for approximate global efficiency and therefore a designer should not worry about the effect of the existence of other projects/topics when splitting value and should simply share value based on local information at each project.

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A Proofs for Section 4

Theorem 15 (Strong Nash Equilibria in Monotone Games) *Any strong Nash equilibrium of a monotone utility sharing game (if one exists) that satisfies the Vickrey condition achieves at least $1/2$ of the optimal social welfare. Furthermore, if the game satisfies k-approximate Vickrey condition, then any strong Nash equilibrium (if one exists) achieves a $k+1$ fraction of the optimal social welfare. If the game is strongly monotone and satisfies the k-approximate Vickrey condition then any strong Nash equilibrium achieves a k fraction of the optimal social welfare.*

Proof. Monotone Games. We focus on the case where the game satisfies the exact Vickrey condition. The approximate version can be proved with similar reasoning. Consider a strategy profile x that is a strong Nash equilibrium. Since it is a strong Nash equilibrium for any coalitional deviation from x, there exists some player who is getting at least as much utility at x as he is getting in the deviating strategy profile.

Let x^* be the optimal strategy profile. If all players deviate to s^* then there is a player i who is blocking the deviation, i.e. $u_i(x) \geq u_i(x^*)$. Without loss of generality, let this person be player 1. Similarly, if players $\{2, \ldots, n\}$ deviate to playing their strategy in x^* then there exists some player, obviously different than 1 who is blocking the deviation. Without loss of generality, by reordering we can assume that this player is 2. Using similar reasoning we can reorder the players such that if players $\{k, \ldots, n\}$ deviate to their strategy in the optimal strategy profile x^* then player k is the one blocking the deviation. That is player k's utility at the Strong Nash Equilibrium is at least his utility in the deviating strategy profile. Let N_k be the set of player $\{k, \ldots, n\}$. Let $(x^*_{N_k}, x_{-N_k})$ be the strategy profile where players $\{k, \ldots, n\}$ play their optimal strategy and players $\{1, \ldots, k-1\}$ play x. By our assumption on the ordering of the players we thus have that for each player k: $u_k(x) \geq u_k(x^*_{N_k}, x_{-N_k})$.

The Vickrey condition states that each players utility at any strategy profile is at least his marginal contribution. Therefore, we obtain

$$u_k(x^*_{N_k}, x_{-N_k}) \geq SW(x^*_{N_k}, x_{-N_k}) - SW(x^{out}_k, x^*_{N_{k+1}}, x_{-N_k}) \tag{11}$$

In addition, by the monotonicity assumption the social welfare can only increase when a player enters the game with any strategy. Therefore,

$$SW(x^{out}_k, x^*_{N_{k+1}}, x_{-N_k}) \leq SW(x_k, x^*_{N_{k+1}}, x_{-N_k}) = SW(x^*_{N_{k+1}}, x_{-N_{k+1}}) \tag{12}$$

Combining all the above, we obtain a telescoping sum which gives the desired result

$$\begin{aligned}
SW(x) = \sum_{k=1}^{n} u_k(x) &\geq \sum_{k=1}^{n} u_k(x^*_{N_k}, x_{-N_k}) \\
&\geq \sum_{k=1}^{n} SW(x^*_{N_k}, x_{-N_k}) - SW(x^{out}_k, x^*_{N_{k+1}}, x_{-N_k}) \\
&\geq \sum_{k=1}^{n} SW(x^*_{N_k}, x_{-N_k}) - SW(x^*_{N_{k+1}}, x_{-N_{k+1}}) \\
&= SW(x^*) - SW(x)
\end{aligned}$$

Thus, $SW(x) \geq \frac{1}{2} SW(x^*)$. The k-approximate Vickrey version follows along similar lines.

Strongly Monotone Games. Similar to the previous part we can reorder the players such that: $u_i(x) \geq u_i(x^*_{N_i}, x_{-N_i})$. By strong monotonicity we know that if any player goes out of the game then the utility of each other player can only decrease. Therefore, $u_i(x^*_{N_i}, x_{-N_i}) \geq u_i(x^*_{N_i}, x^{out}_{-N_i})$.

Then the k-approximate Vickrey condition implies that:

$$u_i(x^*_{N_i}, x^{out}_{-N_i}) \geq \frac{1}{k} \left(SW(x^*_{N_i}, x^{out}_{-N_i}) - SW(x^*_{N_{i+1}}, x^{out}_{-N_{i+1}}) \right)$$

By combining the above we again obtain a telescoping sum:

$$SW(x) = \sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(x^*_{N_i}, x^{out}_{-N_i}) \geq \frac{1}{k}\sum_{i=1}^{n} SW(x^*_{N_i}, x^{out}_{-N_i}) - SW(x^*_{N_{i+1}}, x^{out}_{-N_{i+1}})$$
$$= \frac{1}{k}\left(SW(x^*) - SW(x^{out})\right) = \frac{1}{k}SW(x^*)$$

■

Theorem 16 (Strong Nash Equilibria in Potential Games) *If a utility sharing game is a potential game with non-negative utilities such that $\alpha SW(x) \leq \Phi(x) \leq \beta SW(x)$, for every $x \in X$ and $\alpha, \beta > 0$, then it has strong price of anarchy at most $\frac{\beta+1}{\alpha}$.*

Proof. Let x be a strong Nash equilibrium and x^* the optimal strategy profile. Consider the players in the same order as in Theorem 1 such that for all i: $u_i(x) \geq u_i(x^*_{N_i}, x_{-N_i})$. The strategy profile $(x^*_{N_i}, x_{-N_i})$ can be thought of as resulting from $(x^*_{N_{i+1}}, x_{-N_{i+1}})$, through a unilateral deviation of player i from x_i to x^*_i. Hence, by the definition of the potential function and the fact that utilities are non-negative, we have

$$\begin{aligned} u_i(x^*_{N_i}, x_{-N_i}) &= \Phi(x^*_{N_i}, x_{-N_i}) - \Phi(x^*_{N_{i+1}}, x_{-N_{i+1}}) + u_i(x^*_{N_{i+1}}, x_{-N_{i+1}}) \\ &\geq \Phi(x^*_{N_i}, x_{-N_i}) - \Phi(x^*_{N_{i+1}}, x_{-N_{i+1}}) \end{aligned}$$

Combining with our assumption on the relation between potential and social welfare we obtain the desired result:

$$SW(x) \geq \sum_i u_i(x^*_{N_i}, x_{-N_i}) \geq \sum_i \Phi(x^*_{N_i}, x_{-N_i}) - \Phi(x^*_{N_{i+1}}, x_{-N_{i+1}})$$
$$= \Phi(x^*) - \Phi(x) \geq \alpha SW(x^*) - \beta SW(x).$$

■

Theorem 17 (Nash Equilibria in Monotone Games) *Consider a monotone utility sharing game that satisfies the k-approximate Vickrey condition. If the strategy space of each player is a subset of some vector space and the social welfare function satisfies the diminishing marginal property then any Nash equilibrium achieves at least a $k+1$ fraction of the optimal social welfare. If the game is strongle monotone then any Nash equilibrium achieves at least a k fraction of the optimal social welfare.*

Proof. Monotone Games. Since monotone valid utility games where initially defined for games where the social welfare was a function on sets of strategies, we present here for completeness the whole proof adapted to the case when social welfare and utilities are defined on a vector space and the social welfare satisfies the decreasing marginal property.

By the decreasing marginal returns property we have:

$$\forall x, y, z \in R_+^{|N_j|} : SW(x+y) - SW(y) \geq SW(x+y+z) - SW(y+z)$$

The marginal contribution property of the mechanism implies:

$$\forall x : u_i(x) \geq SW(x) - SW(0, x_{-i})$$

Let x be a Nash Equilibrium and x^* the optimal outcome. Also let

$$m_i = (x^*_1 + x_1, \ldots, x^*_i + x_i, x_{i+1}, \ldots, x_n)$$

i.e. players up to i implement both their Nash and optimal strategies. Although this might not be a valid strategy for them it is not a problem as this vectors are used as an intermediate step and we never consider a deviation to such a strategy. Also let $t_i = (x_i^*, 0_{-i})$, $e_i = (0, x_{-i})$ and $r_i = (x_1^*, \ldots, x_{i-1}^*, x_i, 0, \ldots, 0)$. Then, we have

$$\begin{aligned}
SW(x) = \sum_{i \in N} u_i(x) \geq & \sum_{i \in N} u_i(x_i^*, x_{-i}) \\
\geq & \sum_{i \in N} SW(x_i^*, x_{-i}) - SW(0, x_{-i}) \\
= & \sum_{i \in N} SW(t_i + e_i) - SW(e_i) \\
\geq & \sum_{i \in N} SW(t_i + e_i + r_i) - SW(e_i + r_i) \\
= & \sum_{i \in N} SW(m_i) - SW(m_{i-1}) \\
= & \ SW(m_n) - \sum_i SW(m_0) \\
= & \ SW(x^* + x) - SW(x) \geq SW(x^*) - SW(x)
\end{aligned}$$

Thus $SW(x) \geq \frac{1}{2} SW(x^*)$. The k-approximate Vickrey version follows along similar lines.

Strongly Monotone Games. Let x be a Nash Equilibrium and x^* the optimal strategy profile. By the definition of Nash Equilibrium and by strong monotonicity we have:

$$\forall i : u_i(x) \geq u_i(x_i^*, x_{-i}) \geq u_i(x_i^*, x_{-i}^{out})$$

The diminishing marginal property of the social welfare means that the marginal increase in social welfare that a player makes is smaller when more players participate in the game. More formally, consider any player i and let $S \subseteq T$ be two sets of players not including i. Then for any strategy profile x:

$$SW(x_i, x_{-S}, x_S^{out}) - SW(x_i^{out}, x_{-S}, x_S^{out}) \geq SW(x_i, x_{-T}, x_T^{out}) - SW(x_i^{out}, x_{-T}, x_T^{out})$$

We let again N_i be the set of player $\{i, \ldots, n\}$. Combining submodularity with the approximate Vickrey condition we obtain:

$$\begin{aligned}
SW(x) = & \sum_i u_i(x) \geq \sum_i u_i(x_i^*, x_{-i}^{out}) \geq \frac{1}{k} \sum_i SW(x_i^*, x_{-i}^{out}) - SW(x_i^{out}, x_{-i}^{out}) \\
\geq & \frac{1}{k} \sum_i SW(x_i^*, x_{N_{i+1}}^*, x_{-N_i}^{out}) - SW(x_i^{out}, x_{N_{i+1}}^*, x_{-N_i}^{out}) \\
= & \frac{1}{k} \sum_i SW(x_{N_i}^*, x_{-N_i}^{out}) - SW(x_{N_{i+1}}^*, x_{-N_{i+1}}^{out}) \\
= & \frac{1}{k} (SW(x^*) - SW(x^{out})) = \frac{1}{k} SW(x^*)
\end{aligned}$$

■

A.1 Efficiency of Nash Equilibrium in Submodular Potential Games

In this section we provide some supplementary results on the efficiency of Nash Equilibria in potential games where the potential is a submodular function.

Such an assumption is satisfied for example for all congestion games where resources are associated with decreasing utility functions rather than cost functions. A utility congestion game consists of a set of players and a set of resources R. Each players strategy space consists of subsets of the resources. Each resource r is associated with a utility function π_r that depends only on the number of players n_r using the resource. The utility of each player is the sum of his utilities from each resource in his strategy. A utility congestion game is known to admit Rosenthal's potential: $\Phi(s) = \sum_r \sum_{k=1}^{n_r} \pi_r(k)$. Moreover, the social welfare ends up being $SW(s) = \sum_r n_r \pi_r(n_r)$. It is easy to see that if π_r are decreasing functions then $\Phi(s) \geq SW(s)$. Thus, by the submodularity of Φ we obtain the following theorem:

Theorem 18 *In any utility congestion game with non-negative decreasing utilities the potential at any Nash Equilibrium is at least half the maximum potential and if $\Phi(x) \leq \beta SW(x)$ then the PoA is at most $\beta + 1$.*

Proof. Since $\pi_r(k)$ are decreasing we have

$$\Phi(x) = \sum_{r\in R}\sum_{t=1}^{n_r(x)} \pi_r(t) \geq \sum_{r\in R}\sum_{t=1}^{n_r(x)} \pi_r(n_r(x)) = \sum_{r\in R} n_r(x)\pi_r(n_r(x)) = \sum_{i\in N} u_i(x)$$

Since $\pi_r(k) \geq 0$ we have $\forall x : u_i(x) \geq 0$ and therefore:

$$u_i(x) \geq \Phi(x) - \Phi(x_i', x_{-i}) + u_i(x_i', x_{-i}) \geq \Phi(x) - \Phi(x_i', x_{-i})$$

Moreover, since $\pi_r(k)$ are decreasing we also have that $\Phi(x)$ is in some sense a monotone submodular function as described below.

If we view the potential function as a function on multisets, where a player playing more than one strategies means adding to the resources the extra congestion that these strategies imply. Thus, we can think of congestion $n_r(\mathcal{X})$ as a set function that counts how many strategies $x \in \mathcal{X}$ use the resource r (double counting if a strategy is more than one times in $\mathcal{X}$), i.e. the multiplicity function of $\mathcal{X}$. Let S, T be two multisets of strategies. Let $S \uplus T$ be the multiset sum of S and T. Then:

$$\Phi(S \uplus T) = \sum_{r\in R}\sum_{t=1}^{n_r(S\uplus T)} \pi_r(t) \geq \sum_{r\in R}\sum_{t=1}^{n_r(S)} \pi_r(t) = \Phi(S)$$

Thus $\Phi(S)$ satisfies monotonicity. Moreover, let $S \subset T$ and let W be another multiset of strategies disjoint from S, T. By $S \subseteq T$ we have $\forall r \in R : n_r(S) \geq n_r(T)$. Also it holds that $\forall r : n_r(W \uplus S) = n_r(W) + n_r(S)$. Then using the decreasing property of $\pi_r(t)$:

$$\begin{aligned}\Phi(W \uplus S) - \Phi(S) =& \sum_{r\in R}\sum_{t=n_r(S)+1}^{n_r(W\uplus S)} \pi_r(t) = \sum_{r\in R}\sum_{t=n_r(S)+1}^{n_r(S)+n_r(W)} \pi_r(t)\\ \geq& \sum_{r\in R}\sum_{t=n_r(T)+1}^{n_r(T)+n_r(W)} \pi_r(t) = \Phi(W \uplus T) - \Phi(T)\end{aligned} \tag{13}$$

Thus $\Phi(s)$ satisfies submodularity. Thus we can apply similar analysis as in monotone valid utility games. At points we use same notation for strategy profiles and sets of strategies but it is clear from the context. For example, we use notation such as NE_{-i} to denote the multiset that consists of the multiset sum of the strategies of all players except i, when strategies are viewed as sets of resources. Similarly we use $\text{NE}_{<i}$ ($\text{NE}_{>i}$) to denote the above multiset but for the strategies of all the players less (greater) than i.

$$\begin{aligned}
SW(\text{NE}) \geq \sum_i u_i(\text{NE}) \geq \sum_i u_i(\text{OPT}_i, \text{NE}_{-i}) \geq \sum_i \Phi(\text{OPT}_i, \text{NE}_{-i}) - \Phi(OUT_i, \text{NE}_{-i})\\
=\sum_i \Phi(\text{OPT}_i \uplus \text{NE}_{-i}) - \Phi(\text{NE}_{-i})\\
\geq \sum_i \Phi(\text{OPT}_i \uplus \text{NE}_{-i} \uplus \text{NE}_i \uplus \text{OPT}_{<i}) - \Phi(\text{NE}_{-i} \uplus \text{NE}_i \uplus \text{OPT}_{<i})\\
=\sum_i \Phi(\text{OPT}_{\leq i} \uplus \text{NE}_{\leq i} \uplus \text{NE}_{>i}) - \Phi(\text{OPT}_{\leq i-1} \uplus \text{NE}_{\leq i-1} \uplus \text{NE}_{>i-1})\\
=\Phi(\text{OPT}_{\leq n} \uplus \text{NE}_{\leq n}) - \Phi(\text{NE}_{>0}) \geq \Phi(\text{OPT}) - \Phi(\text{NE})\\
\geq SW(\text{OPT}) - \beta SW(\text{OPT})
\end{aligned}$$

Also using similar analysis with the potential maximizer instead of OPT we obtain:

$$\Phi(\text{NE}) \geq SW(\text{NE}) \geq \Phi(\text{POT-MAX}) - \Phi(\text{NE})$$

■

B Proofs for Section 5

Diminishing Marginal Value Functions and Path-Generated Methods of Sharing. One can think of a fixed-path generated method of sharing the surplus as follows: given the vector of efforts x_j break apart each player's effort in small units of effort dx_j^i. Now, order all the units of effort of all the players in an arbitrary way possibly mixing the units of different players. Notice that as the unit division goes to 0 the above corresponds to an arbitrary path in $\mathbb{R}^{|N_j|}$ from 0 to x_j. Let T be the total number of effort units $\{dx_1^{i_1}, \ldots, dx_T^{i_T}\}$. Moreover, let $x_t = (\sum_{k\leq t: i_k = i} dx_k^{i_k})_{i\in N}$ be the point of $\mathbb{R}^N$ that corresponds to adding up all the effort units up to unit t, vector-wise. The share of each player is then $\phi_{ij}(x_j)v_j(x_j) = \sum_{k:i_k=i} v_j(x_t) - v_j(x_{t-1})$. Now it is easy to see that if $v_j(x_j)$ is a concave function then each player i gets at least $\partial_i v_j(x_j)$. This is because the latter quantity corresponds to the worst case path for player i in which all of his units are ordered last in the fixed path. If any of his units moves up in the ordering then it can only have higher marginal contribution at the value at the point it is accounted and hence this can only increase player i's share. Since, every fixed-path generated sharing method has the marginal contribution property then it is easy to see that every convex combination of fixed path routing methods also satisfies it.

Corollary 19 *Any convex combination fixed-path generated method of sharing the value of a project (e.g. Shapley-Shubik) satisfies the Vickrey condition, when value functions satisfy the decreasing marginal returns property.*

Lemma 20 *The proportional sharing rule satisfies the Vickrey condition when the value functions are increasing and concave in the total effort and $\tilde{v}_j(0) = 0$.*

Proof. Since $\tilde{v}_j(X_j)$ is concave and $\tilde{v}_j(0) = 0$ then for any $y \in [0, X_j] : v_j(y) \geq y\frac{\tilde{v}_j(X_j)}{X_j}$. By setting $y = X_j - x_{ij}$ we obtain:

$$\phi_{ij}(x_j)v_j(x_j) = x_{ij}\frac{\tilde{v}_j(X_j)}{X_j} \geq \tilde{v}_j(X_j) - \tilde{v}_j(X_j - x_{ij}) = \partial_i v_j(x_j)$$

Which implies the property. ■

Lemma 21 *If the players participating in a project can be categorized in skill groups such that the value function $v_j(x_j)$ satisfies the decreasing marginal contribution property within each group, then the degree of substitutability of the value function $\mathcal{S}_{v_j}$ is at most $|\mathcal{G}_j|$*

Proof. We need to show that $\sum_j \partial_j v_j(x_j) \leq |G_j| v_j(x_j)$. Observe that

$$\sum_{j \in M} \partial_j v_j(x_j) = \sum_{g \in G_j} \sum_{k \in g} \partial_k v_j(x_j)$$

Within a skill group the value function satisfies the diminishing marginal contribution property when effort levels of players outside the skill group are fixed. For a specific skill group g that contributes to project j consider an arbitrary ordering of the player $\{1, \ldots, n_g\}$. By the diminishing marginal contribution property: $\forall k \in g : \partial_k v_j(x_j) \leq \partial_k v_j(0, \ldots, 0, x_{k,j}, \ldots, x_{n_g,j}, x_j^{-g})$. Then summing over all players in the group we obtain: $\sum_{k \in g} \partial_k v_j(x_j) \leq v_j(x_j) - v_j(0, \ldots, 0, x_j^{-g}) \leq v_j(x_j)$. This then directly implies: $\sum_{g \in G_j} \sum_{k \in g} \partial_k v_j(x_j) \leq |G_j| v_j(x_j)$ which completes the proof. ∎

B.1 Relaxing the Coalitional Stability Assumptions

Theorem 22 *If the project value functions are convex, the feasibility constraints have the form of a budget constraint and the sharing rule satisfies the k-approximate Vickrey condition, then any setwise Nash equilibrium achieves a $k+1$ fraction of the optimal social welfare. If the sharing rule induces a strongly monotone game, then any setwise Nash equilibrium achieves a k fraction of the optimal social welfare.*

Proof. Since the project value functions are convex, the social welfare is also convex. Hence, the optimization problem is convex and therefore there must be an extreme optimal outcome where each player puts his whole effort on a single project. Denote such an optimal outcome with x^O and let N_j^* be the set of players that put their budget on project j at x^O. Let x^S be a setwise Nash equilibrium. Using similar arguments as in the proof of Theorem 1 we can prove that: $\sum_{i \in N_j^*} u_i(x^S) \geq \frac{1}{\gamma+1} v_j(x^O)$, for every $j \in M$, and if the utilities induce a strongly monotone game then $\sum_{i \in N_j^*} u_i(x^S) \geq \frac{1}{\gamma} v_j(x^O)$, for every $j \in M$. Summing over all projects $j \in M$ and since N_j^*, $j \in M$, is a disjoint partition of N, we establish the theorem. ∎

C Local Smoothness and Total Effort Games

We will present the local smoothness framework briefly, adapted to a utility maximization problem instead of a cost minimization. We also slightly generalize the framework by requiring that the player cost functions be continuously differentiable only at equilibrium points.

Consider a utility maximization game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where each player's strategy space is a continuous convex subset of a Euclidean space $\mathbb{R}^{m_i}$. Let $u_i : \times_{i \in N} S_i \to \mathbb{R}$ be the utility function for a player i, which is assumed to be continuous and concave in x_i for each fixed value of x_{-i}. The concavity and continuity assumptions lead to existence of a Pure Nash Equilibrium using the classical result of Rosen [34]. Also for a strategy profile $x \in \times_{i \in N} S_i$ we denote with $SW(x) = \sum_{i \in N} u_i(x)$.

We start by presenting the definitions of smoothness [35] and local-smoothness [37] for utility maximization games. We then examine their connection and how local smoothness is applied to our effort market setting.

Definition 5 *A utility maximization game is (λ, μ)-smooth iff for every two strategy profiles x and y:*

$$\sum_{i \in N} u_i(y_i, x_{-i}) \geq \lambda SW(y) - \mu SW(x)$$

Roughgarden [35] showed that any (λ, μ)-smooth game has PoA at most $(1+\mu)/\lambda$, which carries over to every coarse correlated equilibrium too. Local smoothness is a better version of smoothness for the case of continuous and convex strategy spaces, since as we'll see it has the potential of giving better PoA bounds than smoothness.

Definition 6 *A utility maximization game is locally (λ, μ)-smooth with respect to a strategy profile x^* iff for every strategy profile x at which $u_i(x)$ are continuously differentiable:*

$$\sum_{i \in N}[u_i(x) + \nabla_i u_i(x) \cdot (x_i^* - x_i)] \geq \lambda SW(x^*) - \mu SW(x)$$

where $\nabla_i u_i \equiv (\frac{\partial u_i}{\partial x_{i,1}}, \ldots, \frac{\partial u_i}{\partial x_{i,m_i}})$.

Observe that for a concave differentiable function v it holds that $v(y) \leq v(x) + \nabla v(x) \cdot (y - x)$, hence from this observation we can derive the following lemma that connects smoothness with local smoothness:

Lemma 23 *If a concave utility maximization game is (λ, μ)-smooth then it is also locally (λ, μ)-smooth with respect to any strategy profile y.*

Proof.

$$\sum_{i \in N}[u_i(x) + \nabla_i u_i(x) \cdot (y_i - x_i)] \geq \sum_{i \in N} u_i(y_i, x_{-i}) \geq \lambda SW(y) - \mu SW(x)$$

■

Theorem 24 *If a utility maximization game is locally (λ, μ)-smooth with respect to a strategy profile x^* and $u_i(x)$ is continuously differentiable at every Nash equilibrium x of the game then $SW(x) \geq \frac{\lambda}{1+\mu} SW(x^*)$*

Proof. The proof is the equivalent of [37] for the case of maximization games. The key claim is that if $\vec{x}$ is a Nash Equilibrium then

$$\nabla_i u_i(x) \cdot (x_i^* - x_i) \leq 0$$

Given this claim then we obtain the theorem, since:

$$SW(x) \geq \sum_{i \in N} u_i(x) + \nabla_i u_i(x) \cdot (x_i^* - x_i) \geq \lambda SW(x^*) - \mu SW(x)$$

To prove the claim define $x^\epsilon = ((1-\epsilon)x_i + \epsilon x_i^*, x_{-i})$ and observe that: $\lim_{\epsilon \to 0} \frac{1}{\epsilon}(u_i(x^\epsilon) - u_i(x)) = \nabla_i u_i(x) \cdot (x_i^* - x_i)$. Thus if $\nabla_i u_i(x) \cdot (x_i^* - x_i) > 0$ then for some ϵ_0 it holds that $u_i(x^\epsilon) - u_i(x) > 0$, which means that i has a profitable deviation which contradicts the fact that x is a Nash Equilibrium. ■

Lemma 23 shows that local smoothness can possibly achieve better bounds on the PoA. If (λ_1, μ_1) where the parameters that minimized $(1 + \mu)/\lambda$ subject to the game being (λ, μ) smooth, i.e. $(1 + \mu_1)/\lambda_1$ was the best PoA bound we could get using smoothness, then by 23 we know that the game is also locally (λ_1, μ_1)-smooth with respect to the optimum outcome. Hence, if (λ_2, μ_2) where the parameters that minimized $(1 + \mu)/\lambda$, subject to the game being locally (λ, μ)-smooth then we know that $(1 + \mu_2)/\lambda_2 \leq (1 + \mu_1)/\lambda_1$. Therefore, the best possible bound on the PoA achieved by local smoothness can only be better than that achieved by smoothness.

C.1 Concave Functions of Total Effort and Proportional Sharing

Now we switch to the specific context of Total Effort Market Games which is a special case of the above continuous strategy space games, assuming that the feasibility constraints of the players define a convex set and the value functions satisfy the concavity and differentiability assumptions of the previous section. We will also assume that the proportional sharing rule is used.

In the theorem we will show in this section we will also use a lemma of [37]:

Lemma 25 *Let $x_i, y_i \geq 0$ and $x = \sum_i x_i$, $y = \sum_i y_i$ then: $\sum_i x_i(y_i - x_i) \leq k(x, y)$ where:*

$$k(x,y) = \begin{cases} \frac{y^2}{4} & x \geq y/2 \\ x(y-x) & x < y/2 \end{cases}$$

First we formulate a lemma that applies to any concave function:

Theorem 26 *Assume $v_j(X_j)$ are concave functions of total effort with $v_j(0) = 0$ and are continuously differentiable at any equilibrium point of the game defined by the proportional sharing scheme. Let $\bar{v}_j(X_j) = \frac{v_j(X_j)}{X_j}$. Let x^* be a strategy profile. If for all strategy profiles x and for all $j \in M$:*

$$X_j^* \bar{v}_j(X_j) + \bar{v}_j'(X_j) k(X_j, X_j^*) \geq \lambda X_j^* \bar{v}_j(X_j^*) - \mu X_j \bar{v}_j(X_j)$$

then the game is locally (λ, μ)-smooth with respect to x^*.

Proof. Let x be an equilibrium effort vector and x^* some other outcome. From the definition of the proportional sharing scheme:

$$u_i(x) = \sum_{j \in M_i} x_{ij} \frac{v_j(X_j)}{X_j} = \sum_{j \in M_i} x_{ij} \bar{v}_j(X_j)$$

$$\nabla_i u_i(x) \cdot (x_i^* - x_i) = \sum_{j \in M_i} \frac{\partial (x_{ij} \bar{v}_j(X_j))}{\partial x_{ij}} (x_{ij}^* - x_{ij}) = \sum_{j \in M_i} (\bar{v}_j(X_j) + x_{ij} \bar{v}_j'(X_j))(x_{ij}^* - x_{ij})$$

Therefore, we have

$$\begin{aligned} SW(x) \geq & \sum_{i \in N} u_i(x) + \nabla_i u_i(x) \cdot (x_i^* - x_i) \\ = & \sum_{i \in N} \sum_{j \in M_i} x_{ij} \bar{v}_j(X_j) + (\bar{v}_j(X_j) + x_{ij} \bar{v}_j'(X_j))(x_{ij}^* - x_{ij}) \\ = & \sum_{i \in N} \sum_{j \in M_i} x_{ij}^* \bar{v}_j(X_j) + \bar{v}_j'(X_j) x_{ij} (x_{ij}^* - x_{ij}) \\ = & \sum_{j \in M} \left(X_j^* \bar{v}_j(X_j) + \bar{v}_j'(X_j) \sum_i [x_{ij}(x_{ij}^* - x_{ij})] \right) \\ \geq & \sum_{j \in M} \left(X_j^* \bar{v}_j(X_j) + \bar{v}_j'(X_j) k(X_j, X_j^*) \right) \end{aligned}$$

where the last inequality follows from the lemma and the fact that $\bar{v}_j(X_j)$ is a non-increasing function, hence $\bar{v}_j'(X_j) \leq 0$. Combining the last above inequality with the assumption of the theorem, we establish the assertion of the theorem.

Indeed, $\bar{v}_j(X_j)$ is non-increasing because $v_j(X_j)$ is concave with $v_j(0) = 0$. This means that the line connecting $(0, 0)$ with $(x, v_j(x))$ is below $v_j(y)$ for any $y \in [0, x]$. Hence, $y \frac{v_j(x)}{x} \leq v_j(y) \implies \frac{v_j(x)}{x} \leq \frac{v_j(y)}{y}$. Thus for any $y \leq x \implies \frac{v_j(x)}{x} \leq \frac{v_j(y)}{y}$ which implies $\bar{v}_j(x)$ is decreasing. ■

C.2 Fractional Exponent Functions

We now cope with the natural case when the value functions have the form $q_j X_j^{\alpha}$ for some $\alpha \in (0,1)$. First, we need a lemma characterizing differentiability at equilibria:

Lemma 27 *If $v_j(X_j) = q_j X_j^{\alpha}$ for some $\alpha \in (0,1)$ then for any equilibrium x of the game it holds that $X_j > 0$ for all $j \in M$. Thus, $u_i(x)$ is continuously differentiable at any Nash equilibrium.*

Proof. Assume that x is an equilibrium where $X_{j_1} = 0$ for some j_1. Now consider a player that participates at j_1. Consider the deviation where he moves a small amount of effort ϵ from a project j_2 to j_1. Let x^{ϵ} be the strategy vector after the deviation.

$$u_i(x^{\epsilon}) - u_i(x) = q_{j_1}\epsilon^{\alpha} - (\phi_{i,j_2}(X_{j_2})v_{j_2}(X_{j_2}) - \phi_{i,j_2}(X_{j_2}-\epsilon)v_{j_2}(X_{j_2}-\epsilon)) \implies$$
$$\frac{u_i(x^{\epsilon}) - u_i(x)}{\epsilon} = \frac{q_{j_1}\epsilon^{\alpha}}{\epsilon} - \frac{\phi_{i,j_2}(X_{j_2})v_{j_2}(X_{j_2}) - \phi_{i,j_2}(X_{j_2}-\epsilon)v_{j_2}(X_{j_2}-\epsilon)}{\epsilon}$$

As $\epsilon \to 0$, $\frac{q_{j_1}\epsilon^{\alpha}}{\epsilon} \to \infty$ and $\frac{\phi_{i,j_2}(X_{j_2})v_{j_2}(X_{j_2}) - \phi_{i,j_2}(X_{j_2}-\epsilon)v_{j_2}(X_{j_2}-\epsilon)}{\epsilon} \to \left.\frac{\partial \phi_{i,j_2}(X_{j_2})v_{j_2}(X_{j_2})}{\partial x_{i,j_2}}\right|_{x=X_{j_2}} = ct$. Thus for some ϵ_0 it has to be that $\frac{q_{j_1}\epsilon^{\alpha}}{\epsilon} > \frac{\phi_{i,j_2}(X_{j_2})v_{j_2}(X_{j_2}) - \phi_{i,j_2}(X_{j_2}-\epsilon)v_{j_2}(X_{j_2}-\epsilon)}{\epsilon}$. Thus x^{ϵ_0} is a profitable deviation for i. ■

Theorem 28 *Suppose that the project value functions are of the form $v_j(x_j) = q_j X_j^{\alpha}$, for $0 \le \alpha \le 1$ and $q_j > 0$. Then, the proportional sharing rule has the price of anarchy at most $\frac{2-\alpha}{2^{1-\alpha}}$. Furthermore, the worst-case price of anarchy is $1/[\log(2)2^{\frac{1}{\log(2)}-1}] \approx 1.0615$.*

Proof. We shall compute the value PoA(α) that minimizes $\frac{1+\mu}{\lambda}$ over $\mu, \lambda \ge 0$ subject to

$$y\hat{v}(x) + \hat{v}'(x)k(x,y) \ge \lambda y \hat{v}(y) - \mu x \hat{v}(x) \text{ for every } x, y \ge 0 \tag{14}$$

where $\hat{v}(x) = v(x)/x$, $v(x) = qx^{\alpha}$, $q > 0$ and $0 < \alpha \le 1$. Without loss of generality, we can assume $q = 1$. Recall that $k(x,y) = \frac{1}{4}y^2$, for $y/x \le 2$, and $k(x,y) = x(y-x)$, otherwise. Using this and defining $z = y/x$, it is easily showed that condition (14) is equivalent to $h_{\lambda,\mu}(z) \ge 0$, for every $z \ge 0$, where

$$h_{\lambda,\mu}(z) = \begin{cases} z - \frac{1-\alpha}{4}z^2 - \lambda z^{\alpha} + \mu, & z \le 2 \\ \alpha z - \lambda z^{\alpha} + \mu, & z > 2. \end{cases}$$

We proceed with considering two different cases.

Case 1: $z \le 2$. In this case, we note that there exist z_1^0 and z_2^0 such that $z_2^0 > z_1^0 > 0$ and $h_{\lambda,\mu}(z)$ is decreasing on $[0, z_1^0) \cup (z_2^0, \infty)$ and increasing on (z_1^0, z_2^0). Now, note that $h'_{\lambda,\mu}(2) \ge 0$ is equivalent to $\lambda \le 2^{1-\alpha}$. Hence, if $\lambda \le 2^{1-\alpha}$, then $\inf_{z\in[0,2]} h_{\lambda,\mu}(z) = h_{\lambda,\mu}(\xi(\lambda))$ where $\xi(\lambda)$ is the smallest positive value z such that $h'_{\lambda,\mu}(z) = 0$. On the other hand, if $\lambda > 2^{1-\alpha}$, we claim that $h_{\lambda,\mu}(z)$ is decreasing on $[0,2]$, and hence $\inf_{z\in[0,2]} h_{\lambda,\mu}(z) = h_{\lambda,\mu}(2) = 1 + \mu + \alpha - 2^{\alpha}\lambda$.

We showed that in the present case, condition (14) is equivalent to: if $\lambda \le 2^{1-\alpha}$, the condition is

$$\mu \ge \xi(\lambda)^{\alpha} + \frac{1-\alpha}{4}\xi(\lambda)^2 - \xi(\lambda)$$

otherwise, if $\lambda > 2^{1-\alpha}$, then the condition is

$$\mu \ge 2^{\alpha}\lambda - 1 - \alpha.$$

Case 2: $z > 2$. In this case, we note that $h_{\lambda,\mu}(z)$ is a concave function with unique minimum value over positive values at $z = \lambda^{\frac{1}{1-\alpha}}$. Therefore, if $\lambda \leq 2^{1-\alpha}$, then $\inf_{z>2} h_{\lambda,\mu}(z) = h_{\lambda,\mu}(2) = 1 + \mu + \alpha - 2^\alpha \lambda$, and otherwise, $\inf_{z>2} h_{\lambda,\mu}(z) = h_{\lambda,\mu}(\lambda^{\frac{1}{1-\alpha}}) = 1 + \mu - \alpha - (1-\alpha)\lambda^{\frac{1}{1-\alpha}}$. Therefore, we have that in the present case, condition (14) is equivalent to: if $\lambda \leq 2^{1-\alpha}$, then the condition is

$$\mu \geq 2^\alpha \lambda - 1 - \alpha$$

otherwise, if $\lambda > 2^{1-\alpha}$, the condition is

$$\mu \geq (1-\alpha)\lambda^{\frac{1}{1-\alpha}} - 1 + \alpha.$$

Now note that

$$\mathrm{PoA}(\alpha) = \mathrm{PoA}_1(\alpha) \wedge \mathrm{PoA}_2(\alpha)$$

where

$$\begin{aligned}
\mathrm{PoA}_1(\alpha) &= \inf_{\lambda \leq 2^{1-\alpha}} \max\left\{\frac{1 + \xi(\lambda)^\alpha + \frac{1-\alpha}{4}\xi(\lambda)^2 - \xi(\lambda)}{\lambda}, \frac{2^\alpha\lambda - \alpha}{\lambda}\right\} \\
\mathrm{PoA}_2(\alpha) &= \inf_{\lambda > 2^{1-\alpha}} \max\left\{\frac{2^\alpha\lambda - \alpha}{\lambda}, \frac{(1-\alpha)\lambda^{\frac{1}{1-\alpha}} + \alpha}{\lambda}\right\}.
\end{aligned}$$

For $\mathrm{PoA}_2(\alpha)$, the minimum over all positive values of λ is at the smallest value of λ at which the two functions under the maximum operator intersect and this is at $\lambda = 2^{1-\alpha}$. Therefore,

$$\mathrm{PoA}_2(\alpha) = \frac{2^\alpha\lambda - \alpha}{\lambda}\Big|_{\lambda = 2^{1-\alpha}} = \frac{2-\alpha}{2^{1-\alpha}}.$$

It remains only to show that $\mathrm{PoA}_1(\alpha) \geq \mathrm{PoA}_2(\alpha)$ and thus $\mathrm{PoA}(\alpha) = \mathrm{PoA}_2(\alpha)$. It is convinient to use an upper bound for the first term that appears under the maximum operator in the definition of $\mathrm{PoA}_1(\alpha)$. To this end, we go back to our analysis of Case 1 and note that $h_{\lambda,\mu}(z) \geq z - \lambda z^\alpha + \mu + \alpha - 1$, for every $0 \leq z \leq 2$. Requiring that the right-hand side is greater or equal zero for every $z \in [0,2]$ is a sufficient condition for $h_{\lambda,\mu}(z) \geq 0$ to hold for every $z \in [0,2]$ and it yields

$$\mu \geq (1-\alpha)[\alpha^{\frac{\alpha}{1-\alpha}}\lambda^{\frac{1}{1-\alpha}} + 1].$$

We thus have

$$\mathrm{PoA}_1(\alpha) \leq \inf_{\lambda \leq 2^{1-\alpha}} \max\left\{\frac{1 + (1-\alpha)[\alpha^{\frac{\alpha}{1-\alpha}}\lambda^{\frac{1}{1-\alpha}} + 1]}{\lambda}, \frac{2^\alpha\lambda - \alpha}{\lambda}\right\}.$$

Now, it is easy to check that the first term under the maximum operator is greater or equal than the second term for every $\lambda \leq 2^{1-\alpha}$, hence

$$\mathrm{PoA}_1(\alpha) \leq \inf_{\lambda \leq 2^{1-\alpha}} \frac{2 - \alpha + (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\lambda^{\frac{1}{1-\alpha}}}{\lambda}.$$

It can be readily checked that the right-hand side is non-increasing and hence the infimum is achieved at $\lambda = 2^{1-\alpha}$ with the value

$$\frac{2 - \alpha + 2(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{2^{1-\alpha}}$$

which indeed is greater than or equal to $(2-\alpha)/2^{1-\alpha} = \mathrm{PoA}_2(\alpha)$. ■

C.3 Exponential Value Functions

If $v_j(X_j) = q_j(1 - p^{X_j})$ then to show that the resulting game is locally (λ, μ) smooth and hence has PoA $\leq \lambda/(1+\mu)$ we need to show the following inequality holds for all $x, y \geq 0$ and $p \in (0,1)$

$$y\frac{1-p^x}{x} + \frac{-xp^x\log(p) - (1-p^x)}{x^2}k(x,y) \geq \lambda(1-p^y) - \mu(1-p^x) \tag{15}$$

This is the condition of Theorem 26 instantiated for our specific function form. Moreover, $k(x,y) = y^2/4$ when $x \geq y/2$ and $x(y-x)$ otherwise.

Theorem 29 *For the family of exponential value functions $v(x) = q(1-p^x)$, for arbitrary constants $q > 0$ and $0 < p < 1$, we have that PoA ≤ 2.*

We first note that function $v(x) = 1 - e^{-x}$ is *scale-invariant*, i.e. if $v(x)$ is locally (λ, μ)-smooth then $a \cdot v(b \cdot x)$ is also locally (λ, μ)- smooth, for any constants $a > 0$ and $b > 0$. Hence, it suffices to consider the function $v(x) = 1 - e^{-x}$ as our original function is given by $v_w(x) = q_w v(-\log(p_w) \cdot x)$.

We will show that the set of parameters (λ, μ) for which $v(x)$ is locally (λ, μ)-smooth is given by $\mathcal{L} = \{\mathbb{R}_+^2 : \lambda \leq \mu \text{ and } \lambda \leq 1\}$. That is, we will show that the following two conditions are necessary and sufficient:

$$\lambda \leq 1 \tag{16}$$

$$\lambda \leq \mu \tag{17}$$

We know that PoA $\leq (1+\mu)/\lambda$, thus in view of the above inequalities

$$\text{PoA} \leq \max\left\{1+\mu, \frac{1+\mu}{\mu}\right\}, \text{ for every } \mu \geq 0.$$

The minimum value in the right hand side is 2 achieved for $\mu = 1$.

For the value function $v(x) = 1 - e^{-x}$ the local smoothness condition corresponds to:

$$y\frac{1-e^{-x}}{x} + \frac{xe^{-x} - (1-e^{-x})}{x^2}k(x,y) \geq \lambda(1-e^{-y}) - \mu(1-e^{-x}), \text{ for every } x, y \geq 0. \tag{18}$$

In the following we separately consider two different cases.

Case 1: $x < y/2$. In this case, we have

$$\begin{aligned}\lambda(1-e^{-y}) - (1+\mu) &\leq y\frac{1-e^{-x}}{x} + \frac{xe^{-x} - (1-e^{-x})}{x}x(y-x) + \mu(1-e^{-x}) \\ &= e^{-x}[y - x - (1+\mu)] := f(x).\end{aligned}$$

The above inequality is thus equivalent to

$$\lambda(1-e^{-y}) - (1+\mu) \leq \inf_{x \in [0, y/2)} f(x).$$

It is easy to notice that $f'(x) \leq 0$ iff $x \leq y - \mu$. Thus we can distinguish the following two subcases.
Subcase 1.a: $y \geq 2\mu$. In this case $y/2 \leq y - \mu$, and thus $f'(x) \leq 0$, for $0 \leq x < y/2$. Hence,

$$\inf_{x \in [0, y/2)} f(x) = f(y/2) = e^{-y/2}[\frac{1}{2}y - (1+\mu)].$$

It follows that we must have $\lambda(1-e^{-y})-(1+\mu)\leq f(y/2)$, for every $y\geq 2\mu$, i.e.

$$\lambda-(1+\mu)\leq \inf_{y\geq 2\mu} g(y) \tag{19}$$

where

$$g(y)=\lambda e^{-y}+\frac{1}{2}ye^{-y/2}-(1+\mu)e^{-y/2}.$$

Since $g(0)=\lambda-(1+\mu)\leq 0$, it is immediate that for the condition (19) to hold, it is necessary that $g'(0)\geq 0$, which is equivalent to $2\lambda\leq\mu+2$. Sufficiency follows by the following arguments. Note that under the condition $g'(0)\geq 0$, the function $g(y)$ has a unique maximum at y^* so that $g'(y)\geq 0$ for $y\leq y^*$ and $g'(y)\leq 0$, for $y>y^*$. Suppose that it holds $g(y^*)\geq 0$, then we have that condition (19) holds in view of the fact that $\lim_{y\to\infty} g(y)=0$. To contradict, suppose that $g(y^*)<0$, then this would mean that $g'(y)\geq 0$, for every $y\geq 0$, which contradicts existence of y^*. We have thus showed that (19) holds iff

$$2\lambda\leq\mu+2$$

which is implied by the stronger condition (16), that is showed to be necessary later.
Subcase 1.b: $y<2\mu$. Suppose first that $y\leq\mu$, then $f'(x)\geq 0$, for every $x\geq 0$, and thus $\inf_{x\in[0,y/2)} f(x)=f(0)=y-(1+\mu)$. In the given case condition (18) reads as $\lambda(1-e^{-y})-(1+\mu)\leq y-(1+\mu)$, i.e. $\inf_{0\leq y\leq\mu} g(y)\geq 0$ where $g(y)=y-\lambda(1-e^{-y})$. Since $g(0)=0$, it must hold $g'(0)\geq 0$, which is equivalent to

$$\lambda\leq 1.$$

This is the above asserted condition (16), under which it obviously holds $g(y)\geq y-(1-e^{-y})=e^{-y}-(1-y)\geq 0$, for every $y\geq 0$.

Suppose now $\mu\leq y\leq 2\mu$. In this case $f'(x)\leq 0$, for $0\leq x\leq y-\mu$, and $f'(x)>0$, otherwise. Since $\inf_{x\in[0,y/2)} f(x)=f(y-\mu)=-e^{-y+\mu}$, we have the condition,

$$\lambda-(1+\mu)\leq \inf_{\mu\leq y\leq 2\mu} e^{-y}[\lambda-e^{\mu}]$$

which is equivalent to $\lambda(1-e^{-\mu})\leq\mu$ and this is implied by (17), showed to be necessary in the following.

Case 2: $x\geq y/2$. In this case it is convenient to use the following change of variables $z=y/(2x)$. Notice that $0\leq z\leq 1$. Condition (18) can be written in the following equivalent form

$$2(1-e^{-x})z+(xe^{-x}-(1-e^{-x}))z^2\geq\lambda(1-e^{-2xz})-\mu(1-e^{-x}),\ \text{for } x\geq 0 \text{ and } 0\leq z\leq 1$$

or, alternatively, $f(x,z)\leq 0$, for $x\geq 0$ and $0\leq z\leq 1$ where

$$f(x,z)=\lambda(1-e^{-2xz})-\mu(1-e^{-x})-2(1-e^{-x})z+e^{-x}(e^x-(1+x))z^2.$$

Let us fix $0\leq z\leq 1$ and consider $g(x)=f(x,z)$. We observe

$$\begin{aligned} g(0) &= 0 \\ g(+\infty) &= \lim_{x\to\infty} g(x)=\lambda-(1+\mu)+(1-z)^2. \end{aligned}$$

From the last fact, notice that it must hold

$$\lambda\leq\mu.$$

This is the above asserted condition (17).

We will consider the first derivative of $g(x)$ which is

$$g'(x) = e^x[2\lambda z e^{(1-2z)x} - (\mu + 2z - xz^2)].$$

Since $g(0) = 0$, it must hold $g'(0) \le 0$, as otherwise $g(x) = f(x,z) > 0$, for some $x > 0$. The condition $g'(0) \le 0$ is equivalent to $2\lambda z \le \mu + 2z$, thus, $(\lambda - 1)z \le \mu/2$ must hold for every $0 \le z \le 1$. The condition is trivially true if $\lambda \le 1$ and otherwise holds true iff $2\lambda \le \mu + 2$, which is implied by (16).

By inspection of the function $g'(x)$ it is readily observed that under $g'(0) \le 0$ the function $g(x)$ is decreasing on $[0, a)$ and increasing on (a, ∞) where $a \ge 0$ is such that $g'(a) = 0$. Combined with the fact $g(0) = 0$, we conclude that

$$\sup_{x \in [0,\infty)} g(x) = \max(0, g(+\infty)).$$

We already noted that $\lim_{x\to\infty} f(x,z) \le 0$, for every $0 \le z \le 1$ iff $\lambda \ge \mu$. Hence, $f(x,z) \le 0$ holds for every $x \ge 0$ and $0 \le z \le 1$ iff $\lambda \le \mu$.

C.4 Tight Lower Bound for Exponential Value Functions

Example 30 *Consider the following instance: there are n players and n projects. Every player participates in every project. Each player has a budget of effort of 1 that he can split arbitrarily among his projects. Project 1 has value function $v_1(x) = 1 - e^{-\alpha x}$. The rest of the $n-1$ projects have value function $q(1 - e^{-\beta x})$. We still assume that value is shared proportional to the effort. We will define α, β and q in such a way that the unique equilibrium will be for all players to invest their whole budget on project 1, while the optimal will be for each players' efforts to be spread out among all the projects. A good approximation to the optimal is when each player picks a different project and devotes his whole effort.*

Since value functions are continuous increasing and concave, a necessary and sufficient condition for a strategy profile to be an equilibrium is that

$$x_{ij} > 0 \implies \forall j' : \frac{\partial u_{ij}(x)}{\partial x_{ij}} \ge \frac{\partial u_{ij'}(x)}{\partial x_{ij'}}$$

In words, the marginal share of a player is equal for all the projects that he puts positive effort and at least as much as the marginal share on the projects he puts zero effort. Since we want the equilibrium to be all players putting their effort on project 1 we need that:

$$\left(\frac{x}{x+n-1}\left(1 - e^{-a(x+n-1)}\right)\right)'\Big|_{x=1} > \left(q(1 - e^{-\beta x})\right)'\Big|_{x=0}$$

Observe that since $1 - e^{-a(x+n-1)}$ is an increasing function the derivative on the left hand side is at least as much as the derivative of $x/(x+n-1)$ evaluated at 1, which is just $(n-1)/n^2$. Thus a sufficient condition for the above inequality to hold is:

$$\frac{n-1}{n^2} \ge q\beta$$

If we let $q = \frac{n-1}{\beta n^2}$ then we satisfy the required conditions. In addition α is free to take any value. We will let $\alpha \to \infty$. The social welfare of the equilibrium will be:

$$SW(x) = 1 - e^{-\alpha n} \to 1$$

On the other hand if all players are split among different projects, then the social welfare will be:

$$SW(x^*) = 1 - e^{-\alpha} + (n-1)q(1-e^{-\beta}) = 1 - e^{-\alpha} + \frac{(n-1)^2}{\beta n^2}(1-e^{\beta}) \to 1 + \left(1 - \frac{1}{n}\right)^2 \frac{1-e^{-\beta}}{\beta}$$

If we also let $\beta \to 0$ we will have:

$$\frac{SW(x^*)}{SW(x)} \to 1 + \left(1 - \frac{1}{n}\right)^2$$

As $n \to \infty$, the above ratio converges to 2.

D Proofs for Section 6

Theorem 31 *Consider a Common Value Sharing Game with Production Costs and a sharing scheme that satisfies locally the Vickrey condition. Let x be an Strong Nash Equilibrium and x^* be the outcome that maximizes social welfare. Then $SW(x) + V(x) \geq SW(x^*)$. For diminishing marginal value functions the latter holds even if x is a pure Nash equilibrium.*

Proof. For the general case, using similar reasoning as in Theorem 1 we can conclude that:

$$\begin{aligned} SW(x) \geq & \sum_{i \in N} u_i(x^*_{N_k}, x_{-N_k}) \\ = & \sum_{i \in N} \left(\sum_{j \in M} \phi_{i,j}(x^*_{N_k}, x_{-N_k}) v_j(x^*_{N_k}, x_{-N_k}) - c_i(x^*_i) \right) \\ \geq & \sum_{i \in N} \left(V(x^*_{N_k}, x_{-N_k}) - V(x^{out}_k, x^*_{N_{k+1}}, x_{-N_k}) \right) - C(x^*) \\ = & V(x^*) - V(x) - C(x^*) = SW(x^*) - V(x) \end{aligned}$$

Using similar reasoning as in the previous theorem and as in the proof of Theorem 26 we obtain the result for the case of diminishing marginal value functions. ■

D.1 Linear Efforts Lead to Inefficiency

Example 32 *Consider a single project with value $v(X) = \sqrt{X}$. Moreover, each player pays a cost of 1 per unit of effort, i.e. $c_i(x_i) = x_i$. The global optimum is the solution to the unconstraint optimization problem:*

$$\max_{X \in \mathbb{R}_+} \sqrt{X} - X$$

which leads to $X^* = 1/4$ and therefore $SW(X^*) = 1/4$. On the other hand each players optimization problem is:

$$\max_{x_i \in \mathbb{R}_+} x_i \frac{1}{\sqrt{X}} - x_i$$

By symmetry and elementary calculus we obtain that at the unique equilibrium the total effort is: $X = \left(\frac{n-1/2}{n}\right)^2$. Hence,

$$SW(X) = \frac{n-1/2}{n} - \left(\frac{n-1/2}{n}\right)^2 = \frac{2n-1}{4n^2}$$

Leading to $PoA = \frac{n^2}{2n-1} = \Omega(n)$.

Proposition 33 *Assume that the cost function is linear $c(X) = X$ and the value function $v(X)$ is such that there exists $t > 0$ such that $X > v(X)$, for every $X > t$. Then, POA = $\Omega(n)$.*

Proof. Social welfare is given by $\mathrm{SW}(x) = v(X) - \sum_{i=1}^{n} c(x_i)$ and socially optimal allocation is such that $x_i = X/n$ for every player i, and

$$v'(X) - c'(X/n) = 0. \tag{20}$$

Due to the symmetry, we use the notation $\mathrm{SW}(X) = v(X) - nc(X/n)$.

Furthermore, a Nash equilibrium allocation x is such that for every i, x_i maximizes

$$\frac{x_i}{X}v(X) - c(x_i).$$

The first order optimality condition reads as, for every i,

$$\frac{v(X)}{X} + x_i\frac{v'(X)X - v(X)}{X^2} - c'(x_i) = 0.$$

Therefore, $x_i = X/n$ for every i, and

$$v'(X) - c'(X/n) + \left(1 - \frac{1}{n}\right)\frac{v(X) - v'(X)X}{X} = 0. \tag{21}$$

It is not difficult that the Nash equilibrium of the game is optimal allocation for a (virtual) social welfare function defined as follows

$$\hat{\mathrm{SW}}(X) = \frac{1}{n}v(X) + \left(1 - \frac{1}{n}\right)\int_0^X \frac{v(y)}{y}dy - nc(X/n).$$

As an aside remark, note since $v(X)$ is a concave function, $v(X)/X$ is non-increasing over $X \geq 0$ and hence, $\mathrm{SW}(X) \leq \hat{\mathrm{SW}}(X)$, for $x \geq 0$.

In view of the identity (21), we have

$$\begin{aligned} \mathrm{SW}(X) &= v(X) - nc(X/n) \\ &= v(X) - nc(X/n) - X\left(v'(X) - c'(X/n) + \left(1 - \frac{1}{n}\right)\frac{v(X) - v'(X)X}{X}\right) \\ &= \frac{v(X) - v'(X)X}{n} + n\left((X/n)c'(X/n) - c(X/n)\right). \end{aligned} \tag{22}$$

We observe that $(X/n)c'(X/n) - c(X/n) = 0$ if and only if $c(X)$ is a linear function and in this case at Nash equilibrium, it holds

$$\mathrm{SW}(X) = \frac{v(X) - v'(X)X}{n}.$$

From (20) we have that optimum social welfare is a constant, independent of n. From (21) and the fact $X \leq t$, we have that in Nash equilibrium

$$\mathrm{SW}(X) \leq \frac{\max_{X\in[0,t]}\{v(X) - v'(X)X\}}{n} = \frac{v(t) - v'(t)t}{n} = O(1/n).$$

The result follows. ■

Remark 34 We observe that socially optimal and Nash equilibrium allocations are determined by (20) and (21), respectively, where for asymptotically large n, the behaviors of the functions $v'(x)$ for large x and $c'(x)$ for small x play a key role.

D.2 Total Effort Games, Cournot Oligopolies and the Elasticity of Costs

Theorem 35 *If the elasticity of the cost functions is at least $1+\mu$ and the elasticity of value functions is at most $1-\lambda$, then the social welfare at equilibrium is at least $\frac{\mu+\lambda/n}{1+2\mu+\lambda/n}$ of the optimal. In addition production at equilibrium is at least $\frac{1}{2}\frac{\lambda+\mu}{1+\mu}$ of the production at the social welfare maximizing outcome.*

Proof. For the first part of the theorem we will prove that if x is a Nash Equilibrium then

$$(1+\mu)C(x) \leq (1-\frac{\lambda}{n})V(x) \tag{23}$$

Since x is an equilibrium it must hold that for each player i:

$$x_{ij} > 0 \implies c_i'(X_i) = \frac{\partial}{\partial x_{ij}}\left(x_{ij}\frac{v_j(X_j)}{X_j}\right) \tag{24}$$

Thus we observe that the partial derivative of the right hand side is the same for all projects j in which player i is putting positive effort. Multiplying both sides with X_i and using the above fact we obtain:

$$\begin{aligned}
X_i c_i'(X_i) =&\ X_i\frac{\partial}{\partial x_{ij}}\left(x_{ij}\frac{v_j(X_j)}{X_j}\right) = \sum_{j'\in M} x_{ij'}\frac{\partial}{\partial x_{ij}}\left(x_{ij}\frac{v_j(X_j)}{X_j}\right)\\
=&\ \sum_{j'\in M} x_{ij'}\frac{\partial}{\partial x_{ij'}}\left(x_{ij'}\frac{v_{j'}(X_{j'})}{X_{j'}}\right)\\
=&\ \sum_{j\in M} x_{ij}\frac{v_j(X_j)}{X_j} + x_{ij}^2\frac{X_j v_j'(X_j) - V_j(X_j)}{X_j^2}
\end{aligned}$$

Now summing over all players and using the μ-convexity of $c_i()$ and λ-concavity of $v_j()$ we obtain:

$$\begin{aligned}
(1+\mu)\sum_{i\in N} c_i(X_i) \leq&\ \sum_{i\in N} X_i c_i'(X_i) \leq \sum_{j\in M}\sum_{i\in N} x_{ij}\frac{v_j(X_j)}{X_j} + x_{ij}^2\frac{X_j v_j'(X_j) - v_j(X_j)}{X_j^2}\\
=&\ \sum_{j\in M}\left(v_j(X_j) + \frac{X_j v_j'(X_j) - v_j(X_j)}{X_j^2}\sum_{i\in N} x_{ij}^2\right)\\
\leq&\ \sum_{j\in M}\left(v_j(X_j) + \frac{(1-\lambda)v_j(X_j) - v_j(X_j)}{X_j^2}\sum_{i\in N} x_{ij}^2\right)\\
=&\ \sum_{j\in M}\left(v_j(X_j) - \lambda\frac{v_j(X_j)}{X_j^2}\sum_{i\in N} x_{ij}^2\right)\\
\leq&\ \sum_{j\in M}\left(v_j(X_j) - \lambda\frac{v_j(X_j)}{X_j^2}\frac{X_j^2}{n}\right)\\
=&\ (1-\frac{\lambda}{n})\sum_{j\in M} v_j(X_j)
\end{aligned}$$

Having proved the above we can conclude that if x is a Nash Equilibrium:

$$SW(x) = V(x) - C(x) \geq V(x) - \frac{1}{1+\mu}(1-\frac{\lambda}{n})V(x) = \frac{\mu+\lambda/n}{1+\mu}V(x)$$

Combining the above with Theorem 13 we obtain:

$$SW(x) + \frac{\mu+1}{\mu+\lambda/n} SW(x) \geq SW(x) + V(x) \geq SW(x^*) \tag{25}$$

By rearranging we obtain the first part of the theorem.

Now we move on to the second part. For that we will prove that if x^* is the social welfare maximizing outcome then:

$$(1+\mu)C(x^*) \leq (1-\lambda)V(x^*) \tag{26}$$

The social welfare maximizing outcome is satisfies the following constraints:

$$x_{ij} > 0 \implies v'_j(X_j) = c'_i(X_i) \tag{27}$$

Thus we observe that all projects in which a player puts positive effort have the same marginal value. Multiplying the above inequality with X_j and using the above property we obtain:

$$X_i c'_i(X_i) = X_i v'_j(X_j) = \sum_{j' \in M} x_{ij'} v'_j(X_j) = \sum_{j' \in M} x_{ij'} v'_{j'}(X_{j'})$$

Summing over all players and using the μ-convexity and λ-concavity property we obtain:

$$(1+\mu)C(x) \leq \sum_{i \in N} X_i c'_i(X_i) = \sum_{i \in N} \sum_{j \in M} x_{ij} v'_j(X_j) = \sum_{j \in M} X_j v'_j(X_j) \leq (1-\lambda) \sum_{j \in M} v_j(X_j)$$

Now using Theorem 13 we obtain that for any Nash Equilibrium:

$$\begin{aligned} 2V(x) \geq\ & SW(x) + V(x) \geq SW(x^*) = V(x^*) - C(x^*) \\ \geq\ & V(x^*) - \frac{1-\lambda}{1+\mu} V(x^*) = \frac{\mu+\lambda}{\mu+1} V(x^*) \end{aligned} \tag{28}$$

■

E Axiomatic Approach

Many papers in the surplus sharing literature have studied how the effort exerted by players affects their share. Much of this work has focused on an axiomatic approach [39, 10, 28], proposing various axioms that are desirable in a sharing scheme. One fundamental axiom is the Axiom of Monotonicity [26], which states that a player's share must be monotonously increasing in his effort. Recently, Moulin [28] also introduced a group-deviation version of the above axiom and the Axiom of Solidarity which determines how a player's change in effort affects the shares of the other players.

We observe here that the latter effect is more important for global efficiency than the monotonicity axiom. The Axiom of Solidarity states that when a player changes his effort, the shares of the other players must all change in same direction (i.e. either all of them increase or all of them decrease). We now propose a refinement of the axiom of solidarity that has a natural economic interpretation and also is sufficient to guarantee reasonable global efficiency in a multiple project environment.

Definition 7 (Axiom of Elimination) *We say that a sharing scheme $(\phi_i)_{i \in N}$ satisfies the Axiom of Elimination if for any subset $C \subseteq 2^N$ and for any $i \notin C$ we have: $\phi_i^{(N,v)} \leq \phi_i^{(N-C,v)}$ if v is submodular and $\phi_i^{(N,v)} \geq \phi_i^{(N-C,v)}$ if v is supermodular.*

Several well known sharing schemes satisfy this axiom, such as the Shapley value. For the submodular case, this axiom requires that if we remove some players from the game then the imputation of each of the rest of the players can only increase. In a submodular setting, in some sense, players are substitutes. Hence, by removing players from the game we are removing competition for the remaining players. Thereby we expect the remaining players to achieve more surplus. On the contrary in convex value functions it is implied that players are complements. Hence, by removing players we are removing collaborators and hence in the resulting game each player should achieve smaller surplus.

Theorem 36 *The game defined by a sharing scheme that satisfies the Axiom of Elimination is a monotone valid utility game when value functions satisfy the decreasing marginal returns property. Hence it has PoA* ≤ 2.

Proof of Theorem 36: By the Elimination Axiom for $C = \{i\}$ we have:

$$\forall j \in N : \phi_j^{(N,v)} \leq \phi_j^{(N-i,v)}$$

Summing over all j:

$$\sum_{j\neq i} \phi_j^{(N,v)} \leq \sum_{j\neq i} \phi_j^{(N-i,v)} = v(N-i)$$

Moreover since $\phi_i^{(N,v)} = v(N) - \sum_{j\neq i} \phi_j^{(N,v)}$ we obtain:

$$\phi_i^{(N,v)} \geq v(N) - v(N-i)$$

Thus the utility that a player gets from each project is at least his marginal contribution. Thereby, the overall utility of a player is at least his marginal contribution to the social welfare. ■

F Discussion and Extensions

F.1 Importance of Monotonicity

Without the assumption that project value functions are monotonic, we cannot guarantee a constant price of anarchy. To see this consider the case of a single project with value function $v(X)$ of total effort that is differentiable, concave, $v(0) = 0$, and it is single peaked, i.e. the function is increasing for $0 \leq X < X^*$ and decreasing for $X > X^*$, for some $X^* > 0$. Without loss of generality, let us assume that $v(X^*) = 1$ and $v(1) = 0$. We assume that $|v'(1)| < \infty$. In this case, the maximum social welfare is $V(X^*) = 1$. The payoff of player i is given by

$$u_i(x) = \frac{x_i}{X} v(X)$$

where $X = \sum_{j\in N} x_j$.

The game has a unique Nash equilibrium at which $\frac{\partial}{\partial x_i} u_i(x) = 0$, i.e.

$$\left(1 - \frac{1}{n}\right) v(X) + \frac{1}{n} v'(X) X = 0.$$

It is easy note from the last identity that it must hold $X^* < X \leq 1$. By concavity, we have $v(X) \geq -v'(X)(1-X)$, which combined with the Nash equilibrium condition yields

$$X \geq 1 - \frac{1}{n}.$$

Therefore, the price of anarchy is

$$PoA(n) = \frac{V(X^*)}{V(X)} \geq \frac{1}{v(1-\frac{1}{n})} \geq \frac{1}{-v'(1)} n = \Theta(n).$$

F.2 Heterogeneous Valuations

In this section we consider the case where each player values differently the share that he gets from different projects. More formally each player has some value $a_{i,j}$ per unit of wealth that he gets from each project that he participates in. Then his total utility is:

$$u_i(x) = \sum_{j \in M_i} a_{i,j} \phi_{i,j}(x_j) v_j(x_j) \tag{29}$$

In what we considered so far we assumed that $a_{i,j} = 1$ for any $i \in N$ and $j \in M_i$.

In this scenario the social welfare $SW(x) = \sum_i u_i(x)$ is not equal to the total value produced $V(x) = \sum_j v_j(x_j)$. Hence, in such settings we will be interested both in measuring the social welfare and the production at equilibrium as compared to the social welfare or production maximizing outcome.

Theorem 37 *Suppose that $a_{i,j} \in [A, B]$ for all $i \in N$ and $j \in M_i$. If the sharing rule awards to each player at least his marginal contribution to the produced value, then the production at any Stronga Nash Equilibrium is at least $\frac{A}{A+B}$ of the optimal and the social welfare is at least $\frac{A}{2B}$ of the optimal. If the effort game is a concave effort game then the latter hold at every Nash Equilibrium of the game.*

Interestingly we also prove that the equal sharing rule doesn't degrade with the heterogeneity in the market. Thereby in a heterogeneous market it the equal sharing rule might outperform more fair redistribution mechanisms.

Theorem 38 *For any $a_{i,j}$ the equal sharing rule has price of anarchy at most k when value functions are concave and and strong price of anarchy at most k for general value functions.*