

# FINITE TREES AND THE NECESSARY USE OF LARGE CARDINALS

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We introduce insertion domains that support the placement of new, higher, vertices into finite trees. We prove that every nonincreasing insertion domain has an element with simple structural properties in the style of classical Ramsey theory. This result is proved using standard large cardinal axioms that go well beyond the usual axioms for mathematics. We also establish that this result cannot be proved without these large cardinal axioms. We also introduce insertion rules that specify the placement of new, higher, vertices into finite trees. We prove that every insertion rule greedily generates a tree with these same structural properties; and every decreasing insertion rule generates (or admits) a tree with these same structural properties. It is also necessary and sufficient to use the same large cardinals (in the precise sense of Corollary D.25). The results suggest new areas of research in discrete mathematics called "Ramsey tree theory" and "greedy Ramsey theory" which demonstrably require more than the usual axioms for mathematics.

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## A. STATEMENT OF RESULTS

## A1. TREES AND INSERTION DOMAINS

We begin with the concrete representation of finite trees that is used throughout the paper. A partial ordering is a pair  $(X, \leq)$ , where  $X$  is a nonempty set, and  $\leq$  is reflexive, transitive, and antisymmetric. The ancestors of  $x$  in  $X$  are just the  $y < x$ .

For the purposes of this paper, a tree  $T = (V, \leq)$  is a partial ordering with a minimum element, where  $V$  is finite, and the ancestors of any  $x \in V$  are linearly ordered under  $\leq$ . The minimum element of  $T$  is called the root of  $T$ , and is written  $r(T)$ . A tree is said to be trivial if and only if it has exactly one vertex, which must be its root.  $V = V(T)$  represents the set of all vertices of the tree  $T = (V, \leq)$ .

In a tree  $T$ , if  $x < y$  and for no  $z$  is  $x < z < y$ , then we say that  $y$  is a child of  $x$  and  $x$  is the parent of  $y$ . Every vertex has at most one parent. However, vertices may have zero or more children. We write  $p(x, T)$  for the parent of  $x$  in  $T$ .

We use  $\text{Ch}(T) = V(T) \setminus \{r(T)\}$  for the set of all children of  $T$ .

We write  $T_1 \subseteq T_2$  if and only if

- i)  $r(T_1) = r(T_2)$ ;
- ii) for all  $x \in \text{Ch}(T_1)$ ,  $p(x, T_1) = p(x, T_2)$ .

This is a partial ordering on trees. Note that if  $T_1 \subseteq T_2$  then no parent/child bond is broken by going from  $T_1$  to  $T_2$ .

Let  $N$  be the set of all nonnegative integers. For  $k \geq 1$ , let  $[N]^k$  be the set of all  $k$  element subsets of  $N$ . In this paper, we focus on the  $k$ -trees,  $k \geq 1$ . These are the trees  $T$  such that

- i)  $r(T) = \infty$ ;
- ii)  $V(T) \subseteq [N]^k \cup \{\infty\}$ .

Thus there is a special treatment of roots in  $k$ -trees. There is exactly one trivial  $k$ -tree, which consists solely of the root  $\infty$ . We let  $TR(k)$  be the set of all  $k$ -trees.

Note that the  $k$ -trees can be viewed as the forests whose vertices are elements of  $[N]^k$ . I.e., we can join the roots of the components of such a forest by the root  $\infty$ . And given any  $k$ -tree, we can remove the root  $\infty$  and thereby obtain such a forest. The correspondence is exact if the empty forest is allowed.

We use  $\leq^*$  for the standard reverse lexicographic ordering on  $[N]^k$ , in which sets are compared according to the lexicographic ordering on  $N^k$  after the sets are placed in strictly descending order. We extend  $\leq^*$  to  $[N]^k \cup \{\infty\}$  by taking  $\infty$  to be the maximum element. We use  $\geq^*$ ,  $<^*$ , and  $>^*$  in the obvious way.

Let  $T \in TR(k)$ . We say that  $x$  dominates  $T$  if and only if  $x \in [N]^k$  and for all  $y \in \text{Ch}(T)$ ,  $y <^* x$ .

An insertion domain in  $TR(k)$  is a nonempty  $S \subseteq TR(k)$  such that the following holds. For all  $T \in S$  and  $x$  dominating  $T$ , there exists  $T' \in S$  such that  $T \subseteq T'$  and  $\text{Ch}(T') = \text{Ch}(T) \cup \{x\}$ .

We introduce another important partial ordering on  $k$ -trees. We write  $T_1 \leq^* T_2$  if and only if for all  $x \in \text{Ch}(T_1)$ ,  $p(x, T_1) \leq^* p(x, T_2)$ . Note that  $T_1 \leq^* T_2$  implies  $\text{Ch}(T_1) \subseteq \text{Ch}(T_2)$ .

Let  $S \subseteq TR(k)$ . We say that  $S$  is nonincreasing if and only if for all  $T_1, T_2 \in S$ , if  $T_1 \leq^* T_2$  then  $T_1 \subseteq T_2$ . I.e., there are no proper inequalities between elements of  $S$ .

In section C, we prove the following in  $Z = \text{Zermelo set theory}$ .

**THEOREM A1.1.** Let  $k, p \geq 1$ . Every nonincreasing insertion domain in  $TR(k)$  contains a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

The system  $Z$  incorporates a rather substantial amount of abstract set theory - infinitely many uncountable cardinals.

We conjecture that infinitely many uncountable cardinals are required to prove Theorem A1.1. Specifically, we conjecture that Theorem A1.1 cannot be proved in ZC with bounded separation.

Let  $x, y \in [N]^k$ . We say that  $x$  is entirely lower than  $y$  if and only if every element of  $x$  is strictly smaller than every element of  $y$ .

PROPOSITION A1.2. Let  $k, p \geq 1$ . Every nonincreasing insertion domain in  $TR(k)$  contains a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

In section C, we prove Proposition A1.2 in  $ZFC + (\forall k)$  (there exists a  $k$ -subtle cardinal). In section D we prove that Proposition A1.2 (first claim) cannot be proved in  $ZFC + \{\text{there exists a } k\text{-subtle cardinal}\}_k$ , provided the latter is consistent. See [Ba73], [Fr98], and [Fr $\infty$ ] for a discussion of the subtle cardinal hierarchy.

## A2. GREEDY TREES AND INSERTION RULES

An insertion rule on  $TR(k)$  is a function  $f$ , where

- i)  $\text{dom}(f) = \{(T, x) : T \in TR(k) \text{ and } x \text{ dominates } T\}$ ;
- ii) for all  $(T, x) \in \text{dom}(f)$ ,  $f(T, x) \in V(T)$ .

For  $T \in TR(k)$ ,  $x \in [N]^k \setminus \text{Ch}(T)$ , and  $y \in V(T)$ , we write  $T/x, y$  for the  $k$ -tree  $T'$  such that  $\text{Ch}(T') = \text{Ch}(T) \cup \{x\}$  and  $p(x, T') = y$ .  $T'$  is the result of inserting  $x$  as a new vertex in  $T$  at  $y$ ; i.e., with parent  $y$ .

We now see why we use the phrase "insertion rule on  $TR(k)$ ." We can view  $f$  as specifying how we insert  $x$  into  $T$ , where  $x$  dominates  $T \in TR(k)$ . Namely, by inserting  $x$  into  $T$  so as to produce the  $k$ -tree  $T/x, f(T, x)$ .

Let  $f$  be an insertion rule on  $TR(k)$ . We say that a  $k$ -tree is generated by  $f$  if and only if it lies in the least class  $K \subseteq TR(k)$  such that

- i) the trivial  $k$ -tree lies in  $K$ ;
- ii) for all  $T \in K$  and  $x$  dominating  $T$ ,  $T/x, f(T, x) \in K$ .

Note that the  $k$ -trees generated by  $f$  are obtained by successively inserting elements of  $[N]^k$  in  $<^*$ -increasing order into the trivial  $k$ -tree, with parents determined by  $f$ .

In section A3 we present results about the trees generated by certain insertion rules on  $TR(k)$ .

We now introduce the related concept of "greedy generation." The term "greedy" comes from "greedy" algorithms, whereby optimal finite objects are sought in various contexts. In certain important contexts, the standard efficient algorithms proceed by building up the desired optimized object sequentially, where at each stage the construction extends the object built thus far in a relatively optimal way. This kind of construction is used for the standard efficient algorithms in such diverse contexts as minimal spanning trees, Huffman codes, task-scheduling, shortest paths, etc. See [CLR90] and its many references for more information.

The results of this section definitely reflect this "greedy" idea. Moreover, we expect that further results will be obtained with yet closer connections with greedy algorithms of the kind that figure so prominently in the theory of algorithms.

Let  $f$  be an insertion rule on  $TR(k)$ . We say that a  $k$ -tree is greedily generated by  $f$  if and only if it lies in the least class  $M \subseteq TR(k)$  such that

- i) the trivial  $k$ -tree lies in  $M$ ;
- ii) for all  $T \in M$  and  $x$  dominating  $T$ ,  $T/x, \min\{f(T', x) : T' \subseteq T\} \in M$ .

We emphasize that this min is with respect to  $<^*$ .

Note that the  $k$ -trees greedily generated by  $f$  are obtained by successively inserting elements of  $[N]^k$  in  $<^*$ -increasing order into the trivial  $k$ -tree, with parents minimized as determined by  $f$ .

Now consider the following analogs to Theorem A1.1 and Proposition A1.2.

THEOREM A2.1. Let  $k, p \geq 1$ . Every insertion rule on  $TR(k)$  greedily generates a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A2.2. Let  $k, p \geq 1$ . Every insertion rule on  $TR(k)$  greedily generates a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

We make the same conjectures and prove the same claims about Theorem A2.1 and Proposition A2.2 that we make and prove about Theorem A1.1 and Proposition A1.2.

### A3. DECREASING INSERTION RULES

Let  $f$  be an insertion rule on  $TR(k)$ . We say that  $f$  is decreasing if and only if for all  $T_1 \subseteq T_2$  from  $TR(k)$  and  $x$  dominating  $T_2$ , we have  $f(T_1, x) \geq^* f(T_2, x)$ .

Consider the following analogs to Theorem A1.1 and Proposition A1.2.

THEOREM A3.1. Let  $k, p \geq 1$ . Every decreasing insertion rule on  $TR(k)$  generates a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A3.2. Let  $k, p \geq 1$ . Every decreasing insertion rule on  $TR(k)$  generates a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

We make the same conjectures and prove the same claims about Theorem A3.1 and Proposition A3.2 that we make and prove about Theorem A1.1 and Proposition A1.2.

An insertion rule in  $TR(k)$  is a pair  $(S, f)$ , where

- i)  $S$  is a nonempty subset of  $TR(k)$ ;
- ii)  $\text{dom}(f) = \{(T, x) : T \in S \text{ and } x \text{ dominates } T\}$ ;
- iii) for all  $(T, x) \in \text{dom}(f)$ ,  $f(T, x) \in V(T)$ ;
- iv) for all  $(T, x) \in \text{dom}(f)$ ,  $T/x, f(T, x) \in S$ .

We say that  $(S, f)$  admits  $T$  if and only if  $T \in S$ . We say that  $(S, f)$  is decreasing if and only if for all  $T_1 \subseteq T_2$  from  $S$  and  $x$  dominating  $T_2$ , we have  $f(T_1, x) \geq^* f(T_2, x)$ .

Consider the following analogs to Theorem A1.1 and Proposition A1.2.

THEOREM A3.3. Let  $k, p \geq 1$ . Every decreasing insertion rule in  $TR(k)$  admits a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A3.4. Let  $k, p \geq 1$ . Every decreasing insertion rule in  $TR(k)$  admits a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

We make the same metamathematical conjectures and prove the same claims about Theorem A3.3 and Proposition A3.4 that we make and prove about Theorem A1.1 and Proposition A1.2.

#### A4. FINITE FORMS

We now present straightforward finite forms of the above results. For  $k \geq 1$  and  $n \geq 0$ , let  $[n]^k$  be the set of all  $k$  element subsets of  $\{0, \dots, n-1\}$ .

Let  $k, n \geq 1$ . The  $k, n$ -trees are the  $k$ -trees all of whose children lie in  $[n]^k$ . We write  $TR(k, n)$  for the set of all  $k, n$ -trees.

An insertion domain in  $TR(k, n)$  is a nonempty  $S \subseteq TR(k, n)$  such that the following holds. For all  $T \in S$ ,  $x \in [n]^k$ ,  $x$  dominating  $T$ , there exists  $T' \in S$  such that  $T \subseteq T'$  and  $\text{Ch}(T') = \text{Ch}(T) \cup \{x\}$ .

An insertion domain in  $TR(k,n)$  is said to be initial if and only if it includes the trivial  $k$ -tree. This definition is also made for insertion domains in  $TR(k)$ .

An insertion rule on  $TR(k,n)$  is a function  $f$ , where

- i)  $\text{dom}(f) = \{(T,x) : T \in TR(k,n), x \in [n]^k, \text{ and } x \text{ dominates } T\}$ ;
- ii) for all  $(T,x) \in \text{dom}(f)$ ,  $f(T,x) \in V(T)$ .

Let  $f$  be an insertion rule on  $TR(k,n)$ . We say that a  $k,n$ -tree is generated by  $f$  if and only if it lies in the least class  $K \subseteq TR(k,n)$  such that

- i) the trivial  $k$ -tree lies in  $K$ ;
- ii) for all  $T \in K$ ,  $x \in [n]^k$ ,  $x$  dominating  $T$ ,  $T/x, f(T,x) \in K$ .

Let  $f$  be an insertion rule on  $TR(k,n)$ . We say that a  $k,n$ -tree is greedily generated by  $f$  if and only if it lies in the least class  $M \subseteq TR(k,n)$  such that

- i) the trivial  $k$ -tree lies in  $M$ ;
- ii) for all  $T$  in  $M$ ,  $x \in [n]^k$ ,  $x$  dominating  $T$ ,  $T/x, \min\{f(T',x) : T' \subseteq T\} \in M$ .

Let  $f$  be an insertion rule on  $TR(k,n)$ . We say that  $f$  is decreasing if and only if for all  $T_1 \subseteq T_2$  from  $TR(k,n)$ ,  $x \in [n]^k$ ,  $x$  dominating  $T_2$ , we have  $f(T_1,x) \geq^* f(T_2,x)$ .

An insertion rule in  $TR(k,n)$  is a pair  $(S,f)$ , where

- i)  $S$  is a nonempty subset of  $TR(k,n)$ ;
- ii)  $\text{dom}(f) = \{(T,x) : T \in S, x \in [n]^k, \text{ and } x \text{ dominates } T\}$ ;
- iii) for all  $(T,x) \in \text{dom}(f)$ ,  $f(T,x) \in V(T)$ ;
- iv) for all  $(T,x) \in \text{dom}(f)$ ,  $T/x, f(T,x) \in S$ .

We say that  $(S,f)$  admits  $T$  if and only if  $T \in S$ . We say that  $(S,f)$  is decreasing if and only if for all  $T_1 \subseteq T_2$  from  $S$ ,  $x \in [n]^k$ ,  $x$  dominating  $T_2$ , we have  $f(T_1,x) \geq^* f(T_2,x)$ .

We say that  $(S, f)$  is initial if and only if the trivial  $k$ -tree lies in  $S$ . This definition is also made for insertion rules in  $TR(k)$ .

Here are natural finite forms of the previously cited results. Note that "initial" is used in Theorems A1.1' and A3.3', as well as in Propositions A1.2' and A3.4'. This is needed.

THEOREM A1.1'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every nonincreasing initial insertion domain in  $TR(k, n)$  contains a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A1.2'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every nonincreasing initial insertion domain in  $TR(k, n)$  contains a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

THEOREM A2.1'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every insertion rule on  $TR(k, n)$  greedily generates a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A2.2'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every insertion rule on  $TR(k, n)$  greedily generates a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

THEOREM A3.1'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every decreasing insertion rule on  $TR(k, n)$  generates a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A3.2'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every decreasing insertion rule on  $TR(k, n)$  generates a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

THEOREM A3.3'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every decreasing initial insertion rule in  $TR(k, n)$  admits a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A3.4'. For all  $k, p \geq 1$  there exists  $n$  such that the following holds. Every decreasing initial insertion rule in  $TR(k, n)$  admits a  $k, n$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

## B. LOGICAL RELATIONSHIPS

### B1. BASIC IMPLICATIONS AND EQUIVALENCES

In this section, we will establish some implications and equivalences between the statements in sections A1 - A3 within  $RCA_0$ , which is the base system for reverse mathematics; see [Si85a], [Si85b], and [Si88]. We also establish some equivalences involving statements in section A4 within  $RCA_0 + WKL$ , another basic system in reverse mathematics. The results are summarized in Theorem B1.13.

Let  $T \in TR(k)$  and  $x \in [N]^k$ . Let  $T|<^*x$  be the tree obtained by restricting  $T$  to the vertices in  $\{y: y <^* x\} \cup \{\infty\}$ . Let  $T|\leq^*x$  be the  $k$ -tree obtained by restricting  $T$  to the vertices in  $\{y: y \leq^* x\} \cup \{\infty\}$ .

LEMMA B1.1. The following is provable in  $RCA_0$ . Let  $k \geq 1$ . If  $S$  is a nonincreasing insertion domain in  $TR(k)$  then there is a decreasing insertion rule  $(S, f)$  in  $TR(k)$ . If  $(S, f)$  is a decreasing insertion rule in  $TR(k)$  then  $S$  includes a nonincreasing insertion domain.

Proof: Let  $S$  be a nonincreasing insertion domain in  $TR(k)$ . We first claim that for all  $T \in S$  and  $x$  dominating  $T$ , there is a unique  $T' \in S$  such that  $Ch(T') = Ch(T) \cup \{x\}$  and  $T' \subseteq T$ . Since  $S$  is an insertion domain, we have existence. Uniqueness is immediate by nonincreasing.

We define  $f$  as follows. Let  $T \in S$  and  $x$  dominate  $T$ . Let  $T'$  be as above. Set  $f(T, x) = p(x, T')$ . It remains to prove that  $f$  is decreasing. Let  $T_1 \subseteq T_2$  lie in  $S$  and  $x$  dominate  $T_2$ . Let  $T_1'$  and  $T_2'$  be as above. Suppose  $f(T_1, x) <^* f(T_2, x)$ . Then  $p(x, T_1') <^* p(x, T_2')$ . But this contradicts that  $S$  is nonincreasing. Hence  $f(T_1, x) \geq^* f(T_2, x)$ .

For the reverse direction, let  $(S, f)$  be a decreasing insertion rule in  $\text{TR}(k)$ . Obviously  $S$  is an insertion domain. Fix  $T_0 \in S$ , and let  $S'$  be the set of all  $k$ -trees generated by  $f$  from  $T_0$ . I.e.,  $S'$  is the least set of  $k$ -trees such that

- i)  $T_0 \in S'$ ;
- ii) for all  $T \in S'$  and  $x$  dominating  $T$ ,  $T/x, f(T, x) \in S'$ .

Obviously  $S'$  is an insertion domain in  $\text{TR}(k)$ . We claim that for all  $T \in S'$  and  $x \in \text{Ch}(T) \setminus \text{Ch}(T_0)$ , we have  $T|<^*x, T|\leq^*x \in S'$ ,  $x$  dominates  $T_0$ , and  $T|<^*x \subseteq T|\leq^*x \subseteq T$ . This is proved by induction on the construction of the elements of  $S'$ .

To see that  $S'$  is nonincreasing, let  $T \leq^* T'$  from  $S'$ , and suppose  $T \not\subseteq T'$ . Let  $x$  be the  $<^*$ -least element of  $\text{Ch}(T)$  such that  $p(x, T) \neq p(x, T')$ . Then  $x$  dominates  $T_0$  and  $T|<^*x \subseteq T'|<^*x$ . Since  $f$  is decreasing and these two restrictions lie in  $S$ , we have  $f(T|<^*x, x) \geq^* f(T'|<^*x, x)$ . Hence  $p(x, T|\leq^*x) \geq^* p(x, T'|\leq^*x)$ . Since  $T \leq^* T'$ , we have  $p(x, T|\leq^*x) = p(x, T'|\leq^*x) = p(x, T) = p(x, T')$ . This is the desired contradiction.

LEMMA B1.2. Theorems A1.1 and A3.3 are equivalent in  $\text{RCA}_0$ . Propositions A1.2 and A3.4 are equivalent in  $\text{RCA}_0$ .

Proof: Immediate from Lemma B1.1.

LEMMA B1.3. The following is provable in  $\text{RCA}_0$ . Let  $k \geq 1$  and  $S \subseteq \text{TR}(k)$ . The following are equivalent.

- i)  $S$  is the set of all  $k$ -trees generated by some decreasing insertion rule on  $\text{TR}(k)$ ;
- ii)  $S$  is the set of all  $k$ -trees greedily generated by some insertion rule on  $\text{TR}(k)$ .

Proof: Assume i), and let  $S$  be the set of all  $k$ -trees generated by the decreasing insertion rule  $f$  on  $\text{TR}(k)$ . We claim that every element of  $S$  is greedily generated by  $f$ . To see this, by examination of the definitions of generation and

greedy generation it suffices to prove the following. Let  $(T, x) \in \text{dom}(f)$ . Then  $f(T, x) = \min\{f(T', x) : T' \subseteq T\}$ . Since  $f$  is decreasing, this min is realized with  $T' = T$ .

Assume ii) and let  $S$  be the set of all  $k$ -trees greedily generated by the insertion rule  $f$  for  $\text{TR}(k)$ . Define the insertion rule  $g$  for  $\text{TR}(k)$  as follows. Let  $(T, x) \in \text{dom}(f)$ . Define  $g(T, x) = \min\{f(T', x) : T' \subseteq T\}$ .

We claim that  $g$  is decreasing. To see this, let  $T_1 \subseteq T_2$  be from  $\text{TR}(k)$  and  $x$  dominate  $T_2$ . Then the min for  $g(T_1, x)$  is included in the min for  $g(T_2, x)$ . Hence  $g(T_1, x) \geq^* g(T_2, x)$ .

It remains to show that  $S$  is the set of  $k$ -trees generated by  $g$ . This is immediate from the definitions; these two sets are the least subset of  $\text{TR}(k)$  satisfying the same conditions.

LEMMA B1.4. Theorems A2.1 and A3.1 are equivalent in  $\text{RCA}_0$ . Propositions A2.2 and A3.2 are equivalent in  $\text{RCA}_0$ .

Proof: Immediate from Lemma B1.3.

LEMMA B1.5. The following is provable in  $\text{RCA}_0$ . Let  $k \geq 1$ . If  $f$  is a decreasing insertion rule on  $\text{TR}(k)$  then there is a decreasing insertion rule in  $\text{TR}(k)$  which admits exactly the  $k$ -trees generated by  $f$ .

Proof: Let  $k, f$  be as given. Let  $S$  be the set of all  $k$ -trees generated by  $f$ . Let  $g$  be the restriction of  $f$  to  $\{(T, x) : T \in S \text{ and } x \text{ dominates } T\}$ . Then  $(S, g)$  is a decreasing insertion rule in  $\text{TR}(k)$  which admits exactly the  $k$ -trees generated by  $f$ .

LEMMA B1.6. Theorem A3.3 implies Theorem A3.1 in  $\text{RCA}_0$ . Proposition A3.4 implies Proposition A3.2 in  $\text{RCA}_0$ .

Proof: Immediate from Lemma B1.5.

We now consider the finite forms given in section A4. We first give the "initial" forms of Theorems A1.1 and A3.3, and of Propositions A1.2 and A3.4.

THEOREM A1.1\*. Let  $k, p \geq 1$ . Every nonincreasing initial insertion domain in  $\text{TR}(k)$  contains a  $k$ -tree in which all  $k$

element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A1.2\*. Let  $k, p \geq 1$ . Every nonincreasing initial insertion domain in  $TR(k)$  contains a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

THEOREM A3.3\*. Let  $k, p \geq 1$ . Every decreasing initial insertion rule in  $TR(k)$  admits a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

PROPOSITION A3.4\*. Let  $k, p \geq 1$ . Every decreasing initial insertion rule in  $TR(k)$  admits a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors. Moreover, these vertices can be required to have the same number of ancestors.

LEMMA B1.7. The following is provable in  $RCA_0$ . If  $S$  is a nonincreasing initial insertion domain in  $TR(k)$  then there is a decreasing initial insertion rule  $(S, f)$  in  $TR(k)$ . If  $(S, f)$  is a decreasing initial insertion rule in  $TR(k)$  then  $S$  includes an initial nonincreasing insertion domain.

Proof: The first claim is immediate from the first claim of Lemma B1.1 The second claim is proved exactly as the second claim of Lemma B1.1, where we take  $T_0$  to be the trivial  $k$ -tree.

LEMMA B1.8. Theorems A1.1\* and A3.3\* are equivalent in  $RCA_0$ . Propositions A1.2\* and A3.4\* are equivalent in  $RCA_0$ .

Proof: Immediate from Lemma B1.7.

LEMA B1.9. The following is provable in  $RCA_0$ . Let  $k \geq 1$ . If  $f$  is a decreasing insertion rule on  $TR(k)$  then there is a decreasing initial insertion rule in  $TR(k)$  which admits exactly the  $k$ -trees generated by  $f$ . If  $(S, f)$  is a decreasing initial insertion rule in  $TR(k)$  then there is a decreasing insertion rule  $g$  on  $TR(k)$  such that every  $k$ -tree generated by  $g$  lies in  $S$ .

Proof: The first claim has the same proof as Lemma B1.5. For the second claim, let  $(S, f)$  be a decreasing initial insertion

rule in  $\text{TR}(k)$ . We define an insertion rule  $g$  on  $\text{TR}(k)$  as follows. Let  $T \in \text{TR}(k)$  and  $x$  dominate  $T$ . Take  $g(T, x) = \min\{f(T', x) : T' \subseteq T \text{ and } T' \in S\}$ . Because  $(S, f)$  is initial,  $g$  is well defined. To see that  $g$  is decreasing, let  $T_1 \subseteq T_2$  be from  $\text{TR}(k)$  and  $x$  dominate  $T_2$ . Then the min for  $g(T_1, x)$  is included in the min for  $g(T_2, x)$ . Hence  $g(T_1, x) \geq^* g(T_2, x)$ .

We now claim that if  $T \in S$  and  $x$  dominates  $T$  then  $g(T, x) = f(T, x)$ . This is clear since the min defining  $g(T, x)$  is realized with  $T' = T$ .

It is now clear by induction that every  $k$ -tree generated by  $f$  lies in  $S$ .

LEMMA B1.10. Theorems A3.1 and A3.3\* are equivalent in  $\text{RCA}_0$ . Propositions A3.2 and A3.4\* are equivalent in  $\text{RCA}_0$ .

Proof: Immediate from Lemma B1.9.

LEMMA B1.11 Theorems A1.1\*, A2.1, A3.1, and A3.3\* are equivalent in  $\text{RCA}_0$ . Propositions A1.2\*, A2.2, A3.2, and A3.4\* are equivalent in  $\text{RCA}_0$ .

Proof: By Lemmas B1.4, B1.8, and B1.10.

It is clear that Theorems A1.1', A2.1', A3.1', and A3.3' are respectively finite forms of Theorems A1.1\*, A2.1, A3.1, and A3.3\*. And Propositions A1.2', A2.2', A3.2', and A3.4' are respectively finite forms of Propositions A1.2\*, A2.2, A3.2, and A3.4\*.

LEMMA B1.12. Theorems A1.1\*, A2.1, A3.1, and A3.3\* are equivalent, respectively, to their finite forms in section A4 in  $\text{RCA}_0 + \text{WKL}$ . Propositions A1.2\*, A2.2, A3.2, and A3.4\* are equivalent, respectively, to their finite forms in section A4 in  $\text{RCA}_0 + \text{WKL}$ .

Proof: The forward implications are carried out in  $\text{RCA}_0 + \text{WKL}$  as follows. Fix  $k, p \geq 1$ , and suppose there is no  $n$  such that the finite form holds. Then consider nonincreasing initial insertion domains in  $\text{TR}(k, n)$ , insertion rules on  $\text{TR}(k, n)$ , decreasing insertion rules on  $\text{TR}(k, n)$ , or decreasing initial insertion rules in  $\text{TR}(k, n)$ , respectively, for various  $n \geq 0$ , which are counterexamples. These form a finitely branching tree under the appropriate notion of extension. The tree is

infinite since it has nodes living in every  $TR(k,n)$ ,  $n \geq 0$ . Therefore by the Konig tree lemma, it has an infinite path, and this infinite path corresponds to a nonincreasing initial insertion domain in  $TR(k)$ , insertion rule on  $TR(k)$ , decreasing insertion rules on  $TR(k)$ , or decreasing initial insertion rule in  $TR(k)$ , respectively. Then apply the original form to obtain an element, greedily generated  $k$ -tree, generated  $k$ -tree, or admitted  $k$ -tree, respectively, with the applicable Ramsey property. But this violates that the tree consists entirely of counterexamples.

The reverse implications are carried out in  $RCA_0$  as follows. Let  $k, p \geq 1$  and a nonincreasing initial insertion domain in  $TR(k)$ , insertion rule on  $TR(k)$ , decreasing insertion rule on  $TR(k)$ , or decreasing initial insertion rule in  $TR(k)$ , respectively. Let  $n$  be given by the finite form. Then appropriately restrict the object to  $TR(k,n)$ , and apply the finite form.

THEOREM B1.13. Theorems A1.1\*, A2.1, A3.1, A3.3\*, A1.1', A2.1', A3.1', A3.3' are equivalent in  $RCA_0 + WKL$ . Propositions A1.2\*, A2.2, A3.2, A3.4\*, A1.2', A2.2', A3.2', A3.4' are equivalent in  $RCA_0 + WKL$ . Theorems A1.1 and A3.3 are equivalent in  $RCA_0$ . Propositions A1.2 and A3.4 are equivalent in  $RCA_0$ . Both Theorems A1.1 and A3.3 imply all of the above Theorems in  $RCA_0 + WKL$ . Both Propositions A1.2 and A3.4 imply all of the above Propositions in  $RCA_0 + WKL$ . The above equivalences and implications not involving ' statements can be done in  $RCA_0$ . The equivalences and implications involving the Propositions still hold if we state them without "same number of ancestors."

Proof: By Lemmas B1.6, B1.8, B1.11, and B1.12. The final claim is evident in light of the way that the preceding claims were established.

## B2. ADDITIONAL IMPLICATIONS AND EQUIVALENCES

We begin this section by showing that all theorems of sections A1 - A3 are provably equivalent in  $RCA_0$ , and all theorems of sections A1 - A4 are provably equivalent in  $RCA_0 + WKL$ . We conjecture that all propositions of sections A1 - A3 are provably equivalent in  $RCA_0$ , and all propositions of

sections A1 - A4 are provably equivalent in  $RCA_0 + WKL$ . (See the conjectures at the end of section C).

To this end, it will be convenient to use what we call  $k$ -tree assignments. For finite  $A \subseteq [N]^k$ , we say that  $x$  dominates  $A$  if and only if for all  $y \in A$ ,  $y <^* x$ .

A  $k$ -tree assignment is a function  $H$  which assigns to every finite  $A \subseteq [N]^k$ , a  $k$ -tree  $H(A)$  with  $Ch(H(A)) = A$ . We say that  $H$  is decreasing\* if and only if for all finite  $A \subseteq B \subseteq [N]^k$  and  $x$  dominating  $B$ , if  $H(A) \subseteq H(B)$  then  $p(x, H(A \cup \{x\})) \geq^* p(x, H(B \cup \{x\}))$  and  $H(B) \subseteq H(B \cup \{x\})$ .

We introduce the following auxiliary proposition.

PROPOSITION B2.1. Let  $k, p \geq 1$ . Every decreasing\*  $k$ -tree assignment has a value in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors.

For finite  $A \subseteq [N]^k$ , we say that  $x$  dominates  $A$  if and only if for all  $y \in A$ ,  $y <^* x$ . We write  $A|<^*x = \{y \in A: y <^* x\}$  and  $A|\leq^*x = \{y \in A: y \leq^* x\}$ .

LEMMA B2.2. The following is provable in  $RCA_0$ . Let  $k \geq 1$ ,  $H$  be a decreasing\*  $k$ -tree assignment,  $A \subseteq [N]^k$  be finite, and  $x \in [N]^k$ . Then  $H(A|<^*x) \subseteq H(A|\leq^*x) \subseteq H(A)$ . The set of values of every decreasing\*  $k$ -tree assignments is a nonincreasing initial insertion domain in  $TR(k)$ . Every nonincreasing initial insertion domain in  $TR(k)$  contains the set of values of some decreasing\*  $k$ -tree assignment.

Proof: For the first claim, let  $k, H, A, x$  be as given. Prove the conclusion by induction on the cardinality of  $A|<^*x$ .

For the second claim, let  $H$  be a decreasing\*  $k$ -tree assignment and let  $V$  be the set of its values. Then obviously  $V$  is an initial insertion domain in  $TR(k)$ . To see that  $V$  is nonincreasing, let  $H(A) \leq^* H(B)$ , and assume  $H(A) \not\subseteq H(B)$ . Let  $x$  be  $<^*$ -least such that  $p(x, H(A)) \neq p(x, H(B))$ . Then obviously  $p(x, H(A)) \leq \bullet p(x, H(B))$ . But since  $H$  is decreasing\*, since  $H(A|<^*x) \subseteq H(B|<^*x)$ , we have  $p(x, H(A|\leq^*x)) \geq^* p(x, H(B|\leq^*x))$ . Hence  $p(x, H(A)) \geq^* p(x, H(B))$ . Therefore  $p(x, H(A)) = p(x, H(B))$ , which is a contradiction.

Finally, let  $S$  be a nonincreasing initial insertion domain in  $TR(k)$ . By Lemma B1.7, let  $(S, f)$  be a decreasing initial insertion rule in  $TR(k)$ . Let  $A \subseteq TR(k)$  be finite. We define  $H(A)$  to be the unique  $k$ -tree generated by  $f$  with  $Ch(H(A)) = A$ . Then  $H(A)$  is clearly a  $k$ -tree assignment satisfying the second part of decreasing\*, and its set of values is included in  $S$ . To see that  $H(A)$  is decreasing\*, let  $A \subseteq B \subseteq [N]k$  be finite,  $x$  dominate  $B$ , and  $H(A) \subseteq H(B)$ . Then  $p(x, H(AU\{x\})) \geq^* p(x, H(BU\{x\}))$  follows from  $f(H(A), x) \geq^* f(H(B), x)$ .

LEMMA B2.3. Theorem A1.1\* and Proposition B2.1 are equivalent in  $RCA_0$ .

Proof: Immediate from Lemma B2.2.

PROPOSITION B2.4. Let  $k, p, r \geq 1$ . Every decreasing\*  $k$ -tree assignment has a value in which all  $k$  element subsets of some  $p$  element set are vertices with the same number of ancestors, and whose children include  $[r]^k$ .

LEMMA B2.5. Propositions B2.1 and B2.4 are equivalent in  $RCA_0$ .

Proof: The reverse direction is trivial. Now assume Proposition B2.1, and let  $k, p, r \geq 1$  and  $H$  be a decreasing\*  $k$ -tree assignment. Without loss generality, assume  $r > k \geq 1$ . Let  $s = C(r, k)$ . We now define an  $s+k$ -tree assignment  $H'$ . Let  $A \subseteq [N]^{s+k}$ . We must define the  $s+k$ -tree  $H(A)$ .

We write elements of  $[N]^{s+k}$  in the form  $xUy$ , where  $x \in [N]^s$ ,  $y \in [N]^k$ , and  $\max(x) < \min(y)$ .

Let  $A' = \{x \in [N]^s : \text{every element of } x \text{ is the maximum element of some element of } A\}$ . For  $x \in [N]^s$  we define  $A/x = \{y \in [N]^k : xUy \in A\} \cup [r]^k$ . Note that this is a disjoint union.

We use the enumeration  $[r]^k = \{v_1 <^* \dots <^* v_s\}$ .

Let  $xUy \in A$ . We define  $p(xUy, H(A))$  as follows.

case 1.  $x \in A'$ . If  $p(y, H(A/x)) = \infty$  then set  $p(xUy, H'(A)) = \infty$ . If  $p(y, H(A/x)) \in [r]^k$  then let  $p(xUy, H'(A)) = v_i \in [r]^k$ .

Set  $p(xUy, H'(A))$  to be the  $<^*$ -least element of  $A$  whose maximum element is  $x_i$ . If  $p(y, H(A/x)) \in (A/x) \setminus [r]^k$  then set  $p(xUy, H'(A)) = xUp(y, H(A/x))$ .

case 2.  $x \notin A'$ . Set  $p(xUy, H'(A)) = \infty$ .

We now claim that  $H'$  is decreasing\*. First let  $A \subseteq N^{s+k}$  be finite and  $xUy$  dominate  $A$ . We show that  $H'(A) \subseteq H'(AU\{xUy\})$ . Since  $s \geq r$  and  $k \geq 1$ , clearly  $\min(y) \geq r$ .

Let  $x'Uy' \in A$ . We must first show that  $p(x'Uy', H'(A)) = p(x'Uy', H'(AU\{xUy\}))$ . We claim that  $x' \in A'$  if and only if  $x' \in (AU\{xUy\})'$ . To see this, suppose  $x' \in (AU\{xUy\})'$ , and let  $i \in x'$ . Then there exists  $u \in AU\{xUy\}$  such that  $\max(u) = i$ . Since  $\max(u) \leq \max(x') < \max(x'Uy')$  and  $x'Uy' <^* xUy$ , we see that  $u \neq xUy$ . Hence  $u \in A$ .

If  $x' \notin A'$  then by the above equivalence, we have  $x' \notin (AU\{xUy\})'$ . Hence  $p(x'Uy', H'(A)) = p(x'Uy', H'(AU\{xUy\})) = \infty$ . So we may assume  $x' \in A'$ , and so  $x \in (AU\{xUy\})'$ .

Obviously if  $x' \neq x$  then  $A/x' = (AU\{xUy\})/x'$ , in which case  $p(x'Uy', H'(A)) = p(x'Uy', H'(AU\{xUy\}))$  by inspection.

So we may assume  $x' = x$  and  $x \in A'$ . Note that  $(AU\{xUy\})/x = (A/x)U\{y\}$ . Since  $xUy$  dominates  $A$ , it is clear that  $y$  dominates  $A/x$ . Since  $H$  is decreasing\*, we have  $H(A/x) \subseteq H((A/x)U\{y\}) = H((AU\{xUy\})/x)$ . By inspection,  $p(xUy', H'(A)) = p(xUy', H'(AU\{xUy\}))$ .

We now let  $A \subseteq B \subseteq [N]^{s+k}$ ,  $H'(A) \subseteq H'(B)$ , and  $xUy$  dominate  $B$ . We must show that  $p(xUy, H'(AU\{xUy\})) \geq^* p(xUy, H'(BU\{xUy\}))$ .

If  $x \notin (AU\{xUy\})'$  then  $p(xUy, H'(A)) = \infty$ , and we are done. So we may assume that  $x \in (AU\{xUy\})'$ . Hence  $x \in (BU\{xUy\})'$ . We now claim that  $H(A/x) \subseteq H(B/x)$ . To see this, note that every element of  $B/x$  lies in  $[r]^k$  or dominates  $[r]^k$ , and hence since  $H$  is decreasing\*,  $H([r]^k) \subseteq H(A/x)$  and  $H([r]^k) \subseteq H(B/x)$ . Also let  $xUw \in A$ . If  $p(w, H(A/x)) = \infty$  then  $p(xUw, H'(A)) = \infty =$

$p(xUw, H'(B)) = p(w, H(B/x))$ . If  $p(w, H(A/x)) \in (A/x) \setminus [r]^k$  then  $p(xUw, H'(A)) = xUp(w, H(A/x)) = xUp(w, H(B/x)) = p(xUw, H'(B))$ . Finally suppose  $p(w, H(A/x)) = v_i \in [r]^k$ . Then  $p(xUw, H'(A))$  is the  $<^*$ -least element of  $A$  whose maximum element is  $x_i$ . Now  $p(xUw, H'(A)) = p(xUw, H'(B))$ . Hence by inspection,  $p(xUw, H'(B))$  is the  $<^*$ -least element of  $B$  whose maximum element is  $x_j$ , where  $p(w, H(B/x)) = v_j \in [r]^k$ . Therefore  $i = j$  and  $p(w, H(A/x)) = p(w, H(B/x))$ .

Since  $H(A/x) \subseteq H(B/x)$  and  $H$  is decreasing $^*$ , we see that  $p(y, H((AU\{xUy\})/x)) \geq^* p(y, H((BU\{xUy\})/x))$ . If  $p(y, H((AU\{xUy\})/x)) = \infty$  then  $p(xUy, H'(AU\{xUy\})) = \infty$ , and so  $p(xUy, H'(AU\{xUy\})) \geq^* p(xUy, H'(BU\{xUy\}))$ .

Suppose  $p(y, H((AU\{xUy\})/x)) \in ((AU\{xUy\})/x) \setminus [r]^k$ . Since  $p(y, H(AU\{xUy\})/x) \geq^* p(y, H(BU\{xUy\})/x)$ , we have  $p(y, H(BU\{xUy\})/x) \in (BU\{xUy\})/x$ . If  $p(y, H(BU\{xUy\})/x) \in [r]^k$  then obviously  $p(xUy, H'(AU\{xUy\})) \geq^* p(xUy, H'(BU\{xUy\}))$ , since the former has  $\max \geq r$  and the latter has  $\max < r$ . So assume  $p(y, H(BU\{xUy\})/x) \in ((BU\{xUy\})/x) \setminus [r]^k$ . Hence  $p(xUy, H'(AU\{xUy\})) = xUp(y, H((BU\{xUy\})/x)) \geq^* xUp(y, H((BU\{xUy\})/x)) = p(xUy, H'(BU\{xUy\}))$ , and so  $p(xUy, H'(AU\{xUy\})) \geq^* p(xUy, H'(BU\{xUy\}))$ .

Finally, suppose that  $p(y, H((AU\{xUy\})/x)) = v_i \in [r]^k$ . Then  $p(y, H((BU\{xUy\})/x)) = v_j \in [r]^k$  where  $j \leq i$ . Clearly the  $<^*$ -least element of  $A$  whose maximum element is  $v_i$  is  $\geq^*$  the  $<^*$ -least element of  $B$  whose maximum element is  $v_j$ . Hence  $p(xUy, H'(AU\{xUy\})) \geq^* p(xUy, H'(BU\{xUy\}))$ .

This completes the proof that  $H'$  is decreasing $^*$ .

Now choose  $p' \gg k, r, p$  so that we may apply the finite Ramsey theorem to  $p'$  below.

By Proposition B2.1, let  $A \subseteq [N]^{s+k}$  be finite and  $E \subseteq N$  have cardinality  $p'$ , where the  $k$  element subsets of  $E$  are vertices of  $H'(A)$  with the same number of ancestors. Let  $E' =$

$\{E_{3s+1}, \dots, E_{p'}\}$  and  $x = \{E_{2s+1}, \dots, E_{3s}\}$ . Then  $x \in A'$  and  $[E']^k \subseteq A/x$ .

It remains to show that all  $y \in [E']^k$  have the same number of ancestors in the  $k$ -tree  $H(A/x)$ . The ancestors of  $x \cup y$  in  $H'(A)$  appear without repetition as  $x \cup y_1, \dots, x \cup y_t, z, \dots, \infty$ , where  $t \geq 0$ , and  $z$  is either  $\infty$  or the  $<^*$ -least element of  $A$  with  $\max(z) = x_i$ ,  $1 \leq i \leq s$ , where  $p(y_t, H(A/x)) = v_i$ . We know that the length of this sequence is independent of the choice of  $y \in [E']^k$ .

Note that  $y_1, \dots, y_t$  is a listing of the ancestors of  $y$  in  $H(A/x)$  up to the point where the ancestors go into  $[r]^k \cup \{\infty\}$ . It therefore suffices to prove that  $t$  and  $z$  are independent of the choice of  $y \in [E']^k$ . We can't show this, but we can show this for some  $E'' \subseteq E'$  of cardinality  $p$ .

First we apply the finite Ramsey theorem to find  $E'' \subseteq E'$  of cardinality  $p$  such that  $z$  is independent of the choice of  $y \in [E'']^k$ . This can be done since the number of choices of  $z$  is just  $s+1$ . Since the length of the sequence  $x \cup y_1, \dots, x \cup y_t, z, \dots, \infty$  is independent of the choice of  $y \in [E'']^k$ , and since these sequences must agree from  $z$  on, we see that  $t$  is also independent of the choice of  $y \in [E'']^k$ . Finally, note that the number of ancestors of  $y \in [E']^k$  in the  $k$ -tree  $H(A/x)$  is  $t+1$  + the number of ancestors of  $v_i$  in the  $k$ -tree  $H([r]^k)$  if  $z \neq \infty$ ;  $t+1$  otherwise.

LEMMA B2.6. The following is provable in  $\text{RCA}_0$ . Let  $S$  be a nonincreasing insertion domain in  $\text{TR}(k)$ ,  $k \geq 1$ . Then there exists a decreasing\*  $k$ -tree assignment  $H$  and  $r \geq 0$  such that the following holds. Let  $A \subseteq [N]^k$  be finite,  $[r]^k \subseteq A$ . There exists  $T \in S$  such that

- i)  $T$  and  $H(A)$  have the same parent function off of  $[r]^k$ ;
- ii) for all  $x \in \text{Ch}(T) \cap [r]^k$ ,  $p(x, T) \in [r]^k \cup \{\infty\}$ .

Also, let  $H$  be a decreasing\*  $k$ -tree assignment and  $r \geq 1$ . Then  $\{H(A) : [r]^k \subseteq A\}$  is a nonincreasing insertion domain in  $\text{TR}(k)$ .

Proof: Let  $S$  be a nonincreasing insertion domain in  $\text{TR}(k)$ . By Lemma B1.7, let  $(S, f)$  be a decreasing insertion rule in

$TR(k)$ . Let  $T_0 \in S$ ,  $Ch(T_0) \subseteq [r]^k$ ,  $r \geq 1$ . We now define a  $k$ -tree assignment  $H$  as follows. Let  $A \subseteq [N]^k$  be finite. Let  $A' = Ch(T_0) \cup (A \setminus [r]^k)$ . Let  $T$  be the unique  $k$ -tree generated by  $f$  from  $T_0$  with  $Ch(T) = A'$ . Set  $H(A)$  to be the  $k$ -tree such that

- i)  $Ch(H(A)) = A$ ;
- ii) whose parent function on  $A \cap [r]^k$  is identically  $\infty$ ;
- iii) if  $[r]^k \subseteq A$  then the parent function of  $H(A)$  agrees with the parent function of  $T$  on  $A \setminus [r]^k$ ;
- iv) otherwise, the parent function of  $H(A)$  is identically  $\infty$ .

We claim that  $H$  is decreasing\*. Let  $A \subseteq B \subseteq [N]^k$  be finite and  $x$  dominate  $B$ , where  $H(A) \subseteq H(B)$ . We first establish that  $p(x, H(AU\{x\})) \geq^* p(x, H(BU\{x\}))$ . First suppose  $[r]^k \not\subseteq A$ . If  $x \in [r]^k$  then  $p(x, H(AU\{x\})) = \infty$ , and we are done. If  $x \notin [r]^k$  then  $[r]^k \not\subseteq AU\{x\}$ , and so the parent function of  $H(AU\{x\})$  is identically  $\infty$ , and we are done. Also we are done if  $x \in [r]^k$ .

Now suppose that  $[r]^k \subseteq A$  and  $x \notin [r]^k$ . We have  $(AU\{x\})' = A' \cup \{x\}$  and  $(BU\{x\})' = B' \cup \{x\}$ . Let  $T_1$  be the unique  $k$ -tree generated by  $f$  from  $T_0$  with  $Ch(T_1) = A'$ , and  $T_2$  be the unique  $k$ -tree generated by  $f$  from  $T_0$  with  $Ch(T_2) = B'$ .

We claim that  $T_1 \subseteq T_2$ . Since  $H(A) \subseteq H(B)$ , the parent function of  $T_1$  off of  $[r]^k$  is contained in the parent function of  $T_2$  off of  $[r]^k$ . The parent function of  $T_1$  in  $[r]^k$  and the parent function of  $T_2$  in  $[r]^k$  are both the same as the parent function of  $T_0$ . Thus the claim is established.

Let  $T_3$  be the unique  $k$ -tree generated by  $f$  from  $T_0$  with  $Ch(T_3) = A' \cup \{x\}$  and  $T_4$  be the unique  $k$ -tree generated by  $f$  from  $T_0$  with  $Ch(T_4) = B' \cup \{x\}$ . Since  $(S, f)$  is decreasing and  $T_1 \subseteq T_2$ , we have  $f(x, T_1) \geq^* f(x, T_2)$ , and hence  $p(x, T_3) \geq^* p(x, T_4)$ . Therefore  $p(x, H(AU\{x\})) \geq^* p(x, H(BU\{x\}))$ .

We now verify that  $H(B) \subseteq H(BU\{x\})$ . By the definition of generation and that  $x$  dominates  $B$ , we see that  $T_2 \subseteq T_4$ . If  $[r]^k \subseteq B$  then by inspection,  $H(B) \subseteq H(BU\{x\})$ . Suppose  $[r]^k \not\subseteq B$ . If  $x \in [r]^k$  then the parent functions of  $H(B)$  and

$H(BU\{x\})$  are both identically  $\infty$ , and we are done. If  $x \notin [r]^k$  then  $[r]^k \not\subseteq BU\{x\}$ . Hence again the parent functions of  $H(B)$  and  $H(BU\{x\})$  are both identically  $\infty$ , and we are done. This concludes the proof that  $H$  is decreasing\*.

Now let  $A \subseteq [N]^k$  be finite, where  $[r]^k \subseteq A$ . Let  $T$  be the unique  $k$ -tree generated by  $f$  from  $T_0$  such that  $\text{Ch}(T) = A'$ . Then by inspection,  $T$  and  $H(A)$  have the same parent function off of  $[r]^k$ .

For the second claim, note that the set of all values of  $H$  is a nonincreasing initial insertion domain  $S$  in  $\text{TR}(k)$ . Obviously the set of all elements of  $S$  whose children contain  $[r]^k$  is a nonincreasing insertion domain in  $\text{TR}(k)$ , thus establishing the claim.

LEMMA B2.7. Propositions B2.4 and Theorem A1.1 are equivalent in  $\text{RCA}_0$ .

Proof: Assume Proposition B2.4 and let  $S$  be a nonincreasing insertion domain in  $\text{TR}(k)$ . By Lemma B2.6, let  $H$  be a decreasing\*  $k$ -tree assignment and  $r \geq 0$  such that for all finite  $A \subseteq [N]^k$  with  $[r]^k \subseteq A$ , there exists  $T \in S$  such that  $T$  and  $H(A)$  have the same parent function off of  $[r]^k$ , and where for all  $x \in \text{Ch}(T) \cap [r]^k$ ,  $p(x, T) \in [r]^k \cup \{\infty\}$ .

By Proposition B2.4, let  $A \subseteq [N]^k$  and  $E \subseteq N$ , where  $E$  has  $p+r$  elements, and every  $k$  element subset of  $E$  are vertices in  $H(A)$  with the same number of ancestors, and  $[r]^k \subseteq A$ . Let  $T \in S$  be such that  $T$  and  $H(A)$  have the same parent function off of  $[r]^k$ , and where for all  $x \in \text{Ch}(T) \cap [r]^k$ ,  $p(x, T) \in [r]^k \cup \{\infty\}$ . Then  $T$  and  $H(A)$  have the same parent function on  $(E \setminus [r])^k$ . Hence all  $k$  element subsets of  $E \setminus [r]$  are vertices of  $T$  and  $H(A)$ , and have the same ancestors in  $T$  and in  $H(A)$  up through the first point (if any) at which the ancestors enter  $\text{Ch}(T) \cap [r]^k$ ; and then the ancestors in  $T$  stay in  $[r]^k \cup \{\infty\}$ . Hence all  $k$  element subsets of  $E \setminus [r]$  are vertices of  $T$  with the same number of ancestors.

Now assume Theorem A1.1 and let  $H$  be a decreasing\*  $k$ -tree assignment. By Lemma B2.2, the set of values of  $H$  is a nonincreasing initial insertion domain  $S$  in  $\text{TR}(k)$ . Hence  $S' =$  the set of all elements of  $S$  whose children contain  $[r]^k$  is a

nonincreasing insertion domain in  $TR(k)$ . Now apply Theorem A1.1 to  $S'$ .

THEOREM B2.8. Theorems A1.1, A2.1, A3.1, A3.3, A1.1\*, A3.3\*, A1.1', A2.1', A3.1', A3.3', and Propositions B2.1, B2.4 are equivalent in  $RCA_0 + WKL$ . Theorems A1.1, A2.1, A3.1, A3.3, A1.1\*, A3.3\*, and Propositions B2.1, B2.4 are equivalent in  $RCA_0$ .

Proof: By Theorem B1.13 and Lemmas B2.3, B2.5, and B2.7.

We conclude with an implication from the following strengthened form of Proposition B2.1 to Proposition A3.4. This implication will be used in section C.

PROPOSITION B2.1#. Let  $k, p \geq 1$  and  $H$  be a decreasing\*  $k$ -tree assignment. There exists finite  $A \subseteq [N]^k$  and  $E \subseteq N$  of cardinality  $p$  such that the following holds.

i) for all  $x, y \in [E]^k$ ,  $x, y$  are vertices in  $H(A)$  with the same number of ancestors;

ii) let  $x, y \in [E]^k$  have the same first  $i$  elements,  $0 \leq i < k$ . Then every ancestor of  $x$  in  $H(A)$  of the form  $\{x_1, \dots, x_i, z_1, \dots, z_{k-i}\}$  with  $x_i < z_1, \dots, z_{k-i} < x_{i+1}$ , is an ancestor of  $y$  in  $H(A)$  with  $z_1, \dots, z_{k-i} < y_{i+1}$ .

PROPOSITION B2.9. Let  $k, p, r \geq 1$ . Every decreasing\*  $k$ -tree assignment has a value in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors, and whose children include  $[r]^k$ . Moreover, these vertices can be required to have the same number of ancestors.

LEMMA B2.10. Proposition B2.1# implies Proposition B2.9 in  $RCA_0$ .

Proof: We follow the forward direction of the proof of Lemma B2.5. Assume Proposition 2.1#, and let  $k, p, r \geq 1$  and  $H$  be a decreasing\*  $k$ -tree assignment. We construct the decreasing\*  $s+k$ -tree assignment  $H'$  as before. Now choose  $p' \gg k, r, p$  so that we may apply the finite Ramsey theorem to  $p'$  below.

By Proposition B2.1#, let  $A \subseteq [N]^{s+k}$  be finite,  $E \subseteq N$  have cardinality  $p'$ , where

i) for all  $z, w \in [E]^{s+k}$ ,  $z, w$  are vertices in  $H'(A)$  with the same number of ancestors;

ii) let  $z, w \in [E]^{s+k}$  have the same first  $s$  elements. Then every ancestor of  $z$  in  $H'(A)$  of the form  $\{z_1, \dots, z_s, u_1, \dots, u_k\}$  with  $z_s < u_1, \dots, u_k < z_{s+1}$ , is an ancestor of  $w$  in  $H'(A)$  with  $u_1, \dots, u_k < w_{s+1}$ .

Let  $E' = \{E_{3s+1}, \dots, E_p\}$  and  $x = \{E_{2s+1}, \dots, E_{3s}\}$ . Then  $x \in A'$  and  $[E']^k \subseteq A/x$ .

It remains to show that all  $y \in [E']^k$  have the same entirely lower ancestors and number of ancestors in the  $k$ -tree  $H(A/x)$ . As in the proof of the forward direction of Lemma B2.5, we have to replace  $E'$  with a suitable  $E'' \subseteq E'$ .

The ancestors of  $xUy$  in  $H'(A)$  appear without repetition as  $xUy_1, \dots, xUy_t, z, \dots, \infty$ , where  $t \geq 0$ , and  $z$  is either  $\infty$  or the  $<^*$ -least element of  $A$  with  $\max(z) = x_i$ ,  $1 \leq i \leq s$ . We know that the length of this sequence is independent of the choice of  $y \in [E']^k$ .

As in the proof of the forward direction of Lemma B2.5, note that  $y_1, \dots, y_t$  is a listing of the ancestors of  $y$  in  $H(A/x)$  up to the point where the ancestors go into  $[r]^k \cup \{\infty\}$ . Also, the entirely lower ancestors of  $y$  in  $H(A/x)$  consist of the  $y_1, \dots, y_t$  that are entirely lower than  $y$ , together with  $v_i$  (if it exists) and all of its finite ancestors in  $H([r]^k)$ . Then by ii) above, the the entirely lower terms among the  $y_1, \dots, y_t$  are independent of the choice of  $y \in [E']^k$ . It therefore suffices to prove that  $t$  and  $z$  are independent of the choice of  $y \in [E']^k$ . We can't show this, but we can show this for some  $E'' \subseteq E'$  of cardinality  $p$ , just as we did in the proof of the forward direction of Lemma B2.5.

LEMMA B2.11. Propositions B2.9 and A1.2 are equivalent in  $RCA_0$ .

Proof: We follow the proof of Lemma B2.7. Assume Proposition B2.9 and let  $S$  be a nonincreasing insertion domain in  $TR(k)$ . By Lemma B2.6, let  $H$  be a decreasing\*  $k$ -tree assignment and  $r \geq 0$  such that for all finite  $A \subseteq [N]^k$  with  $[r]^k \subseteq A$ , there exists  $T \in S$  such that  $T$  and  $H(A)$  have the same parent

function off of  $[r]^k$ , and where for all  $x \in \text{Ch}(T) \cap [r]^k$ ,  $p(x, T) \in [r]^k \cup \{\infty\}$ .

By Proposition B2.9, let  $A \subseteq [N]^k$  and  $E \subseteq N$ , where  $E$  has  $p+r$  elements, and every  $k$  element subset of  $E$  are vertices in  $H(A)$  with the same number of ancestors and with the same entirely lower ancestors, and  $[r]^k \subseteq A$ . Let  $T \in S$  be such that  $T$  and  $H(A)$  have the same parent function off of  $[r]^k$ , and where for all  $x \in \text{Ch}(T) \cap [r]^k$ ,  $p(x, T) \in [r]^k \cup \{\infty\}$ . Then  $T$  and  $H(A)$  have the same parent function on  $(E \setminus [r])^k$ . Hence all  $k$  element subsets of  $E \setminus [r]$  are vertices of  $T$  and  $H(A)$ , and have the same ancestors in  $T$  and in  $H(A)$  up through the first point (if any) at which the ancestors enter  $\text{Ch}(T) \cap [r]^k$ ; and then the ancestors in  $T$  stay in  $[r]^k \cup \{\infty\}$ . Hence all  $k$  element subsets of  $E \setminus \{r\}$  are vertices of  $T$  with the same number of ancestors, and also with the same entirely lower ancestors.

The reverse direction is proved using Lemma B2.2 according to the final paragraph in the proof of Lemma B2.7.

THEOREM B2.12. Proposition B2.1# implies Propositions A1.2, A2.2, A3.2, A3.4, A1.2\*, A3.4\*, and B2.9 in  $\text{RCA}_0$ . Proposition B2.1# implies Propositions A1.2, A2.2, A3.2, A3.4, A1.2\*, A3.4\*, A1.2', A2.2', A3.2', A3.4' and B2.9 in  $\text{RCA}_0 + \text{WKL}$ . These implications also hold if we state the propositions without "same number of ancestors."

Proof: By Theorem B1.13 and Lemmas B2.10, B2.11.

### C. PROOFS IN $Z$ ; PROOFS USING LARGE CARDINALS

In this section we give a proof in  $Z$  of Proposition B2.1 and a proof in  $ZFC + (\forall k)$  (there exists a  $k$ -subtle cardinal) of Proposition B2.1#. According to Theorem B2.8, this proves all the theorems in sections A1 - A4 in  $Z$ . And according to Theorem B2.12, this proves all the propositions in sections A1 - A4 in  $ZFC + (\forall k)$  (there exists a  $k$ -subtle cardinal). See [Ba73] and [Fr98] for a discussion of the subtle cardinal hierarchy.

We begin by recalling Proposition C of [Fr98]. We simplify the discussion by focusing on the relevant specialization of Proposition C, where  $<_1$  and  $<_2$  are both the lexicographic ordering of  $N^k$ .

A function assignment for  $N^k$  is a mapping  $U$  which assigns to each finite subset  $A$  of  $N^k$ , a unique function

$$U(A):A \rightarrow A.$$

Let  $k \geq 1$ . We write  $<_{lex}$  and  $\leq_{lex}$  indicate the lexicographic ordering of  $N^k$ .

We say that  $U$  is lex #-decreasing if and only if for all finite  $A \subseteq N^k$  and  $x \in N^k$ ,

$$\text{either } U(A) \subseteq U(AU\{x\}) \text{ or there exists } y >_{lex} x \text{ such that } U(A)(y) >_{lex} U(AU\{x\})(y).$$

PROPOSITION C [Fr98]. Let  $k, p > 0$  and  $U$  be a lex #-decreasing function assignment for  $N^k$ . Then some  $U(A)$  has  $\leq k^k$  regressive values on some  $E^k \subseteq A$  of cardinality  $p^k$ .

The following is proved in [Fr98].

THEOREM 4.18 [Fr98]. Proposition C is provable in ZFC +  $(\forall k)$  (there exists a  $k$ -subtle cardinal).

We will be adapting this proof from [Fr98] for our purposes. Accordingly, we introduce the following adaptation of the terminology from [Fr98].

Let  $Y = (Y, \leq')$  be a linearly ordered set. Let  $f$  be a finite partial function from  $N^k$  into  $N^k$  and  $g$  be a finite partial function from  $Y^k$  into  $Y^k$ . We define  $fld(f)$  and  $fld(g)$  as the set of all coordinates of domain and range elements of  $f$  and  $g$ . Thus  $fld(f) \subseteq N$  and  $fld(g) \subseteq Y$ . An order isomorphism from  $f$  to  $g$  is an order preserving  $h:fld(f) \rightarrow fld(g)$  such that for all  $x, y$ ,  $f(x) = y$  if and only if  $g(h(x)) = g(h(y))$ , where  $h$  is extended to  $k$ -tuples coordinatewise.

Let  $U$  be a function assignment for  $N^k$  and  $Y$  be a linearly ordered set. We say that  $f$  is a completion of  $U$  on  $Y$  if and only if

- i)  $f: Y^k \rightarrow Y^k$ ;
- ii) for all finite  $B \subseteq Y^k$  there exists  $B \subseteq C \subseteq X^k$  such that  $f|_C$  is order isomorphic to some  $U(A)$ .

We say that  $U$  has the ordinal completion property if and only if  $U$  has a completion on every well ordered set  $Y$ .

The following crucial lemma (first claim) is an adaption of Theorem 4.9 (or Theorem 4.17) of [Fr98].

LEMMA C.1. For all  $k \geq 1$ , every lex #-decreasing function  $U$  assignment for  $N^k$  has the ordinal completion property. Every  $U(A)$  obeys  $U(A)(x) \leq_{\text{lex}} x$ . Every ordinal completion  $g$  of  $U$  obeys  $g(x) \leq_{\text{lex}} x$ .

Proof: For the first claim, see the proof of Theorem 4.9 in [Fr98]. The first claim also immediately follows from the statement of Theorem 4.17 in [Fr98]. The second claim obviously implies the third claim by the definition of ordinal completion. For the second claim, by Theorem 3.10 of [Fr98],  $U$  is lex end preserving, and hence each  $U(A)$  maps every initial segment of  $A$  under lex into itself.

We now introduce two propositions which provide the link between [Fr98] and this section.

PROPOSITION C.2. Let  $k, p \geq 1$  and  $U$  be a lex #-decreasing function assignment for  $N^k$ . Then there exists finite  $A \subseteq N^k$  and  $E \subseteq N$  of cardinality  $p$ , such that  $E^k \subseteq A$  and  $U(A)$  has the same number of distinct iterates at any two elements of  $E^k$  of the same order type.

PROPOSITION C.3. Let  $k, p \geq 1$  and  $U$  be a lex #-decreasing function assignment for  $N^k$ . Then there exists finite  $A \subseteq N^k$  and  $E \subseteq N$ ,  $E$  of cardinality  $p$ , such that  $E^k \subseteq A$  and the following holds.

- i)  $U(A)$  has the same number of distinct iterates at any two elements of  $E^k$  of the same order type;
- ii) Let  $x, y \in E^k$  be of the same order type and have the same first  $i$  elements,  $0 \leq i < k$ . Then every iterate of  $U(A)$  at  $x$  of the form  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(x_{i+1}, \dots, x_k))$ , is an iterate of  $y$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(y_{i+1}, \dots, y_k))$ .

Here we take the iterates of  $U(A)$  at  $x$  to start with  $x$ . Also  $(\max(x_1, \dots, x_i), \min(x_{i+1}, \dots, x_k))$  refers to the open interval with the given endpoints.

LEMMA C.4. Proposition C.2 is provable in  $Z$ .

Proof: Let  $k, p, U$  be as given. By Lemma C.1,  $U$  has the ordinal completion property.

We need the following combinatorial lemma. For all  $k \geq 1$ , there exists a cardinal  $\lambda$  such that the following holds. Let  $g: \lambda^k \rightarrow \lambda^k$  obey the inequality  $g(x) \leq_{\text{lex}} x$ . Then there exists an infinite set  $S \subseteq \lambda$  such that  $g$  has the same number of distinct iterates at any two elements of  $S^k$  of the same order type.

This is proved by taking  $\lambda$  to be a cardinal such that any map  $F$  from the unordered  $k$ -tuples from  $\lambda$  into  $N$  is constant on all unordered  $k$ -tuples from an infinite set  $S$ . This is afforded by the Erdos-Rado theorem (see [Ka94], p. 74), which involves iterating cardinal exponentiation approximately  $k$  times starting at  $\omega$ . Apply the Ramsey property of  $\lambda$  by choosing  $F$  such that each  $F(x)$  codes the number of distinct iterates of  $g$  at the various  $k$ -tuples from the  $k$  element  $x \subseteq \lambda$ .

Let  $g: \lambda^k \rightarrow \lambda^k$  be a completion of  $U$  on  $\lambda$ . By Lemma C.1,  $g$  obeys  $g(x) \leq_{\text{lex}} x$ . Choose  $S \subseteq \lambda$  according to the combinatorial lemma. Let  $B = \{S_1, \dots, S_p\}$ . Let  $C$  be the closure of  $B^k$  under  $g$ . I.e.,  $C$  consists of all of the iterates of the elements of  $B^k$  by  $g$ . Let  $C \subseteq D \subseteq \lambda^k$  and  $A \subseteq N^k$  be such that  $g|D$  is order isomorphic to  $U(A)$  via  $h$ . Set  $E = h[B]$ . Then the iterates of  $U(A)$  at the elements of  $E^k$  correspond exactly to the iterates of  $g$  at the elements of  $B^k$ . I.e., for all  $x \in B^k$ , if  $x_1, \dots, x_t$  is the listing of the iterates of  $x$  under  $g$  until  $g(x_t) = x_t$ , then  $h(x_1), \dots, h(x_t)$  is the listing of the iterates of  $h(x)$  under  $U(A)$  until  $U(A)(h(x_t)) = h(x_t)$ . But  $t$  depends only on the order type of  $x \in B^k$ . Hence for all  $x \in B^k$ , the number of iterates of  $h(x)$  under  $U(A)$  depends only on the order type of  $x$ . Hence the number of iterates of each  $y \in E^k$  under  $U(A)$  depends only on the order type of  $y$ .

We can obviously carry out the argument in ZC using a well ordering instead of a von Neumann cardinal. And we can get away with Z rather than ZC by the well known conservative extension result: every  $\Pi_1^1$  sentence provable in ZC is provable in Z.

LEMMA C.5. Proposition C.3 is provable in ZFC +  $\{\forall k\}$  (there exists a subtle cardinal of order  $k$ ).

Proof: Let  $k, p, U$  be as given. By Lemma C.1,  $U$  has the ordinal completion property.

By Theorem 2.11 from [Fr98], fix  $\lambda$  to be a  $k$ -ineffable cardinal. By elementary set theory, fix a one-one onto function  $f: \lambda^{<\omega} \rightarrow \lambda$ . Let  $W = \{x > \omega: x < \lambda \text{ and } f[x^{<\omega}] \subseteq x\}$ . Clearly  $W$  is closed and unbounded. Here  $x^{<\omega}$  is the set of all finite sequences from  $x$ . We use  $[\lambda]^k$  and  $[W]^k$  for the set of all  $k$  element subsets of  $\lambda$  and  $W$ .

We need the following combinatorial lemma. Let  $g: \lambda^k \rightarrow \lambda^k$  obey the inequality  $g(x) \leq_{\text{lex}} x$ . Then there exists an infinite set  $V \subseteq \lambda$  such that the following holds.

i)  $g$  has the same number of distinct iterates at any two elements of  $V^k$  of the same order type;

ii) Let  $x, y \in V^k$  have the same order type and the same first  $i$  elements,  $0 \leq i < k$ . Then every iterate of  $g$  at  $x$  of the form  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(x_{i+1}, \dots, x_k))$ , is an iterate of  $y$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(y_{i+1}, \dots, y_k))$ .

To prove this combinatorial lemma, we define an auxiliary regressive function  $h: [\lambda]^k \rightarrow [\lambda]^k$  as follows. For  $A \in [W]^k$ , define  $h'(A) = \{(u, n, w) : n \in \omega, n \text{ codes } v \in A^n, (u, v), (u, w) \in \lambda^k, \max(u) < \min(A), \text{ every coordinate of } w \text{ lies in } (\max(u), \min(v))^k, \text{ and } (u, w) \text{ is an iterate of } g \text{ at } (u, v)\} \times \{(n, m) : \text{if } x \text{ is the element of } A^k \text{ coded by } n \text{ then the number of distinct iterates of } g \text{ at } x \text{ is } m\}$ . Let  $h(A)$  be the coding up of  $h'(A)$  as a subset of  $\min(A)$  using  $f$ . Define  $h(A)$  arbitrarily for  $A \notin [W]^k$ .

Let  $V$  be a stationary subset of  $\lambda$  which is  $h$ -homogeneous. Since  $W$  is closed and unbounded in  $\lambda$ , we may assume that  $V \subseteq$

W. Then  $h$  is constant on  $[V]^k$ . Let  $x, y \in V^k$  be of the same order type and have the same first  $i$  elements,  $0 \leq i < k$ . Let  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  be an iterate of  $g$  at  $x$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(x_{i+1}, \dots, x_k))$ . Then  $\max(x_1, \dots, x_i) < \min(x_{i+1}, \dots, x_k)$ , and so  $\max(x_1, \dots, x_i) = \max(y_1, \dots, y_i) < \min(y_{i+1}, \dots, y_k)$ . Choose  $A, B \in [V]^k$  such that  $\{x_{i+1}, \dots, x_k\}$  forms an initial segment of  $A$  and  $\{y_{i+1}, \dots, y_k\}$  forms an initial segment of  $B$ . Let  $n$  code the order type of  $x_{i+1}, \dots, x_k$  = the order type of  $y_{i+1}, \dots, y_k$ . Then  $(x_1, \dots, x_i, n, z_1, \dots, z_{k-i})$  is coded in  $h(A) = h(B)$ . Hence  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  is an iterate of  $g$  at  $(x_1, \dots, x_i, w)$  with  $z_1, \dots, z_{k-i} < \min(y_{i+1}, \dots, y_k)$ , where  $n$  codes  $w \in B^{k-i}$ . But then  $w = (y_{i+1}, \dots, y_k)$ . Hence  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  is an iterate of  $g$  at  $(x_1, \dots, x_i, y_{i+1}, \dots, y_k)$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(x_{i+1}, \dots, x_k))$  as required by clause ii) in the combinatorial lemma. Clause i) is easily verified by the constancy of  $\{(n, m) : \text{if } x \text{ is the element of } A^k \text{ coded by } n \text{ then the number of distinct iterates of } g \text{ at } x \text{ is } m\}$  over  $A \in [V]^k$ .

Now let  $g: \lambda^k \rightarrow \lambda^k$  be a completion of  $U$  on  $\lambda$ . By Lemma C.1,  $g$  obeys  $g(x) \leq_{\text{lex}} x$ . Choose  $V \subseteq \lambda$  according to the combinatorial lemma. Let  $B = \{V_1, \dots, V_p\}$ . Again let  $C$  be the closure of  $B^k$  under  $g$ . Let  $C \subseteq D \subseteq \lambda^k$  and  $A \subseteq N^k$  be such that  $g|D$  is order isomorphic to  $U(A)$  via  $h$ . Set  $E = h[B]$ . Then as in the proof of Lemma C.4, the iterates of the elements of  $E^k$  by  $U(A)$  correspond exactly to the iterates of the elements of  $B$  by  $g$ . This establishes that the number of iterates of  $U(A)$  at  $y \in E^k$  depends only on the order type of  $y$ . This verifies conclusion i) in Proposition C.3.

Let  $x, y \in B^k$  have same order type and the same first  $i$  elements,  $0 \leq i < k$ . Let  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  be an iterate of  $g$  at  $x$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(x_{i+1}, \dots, x_k))$ . Then by the choice of  $V$ ,  $(x_1, \dots, x_i, z_1, \dots, z_{k-i})$  is also an iterate of  $g$  at  $y$  with  $z_1, \dots, z_{k-i} \in (\max(x_1, \dots, x_i), \min(y_{i+1}, \dots, y_k))$ . But this property is preserved under the order isomorphism  $h$  from  $g|D$  onto  $U(A)$ , since all iterates of  $x, y$  are present in  $D$ . This verifies conclusion ii) in Proposition C.3.

We now show that Propositions C.2 and C.3 respectively imply Propositions B2.1 and B2.1#.

In [Fr98], we often work with  $*$ -decreasing, which is sometimes more convenient than  $\#$ -decreasing. Following the definition given right after the proof of Lemma 3.2 in [Fr98], we say that  $U$  is lex  $*$ -decreasing if and only if the following holds. Let  $A, B \subseteq N^k$  be finite. Suppose that  $x \in A \cap B$ , and for all  $y \in A$ , if  $y <_{\text{lex}} x$  then  $U(A)(y) = U(B)(y)$ . Then  $U(A)(x) \succeq_{\text{lex}} U(B)(x)$ .

For our purposes, it is useful to break lex  $*$ -decreasing into two parts. Let  $U$  be a function assignment for  $N^k$ . We say that  $U$  is weakly lex  $*$ -decreasing if and only if the following holds. Suppose that  $A, B \subseteq N^k$  are finite, and suppose that  $x$  is the lex greatest element of  $A$  and the lex greatest element of  $B$ . Suppose also that for all  $y \in A$ , if  $y <_{\text{lex}} x$  then  $U(A)(y) = U(B)(y)$ . Then  $U(A)(x) \succeq_{\text{lex}} U(B)(x)$ .

We say that  $U$  is singly lex end preserving if and only if for all finite  $A \subseteq N^k$  and  $x \in N^k$ , if  $x >_{\text{lex}}$  every element of  $A$ , then  $U(A) \subseteq U(A \cup \{x\})$ . We say that  $U$  is lex end preserving if and only if for finite  $A, B \subseteq N^k$  and  $x \in N^k$ , if  $A \subseteq B$  and for all  $y \in B \setminus A$ ,  $y$  is  $>_{\text{lex}}$  every element of  $A$ , then  $U(A) \subseteq U(B)$ .

LEMMA C.6. The following is provable in  $\text{RCA}_0$ . Let  $U$  be a function assignment for  $N^k$ . Then the following are equivalent:

- i)  $U$  is lex  $*$ -decreasing;
- ii)  $U$  is lex  $\#$ -decreasing;
- iii)  $U$  is weakly lex  $*$ -decreasing and  $U$  is lex end preserving.
- iv)  $U$  is weakly lex  $*$ -decreasing and  $U$  is singly lex end preserving.

Proof: Assume i). By Theorem 3.10 of [Fr98], we can take  $<_1 = <_2 = <_{\text{lex}}$  on  $N^k$ , and conclude ii).

Assume ii). Again by Theorem 3.10, we have i), and obviously  $U$  is weakly lex  $*$ -decreasing. Also by Theorem 3.10,  $U$  is lex end preserving. Hence iii) holds.

Obviously iii) implies iv). Finally suppose iv). By Lemma 3.2 of [Fr98],  $U$  is lex end preserving.

We will show i). Let  $A \subseteq B \subseteq N^k$  be finite,  $x \in A \cap B$ , and suppose that for all  $y \in A$ , if  $y <_{\text{lex}} x$  then  $U(A)(y) = U(B)(y)$ . Let  $A' = \{y \in A: y \leq_{\text{lex}} x\}$  and  $B' = \{y \in B: y \leq_{\text{lex}} x\}$ . Then by lex end preserving, we have  $U(A') \subseteq U(A)$  and  $U(B') \subseteq U(B)$ . Hence for all  $z \in A'$ , if  $z <_{\text{lex}} x$  then  $U(A')(z) = U(B')(z)$ . Therefore by weak lex \*-decreasing, we have  $U(A')(x) \geq_{\text{lex}} U(B')(x)$ . Hence  $U(A)(x) \geq_{\text{lex}} U(B)(x)$ .

We now show that Propositions C.2 and C.3 respectively imply Propositions B2.1 and B2.1#.

Fix  $k, p \geq 1$  and a decreasing\*  $k$ -tree assignment  $H$ .

For  $k \geq 1$  let  $N^{k>}$  be the set of all strictly decreasing elements of  $N^k$ . For every  $x \in N^{k>}$ , let  $x\#$  be the set of all coordinates of  $x$ . For every  $A \subseteq N^k$ , let  $A\# = \{x\#: x \in A \cap N^{k>}\}$ . For  $x \in [N]^{k>}$ , let  $x^\wedge = \{x_k, \dots, x_1\}$ .

We now define the function assignment  $U$  for  $N^k$  as follows. Let  $A \subseteq N^k$  be finite and  $x \in A$ . We must define  $U(A)(x)$ .

case 1.  $x \in N^{k>}$  and  $p(x\#, H(A\#)) \neq \infty$ . Define  $U(A)(x) = p(x\#, H(A\#))^\wedge$ .

case 2. otherwise. Define  $U(A)(x) = x$ .

LEMMA C.7. The following is provable in  $\text{RCA}_0$ . Suppose  $U(A) \subseteq U(B)$ . Then  $H(A\#) \subseteq H(B\#)$ .

Proof: Let  $A, B$  be as given. Then  $A \subseteq B$  and  $A\# \subseteq B\#$ . Let  $x \in A\#$ . Then  $U(A)(x^\wedge) = p(x^\wedge\#, H(A\#))^\wedge$  if  $p(x^\wedge\#, H(A\#)) \neq \infty$ ;  $x^\wedge$  otherwise. Hence  $U(A)(x^\wedge) = p(x, H(A\#))^\wedge$  if  $p(x, H(A\#)) \neq \infty$ ;  $x^\wedge$  otherwise.

Similarly,  $U(B)(x^\wedge) = p(x, H(B\#))^\wedge$  if  $p(x, H(B\#)) \neq \infty$ ;  $x^\wedge$  otherwise. Suppose  $p(x, H(B\#)) \neq \infty$  and  $p(x, H(A\#)) \neq \infty$ . Then  $p(x, H(A\#))^\wedge = p(x, H(B\#))^\wedge$ .

Suppose  $p(x, H(A\#)) \neq \infty$  and  $p(x, H(B\#)) = \infty$ . Then  $U(A)(x^\wedge) = p(x, H(A\#))^\wedge$  and  $U(B)(x^\wedge) = x^\wedge$ , and so  $p(x, H(A\#))^\wedge = x^\wedge$ , which is a contradiction. Similarly we can discard the case  $p(x, H(B\#)) \neq \infty$  and  $p(x, H(A\#)) \neq \infty$ .

Thus we have shown that  $p(x, H(A\#)) = p(x, H(B\#))$ .

LEMMA C.8. The following is provable in  $\text{RCA}_0$ . The function assignment  $U$  is weakly  $*$ -decreasing.

Proof: We first claim that for all finite  $A \subseteq N^k$  and  $x \in A$ ,  $U(A)(x) \leq_{\text{lex}} x$ . To see this, fix  $A, x$  and suppose that  $x \in N^{k>}$  and  $p(x\#, H(A\#)) \neq \infty$ . Since  $H$  is decreasing $*$ , we have  $p(x\#, H(A\#)) <^* x\#$  and  $p(x\#, H(A\#)) \in A\#$ . Hence  $U(A)(x) = p(x\#, H(A\#)) \wedge <_{\text{lex}} x\# \wedge = x$ . If  $\neg(x \in N^{k>})$  and  $p(x\#, H(A\#)) \neq \infty$ , then  $U(A)(x) = x$ .

Suppose  $A \subseteq B \subseteq N^k$  are both finite, and  $x$  is the lex greatest element of  $A$  and of  $B$ . Also assume that for all  $y \in A$ , if  $y <_{\text{lex}} x$  then  $U(A)(y) = U(B)(y)$ . We wish to prove that  $U(A)(x) \geq_{\text{lex}} U(B)(x)$ .

By the first claim, we may assume without loss of generality that  $x \in N^{k>}$  and  $p(x\#, H(A\#)) \neq \infty$ ; for otherwise,  $U(A)(x) = x \geq_{\text{lex}} U(B)(x)$ .

Now  $A\# \subseteq B\#$  and  $(A \setminus \{x\})\# \subseteq (B \setminus \{x\})\#$ . Also  $(A \setminus \{x\})\# = A\# \setminus \{x\#$ , and  $(B \setminus \{x\})\# = B\# \setminus \{x\#$ . Hence  $U(A\# \setminus \{x\#}) \subseteq U(B\# \setminus \{x\#})$ .

Therefore by Lemma C.7,  $H(A\# \setminus \{x\#}) \subseteq H(B\# \setminus \{x\#})$ . Since  $H$  is decreasing $*$  and  $x\#$  dominates  $B\#$ , we see that  $p(x\#, H(A\#)) \geq^* p(x\#, H(B\#))$ . Since  $p(x\#, H(A\#)) \neq \infty$ , we see that  $p(x\#, H(B\#)) \neq \infty$ . Hence  $U(A)(x) = p(x\#, H(A\#)) \wedge \geq_{\text{lex}} p(x\#, H(B\#)) = U(B)(x)$ .

LEMMA C.9. The following is provable in  $\text{RCA}_0$ . The function assignment  $U$  is singly lex end preserving.

Proof: Let  $A \subseteq N^k$  be finite and  $x \in N^k$  be  $>_{\text{lex}}$  all elements of  $A$ . We must verify that  $U(A) \subseteq U(A \cup \{x\})$ . Let  $y \in A$ . If  $y \notin N^{k>}$  then  $U(A)(y) = U(A \cup \{x\})(y) = y$ . So we may assume  $y \in N^{k>}$ . Note that  $x\#$  is  $>^*$  all elements of  $A\#$ . Since  $H$  is decreasing $*$ ,  $H(A\#) \subseteq H(A\# \cup \{x\#}) = H((A \cup \{x\})\#)$ . Hence  $p(y\#, H(A\#)) = p(y\#, H((A \cup \{x\})\#))$ . Therefore  $U(A)(y) = U(A \cup \{x\})(y)$ .

LEMMA C.10. The following is provable in  $\text{RCA}_0$ . The function assignment  $U$  is  $\#$ -decreasing.

Proof: By Lemmas C.6, C.8 and C.9.

LEMMA C.11. Proposition C.2 implies Proposition B2.1 in  $\text{RCA}_0$ . Proposition C.3 implies Proposition B2.1# in  $\text{RCA}_0$ .

Proof: We have already fixed  $k, p, r \geq 1$  and a decreasing\*  $k$ -tree assignment  $H$ , defined the associated  $U$ , and proved that  $U$  is #-decreasing. Assume Proposition C.2. Fix finite  $A \subseteq \mathbb{N}^k$  and  $E \subseteq \mathbb{N}$ ,  $E$  of cardinality  $p$ , such that  $U(A)$  has the same number,  $t$ , of distinct iterates at the various elements of  $E^{k>}$ . Let  $x \in E^{k>}$  and  $U(A) = F$ . Then the iterates of  $F$  at  $x$  are of the form  $x >_{\text{lex}} F(x) >_{\text{lex}} FF(x) >_{\text{lex}} \dots >_{\text{lex}} F^t(x)$ , where  $F^{t+1}(x) = F^t(x)$ , and  $t \geq 0$  is independent of the choice of  $x \in E^{k>}$ . Also  $x, F(x), \dots, F^t(x) \in \mathbb{N}^{k>}$ . The case  $t = 0$  corresponds to  $x = F(x)$ . Thus the ancestors of  $x\#$  in  $H(A\#)$  are  $p(x\#, H(A\#)) >^* pp(x\#, H(A\#)) >^* \dots >^* p^t(x\#, H(A\#))$  followed by  $\infty$ . These parents are  $F(x)^\wedge, \dots, F^t(x)^\wedge$ . Hence in  $H(A\#)$ , the  $k$  element subsets of  $E$  are vertices with the same number of ancestors. Furthermore, since  $\{0, \dots, r-1\}^k \subseteq A$ , we see that  $[r]^k \subseteq A\# = \text{Ch}(H(A\#))$ . Hence we have derived Proposition B2.1.

Now assume Proposition C.3. We argue analogously to the above. We have that any two elements of  $E^{k>}$  have the same entirely lower ancestors. Let  $x, y \in E^{k>}$ . Then the terms in  $F(x), \dots, F^t(x)$  that are entirely lower than  $x$  are the same as the terms in  $F(y), \dots, F^t(y)$  that are entirely lower than  $y$ . Hence the terms in  $p(x\#, H(A\#))^\wedge, \dots, p^t(x\#, H(A\#))^\wedge$  that are entirely lower than  $x$  are the same as the terms in  $p(y\#, H(A\#))^\wedge, \dots, p^t(y\#, H(A\#))^\wedge$  that are entirely lower than  $y$ . Therefore the terms in  $p(x\#, H(A\#)), \dots, p^t(x\#, H(A\#))$  that are entirely lower than  $x\#$  are the same as the terms in  $p(y\#, H(A\#)), \dots, p^t(y\#, H(A\#))$  that are entirely lower than  $y\#$ . I.e.,  $x\#$  and  $y\#$  have the same entirely lower ancestors in the  $k$ -tree  $H(A\#)$ .

THEOREM C.12. All Theorems in sections A1 - A4 as well as Propositions B2.1 and B2.4 are provable in  $Z$ . All Propositions in sections A1 - A4 as well as Propositions B2.1# and B2.9 are provable in  $ZFC + (\forall k)$  (there exists a  $k$ -subtle cardinal).

Proof: From Lemmas C.4, C.5, C.11, and Theorems B2.8, B2.12.

We conclude this section with the following conjectures.

i) all of the theorems of sections A1 - A4 and Propositions B2.1, B2.4 are equivalent in  $RCA_0 + WKL$ , and in fact equivalent in  $RCA_0 + WKL$  to the 1-consistency of Zermelo set theory with bounded separation;

ii) all of the propositions of sections A1 - A4 and Propositions B2.1#, B2.9, either fully stated or stated without "same number of ancestors," are equivalent in  $RCA_0 + WKL$ , and in fact equivalent in  $RCA_0 + WKL$  to the 1-consistency of  $ZFC + \{\text{there exists a } k\text{-Mahlo cardinal}\}_k$ ;

iii) if we disregard the theorems and propositions of A4, then we can use  $RCA_0$  instead of  $RCA_0 + WKL$  in conjectures i) and ii).

#### D. NECESSITY OF LARGE CARDINALS

In this section, we show that the following proposition cannot be proved in  $ZFC + \{\text{there exists a } k\text{-subtle cardinal}\}_k$ , provided the latter is consistent. See [Ba73] and [Fr98] for a discussion of the subtle cardinal hierarchy.

PROPOSITION D.1. Let  $k, p \geq 1$ . Every decreasing initial insertion rule in  $TR(k)$  generates a  $k$ -tree in which all  $k$  element subsets of some  $p$  element set are vertices with the same entirely lower ancestors.

Proposition D.1 is Proposition A3.4\* without "same number of ancestors." Note that by Theorems B1.13 and B2.12, all Propositions in sections A1 - A4 and Propositions B2.1, B2.4, B2.1#, B2.9 each individually imply Proposition D.1 in  $RCA_0 + WKL$ , even without "same number of ancestors." Thus the same unprovability result holds for these other Propositions.

We will ultimately derive Lemma 5.2 of [Fr97] from Proposition D.1 within  $RCA_0$ . We then obtain our unprovability results from [Fr98].

We now introduce an intermediate Proposition which is a weakening of Lemma 5.2 in [Fr98]. We use the following definitions from [Fr98], Part 5.

For  $k \geq 1$ , let  $\text{FPF}(N^k)$  be the set of all finite partial functions from  $N^k$  into  $N$ ; i.e., functions whose domain is a finite subset of  $N^k$  and whose range is a subset of  $N$ .

For  $A \subseteq N^k$ , we define  $\text{fld}(A)$  to be the set of all coordinates of elements of  $A$ .

For  $x \in N^k$ , we write  $\max(x)$  for the maximum coordinate of  $x$ , and  $\min(x)$  for the minimum coordinate of  $x$ .

Let  $f \in \text{FPF}(N^k)$  and  $x \in N^k$ . We write  $f|_x$  for the restriction of  $f$  to  $\{y: \max(y) < \max(x)\}$ .

Let  $\text{DFNL}(N^k)$  be the set of all  $H: \text{FPF}(N^k) \times N^k \rightarrow N$  such that for all  $f \subseteq g$  from  $\text{FPF}(N^k)$  and  $x \in N^k$ ,

- i)  $H(f, x) \geq H(g, x)$ ;
- ii)  $H(f, x) \in \text{fld}(\text{dom}(f)) \cup \{x_1, \dots, x_n\}$ .

Let  $A \subseteq N^k$  be finite. We define  $\text{RCN}(A, H)$  to be the unique function  $F: A \rightarrow N$  such that for all  $x \in A$ ,  $F(x) = H(F|_x, x)$ . Note that  $F: A \rightarrow \text{fld}(A)$  can be proved by induction on  $\max(x)$  for  $x \in A$ .

(Actually, in [Fr98], we defined  $\text{RCN}(A, H)$  to be the unique function  $F: A \rightarrow \text{fld}(A)$  satisfying this same condition. But it is easy to see that the two definitions are equivalent).

For  $E \subseteq N$ , let  $E^{k<}$  be the set of all increasing  $k$ -tuples from  $E$ .

PROPOSITION D.2. Let  $k, p \geq 1$  and  $H \in \text{DFNL}(N^k)$ . Then there exists finite  $A \subseteq N^k$  and  $E \subseteq N$  of cardinality  $p$  such that  $E^k \subseteq A$ , and either

- i) for all  $x \in E^{k<}$ ,  $\text{RCN}(A, H)(x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{k<}$ ,  $\text{RCN}(A, H)(x) = \text{RCN}(A, H)(y) < E_1$ .

Note that D.2 is weaker than Lemma 5.2 of [Fr98] in that "closed" is missing and the Ramsey property is only over  $E^{k<}$ . However, we will derive Lemma 5.2 of [Fr98] from D.2 after we derive D.2 from D.1.

We fix  $k \geq 1$  and  $H \in \text{DFNL}(\mathbb{N}^k)$  through Theorem D.18.

We say that  $x, y \in \mathbb{N}^k$  have the same order type if and only if for all  $1 \leq i, j \leq k$ ,  $x_i < x_j \leftrightarrow y_i < y_j$ . The order types of  $k$ -tuples are the equivalence classes under this equivalence relation on  $\mathbb{N}^k$ . The equivalence classes have canonical representatives by requiring that the set of coordinates be an initial segment of  $\mathbb{N}$ .

We use a listing without repetition of the order types of the  $k$ -tuples from  $\mathbb{N}$ . We let  $r$  be the number of items in this list.

We are going to apply Proposition D.1 in dimension  $k+r+2$ .

Let  $A \subseteq [\mathbb{N}]^{k+r+2}$  and  $x \in [\mathbb{N}]^{k+r+2}$ . We always write  $x = \{x_1 < \dots < x_{k+r+2}\}$ . For  $1 \leq i \leq j \leq k+r+2$ , we define  $x[i, j] = \{x_i, \dots, x_j\}$ . We define  $\#(A, x)$  to be the number of  $i \leq \min(x)$  such that  $x[2, k+r+2] \cup \{i\} \in A$ . We say that  $x$  dominates  $A$  if and only if for all  $y \in A$ ,  $y <^* x$ .

We define  $\alpha(A, x)$  as follows. If  $2 \leq \#(A, x) \leq r+1$  then  $\alpha(A, x)$  is the unique  $y \in x[2, k+1]^k$  whose order type is  $\#(A, x) - 1$  in the list of order types of  $k$ -tuples, and whose set of coordinates forms a tail in  $x[2, k+1]$ . If  $\#(A, x) \geq r+2$  then  $\alpha(A, x)$  is  $(x_1, x_3, \dots, x_{k+1})$ . If  $\#(A, x) \leq 1$  then  $\alpha(A, x) = \infty$ .

For  $x \in [\mathbb{N}]^{k+r+2}$  let  $\max'(x)$  be the second largest element of  $x$ . Of course,  $\max(\infty)$  and  $\max'(\infty)$  are taken to be undefined.

We define  $*(A, x)$  to be the set of all  $i \in \mathbb{N}$  such that

- i) there exists  $y \in A$  such that  $\max(y) = i$ ;
- ii) for all  $j \in x[k+2, k+r+1]$ , there exists  $w \in A$  with  $\max(w) = j$  and  $\max'(w) = i$ .

LEMMA D.3. The following is provable in  $\text{RCA}_0$ . Let  $A, A' \subseteq [\mathbb{N}]^{k+r+2}$  and  $x, x' \in [\mathbb{N}]^{k+r+2}$ . If  $\#(A, x) = \#(A', x')$  and  $x[2, k+1] = x'[2, k+1]$ , then  $\alpha(A, x) = \alpha(A', x')$ . If  $x[k+2, k+r+1] = x'[k+2, k+r+1]$  then  $*(A, x) = *(A', x')$ . Also, if  $\#(A, x) \geq 2$  then  $\max(\alpha(A, x)) = x_{k+1}$ .

Proof: Let  $A, A', x, x'$  be as given. Suppose  $\#(A, x) = \#(A', x') > 0$  and  $x[2, k+1] = x'[2, k+1]$ . Then  $\alpha(A, x)$  and  $\alpha(A', x')$  are defined, and are both tails of  $x[2, k+1]$  with the same order type. Hence they are equal. Finally, suppose  $x[k+2, k+r+1] = x'[k+2, k+r+1]$ . By inspection,  $* (A, x) = * (A, x')$ . The final claim is also by inspection.

LEMMA D.4. The following is provable in  $\text{RCA}_0$ . Let  $A \subseteq [N]^{k+r+2}$  and  $x \in [N]^{k+r+2}$  dominate  $A$ . Let  $A' = AU\{x\}$ . Then for all  $x' \in A$ ,

- i)  $\#(A', x') = \#(A, x')$ ;
- ii)  $\alpha(A', x') = \alpha(A, x')$ ;
- iii)  $* (A', x') = * (A, x')$ .

Also  $* (A', x) = * (A, x)$ .

Proof: Let  $A, x, x'$  be as given. Since  $\#(A', x')$  and  $\#(A, x')$  count only sets  $\leq^* x'$ , i) is immediate. And ii) follows from i) by Lemma D.3 and  $x' \in A$ . And in evaluating  $*$ , the only  $w$ 's involved are  $<^* x'$ , which establishes iii). The final claim also follows from the fact that  $*$  involves only membership in  $A$  for sets  $w <^* x$ .

LEMMA D.5. The following is provable in  $\text{RCA}_0$ . Let  $A \subseteq B \subseteq [N]^{k+r+2}$  and  $x \in [N]^{k+r+2}$ . Then  $* (A, x) \subseteq * (B, x)$ . Also  $\#(A, x) \leq \#(B, x)$ .

Proof: Let  $A, B, x$  be as given. Let  $i \in * (A, x)$ . Then by inspection,  $i \in * (B, x)$ .

For the second claim, note that the left hand side represents a count on a set, and the right hand side represents a count on a superset.

Let  $T$  be a  $k+r+2$ -tree and  $x \in [N]^{k+r+2}$ . We define the binary relation  $T \langle x \rangle \subseteq N^k \times N$  as follows.  $T \langle x \rangle (y, i)$  if and only if  $y \in N^k$ ,  $i \in N$ , and there exists  $z \in \text{Ch}(T)$  such that

- i)  $z[2, k+1] \subseteq * (\text{Ch}(T), x)$ ;
- ii)  $z[k+2, k+r+2] = x[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y$ ;
- iv) if  $\#(\text{Ch}(T), z) \leq r+1$  then  $\max' (p(z, T)) = i$

v) if  $\#(\text{Ch}(T), z) \geq r+2$  then  $\max(p(z, T)) = i$  and  $z_1 \in *(\text{Ch}(T), x)$ .

Note that by iii),  $\#(\text{Ch}(T), z) \geq 2$ .

LEMMA D.6. The following is provable in  $\text{RCA}_0$ . Let  $T$  be a  $k+r+2$ -tree,  $x, x' \in [N]^{k+r+2}$ ,  $y \in N^k$ , and  $i \in N$ . If  $x[k+2, k+r+2] = x'[k+2, k+r+2]$  then  $T\langle x \rangle = T\langle x' \rangle$ . Either  $T\langle x \rangle$  is empty or there exists  $z \in \text{Ch}(T)$  such that  $T\langle x \rangle = T\langle z \rangle$ . If  $T\langle x \rangle(y, i)$  then  $y \in *(\text{Ch}(T), x)^k$ .

Proof: Let  $T, x, x', y, i$  be as given. Suppose  $x[k+2, k+r+2] = x'[k+2, k+r+2]$ . By Lemma D.3,  $*(\text{Ch}(T), x) = *(\text{Ch}(T), x')$ . Then  $T\langle x \rangle = T\langle x' \rangle$  follows by inspection.

Now suppose that  $T\langle x \rangle$  is nonempty. Let  $z \in \text{Ch}(T)$ ,  $y' \in N^k$ ,  $i' \in N$ , and

- i)  $z[2, k+1] \subseteq *(\text{Ch}(T), x)$ ;
- ii)  $z[k+2, k+r+2] = x[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y'$ ;
- iv) if  $\#(\text{Ch}(T), z) \leq r+1$  then  $\max'(p(z, T)) = i'$
- v) if  $\#(\text{Ch}(T), z) \geq r+2$  then  $\max(p(z, T)) = i'$  and  $z_1 \in *(\text{Ch}(T), x)$ .

By ii) and the first claim, we see that  $T\langle x \rangle = T\langle z \rangle$  as required.

Finally, assume  $T\langle x \rangle(y, i)$ . Let  $z \in \text{Ch}(T)$  be such that

- i)  $z[2, k+1] \subseteq *(\text{Ch}(T), x)$ ;
- ii)  $z[k+2, k+r+2] = x[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y$ ;
- iv) if  $\#(\text{Ch}(T), z) \leq r+1$  then  $\max'(p(z, T)) = i$ ;
- v) if  $\#(\text{Ch}(T), z) \geq r+2$  then  $\max(p(z, T)) = i$  and  $z_1 \in *(\text{Ch}(T), x)$ .

By i), iii), if  $2 \leq \#(\text{Ch}(T), z) \leq r+1$  then  $y \in z[2, k+1]^k \subseteq *(\text{Ch}(T), x)^k$ . Also by i), iii), iv), if  $\#(\text{Ch}(T), z) \geq r+2$  then  $y \in z[1, k+1]^k \subseteq *(\text{Ch}(T), x)^k$ . By iii),  $\#(\text{Ch}(T), z) \geq 2$ .

LEMMA D.7. The following is provable in  $\text{RCA}_0$ . Let  $T$  be a  $k+r+2$ -tree and  $x \in [N]^{k+r+2}$  dominate  $T$ . Let  $w \in V(T)$ ,  $T' = T/x, w$ . If  $x[2, k+1] \subseteq *(Ch(T), x)$ ,  $w \neq \infty$ , and  $2 \leq \#(Ch(T'), x) \leq r+1$ , then  $T' \langle x \rangle = T \langle x \rangle \cup \{(\alpha(Ch(T'), x), \max'(w))\}$ . If  $x[1, k+1] \subseteq *(Ch(T), x)$ ,  $w \neq \infty$ , and  $\#(Ch(T'), x) \geq r+2$ , then  $T' \langle x \rangle = T \langle x \rangle \cup \{(\alpha(Ch(T'), x), \max(w))\}$ . In all other cases,  $T' \langle x \rangle = T \langle x \rangle$ . Furthermore, let  $x' \in Ch(T)$ . If  $x[k+2, k+r+2] \neq x'[k+2, k+r+2]$  then  $T' \langle x' \rangle = T \langle x' \rangle$ . If  $x[k+2, k+r+2] = x'[k+2, k+r+2]$  then  $T' \langle x' \rangle = T' \langle x \rangle$ .

Proof: Let  $T, x, w, T'$  be as given. Then  $T' \langle x \rangle(y, i)$  if and only if  $y \in N^k$ ,  $i \in N$ , and there exists  $z \in Ch(T')$  such that

- i)  $z[2, k+1] \subseteq *(Ch(T'), z)$ ;
- ii)  $z[k+2, k+r+2] = x[k+2, k+r+2]$ ;
- iii)  $\alpha(Ch(T'), z) = y$ ;
- iv) if  $\#(Ch(T'), z) \leq r+1$  then  $\max'(p(z, T')) = i$
- v) if  $\#(Ch(T'), z) \geq r+2$  then  $\max(p(z, T')) = i$  and  $z_1 \in *(Ch(T'), x)$ .

Let  $z \in Ch(T)$ . By Lemma D.4,  $*(Ch(T'), z) = *(Ch(T), z)$ ,  $\alpha(Ch(T'), z) = \alpha(Ch(T), z)$ ,  $\#(Ch(T'), z) = \#(Ch(T), z)$ , and  $*(Ch(T'), x) = *(Ch(T), x)$ . Hence  $z$  obeys i)-v) if and only if  $z$  obeys i)-v) with  $T'$  replaced by  $T$ .

From this we can conclude that  $T' \langle x \rangle(y, i)$  if and only if  $y \in N^k$ ,  $i \in N$ , and the following disjunct holds.  $T \langle x \rangle(y, i)$  or the conjunction of

- i)  $x[2, k+1] \subseteq *(Ch(T), x)$ ;
- ii)  $\alpha(Ch(T'), x) = y$ ;
- iii) if  $\#(Ch(T'), x) \leq r+1$  then  $\max'(w) = i$
- iv) if  $\#(Ch(T'), x) \geq r+2$  then  $\max(w) = i$  and  $x_1 \in *(Ch(T), x)$ .

First suppose that  $x[2, k+1] \subseteq *(Ch(T), x)$ ,  $w \neq \infty$ , and  $2 \leq \#(Ch(T'), x) \leq r+1$ , and  $y, i$  are arbitrary entities. If  $T' \langle x \rangle(y, i)$  then  $T \langle x \rangle(y, i)$  or  $(y = \alpha(Ch(T'), x)$  and  $i = \max'(w))$ . On the other hand,  $\alpha(Ch(T'), x) \in N^k$  and  $\max'(w) \in N$ , and so if  $T \langle x \rangle(y, i)$  or  $(y = \alpha(Ch(T'), x)$  and  $i = \max'(w))$ , then  $T' \langle x \rangle(y, i)$ .

Secondly, suppose that  $x[1, k+1] \subseteq {}^*(\text{Ch}(T'), x)$ ,  $w \neq \infty$ , and  $\#(\text{Ch}(T'), x) \geq r+2$ , and  $y, i$  are arbitrary entities. If  $T'\langle x \rangle(y, i)$  then  $T\langle x \rangle(y, i)$  or  $(y = \alpha(\text{Ch}(T'), x)$  and  $i = \max(w))$ . On the other hand,  $\alpha(\text{Ch}(T'), x) \in N^k$  and  $\max(w) \in N$ , and so if  $T\langle x \rangle(y, i)$  or  $(y = \alpha(\text{Ch}(T'), x)$  and  $i = \max(w))$ , then  $T'\langle x \rangle(y, i)$ .

Thirdly, suppose otherwise. We claim that i)-iv) cannot conjunctively hold. Suppose i)-iv) holds. Then  $w \neq \infty$ . Hence  $2 \leq \#(\text{Ch}(T'), x) \leq r+1$  fails. If  $\#(\text{Ch}(T'), x) \geq r+2$  then  $x_1 \in {}^*(\text{Ch}(T), x)$ , and so  $x[1, k+1] \subseteq {}^*(\text{Ch}(T), x)$ , which is a contradiction. So  $\#(\text{Ch}(T'), x) = 1$ . But this contradicts ii).

Since i)-iv) cannot conjunctively hold, we have  $T'\langle x \rangle = T\langle x \rangle$ .

For the furthermore part of Lemma D.7, let  $x' \in \text{Ch}(T)$ . We have  $T'\langle x' \rangle(y, i)$  if and only if  $y \in N^k$ ,  $i \in N$ , and there exists  $z \in \text{Ch}(T')$  such that

- i)  $z[2, k+1] \subseteq {}^*(\text{Ch}(T'), x')$ ;
- ii)  $z[k+2, k+r+2] = x'[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T'), z) = y$ ;
- iv) if  $\#(\text{Ch}(T'), z) \leq r+1$  then  $\max'(p(z, T')) = i$ ;
- v) if  $\#(\text{Ch}(T'), z) \geq r+2$  then  $\max(p(z, T')) = i$  and  $z_1 \in {}^*(\text{Ch}(T'), x')$ .

Suppose  $x[k+2, k+r+2] \neq x'[k+2, k+r+2]$ . Then the  $z$  above cannot be  $x$ , and so we can rewrite this definition of  $T'\langle x' \rangle$  as follows.  $T'\langle x' \rangle(y, i)$  if and only if  $y \in N^k$ ,  $i \in N$ , and there exists  $z \in \text{Ch}(T)$  such that

- i)  $z[2, k+1] \subseteq {}^*(\text{Ch}(T), x')$ ;
- ii)  $z[k+2, k+r+2] = x'[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y$ ;
- iv) if  $\#(\text{Ch}(T), z) \leq r+1$  then  $\max'(p(z, T')) = i$ ;
- v) if  $\#(\text{Ch}(T), z) \geq r+2$  then  $\max(p(z, T)) = i$  and  $z_1 \in {}^*(\text{Ch}(T), x')$ .

Here we have used Lemma D.4 extensively to replace  $T'$  by  $T$ . But this is just the definition of  $T\langle x' \rangle$ . Hence  $T'\langle x' \rangle = T\langle x' \rangle$ .

On the other hand, suppose  $x[k+2, k+r+2] = x'[k+2, k+r+2]$ . Then by Lemma D.6,  $T' \langle x' \rangle = T' \langle x \rangle$ .

Let  $A \subseteq [N]^{k+r+2}$  and  $x \in A$ . We let  $\beta(A, x) = x_{k+r+3-\#(A, x)}$  if  $2 \leq \#(A, x) \leq r+1$ ;  $\infty$  otherwise.

LEMMA D.8. The following is provable in  $\text{RCA}_0$ . Let  $A, B \subseteq [N]^{k+r+2}$ ,  $x \in [N]^{k+r+2}$ , and  $\beta(A, x) = \beta(B, x) \neq \infty$ . Then  $\alpha(A, x) = \alpha(B, x) \neq \infty$ . Let  $A \subseteq B \subseteq [N]^{k+r+2}$ ,  $x \in [N]^{k+r+2}$ , and  $\beta(A, x), \beta(B, x) \neq \infty$ . Then  $\beta(A, x) \geq \beta(B, x)$ .

Proof: Let  $A, B, x$  be as given. Then  $2 \leq \#(A, x), \#(B, x) \leq r+1$ . Hence  $\beta(A, x) = x_{k+r+3-\#(A, x)} = x_{k+r+3-\#(B, x)}$ , and so  $\#(A, x) = \#(B, x)$ . Therefore  $\alpha(A, x) = \alpha(B, x) \neq \infty$ . Now let  $A, B, x$  again be as given. By Lemma D.5,  $2 \leq \#(A, x) \leq \#(B, x) \leq r+1$ . Therefore  $\beta(A, x) \geq \beta(B, x)$ .

Let  $S$  be the set of all  $k+r+2$ -trees  $T$  such that the following holds for all  $x \in \text{Ch}(T)$ .

1. Suppose  $x[2, k+1] \subseteq *(\text{Ch}(T), x)$  and  $2 \leq \#(\text{Ch}(T), x) \leq r+1$ . Then  $\max(p(x, T)) = \beta(\text{Ch}(T), x)$  and  $\#(\text{Ch}(T), p(x, T)) = 1$ .
2. Suppose  $x[2, k+1] \subseteq *(\text{Ch}(T), x)$  and  $\#(\text{Ch}(T), x) \geq r+2$ . Then  $\max(p(x, T)) \leq x_{k+1}$  and  $\#(\text{Ch}(T), p(x, T)) = 1$ .
3. Otherwise. Then  $p(x, T) = \infty$ .

LEMMA D.9. The following is provable in  $\text{RCA}_0$ . Let  $T_1 \subseteq T_2$  both lie in  $S$  and  $x \in [N]^{k+r+2}$ . Then  $T_1 \langle x \rangle \subseteq T_2 \langle x \rangle$ .

Proof: Let  $T_1, T_2, x$  be as given. Let  $T_1 \langle x \rangle (y, i)$ . Let  $z \in \text{Ch}(T_1)$  be such that

- i)  $z[2, k+1] \subseteq *(\text{Ch}(T_1), x)$ ;
- ii)  $z[k+2, k+r+2] = x[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T_1), z) = y$ ;
- iv) if  $\#(\text{Ch}(T_1), z) \leq r+1$  then  $\max'(p(z, T_1)) = i$ ;
- v) if  $\#(\text{Ch}(T_1), z) \geq r+2$  then  $\max(p(z, T_1)) = i$  and  $z_1 \in *(\text{Ch}(T_1), x)$ .

By Lemma D.5,  $*(\text{Ch}(T_1), x) \subseteq *(\text{Ch}(T_2), x)$ , and  $\#(\text{Ch}(T_1), z) \leq \#(\text{Ch}(T_2), z)$ . Also by Lemma D.3 and ii) above,  $*(\text{Ch}(T_1), x) = *(\text{Ch}(T_1), z)$  and  $*(\text{Ch}(T_2), x) = *(\text{Ch}(T_2), z)$ .

Firstly, suppose  $\#(\text{Ch}(T_1), z) \geq r+2$ . Then  $\#(\text{Ch}(T_2), z) \geq r+2$ , and hence  $\alpha(\text{Ch}(T_1), z) = \alpha(\text{Ch}(T_2), z) = (z_1, z_3, \dots, z_{k+1}) = y$ . Therefore, we can replace  $T_1$  by  $T_2$ . Hence  $T_2 \langle x \rangle (y, i)$ .

Secondly, suppose  $2 \leq \#(\text{Ch}(T_1), z) \leq r+1$ . Since  $T_1 \in S$ , we have  $\max(p(z, T_1)) = \beta(\text{Ch}(T_1), z) = z_{k+r+3-\#(\text{Ch}(T_1), z)}$ . Now if  $\#(\text{Ch}(T_2), z) \geq r+2$  then since  $T_2 \in S$ , we have  $\max(p(z, T_2)) \leq z_{k+1}$ . Since  $T_1 \subseteq T_2$ , this is impossible. Hence  $2 \leq \#(\text{Ch}(T_2), z) \leq r+1$  and  $\max(p(z, T_2)) = \beta(\text{Ch}(T_2), x) = z_{k+r+3-\#(\text{Ch}(T_2), z)}$ . Since  $T_1 \subseteq T_2$ , we see that  $\#(\text{Ch}(T_1), z) = \#(\text{Ch}(T_2), z)$ . By Lemma D.3,  $\alpha(\text{Ch}(T_1), z) = \alpha(\text{Ch}(T_2), z) = y$ . Hence  $T_2 \langle x \rangle (y, i)$  holds with the same witness  $z$ .

The final case is  $\#(\text{Ch}(T_1), z) \leq 1$ . But this is impossible by iii) and  $y \in N^k$ .

We let  $S'$  be the set of all  $T \in S$  such that for all  $x \in \text{Ch}(T)$ ,  $T \langle x \rangle$  is a function of the form  $\text{RCN}(B, H)$  for some finite  $B \subseteq N^k$ .

LEMMA D.10. The following is provable in  $\text{RCA}_0$ . Let  $T \in S'$ . Then for all  $x \in [N]^{k+r+2}$ ,  $T \langle x \rangle$  is a function of the form  $\text{RCN}(B, H)$  for some finite  $B \subseteq N^k$ .

Proof: Let  $T, x$  be as given. By Lemma D.6,  $T \langle x \rangle$  is either empty or is  $T \langle z \rangle$  for some  $z \in \text{Ch}(T)$ . In either case,  $T \langle x \rangle$  is of the form  $\text{RCN}(B, H)$  for some finite  $B \subseteq N^k$ .

LEMMA D.11. The following is provable in  $\text{RCA}_0$ . Let  $A \subseteq [N]^{k+r+2}$  and  $x \in [N]^{k+r+2}$ . Assume that for all  $y \in A$ ,  $\max(x) \geq \max(y)$ . Let  $F = \text{RCN}(A, H)$ . Then  $F \cup \{(x, H(F|x, x))\} = \text{RCN}(A \cup \{x\}, H)$ .

Proof: Let  $A, x, F$  be as given. Let  $F' = F \cup \{(x, H(F|x, x))\}$ . We need to check that for all  $y \in \text{dom}(F')$ ,  $F'(y) = H(F'|y, y)$ . First let  $y \in \text{dom}(F)$ . Then  $F'|y = F|y$  and  $F'(y) = F(y)$ . Hence this follows from  $F = \text{RCN}(A, H)$ . Finally,  $F'(x) = H(F|x, x) = H(F'|x, x)$ .

We now define a function  $f$  ( $S', f$ ) as follows. Let  $T \in S'$  and  $x \in [N]^{k+r+2}$  dominate  $T$ .

- a.  $x[2, k+1] \subseteq {}^*(\text{Ch}(T), x)$  and  $2 \leq \#(\text{Ch}(T) \cup \{x\}, x) \leq r+1$ . Set  $f(T, x)$  to be the  $<^*$  least  $w \in \text{Ch}(T)$  such that  $\max'(w) = H(T \langle x \rangle | \alpha(\text{Ch}(T) \cup \{x\}, x), \alpha(\text{Ch}(T) \cup \{x\}, x))$  and  $\max(w) = \beta(\text{Ch}(T) \cup \{x\}, x)$ .
- b.  $x[2, k+1] \subseteq {}^*(\text{Ch}(T), x)$  and  $\#(\text{Ch}(T) \cup \{x\}, x) \geq r+2$ . Set  $f(T, x)$  to be the  $<^*$  least  $w \in \text{Ch}(T)$  such that  $\max(w) \geq H(T \langle x \rangle | \alpha(\text{Ch}(T) \cup \{x\}, x), \alpha(\text{Ch}(T) \cup \{x\}, x))$ .
- c. Otherwise. Set  $f(T, x) = \infty$ .

We need to show that  $f$  is well defined, and that  $(S', f)$  is an insertion rule in  $\text{TR}(k+r+2)$ .

LEMMA D.12. The following is provable in  $\text{RCA}_0$ . Let  $T$  be a  $k+r+2$ -tree and  $x \in [N]^{k+r+2}$  dominate  $T$ . Then for all  $y \in \text{dom}(T \langle x \rangle)$ ,  $\max(y) \leq x_{k+1}$ .

Proof: Let  $T, x, y$  be as given. Then  $y \in N^k$ . Fix  $z \in \text{Ch}(T)$  and  $i \in N$  such that

- i)  $z[2, k+1] \subseteq {}^*(\text{Ch}(T), x)$ ;
- ii)  $z[k+2, k+r+2] = x[k+2, k+r+2]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y$ ;
- iv) if  $\#(\text{Ch}(T), z) \leq r+1$  then  $\max'(p(z, T)) = i$ ;
- v) if  $\#(\text{Ch}(T), z) \geq r+2$  then  $\max(p(z, T)) = i$  and  $z_1 \in {}^*(\text{Ch}(T), x)$ .

Now  $z <^* x$ . Hence  $z_{k+1} \leq x_{k+1}$ . By iii),  $\#(\text{Ch}(T), z) \geq 2$ . Hence by Lemma D.3,  $\max(y) = \max(\alpha(\text{Ch}(T), z)) = z_{k+1} \leq x_{k+1}$ .

LEMMA D.13. The following is provable in  $\text{RCA}_0$ .  $(S', f)$  is an insertion rule in  $\text{TR}(k+r+2)$ . If case a applies to  $f(T, x)$  then  $\max(f(T, x)) \geq x_{k+2}$ . If case b applies to  $f(T, x)$  then  $\max(f(T, x)) \leq x_{k+1}$ .

Proof: Let  $T \in S'$  and  $x \in [N]^{k+r+2}$  dominate  $T$ . We need to show that  $f(T, x)$  is well defined, and  $T/x, f(T, x) \in S'$ .

We first assume that case a applies to  $f(T, x)$ , and show that  $f(T, x)$  is well defined, that  $T' = T/x, f(T, x) \in S$ , and finally that  $T' \in S'$ .

Let  $i = H(T \langle x \rangle | \alpha(\text{Ch}(T) \cup \{x\}, x), \alpha(\text{Ch}(T) \cup \{x\}, x))$ . Then every coordinate of  $\alpha(\text{Ch}(T) \cup \{x\}, x)$  lies in  $x[2, k+1]$ . Hence  $i \in \text{fld}(\text{dom}(T \langle x \rangle | \alpha(\text{Ch}(T) \cup \{x\}, x))) \cup x[2, k+1]$ . By the inclusion in case a, and the final claim of Lemma D.6,  $i \in {}^*(\text{Ch}(T), x)$ . Hence there exists  $w \in \text{Ch}(T)$  such that  $\max'(w) = i$  and  $\max(w) = \beta(\text{Ch}(T) \cup \{x\}, x)$ . Hence  $f(T, x)$  is well defined.

To see that  $T' = T/x, f(T, x) \in S$ , first let  $x' \in \text{Ch}(T)$ , and suppose that  $x'[2, k+1] \subseteq {}^*(\text{Ch}(T'), x')$ . By Lemma D.4,  ${}^*(\text{Ch}(T'), x') = {}^*(\text{Ch}(T), x')$  and  $\#(\text{Ch}(T'), x') = \#(\text{Ch}(T), x')$ . Hence  $x'[2, k+1] \subseteq {}^*(\text{Ch}(T), x')$ . Firstly, assume that  $2 \leq \#(\text{Ch}(T'), x') \leq r+1$ . Then  $2 \leq \#(\text{Ch}(T), x') \leq r+1$ . Since  $T \in S$ , we have  $\max(p(x', T)) = \beta(\text{Ch}(T), x')$  and  $\#(\text{Ch}(T), p(x', T)) = 1$ . Using Lemma D.4, we see that  $\max(p(x', T')) = \max(p(x', T)) = \beta(\text{Ch}(T), x') = \beta(\text{Ch}(T'), x')$  and  $\#(\text{Ch}(T'), p(x', T')) = \#(\text{Ch}(T), p(x', T)) = 1$ . Secondly, assume that  $\#(\text{Ch}(T'), x') \geq r+2$ . Hence  $\#(\text{Ch}(T), x') \geq r+2$ . Since  $T \in S$ , we have  $\max(p(x', T)) \leq x_{k+1}$  and  $\#(\text{Ch}(T), x') = 1$ . Using Lemma D.4, we see that  $\max(p(x', T')) = \max(p(x', T)) \leq x_{k+1}$  and  $\#(\text{Ch}(T'), x') = \#(\text{Ch}(T), x') = 1$ . Thirdly, assume that  $\#(\text{Ch}(T'), x') \leq 1$  or  $x'[2, k+1] \not\subseteq {}^*(\text{Ch}(T'), x')$ . Hence  $\#(\text{Ch}(T), x') \leq 1$  or  $x'[2, k+1] \not\subseteq {}^*(\text{Ch}(T), x')$ . Since  $T \in S$ , we have  $p(x', T) = \infty$ . Hence  $p(x', T') = \infty$ .

To complete the proof that  $T' = T/x, f(T, x) \in S$  in this case a, we now let  $x' = x$ . According to case a,  $x[2, k+1] \subseteq {}^*(\text{Ch}(T), x)$ . By Lemma D.4,  $x[2, k+1] \subseteq {}^*(\text{Ch}(T'), x)$ . Let  $w = f(T, x) = p(x, T')$ . Then  $w \in \text{Ch}(T)$  and  $\max(w) = \beta(\text{Ch}(T'), x)$ . Also by the  $\langle {}^* \rangle$  minimality of  $w$ , we see that  $\#(\text{Ch}(T), w) = 1$ . Using Lemma D.4, we have  $\#(\text{Ch}(T'), w) = 1$ . This completes the proof that in case a,  $T' \in S$ .

Finally, we must show that  $T' \in S'$ , in case a. We first show that  $T' \langle x \rangle$  is a function of the required form. Let  $w = f(T, x)$ . We are in case a, and so  $x[2, k+1] \subseteq {}^*(\text{Ch}(T), x)$ ,  $w \neq$

$\infty$ , and  $2 \leq \#(\text{Ch}(T'), x) \leq r+1$ . So by Lemma D.7,  $T' \langle x \rangle = T \langle x \rangle \cup \{(\alpha(\text{Ch}(T'), x), \max'(w))\}$ .

By  $T \in S'$  and Lemma D.10, we see that  $T \langle x \rangle$  is a function of the required form. Now according to case a,  $\max'(w) = H(T \langle x \rangle | \alpha(\text{Ch}(T'), x), \alpha(\text{Ch}(T'), x))$ . By Lemma D.3,  $\max(\alpha(\text{Ch}(T'), x)) = x_{k+1}$ . By Lemma D.12, for all  $y \in \text{dom}(T \langle x \rangle)$ ,  $\max(y) \leq x_{k+1} = \max(\alpha(\text{Ch}(T'), x))$ . By setting  $F = T \langle x \rangle$  and  $x = \alpha(\text{Ch}(T'), x)$  in Lemma D.11, we see that  $T' \langle x \rangle$  is a function of the required form.

Now let  $x' \in \text{Ch}(T)$ . By Lemma D.7,  $T' \langle x' \rangle = T \langle x' \rangle$  or  $T' \langle x \rangle$ . In either case,  $T' \langle x' \rangle$  is a function of the required form. We have now completed the proof of this lemma under case a.

We secondly assume that case b applies to  $f(T, x)$ , and show that  $f(T, x)$  is well defined, that  $T' = T/x, f(T, x) \in S$ , and finally that  $T' \in S'$ .

To see that  $f(T, x)$  is defined, note that  $\alpha(\text{Ch}(T) \cup \{x\}, x) = (x_1, x_3, \dots, x_{k+1})$ . Hence  $H(T \langle x \rangle | \alpha(\text{Ch}(T) \cup \{x\}, x), \alpha(\text{Ch}(T) \cup \{x\}, x)) \leq \max(\alpha(\text{Ch}(T) \cup \{x\}, x)) = x_{k+1}$ . Now  $x_{k+1} \in {}^*(\text{Ch}(T), x)$ . Hence there exists  $w \in \text{Ch}(T)$  such that  $\max(w) = x_{k+1} \geq H(T \langle x \rangle | \alpha(\text{Ch}(T) \cup \{x\}, x), \alpha(\text{Ch}(T) \cup \{x\}, x))$ . This establishes that  $f(T, x)$  is defined and also that  $\max(f(T, x)) \leq x_{k+1}$ . (A subtle point here is that we only get  $\geq$  and not  $=$ , as it is in case a. This is because the H-expression may be  $x_1$ . This does not happen in case a because the H-expression lies in  ${}^*(\text{Ch}(T), x)$  as a consequence of the fact that, in case a, every coordinate of  $\alpha(\text{Ch}(T) \cup \{x\}, x)$  lies in  $x[2, k+1]$ . In case b, the first coordinate of  $\alpha(\text{Ch}(T) \cup \{x\}, x)$  is  $x_1$ , which may not be in  ${}^*(\text{Ch}(T), x)$ ).

To see that  $T' = T/x, f(T, x) \in S$ , first let  $x' \in \text{Ch}(T)$  in the definition of  $S$ . We argue exactly as in case a without change.

To complete the proof that  $T' = T/x, f(T, x) \in S$ , we let  $x' = x$ . By Lemma D.4,  ${}^*(\text{Ch}(T'), x) = {}^*(\text{Ch}(T), x)$ . We have already verified that  $\max(f(T, x)) \leq x_{k+1}$ . Also, by the  $\langle {}^*$  minimality of  $w = f(T, x)$ , we see that  $\#(\text{Ch}(T'), w) = 1$ .

Finally, we must show that  $T' \in S'$ . We first show that  $T' \langle x \rangle$  is a function of the required form. Let  $w = f(T, x)$ . We are in case b, and so  $x[2, k+1] \subseteq^* (\text{Ch}(T), x)$ ,  $w \neq \infty$ , and  $\#(\text{Ch}(T'), x) \geq r+2$ . Assume  $x[1, k+1] \subseteq^* (\text{Ch}(T), x)$ . By Lemma D.7,  $T' \langle x \rangle = T \langle x \rangle \cup \{(\alpha(\text{Ch}(T'), x), \max(w))\}$ . We claim that  $H(T \langle x \rangle | \alpha(\text{Ch}(T'), x), \alpha(\text{Ch}(T'), x))$  lies in  $^*(\text{Ch}(T), x)$ . To see this, note that it must be a coordinate of  $\text{dom}(T \langle x \rangle)$  or among  $x_1, \dots, x_{k+1}$ . (It could be  $x_1$ , and so we need  $x_1 \in ^*(\text{Ch}(T), x)$  here). Therefore by Lemma D.6, it lies in  $^*(\text{Ch}(T), x)$ . Hence by this case b, we see that  $\max(w) = H(T \langle x \rangle | \alpha(\text{Ch}(T'), x), \alpha(\text{Ch}(T'), x))$ . So for all  $y \in \text{dom}(T \langle x \rangle | \alpha(\text{Ch}(T'), x))$ ,  $\max(y) \leq \max(\alpha(\text{Ch}(T'), x)) = x_{k+1}$ . Hence  $T' \langle x \rangle$  is a function of the required form by Lemmas D.11 and D.12. Finally, assume  $x[1, k+1] \not\subseteq^* (\text{Ch}(T), x)$ . By Lemma D.7,  $T' \langle x \rangle = T \langle x \rangle$ . Hence  $T' \langle x \rangle$  is of the required form by  $T \in S'$  and Lemma D.10.

We now show that  $T' \langle x' \rangle$  is of the required form for  $x' \in \text{Ch}(T)$ . By Lemma D.7,  $T' \langle x' \rangle = T \langle x' \rangle$  or  $T' \langle x \rangle$ .  $T \langle x' \rangle$  is of the required form by  $T \in S'$ , and we have shown that  $T' \langle x \rangle$  is of the required form.

We thirdly assume that case c applies to  $f(T, x)$ . Then  $f(T, x) = \infty$ . Also since  $x[2, k+1] \not\subseteq^* (\text{Ch}(T), x)$  and  $T \in S$ , we have  $T' = T/x, \infty \in S$ . In addition, by Lemma D.7,  $T' \langle x \rangle = T \langle x \rangle$ , and hence  $T' \langle x \rangle$  is a function of the required form (using Lemma D.10 and  $T \in S'$ ). And for  $x' \in \text{Ch}(T)$ ,  $T' \langle x' \rangle = T \langle x' \rangle$  or  $T' \langle x \rangle$  by Lemma D.7. Therefore  $T' \in S'$ .

For the second claim of the lemma, note that by the third paragraph of the proof,  $f(T, x)$  is a  $\langle^*$ -min over  $w$  with  $\max(w) = \beta(\text{Ch}(T) \cup \{x\}, x) \geq x_{k+2}$ . For the final claim of the lemma, note that in case b, we have already shown that  $f(T, x) \leq x_{k+1}$  when we showed that  $f(T, x)$  is well defined.

LEMMA D.14. The following is provable in  $\text{RCA}_0$ .  $(S', f)$  is a decreasing insertion rule in  $\text{TR}(k+r+2)$ .

Proof: Let  $T_1 \subseteq T_2$  both lie in  $S'$  and  $x \in [N]^{k+r+2}$  dominate  $T_2$ . By Lemma D.9,  $T_1 \langle x \rangle \subseteq T_2 \langle x \rangle$ .

First suppose that case a applies to  $f(T_1, x)$ . By Lemmas D.5 and D.13,  $x[2, k+1] \subseteq {}^*(\text{Ch}(T_2), x)$ , and  $\max(f(T_1, x)) \geq x_{k+2}$ . By Lemma D.13, if  $\#(\text{Ch}(T_2) \cup \{x\}, x) \geq r+2$  then  $\max(f(T_2, x)) \leq x_{k+1}$ , in which case  $f(T_1, x) >^* f(T_2, x)$ . So we may assume that  $2 \leq \#(\text{Ch}(T_1) \cup \{x\}, x), \#(\text{Ch}(T_2) \cup \{x\}, x) \leq r+1$ . By definition, we have  $f(T_1, x)$  is the  $<^*$  least  $w \in \text{Ch}(T_1)$  such that  $\max'(w) = H(T_1 \langle x \rangle | \alpha(\text{Ch}(T_1) \cup \{x\}, x), \alpha(\text{Ch}(T_1) \cup \{x\}, x))$  and  $\max(w) = \beta(\text{Ch}(T_1) \cup \{x\}, x)$ . And  $f(T_2, x)$  to be the  $<^*$  least  $w \in \text{Ch}(T_2)$  such that  $\max'(w) = H(T_2 \langle x \rangle | \alpha(\text{Ch}(T_2) \cup \{x\}, x), \alpha(\text{Ch}(T_2) \cup \{x\}, x))$  and  $\max(w) = \beta(\text{Ch}(T_2) \cup \{x\}, x)$ . We wish to compare these two min's.

case 1.  $\beta(\text{Ch}(T_1) \cup \{x\}, x) = \beta(\text{Ch}(T_2) \cup \{x\}, x)$ . By Lemma D.8,  $\alpha(\text{Ch}(T_1) \cup \{x\}, x) = \alpha(\text{Ch}(T_2) \cup \{x\}, x) \neq \infty$ . Therefore since  $H \in \text{DFNL}(N^k)$  and  $T_1 \langle x \rangle \subseteq T_2 \langle x \rangle$ , we see that  $H(T_1 \langle x \rangle | \alpha(\text{Ch}(T_1) \cup \{x\}, x), \alpha(\text{Ch}(T_1) \cup \{x\}, x)) \geq H(T_2 \langle x \rangle | \alpha(\text{Ch}(T_2) \cup \{x\}, x), \alpha(\text{Ch}(T_2) \cup \{x\}, x))$ . If  $>$  holds then every  $w$  in the first min is  $>^*$  every  $w$  in the second min. And if  $=$  holds then every  $w$  in the first min is some  $w$  in the second min. In either case,  $f(T_1, x) \geq^* f(T_2, x)$ .

case 2.  $\beta(\text{Ch}(T_1) \cup \{x\}, x) > \beta(\text{Ch}(T_2) \cup \{x\}, x)$ . Then every  $w$  in the first min is  $>^*$  every  $w$  in the second min. Hence  $f(T_1, x) >^* f(T_2, x)$ .

By Lemma D.8, cases 1 and 2 are inclusive.

Secondly, suppose case b applies to  $f(T_1, x)$ . By Lemma D.5,  $x[2, k+1] \subseteq {}^*(\text{Ch}(T_2), x)$ , and  $\#(\text{Ch}(T_1), x) \leq \#(\text{Ch}(T_2), x)$ , and so case b applies to  $f(T_2, x)$ . Now  $f(T_1, x)$  is the  $<^*$  least  $w \in \text{Ch}(T_1)$  such that  $\max(w) \geq H(T_1 \langle x \rangle | \alpha(\text{Ch}(T_1) \cup \{x\}, x), \alpha(\text{Ch}(T_1) \cup \{x\}, x))$ . And  $f(T_2, x)$  is the  $<^*$  least  $w \in \text{Ch}(T_2)$  such that  $\max(w) \geq H(T_2 \langle x \rangle | \alpha(\text{Ch}(T_2) \cup \{x\}, x), \alpha(\text{Ch}(T_2) \cup \{x\}, x))$ . In this case b,  $\alpha(\text{Ch}(T_1) \cup \{x\}, x) = \alpha(\text{Ch}(T_2) \cup \{x\}, x) = (x_1, x_3, \dots, x_{k+1})$ . Hence the first H-expression is  $\geq$  the second H-expression by  $H \in \text{DFNL}(N^k)$ . Therefore every  $w$  in the first min is a  $w$  in the second min. Therefore  $f(T_1, x) \geq^* f(T_2, x)$ .

Thirdly, suppose case c applies to  $f(T_1, x)$ . Then  $f(T_1, x) = \infty \geq^* f(T_2, x)$ .

LEMMA D.15. The following is provable in  $\text{RCA}_0$ . Every vertex in every  $T \in S$  has at most two ancestors.

Proof: Let  $x \in V(T)$ ,  $T \in S$ . If  $x = \infty$  then we are done. Assume  $x \in \text{Ch}(T)$ . If  $\#(\text{Ch}(T), x) = 1$  then  $p(x, T) = \infty$ , and so  $x$  has exactly one ancestor. If  $\#(\text{Ch}(T), x) > 1$  then by inspection,  $\#(\text{Ch}(T), p(x, T)) \leq 1$ , and so  $p(x, T)$  has at most one ancestor. Hence  $x$  has exactly two ancestors.

Recall that we have fixed  $k \geq 1$  and defined  $r$  in terms of  $k$ . We now fix  $p \geq 1$  for Proposition D.2. We also fix  $q \gg k, r, p$ .

LEMMA D.16. The following is provable in  $\text{RCA}_0$ . Let  $T \in S$  and  $E$  be a  $4q$  element subset of  $N$ , where  $[E]^{k+r+2} \subseteq \text{Ch}(T)$ . Then  $E[q, 4q-r-1]^k \subseteq \text{dom}(T \langle E[4q-k-r-1, 4q] \rangle)$ .

Proof: Let  $T, E$  be as given, and let  $y \in E[q, 4q-r-1]^k$ . It suffices to verify that there exists  $z \in \text{Ch}(T)$  such that

- i)  $z[2, k+1] \subseteq *(\text{Ch}(T), E[4q-k-r-1, 4q])$ ;
- ii)  $z[k+2, k+r+2] = E[4q-r, 4q]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y$ ;
- iv)  $\#(\text{Ch}(T), z) \leq r+1$  and  $p(z, T) \neq \infty$ .

Let  $1 \leq j \leq r$  be the index of the order type of the  $k$ -tuple  $y$ . Let the coordinates of  $y$  be listed in strictly increasing order as  $E_{m_1} < \dots < E_{m_t}$ ,  $1 \leq t \leq k$ . Note that  $q \leq m_1, \dots, m_t \leq 4q-r-1$ .

Let  $z' = \{E_{m_1-(k-t)}, \dots, E_{m_1-1}, E_{m_1}, \dots, E_{m_t}, E_{4q-r}, \dots, E_{4q}\}$ , which is in strictly increasing order, and has exactly  $k+r+1$  elements. If we insert any of  $E_1, \dots, E_{m_1-(k-t+1)}$  to  $z'$ , we obtain an element of  $\text{Ch}(T)$ . Therefore we can let  $n$  be such that  $z' \cup \{n\} \in \text{Ch}(T)$  and  $\#(\text{Ch}(T), z' \cup \{n\}) = j+1$ . Set  $z = z' \cup \{n\}$ .

Obviously  $z[k+2, k+r+2] = E[4q-r, 4q]$ ,  $2 \leq \#(\text{Ch}(T), z) \leq r+1$ , and  $\alpha(\text{Ch}(T), z) = y$ . Also,  $z[2, k+1] \subseteq *(\text{Ch}(T), z)$  since all  $k+r+2$  element subsets of  $E$  lie in  $\text{Ch}(T)$ . Since  $T \in S$ , we see that  $p(z, T) \neq \infty$ .

LEMMA D.17. The following is provable in  $\text{RCA}_0$ . Let  $T \in S'$  and  $E$  be a  $4q$  element subset of  $N$ , where all  $k+r+2$  element subsets of  $E$  are vertices with the same entirely lower ancestors. Let  $E^*$  consist of every other element of  $E[q, 4q-r-1]$ , starting with  $E_q$ . Then either

- i) for all  $x \in E^{*k<}$ ,  $T \langle E[4q-k-r-1, 4q] \rangle (x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{*k<}$ ,  $T \langle E[4q-k-r-1, 4q] \rangle (x) = T \langle E[4q-k-r-1, 4q] \rangle (y) < E_1$ .

Proof: Let  $T, E, E^*$  be as given. Write  $F = T \langle E[4q-k-r-1, 4q] \rangle$ . Let  $x, y \in E^{*k<}$ . Assume  $F(x) < \min(x)$ . It suffices to prove that  $F(x) = F(y) < E_1$ .

Let  $F(x) = i$ . Fix  $z \in \text{Ch}(T)$  such that

- i)  $z[2, k+1] \subseteq *( \text{Ch}(T), E[4q-k-r-1, 4q] )$ ;
- ii)  $z[k+2, k+r+2] = E[4q-r, 4q]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = x$ ;
- iv) if  $\#(\text{Ch}(T), z) \leq r+1$  then  $\max'(p(z, T)) = i$ ;
- v) if  $\#(\text{Ch}(T), z) \geq r+2$  then  $\max(p(z, T)) = i$  and  $z_1 \in *( \text{Ch}(T), E[4q-k-r-1, 4q] )$ .

Note that since  $x = \alpha(\text{Ch}(T), z)$  is strictly increasing,  $\alpha(\text{Ch}(T), z) = \{x_1, \dots, x_k\} = \{z_2, \dots, z_{k+1}\}$  or  $\{z_1, z_3, \dots, z_{k+1}\}$ . If the first disjunct holds, set  $w$  to be obtained from  $z$  by removing  $z_1$  and inserting any element of  $E$  strictly between  $x_1$  and  $x_2$ . If the second disjunct holds, set  $w = z$ . Then  $\#(\text{Ch}(T), w) \geq r+2$  and  $\alpha(\text{Ch}(T), w) = (w_1, w_3, \dots, w_{k+1}) = x$ . Also note that  $w[2, k+1] \subseteq *( \text{Ch}(T), w )$ . Since  $T \in S$ , we see that  $p(w, T) \neq \infty$ .

We now claim that  $F(x) = \max(p(w, T))$ . To witness this, we use  $w$  itself. Thus it suffices to verify that  $x \in [N]^{k+r+2}$ ,  $\max(p(w, T)) \in N$ , and

- i)  $w[2, k+1] \subseteq *( \text{Ch}(T), E[4q-k-r-1, 4q] )$ ;
- ii)  $w[k+2, k+r+2] = E[4q-r, 4q]$ ;
- iii)  $\alpha(\text{Ch}(T), w) = x$ ;
- iv) if  $\#(\text{Ch}(T), w) \leq r+1$  then  $\max'(p(w, T)) = \max(p(w, T))$ ;
- v) if  $\#(\text{Ch}(T), w) \geq r+2$  then  $\max(p(w, T)) = \max(p(w, T))$  and  $w_1 \in *( \text{Ch}(T), E[4q-k-r-1, 4q] )$ .

But this is immediate, using  $\#(\text{Ch}(T), x) \geq r+2$ .

Thus we have shown that  $F(x) = \max(p(w, T)) < \min(x) = x_1 \leq w_1$ . Thus by Lemma D.15,  $p(w, T)$  is the unique entirely lower ancestor of  $w$ . Hence for all  $v \in [E]^{k+r+2}$ ,  $p(w, T)$  is the unique entirely lower ancestor of  $v$ . Therefore, for all  $v \in [E]^{k+r+2}$ ,  $p(v, T) = p(w, T)$ . We can take  $v$  so that  $v_1 = \min(E)$  and see that  $\max(p(w, T)) < E_1$ .

Now let  $y \in E^{*k<}$ . We wish to verify that  $F(y) = \max(p(w, T))$ . It suffices to prove that there exists  $z \in \text{Ch}(T)$  such that

- i)  $z[2, k+1] \subseteq *( \text{Ch}(T), E[4q-k-r-1, 4q] )$ ;
- ii)  $z[k+2, k+r+2] = E[4q-r, 4q]$ ;
- iii)  $\alpha(\text{Ch}(T), z) = y$ ;
- iv)  $\#(\text{Ch}(T), z) \geq r+2$  and  $\max(p(z, T)) = \max(p(w, T))$  and  $z_1 \in *( \text{Ch}(T), E[4q-k-r-1, 4q] )$ .

Let  $z$  be the result of taking the set of coordinates of  $y$ , together with the elements of  $E[4q-r, 4q]$ , and inserting an element from  $E$  between the first two coordinates of  $y$ . Then  $z \in [E]^{k+r+2}$ , and so  $p(z, T) = p(w, T)$ . Hence i)-iv) holds for  $z$ . Therefore  $F(y) = \max(p(w, T)) < E_1$ .

LEMMA D.18. Proposition D.1 implies Proposition D.2 in  $\text{RCA}_0$ .

Proof: Apply Proposition D.1 to the decreasing insertion rule  $(S', f)$  in  $\text{TR}(k+r+2)$ , according to Lemma D.14. Let  $T \in S'$  and  $E$  be a  $4q$  element subset of  $N$ , where all  $k+r+2$  element subsets of  $E$  are vertices in  $T$  with the same entirely lower ancestors. By Lemma D.16,  $E[q, 4q-r-1]^k \subseteq \text{dom}(T \langle E[4q-k-r-1, 4q] \rangle)$ . Since  $T \in S'$ , let  $T \langle E[4q-k-r-1, 4q] \rangle = \text{RCN}(A, H)$ . Then by Lemma D.17,  $E^{*k} \subseteq A$  and either

- i) for all  $x \in E^{*k<}$ ,  $\text{RCN}(A, H)(x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{*k<}$ ,  $\text{RCN}(A, H)(x) = \text{RCN}(A, H)(y) < E^*_1$ .

This completes the derivation of Proposition D.2 since  $E^*$  has at least  $q \gg p$  elements.

For  $x, y \in N^k$ , we write  $x \subseteq y$  if and only if every coordinate of  $x$  is a coordinate of  $y$ .

We say that  $A \subseteq N^k$  is closed if and only if for all  $x \subseteq y$  with  $y \in A$ , we have  $x \in A$ .

PROPOSITION D.19. Let  $k, p \geq 1$  and  $H \in \text{DFNL}(N^k)$ . Then there exists finite closed  $A \subseteq N^k$  and  $E \subseteq N$  of cardinality  $p$  such that  $E^k \subseteq A$ , and either

- i) for all  $x \in E^{k<}$ ,  $\text{RCN}(A, H)(x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{k<}$ ,  $\text{RCN}(A, H)(x) = \text{RCN}(A, H)(y) < E_1$ .

LEMMA D.20. Proposition D.2 implies Proposition D.19 in  $\text{RCA}_0$ .

Proof: Assume Proposition D.2, and let  $k, p \geq 1$  and  $H \in \text{DFNL}(N^k)$ . For any finite  $A \subseteq N^k$ , let  $A' = \{x \in A : (\forall y \subseteq x)(y \in A)\}$ . Obviously,  $A'$  is closed. Define  $H' : \text{FPF}(N^k) \times N^k \rightarrow N$  by  $H'(f, x) = H(f|_{\text{dom}(f)'}, x)$ . Using the obvious law  $A \subseteq B \rightarrow A' \subseteq B'$ , we see that  $H' \in \text{DFNL}(N^k)$ .

We claim that for all finite  $A \subseteq N^k$ ,  $\text{RCN}(A', H) \subseteq \text{RCN}(A, H')$ . To see this, let  $f = \text{RCN}(A, H')$ . By definition, for all  $x \in A$ ,  $f(x) = H'(f|x, x) = H(f|x|_{\text{dom}(f|x)'}, x)$ . Now  $\text{dom}(f|x)' = \{z \in \text{dom}(f|x) : (\forall y \subseteq z)(y \in \text{dom}(f|x))\}$ . Suppose  $z \in \text{dom}(f|x)'$ . Then  $z \in \text{dom}(f|x)$  and  $z \in A'$ . I.e.,  $z \in A'$  and  $\max(z) < \max(x)$ . And conversely, suppose  $z \in A'$  and  $\max(z) < \max(x)$ . Then  $z \in \text{dom}(f|x)$ , and every  $w \subseteq z$  lies in  $A$ . Hence every  $w \subseteq z$  lies in  $\text{dom}(f|x)$ . So  $z \in \text{dom}(f|x)'$ .

We have shown that  $z \in \text{dom}(f|x)'$  if and only if  $z \in A'$  and  $\max(z) < \max(x)$ . Therefore we can continue the equality chain: for all  $x \in A$ ,  $f(x) = H'(f|x, x) = H(f|x|_{\text{dom}(f|x)'}, x) = H(f|_{A'|}, x)$ .

Now let  $g = f|_{A'}$ . Then for all  $x \in A'$ ,  $g(x) = f(x) = H(g|x, x)$ . Therefore  $g = \text{RCN}(A', H)$ , thereby establishing the claim.

By Proposition D.2, fix finite  $A \subseteq N^k$  and  $E \subseteq N$  of cardinality  $p$  such that  $E^k \subseteq A$ , and either

- i) for all  $x \in E^{k<}$ ,  $\text{RCN}(A, H')(x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{k<}$ ,  $\text{RCN}(A, H')(x) = \text{RCN}(A, H')(y) < E_1$ .

Note that  $E^k \subseteq A'$ . Hence by  $\text{RCN}(A', H) \subseteq \text{RCN}(A, H')$ , either

- i) for all  $x \in E^{k<}$ ,  $\text{RCN}(A', H)(x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{k<}$ ,  $\text{RCN}(A', H)(x) = \text{RCN}(A', H)(y) < E_1$ .

Let  $f$  be a nonempty partial function from  $N^k$  into  $N$ , and  $E \subseteq N$ . We say that  $f$  is regressively regular over  $E$  if and only if the following holds.

- i)  $E^k \subseteq \text{dom}(f)$ ;
- ii) Let  $x, y \in E^k$  have the same order type. If  $f(x) < x_1$  then  $f(y) < y_1$  and  $f(x) = f(y)$ .

PROPOSITION D.21. Let  $k, p \geq 1$  and  $H \in \text{DFNL}(N^k)$ . Then there exists finite closed  $A$  such that  $\text{RCN}(A, H)$  is regressively regular over some  $E$  of cardinality  $p$ .

LEMMA D.22. Proposition D.19 implies Proposition D.21 in  $\text{RCA}_0$ .

Proof: Assume Proposition D.19, and let  $k, p \geq 1$  and  $H \in \text{DFNL}(N^k)$ . For  $x \in N^k$  define  $x\# = (x, \max(x), \dots, \max(x)) \in N^{2k}$ . For  $A \subseteq N^{2k}$ , define  $A^* = \{x \in N^k : x\# \in A\}$ . For  $f \in \text{FPF}(N^{2k})$ ,  $\text{dom}(f) = A$ , define  $f^* : A^* \rightarrow N$  by  $f^*(x) = f(x\#)$ .

We define  $H^* : \text{FPF}(N^{2k}) \times N^{2k} \rightarrow N$  as follows. Let  $f \in \text{FPF}(N^{2k})$  and  $x \in N^{2k}$ . We assume a fixed listing without repetition of the order types of  $k$ -tuples from  $N$ .

case 1.  $x = y\#$ . Define  $H^*(f, x) = H(f^*, y)$ .

case 2.  $x \in N^{2k<}$ , and there exists  $y \in \{x_1, \dots, x_k\}^k$  and  $z \in \{x_{k+1}, \dots, x_{2k}\}^k$  of the same order type,  $\alpha$ , such that

- i)  $f^*(y) < \min(y)$ ,  $f^*(y) \neq f^*(z)$ , and  $f^*(y) \in \text{fld}(\text{dom}(f))$ ;
- ii) the coordinates of  $y$  form an initial segment of  $\{x_1, \dots, x_k\}$ ;
- iii) the coordinates of  $z$  form an initial segment of  $\{x_{k+1}, \dots, x_{2k}\}$ .

Choose the first possible order type,  $\alpha$ , and define  $H^*(f, x) = f^*(y)$ . This is well defined since  $y, z$  are completely determined by the order type,  $\alpha$ .

case 3. otherwise. Define  $H^*(f, x) = \max(x)$ .

To verify that  $H^* \in \text{DFNL}(N^{2k})$ , first observe that in case 2,  $H^*(f, x) \in \text{fld}(\text{dom}(f))$ , and in case 3,  $H^*(f, x) \in \{x_1, \dots, x_{2k}\}$ . In case 1,  $H^*(f, x) = H(f^*, y) \in \text{fld}(\text{dom}(f^*)) \cup \{y_1, \dots, y_k\} \subseteq \text{fld}(\text{dom}(f)) \cup \{x_1, \dots, x_{2k}\}$ .

Suppose  $f \subseteq g$ . If case 1 applies to  $H^*(f, x)$ , then case 1 applies to  $H^*(g, x)$ , and so  $H^*(f, x) = H^*(g, x)$ .

Now suppose case 2 applies to  $H^*(f, x)$ . Then case 2 applies to  $H^*(g, x)$ . Also, obviously  $f^* \subseteq g^*$ . Hence every order type  $\alpha$  in case 2 applied to  $H^*(f, x)$  is an order type  $\alpha$  in case 2 applied to  $H^*(g, x)$ . Therefore  $H^*(f, x) \geq H^*(g, x)$ .

Finally, suppose case 3 applies to  $H^*(f, x)$ . Then case 2 or case 3 applies to  $H^*(g, x)$ . Hence  $H^*(g, x) \leq \max(x) = H^*(f, x)$ . This completes the verification that  $H^* \in \text{DFNL}(N^{2k})$ .

By Proposition D.19, fix finite closed  $A \subseteq N^{2k}$  and  $E \subseteq N$  of cardinality  $q \gg k, p$  such that  $E^{2k} \subseteq A$ , and either

- i) for all  $x \in E^{2k<}$ ,  $\text{RCN}(A, H^*)(x) \geq x_1$ ; or
- ii) for all  $x, y \in E^{2k<}$ ,  $\text{RCN}(A, H^*)(x) = \text{RCN}(A, H^*)(y) < E_1$ .

Let  $f = \text{RCN}(A, H^*)$ . We need  $q \gg k, p$  in order to apply the finite Ramsey theorem.

We wish to find  $V \subseteq E$  of cardinality  $p$  such that the following holds. Let  $x, y \in V^k$  be of the same order type. If  $f^*(x) < x_1$  then  $f^*(x) = f^*(y) < V_1$ .

For all  $x \in E^{2k<}$ , color  $x$  as the first order type  $\alpha$  of  $k$ -tuples from  $N$  such that there exists  $y \in \{x_1, \dots, x_k\}^k$  and  $z \in \{x_{k+1}, \dots, x_{2k}\}^k$  of this order type  $\alpha$ , where

- i)  $f^*(y) < \min(y)$ ,  $f^*(y) \neq f^*(z)$ , and  $f^*(y) \in \text{fld}(\text{dom}(f))$ ;
- ii) the coordinates of  $y$  form an initial segment of  $\{x_1, \dots, x_k\}$ ;

iii) the coordinates of  $z$  form an initial segment of  $\{x_{k+1}, \dots, x_{2k}\}$ .

If there is no such order type  $\alpha$ , then color  $x$  with a default color.

By the finite Ramsey theorem, fix  $E' \subseteq E$  of cardinality  $p+k$  such that all  $x \in E'^{2k}$  have the same color.

Let  $y \in \{E'_1, \dots, E'_p\}^k$  be such that  $f^*(y) < \min(y)$ . Let  $z$  be the unique element of  $\{E'_{p+1}, \dots, E'_{p+k}\}^k$  of the same order type as  $z$  whose coordinates form an initial segment of  $\{E'_{p+1}, \dots, E'_{p+k}\}$ . We wish to prove that  $f^*(y) = f^*(z)$ .

Suppose  $f^*(y) \neq f^*(z)$ . Then let  $x \in E'^{2k}$  be such that the coordinates of  $y$  form an initial segment of  $\{x_1, \dots, x_k\}$ , and  $\{x_{k+1}, \dots, x_{2k}\} = \{E'_{p+1}, \dots, E'_{p+k}\}$ .

We claim that case 2 applies to  $H^*(f|x, x)$ . To see this, we need only verify that  $f^*(y) \in \text{fld}(\text{dom}(f|x))$ . Since  $f = \text{RCN}(A, H^*)$  and  $H^* \in \text{DFNL}(N^{2k})$ , we have  $f^*(y) = f(y\#) = H^*(f|y\#, y\#) \in \text{fld}(\text{dom}(f|y\#)) \cup \{y_1, \dots, y_k\} \subseteq \text{fld}(\text{dom}(f|x))$ , since  $\max(y\#) < \max(x)$ .

By the homogeneity of  $E'$ , we can fix  $\alpha$  to be the constant color; i.e., order type. Also by homogeneity, for all  $u \in E'^{2k}$ , case 2 applies to  $H^*(f|u, u)$ .

We have  $H^*(f|x, x) = f(x) = f^*(y) < \min(y) = x_1$ . Therefore, for all  $x' \in E'^{2k}$ ,  $f(x) = f(x') < E_1$ . Thus we have the following equalities and inequalities:

- 1)  $f(E'_1, \dots, E'_{2k}) = f(E'_{k+1}, \dots, E'_{2k}, E'_{p+1}, \dots, E'_{p+k})$ ;
- 2)  $f^*(E'_1, \dots, E'_k) \neq f^*(E'_{k+1}, \dots, E'_{2k})$  by examining  $f(E'_1, \dots, E'_{2k})$ ;
- 3)  $f(E'_1, \dots, E'_{2k}) = f^*(E'_1, \dots, E'_k)$  by examining  $f(E'_1, \dots, E'_{2k})$ ;
- 4)  $f(E'_{k+1}, \dots, E'_{2k}, E'_{p+1}, \dots, E'_{p+k}) = f^*(E'_{k+1}, \dots, E'_{2k})$  by examining  $f(E'_{k+1}, \dots, E'_{2k})$ .

This is a contradiction. Hence  $f^*(y) = f^*(z)$ .

Thus we have found  $V \subseteq E$  of cardinality  $p$  such that the following holds. Let  $x, y \in V^k$  be of the same order type. If  $f^*(x) < x_1$  then  $f^*(x) = f^*(y)$ . Namely, take  $V = \{E'_1, \dots, E'_p\}$ . Clearly  $V^k \subseteq \text{dom}(f^*) = A^*$ , and  $A^*$  is closed.

We would like to strengthen the above with  $f^*(x) = f^*(y) < V_1$ . This can be done by a slight modification of the argument. I.e., we can find  $V$  as in the previous paragraph of cardinality  $\gg p$ , and then apply the finite Ramsey theorem again to the attribute  $f^*(x) < x_1$ , obtaining  $V' \subseteq V$  of cardinality  $p$ .

So it only remains to show that  $f^* = \text{RCN}(A^*, H)$ . By definition, for all  $x \in A$ ,  $f(x) = H^*(f|x, x)$ . Let  $y \in A^*$ . Then  $f(y\#) = f^*(y) = H^*(f|y\#, y\#) = H((f|y\#)^*, y) = H(f^*|y, y)$ , since case 1 applies to  $H^*(f|y\#, y\#)$ . Therefore  $f^* = \text{RCN}(A^*, H)$ .

We have used the systems  $\text{RCA}_0$  and  $\text{RCA}_0 + \text{WKL}$  of reverse mathematics. We now use the stronger system  $\text{ACA}_0$  (see [Si85a], [Si85b], and [Si88]).

LEMMA D.23. Proposition D.21 implies the consistency of  $\text{ZFC} + \{\text{there exists a subtle cardinal of order } k\}_k$  in  $\text{ACA}_0$ .

Proof: Proposition D.21 appears as Lemma 5.2 in [Fr98], which is shown to imply the consistency of this system in [Fr98] in  $\text{ZFC}$ . An inspection of the proof shows that only  $\text{ACA}_0$  is actually used.

We conjecture that Part 5 in [Fr98] can be modified so as to produce a proof that Proposition D.21 implies the consistency of  $\text{ZFC} + \{\text{there exists a subtle cardinal of order } k\}_k$  in  $\text{RCA}_0 + \text{WKL}$ . If this conjecture holds, then  $\text{ACA}_0$  can also be replaced by  $\text{RCA}_0 + \text{WKL}$  in Theorem D.24.

THEOREM D.24. Proposition D.1 implies the consistency of  $\text{ZFC} + \{\text{there exists a subtle cardinal of order } k\}_k$  in  $\text{ACA}_0$ . If  $\text{ZFC} + \{\text{there exists a subtle cardinal of order } k\}_k$  is consistent then Proposition D.1 cannot be proved in  $\text{ZFC} + \{\text{there exists a subtle cardinal of order } k\}_k$ . In fact, Proposition D.1 cannot be proved from any consistent subset of the theorems of  $\text{ZFC} + \{\text{there exists a subtle cardinal of order } k\}_k$  that derives  $\text{ACA}_0$ . (Here  $\text{ACA}_0$  is formalized in set theory in the obvious way using hereditarily finite sets in place of

natural numbers). The same results hold for Propositions A1.2, A2.2, A3.2, A3.4, A1.2\*, A3.4\*, A1.2', A2.2', A3.2', A3.4', B2.1#, and B2.9 even if stated without "same number of ancestors."

Proof: The first claim is by Lemmas D.18, D.20, D.22, and Lemma D.23. The second claim follow from the first claim by the Gödel second incompleteness theorem.

For the third claim, let  $T$  be any consistent subset of the theorems of  $ZFC + \{\text{there exists a } k\text{-subtle cardinal}\}_k$  that derives  $ACA_0 + \text{Proposition D.1}$ . Since  $ACA_0$  is finitely axiomatizable, let  $T'$  be a finite subset of  $T$  that derives  $ACA_0 + \text{Proposition D.1}$ . Then  $ACA_0 + T'$  is consistent since its theorems are a subset of the theorems of  $T$ . By the first claim,  $ACA_0 + T'$  proves the consistency of  $ZFC + \{\text{there exists a } k\text{-subtle cardinal}\}_k$ . Now  $T'$  and  $ACA_0$  comprise a finite set of theorems of  $ZFC + \{\text{there exists a } k\text{-subtle cardinal}\}_k$ , and hence also  $ACA_0 + T'$  proves this fact. Hence  $ACA_0 + T'$  proves its own consistency. This contradicts the consistency of  $ACA_0 + T'$  by the Gödel second incompleteness theorem.

The final claim is from Theorem B2.12.

COROLLARY D.25. It is necessary and sufficient to use subtle cardinals of every finite order in order to derive the Propositions listed at the end of Theorem D.24 in the following sense. The Propositions are provable in  $ZFC + (\forall k)(\text{there exists a } k\text{-subtle cardinal})$ . And any formal system containing  $ACA_0$  that derives any of these Propositions is a formal system in which  $ZFC + \{\text{there exists a } k\text{-subtle cardinal}\}_k$  is interpretable. (Here  $ACA_0$  is formalized in set theory in the obvious way using hereditarily finite sets in place of natural numbers).

Proof: The first claim is from Theorem C.12. The second claim is from Theorem D.24 and the usual completeness proof for predicate calculus.

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