Fault Tolerant Routing in the Star and Pancake Interconnection Networks*

L. Gargano U. Vaccaro A. Vozella
Dipartimento di Informatica ed Applicazioni
Università di Salerno
84081 Baronissi (SA), Italy

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1 Introduction

Consider a communication network modeled by an undirected graph $G = (V, E)$, where the vertices in V represent the processors of the network and the edges in E correspond to the communication lines. A *routing* ρ in G is a function assigning to ordered pairs of vertices $(\mathbf{x}, \mathbf{y}) \in V \times V$ a fixed simple path $\rho(\mathbf{x}, \mathbf{y})$ from $\mathbf{x}$ to $\mathbf{y}$; ρ is called *bidirectional* if $\rho(\mathbf{x}, \mathbf{y})$ and $\rho(\mathbf{y}, \mathbf{x})$ use the same path for every vertex $\mathbf{x}$ and $\mathbf{y}$. Given $\mathbf{x}, \mathbf{y} \in V$, the distance between $\mathbf{x}$ and $\mathbf{y}$ in G, denoted by $d_G(\mathbf{x}, \mathbf{y})$, is the length of the shortest path in G between $\mathbf{x}$ and $\mathbf{y}$. The diameter $D(G)$ is the maximum of $d_G(\mathbf{x}, \mathbf{y})$ over all vertices $\mathbf{x}$ and $\mathbf{y}$. Let F be a subset of $V \cup E$, F will be called the set of *faults*. Given a routing ρ and a set of faults F, the surviving route graph $R(G, \rho)/F$ is the directed graph consisting of all non faulty vertices of G and with an edge connecting vertices $\mathbf{x}$ and $\mathbf{y}$ iff $\rho(\mathbf{x}, \mathbf{y})$ does not contain elements of F. Note that for bidirectional routing ρ, $R(G, \rho)/F$ is an undirected graph.

The seminal paper of Dolev *et al.* [5] initiated the study of the diameter $D(R(G, \rho)/F)$ and started a fruitful line of research [4], [6], [7], [8],[11], [10], [16], [20], [21]. The motivations in [5] were based on the observation that in a communication network with a fixed routing, the time required to send a message along a route is often dominated by the message processing at the ends of the path. Under this assumption, the diameter of the surviving route graph gives a good estimate of the time to complete broadcast in presence of faults. We refer the reader to [4], [5] and [16] for additional motivations and discussion of the problem.

One of the most interesting results given in [5] is that for *any* minimal length routing ρ on the n-dimensional hypercube H_n the diameter $D(R(H_n, \rho)/F)$ is at most 3, as long as $|F| < n$. In [15] a

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for any minimal length routing ρ on G the diameter $D(R(G,\rho)/F)$ is at most 2, as long as F consists of faulty nodes only and $|F|$ is strictly less than the connectivity of G. In spite of the considerable progress that has been made on the problem of bounding $D(R(G,\rho)/F)$ (see [4], [6], [7], [8],[11], [10], [16], [20], [21]) no other graphs are known to exhibit such an extremely good performance. The main scope of this paper is to show that a recently proposed interconnection network enjoys similar properties. We will prove that if S_n is the n-star graph then for *any* minimal length routing ρ the diameter $D(R(S_n,\rho)/F)$ is at most 3, as long as $|F| < n$. Moreover, we will prove that if P_n is the n-pancake graph then $D(R(P_n,\rho)/F) \leq 4$, as long as $|F| < n$ and ρ belongs to a particular class of routings. The n-star graph and the n-pancake graph were proposed in [2], [3] and have recently received much attention because of their rich structure, small diameter and symmetry property (cf., [1], [12], [13], [14], [17], [18], [19]). Our results confirm the goodness of their topology.

2 Fault tolerant routing in the star graph

Let Σ_n be the set (group) of all permutations of symbols $1, 2, \ldots, n$. Given $\mathbf{u} = u_1 \ldots u_n \in \Sigma_n$ and $i \in \{2, \ldots, n\}$, let us denote by $\mathbf{u}\langle i \rangle = u_i \ldots u_1 \ldots u_n$ the element of Σ_n obtained by permuting the first symbol u_1 of $\mathbf{u}$ with the i-th u_i. The n-star graph S_n is a graph with vertex set $V(S_n) = \Sigma_n$ and edge set

$$E(S_n) = \{(\mathbf{u}, \mathbf{u}\langle i \rangle) : \mathbf{u} \in \Sigma_n, 2 \leq i \leq n\}.$$

S_n has $n!$ nodes, is a Cayley graph, is regular of degree $n-1$, is both edge and vertex symmetric, has diameter equal to $\lceil 3(n-1)/2 \rceil$ and has connectivity $n-1$ (see [1], [2], [3] for a proof of these results). Before proving the main result of this section, let us state the following auxiliary lemmas.

Let the set of faulty elements be $F = F_V \cup F_E$, with $F_V \subset \Sigma_n = V(S_n)$ and $F_E \subset E(S_n)$. Define the set

$$\mathcal{V}(F) = F_V \cup \{\mathbf{v} \in \Sigma_n \ : \ \text{there exists } \mathbf{u} \in \Sigma_n \text{ such that } (\mathbf{u}, \mathbf{v}) \in F_E\}$$

Lemma 1 *Let $F \subset V(S_n) \cup E(S_n)$, $|F| \leq n-2$. For any $a \in \{1, \ldots, n\}$ there exists an index i, with $2 \leq i \leq n$, such that $f_i \neq a$ for each $\mathbf{f} \in \mathcal{V}(F)$.*

Proof. Let $F = F_V \cup F_E$ and consider $a \in \{1, \ldots, n\}$. It is obvious that there exist at most $|F_V|$ indices j, with $1 \leq j \leq n$, such that $f_j = a$ for some $\mathbf{f} \in F_V$. Consider now the remaining vertices of $\mathcal{V}(F)$. For each edge $(\mathbf{u}, \mathbf{v}) \in F_E$ the two vertices $\mathbf{u}, \mathbf{v} \in \mathcal{V}(F)$ have the property that there exists exactly one index $j > 1$ such that either $u_j = v_1 = a$ or $u_1 = v_j = a$ or $u_j = v_j = a$. Therefore, there exist at most $|F_E|$ indices j, with $j \geq 2$, such that $f_j = a$ for some $\mathbf{f} \in \mathcal{V}(F) - F_V$. Hence, there exist at most $|F_V| + |F_E| \leq n-2$ indices j, with $2 \leq j \leq n$, such that $f_j = a$ for some $\mathbf{f} \in \mathcal{V}(F)$. □

minimal length paths from $\mathbf{u}$ *to* $\mathbf{v}$ *go through vertices having* i*-th component equal to* u_i*.*

Proof. Since S_n is a Cayley graph, all paths from $\mathbf{u}$ to $\mathbf{v}$ are in one-to-one correspondence with the paths from $\mathbf{v}^{-1}\mathbf{u}$ to the identity permutation $\mathbf{1} = 1 \dots n$. Moreover, a path from $\mathbf{u}$ to $\mathbf{v}$ goes only through vertices $\mathbf{x}$ such that $x_i = u_i$ if and only if the corresponding path from $\mathbf{v}^{-1}\mathbf{u}$ to $\mathbf{1}$ goes only through vertices $\mathbf{y}$ such that $y_i = i$. Therefore, let us suppose that $\mathbf{u}$ is arbitrary, with $u_i = i \neq 1$, and $\mathbf{v}$ is the identity permutation $\mathbf{1}$. Let $\psi(\mathbf{u})$ be the number of invariances (i.e., the number of u_k such that $u_k = k$) and $\eta(\mathbf{u})$ be the number of cycles (including invariances) in the permutation $\mathbf{u}$. It has been shown in [1] that the length $d_{S_n}(\mathbf{u}, \mathbf{1})$ of the shortest path between $\mathbf{u}$ and $\mathbf{1}$ is

$$d_{S_n}(\mathbf{u}, \mathbf{1}) = \begin{cases} n + \eta(\mathbf{u}) - 2\psi(\mathbf{u}) & \text{if } u_1 = 1, \\ n + \eta(\mathbf{u}) - 2\psi(\mathbf{u}) - 2 & \text{if } u_1 \neq 1. \end{cases} \tag{1}$$

Suppose now, by contradiction, that there exists a minimal length path from $\mathbf{u}$ to $\mathbf{1}$ that contains a vertex having i-th component different from i. Such a path can be depicted as follows

$$\begin{array}{rcl} \mathbf{u} & = & u_1 \dots i \dots u_n \\ & \vdots & \\ \mathbf{x} & = & x_1 \dots i \dots x_n \\ \mathbf{y} & = & i \dots x_1 \dots x_n \\ & \vdots & \\ \mathbf{w} & = & i \dots w_1 \dots w_n \\ \mathbf{z} & = & w_1 \dots i \dots w_n \\ & \vdots & \\ \mathbf{1} & = & 1 \dots i \dots n \end{array}$$

where $\mathbf{x}$ may be equal to $\mathbf{u}$ and $\mathbf{z}$ to $\mathbf{1}$. It is easy to check that

$$\eta(\mathbf{x}) = \eta(\mathbf{y}) + 1 \quad \text{and} \quad \psi(\mathbf{x}) = \psi(\mathbf{y}) + \begin{cases} 2 & \text{if } x_1 = 1; \\ 1 & \text{if } x_1 \neq 1. \end{cases}$$

Therefore, since $y_1 = i \neq 1$, we have

$$\begin{aligned} d_{S_n}(\mathbf{y}, \mathbf{1}) &= n + \eta(\mathbf{y}) - 2\psi(\mathbf{y}) - 2 \\ &= n + \eta(\mathbf{x}) - 2\psi(\mathbf{x}) + \begin{cases} 1 & \text{if } x_1 = 1; \\ -1 & \text{if } x_1 \neq 1. \end{cases} \\ &> d_{S_n}(\mathbf{x}, \mathbf{1}) \end{aligned}$$

that contradicts the minimality of the path. □

Next theorem shows that if the set of faulty elements of the n-star graph S_n has size strictly less than the connectivity of S_n then $D(R(S_n, \rho)/F) \leq 3$ for *any* minimal length routing ρ.

routing ρ.

Proof. Let ρ be an arbitrary minimal length routing on S_n and $\mathbf{u}, \mathbf{v}$ be two vertices of $R(S_n, \rho)/F$. Consider the following two subcases.

a) $u_1 = v_1$. By Lemma 1, there exists i, with $2 \leq i \leq n$, such that for each $\mathbf{f} \in \mathcal{V}(F)$

$$f_i \neq u_1. \tag{2}$$

We show that $(\mathbf{u}, \mathbf{u}\langle i\rangle), (\mathbf{u}\langle i\rangle, \mathbf{v}\langle i\rangle), (\mathbf{v}\langle i\rangle, \mathbf{v})$ is a fault–free path of length 3 in $R(S_n, \rho)/F$. Indeed, $\mathbf{u}, \mathbf{v} \notin F$ by hypothesis and $\mathbf{u}\langle i\rangle, \mathbf{v}\langle i\rangle \notin \mathcal{V}(F)$ by (2). Moreover, $(\mathbf{u}, \mathbf{u}\langle i\rangle) \in E(S_n)$ and, trivially, any minimal length routing ρ in S_n goes from $\mathbf{u}$ to $\mathbf{u}\langle i\rangle$ through the edge $(\mathbf{u}, \mathbf{u}\langle i\rangle)$. Analogously for $\mathbf{v}\langle i\rangle$ and $\mathbf{v}$. Finally, since the vertices $\mathbf{u}\langle i\rangle$ and $\mathbf{v}\langle i\rangle$ coincide on the i-th component (which is $u_1 = v_1$), by Lemma 2 any minimal length routing will go from $\mathbf{u}\langle i\rangle$ to $\mathbf{v}\langle i\rangle$ through a sequence of vertices all having the i-th component equal to u_1. Therefore, by (2), all minimal paths from $\mathbf{u}\langle i\rangle$ to $\mathbf{v}\langle i\rangle$ avoid vertices in $\mathcal{V}(F)$ and, therefore, avoid both vertices in F_V and edges in F_E.

b) $u_1 \neq v_1$. By Lemma 1, there exist two indices i and j (possibly $i = j$) such that for each $\mathbf{f} \in \mathcal{V}(F)$

$$f_i \neq u_1 \text{ and } f_j \neq v_1. \tag{3}$$

Moreover, let $h \geq 2$ such that $u_1 = v_h$. Let us first suppose that $h \neq i$ and $h \neq j$. Consider the vertices $\mathbf{v}' = \mathbf{v}\langle j\rangle$ and $\mathbf{v}'' = \mathbf{v}'\langle h\rangle$. We claim that $(\mathbf{u}, \mathbf{u}\langle i\rangle), (\mathbf{u}\langle i\rangle, \mathbf{v}''\langle i\rangle), (\mathbf{v}''\langle i\rangle, \mathbf{v})$ is a fault–free path of length 3 in $R(S_n, \rho)/F$. Indeed, $\mathbf{u}, \mathbf{v} \notin F$ and $(\mathbf{u}, \mathbf{u}\langle i\rangle) \in E(S_n)$. Moreover, the i-th component of $\mathbf{u}\langle i\rangle$ is u_1 and the i-th component of $\mathbf{v}''\langle i\rangle = \mathbf{v}\langle j\rangle\langle h\rangle\langle i\rangle$ is $v_h = u_1$. Therefore, by (3), $\mathbf{u}\langle i\rangle, \mathbf{v}''\langle i\rangle \notin \mathcal{V}(F)$ and, by Lemma 2, any minimal length routing from $\mathbf{u}\langle i\rangle$ to $\mathbf{v}''\langle i\rangle$ does not contain vertices of $\mathcal{V}(F)$. Moreover, it is easy to see that to go from $\mathbf{v}''\langle i\rangle = \mathbf{w}$ to $\mathbf{v}$, any minimal length routing ρ will go trough the sequence of vertices $\mathbf{w}\langle i\rangle = \mathbf{v}''$, $\mathbf{v}''\langle h\rangle = \mathbf{v}'$, $\mathbf{v}'\langle j\rangle = \mathbf{v}$. Since both $\mathbf{v}' = \mathbf{v}\langle j\rangle$ and $\mathbf{v}'' = \mathbf{v}\langle j\rangle\langle h\rangle$ have the j-th component equal to v_1, by (3) we get $\mathbf{v}', \mathbf{v}'' \notin \mathcal{V}(F)$. Therefore, $\rho(\mathbf{v}''\langle i\rangle, \mathbf{v})$ does not contain faulty elements.

If $h = j$ let $\mathbf{v}' = \mathbf{v}\langle j\rangle$ and consider the path $(\mathbf{u}, \mathbf{u}\langle i\rangle), (\mathbf{u}\langle i\rangle, \mathbf{v}'\langle i\rangle), (\mathbf{v}'\langle i\rangle, \mathbf{v})$; if $h = i$ consider the path $(\mathbf{u}, \mathbf{u}\langle i\rangle), (\mathbf{u}\langle i\rangle, \mathbf{v})$. In the same way as above one can show that these paths do not contain vertices in $\mathcal{V}(F)$ and, therefore, avoid both vertices in F_V and edges in F_E. □

3 Fault tolerant routing in the pancake graph

For each $\mathbf{u} \in \Sigma_n$ denote by $\mathbf{u}(i)$, $2 \leq i \leq n$, the element of Σ_n obtained from $\mathbf{u}$ by reversing the prefix of length i of $\mathbf{u}$, that is, if $\mathbf{u} = u_1 u_2 \ldots u_{i-1} u_i u_{i+1} \ldots u_n$ then $\mathbf{u}(i) = u_i u_{i-1} \ldots u_2 u_1 u_{i+1} \ldots u_n$. For the element $\mathbf{x}(j) \in \Sigma_n$, where $\mathbf{x} = \mathbf{u}(i)$. Clearly, if $i = j$ then $\mathbf{u}(i)(j) = \mathbf{u}$. The n-pancake graph P_n is a graph with $n!$ vertices having vertex set $V(P_n) = \Sigma_n$ and edge set

$$E(P_n) = \{(\mathbf{u}, \mathbf{u}(i)) : \mathbf{u} \in \Sigma_n, 2 \leq i \leq n\}.$$

P_n is a Cayley graph, is regular of degree $n-1$, is vertex symmetric and has diameter upper bounded by $\lceil 5(n+1)/3 \rceil$ (see [1], [9]). Finally, denote by Ω the class of routings ρ on P_n satisfying:

1. if $\mathbf{x}, \mathbf{y} \in \Sigma_n$, $x_n = y_n$ then $\rho(\mathbf{x}, \mathbf{y})$ contains only vertices $\mathbf{u}$ such that $u_n = x_n$;

2. if $\mathbf{x}, \mathbf{y} \in \Sigma_n$, $\mathbf{y} = \mathbf{x}(i)$ then the path from $\mathbf{x}$ to $\mathbf{y}$ is $\rho(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{x}(i))$; if $\mathbf{y} = \mathbf{x}(i)(j)$ then the path from $\mathbf{x}$ to $\mathbf{y}$ is $\rho(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{x}(i))(\mathbf{x}(i), \mathbf{y})$.

Informally, Ω consists of those routings that proceed from $\mathbf{x}$ to $\mathbf{y}$ by moving along vertices in the smallest "sub"–pancake which $\mathbf{x}$ and $\mathbf{y}$ belong to. A routing in this class is the one presented by Akers and Krishnamurty in [1]. Moreover, the routing algorithm by Gates and Papadimitriou [9] can be modified (simply applying the algorithm only to the first $n-1$ components of $\mathbf{x}$ whenever $x_n = y_n$) so to obtain a routing in Ω that preserves the worst–case upper bound of $\lceil (5n+5)/3 \rceil$ steps.

Let the set of faults $F \subset V(P_n) \cup E(P_n)$ be $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_{|F|}\}$, where each $\mathbf{f}_i$ is either a faulty vertex or a faulty edge. Let F_n be a subset of $\{1, \ldots, n\}$ constructed according the following procedure

1. Define $F_n^{(0)} = \emptyset$

2. For each $i = 1, \ldots, |F|$ consider the faulty element $\mathbf{f}_i$:
if $\mathbf{f}_i = \mathbf{v} = v_1 \ldots v_n \in V(P_n)$ then add v_n to $F_n^{(i-1)}$, that is, put $F_n^{(i)} = F_n^{(i-1)} \cup \{v_n\}$,
if $\mathbf{f}_i = (\mathbf{v}, \mathbf{v}(j)) \in E(P_n)$, $2 \leq j \leq n-1$, then add v_n to $F_n^{(i-1)}$, i.e., put $F_n^{(i)} = F_n^{(i-1)} \cup \{v_n\}$,
if $\mathbf{f}_i = (\mathbf{v}, \mathbf{v}(n)) \in E(P_n)$ then add either v_1 or v_n to $F_n^{(i-1)}$, i.e., put either
$F_n^{(i)} = F_n^{(i-1)} \cup \{v_1\}$ or $F_n^{(i)} = F_n^{(i-1)} \cup \{v_n\}$;

3. Put $F_n = F_n^{(|F|)}$.

Since $|F_n^{(i)}| - |F_n^{(i-1)}| \leq 1$, at each step the procedure adds one new element to F_n at most, thus $|F_n| \leq |F|$. Moreover, it is immediate to see that for each $\mathbf{u} \in \Sigma_n$

$$u_n \notin F_n \quad \Rightarrow \quad \mathbf{u} \notin F \text{ and } (\mathbf{u}, \mathbf{u}(i)) \notin F \text{ for each } i = 2, \ldots, n-1 \tag{4}$$

$$u_1,\ u_n \notin F_n \quad \Rightarrow \quad (\mathbf{u}, \mathbf{u}(n)) \notin F \tag{5}$$

Lemma 3 *Let $F \subset V(P_n) \cup E(P_n)$, $|F| < n-1$, $\mathbf{u} \notin F$, and $u_n \in F_n$. Either*

$$u_1 \notin F_n \quad \textit{and} \quad (\mathbf{u}, \mathbf{u}(n)) \notin F \tag{6}$$

$$u_h \notin F_n, \quad \mathbf{u}(h) \notin F, \quad (\mathbf{u}, \mathbf{u}(h)) \notin F, \quad \textit{and} \quad (\mathbf{u}(h), \mathbf{u}(h)(n)) \notin F. \tag{7}$$

Proof Let us notice that, since $|F_n| \leq |F| \leq n-2$, there exist $a, b \in \{1, \ldots, n\}, a \neq b$, such that $a, b \notin F_n$. Therefore, we can find two indices i, j, with $i \neq j$, $2 \leq i \leq n-1$, $1 \leq j \leq n-1$, such that $u_i, u_j \notin F_n$. If either i satisfies (7) or j is larger than 1 and satisfies (7) or j is equal to 1 and satisfies (6) then the lemma is proved; otherwise there exist $\mathbf{f}', \mathbf{f}'' \in F$ such that

$$\mathbf{f}' \in \{\mathbf{u}(i),\ (\mathbf{u}, \mathbf{u}(i)),\ (\mathbf{u}(i), \mathbf{u}(i)(n))\} \text{ and } \mathbf{f}'' \in \begin{cases} \{(\mathbf{u}, \mathbf{u}(n))\} & \text{if } j = 1 \\ \{\mathbf{u}(j),\ (\mathbf{u}, \mathbf{u}(j)),\ (\mathbf{u}(j), \mathbf{u}(j)(n))\} & \text{if } j \neq 1 \end{cases} \tag{8}$$

Notice that vertices $\mathbf{u}$, $\mathbf{u}(i)$, and $\mathbf{u}(j)$ have the n–th component equal to u_n and vertices $\mathbf{u}(i)(n)$ and $\mathbf{u}(j)(n)$ have the n–th component equal to u_i and u_j, respectively, with $u_i, u_j \notin F_n$. Then in correspondence of both $\mathbf{f}'$ and $\mathbf{f}''$ at point **2.** of the procedure that constructs F_n the added symbol was u_n. We can then conclude that $|F_n - \{u_n\}| \leq |F| - 2 \leq n-4$. Therefore, there exists $u_k \notin F_n$ for some index k such that $k \neq n, i, j$. If either k is equal to 1 and satisfies (6) or k is larger then 1 and satisfies (7) then the lemma is proved; otherwise we keep repeating this procedure. In this way either we will find an index $h \in \{1, \ldots, n-1\}$ satisfying (6) or (7) (and the lemma is proved) or we will get $u_1, \ldots, u_{n-1} \notin F_n$ and

$$(\mathbf{u}, \mathbf{u}(n)) \in F \quad \text{and} \quad \{\mathbf{u}(h),\ (\mathbf{u}, \mathbf{u}(h)),\ (u(h), \mathbf{u}(h)(n))\} \cap F \neq \emptyset, \quad \text{for each } h = 2, \ldots, n-1. \tag{9}$$

But (9) implies $|F| \geq n-1$ that contradicts the hypothesis of the lemma. □

We are now ready to prove the main result of this section.

Theorem 2 *Let $F \subset \Sigma_n$, $|F| < n-1$. Then $D(R(P_n, \rho)/F) \leq 4$ for any routing $\rho \in \Omega$.*

Proof. Let $\rho \in \Omega$ and $\mathbf{u}$ and $\mathbf{v}$ be two vertices of $R(P_n, \rho)/F$. Consider the following cases:

a) $u_n \notin F_n$.

a$_1$) Suppose $v_n \notin F_n$ and let i be such that $u_i = v_n$. We claim that in $R(P_n, \rho)/F$ the path $(\mathbf{u}, \mathbf{u}(i)(n))(\mathbf{u}(i)(n), \mathbf{v})$ is fault–free. Indeed, for each routing $\rho \in \Omega$

$$\rho(\mathbf{u}, \mathbf{u}(i)(n)) = \begin{cases} (\mathbf{u}, \mathbf{u}(n)) & \text{if } i = 1 \\ (\mathbf{u}, \mathbf{u}(i))(\mathbf{u}(i), \mathbf{u}(i)(n)) & \text{if } 2 \leq i \leq n-1 \\ \text{empty} & \text{if } i = n. \end{cases}$$

Since the n-th component of $\mathbf{u}$ and $\mathbf{u}(i)$ is $u_n \notin F_n$ and that of $\mathbf{u}(i)(n)$ is $u_i \notin F_n$, Therefore, by (4) and (5), the path $\rho(\mathbf{u}, \mathbf{u}(i)(n))$ avoids nodes and edges in F. Moreover, by definition of routings in Ω, $\rho(\mathbf{u}(i)(n), \mathbf{v})$ [$\rho(\mathbf{u}, \mathbf{v})$ if $i = n$] goes through a sequence of vertices all having last component equal to $u_i = v_n \notin F_n$ and, by (4) and (5) avoids nodes and edges in F.

$(\mathbf{v}, \mathbf{v}(h)) \notin F$ if $h \neq 1$, and $(\mathbf{v}(h), \mathbf{v}(h)(n)) \notin F$. Moreover, by (4) we get $\mathbf{v}(h)(n) \notin F$ since its n–component is $v_h \notin F_n$. Noticing that for any routing $\rho \in \Omega$

$$\rho(\mathbf{v}(h)(n), \mathbf{v}) = \begin{cases} (\mathbf{v}(n), \mathbf{v}) & \text{if } h = 1 \\ (\mathbf{v}(h)(n), \mathbf{v}(h))(\mathbf{v}(h), \mathbf{v}) & \text{if } h \neq 1 \end{cases}$$

we get that $\rho(\mathbf{v}(h)(n), \mathbf{v})$ avoids faults in F. Moreover, a fault–free path from $\mathbf{u}$ to $\mathbf{v}(h)(n)$ in $R(P_n, \rho)/F$ can be obtained from case $\mathbf{a_1}$) since the n-th component of $\mathbf{v}(h)(n)$ is $v_h \notin F_n$.

b) $u_n \in F_n$. By Lemma 3 there exists an index $h < n$, such that $u_h \notin F_n$, $\mathbf{u}(h) \notin F$, $(\mathbf{u}, \mathbf{u}(h)) \notin F$ if $h \neq 1$, and $(\mathbf{u}(h), \mathbf{u}(h)(n)) \notin F$. By (4) we get $\mathbf{u}(h)(n) \notin F$. Since any routing $\rho \in \Omega$ uses the path

$$\rho(\mathbf{u}, \mathbf{u}(h)(n))) = \begin{cases} (\mathbf{u}, \mathbf{u}(n)) & \text{if } h = 1 \\ (\mathbf{u}, \mathbf{u}(h))(\mathbf{u}(h), \mathbf{u}(h)(n)) & \text{if } h \neq 1 \end{cases}$$

we get that $\rho(\mathbf{u}, \mathbf{u}(h)(n)))$ avoids faults in F. The fault–free path from $\mathbf{u}(h)(n)$ to $\mathbf{v}$ can be obtained from case **a**) since the n-th component of $\mathbf{u}(h)(n)$ is $u_h \notin F_n$. □

We conclude this section with the following conjecture that would imply the extension of the previous result to *all* minimal length routing on P_n.

Conjecture *The class Ω contains all minimal length routings on P_n.*

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