

Hiding and Behaviour: an Institutional Approach

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Abstract

Theories with hidden sorts provide a setting to study the idea of behaviour and behavioural equivalence of elements. But there are variants on the notion of theory: many sorted algebras, order sorted algebras and so on; we would like to use the theory of institutions to develop ideas of some generality. We formulate the notion of behavioural equivalence in a more abstract and categorical way, and we give a general explication of "hiding" in an institution. We use this show that both hidden many sorted algebras and hidden order sorted algebras yield institutions.

1.1 Introduction

An institution [5] is an abstract definition of a logic system, which has proved useful in studying specification languages. It consists of a category of signatures with functors defining sentences and models for each signature, together with a satisfaction relation between sentences and models. Our aim is to explicate the notion of object ¹ in the context of institutions, or since the notion of object is very rich, we should more modestly seek to contribute to such an explication. The aspects which we study are:

- The unary nature of objects - they have unary operations on a "state" which produce a new state;
- This state is "hidden" - different states can only be distinguished by their "visible" behaviour

¹It turns out that the world is full of objects; here we mean 'object-oriented' objects

Our work takes as its starting point Goguen’s representation of objects by “hidden many sorted algebras” [3, 7], with which he seeks to capture the two aspects described above. A hidden many sorted signature has “hidden” and “visible” sorts. The visible sorts and the operations on them have a fixed interpretation for all algebras, but the interpretation of hidden sorts may vary from algebra to algebra. The operations on hidden sorts are unary in the sense that they take at most one argument of hidden sort - they cannot act on two states.

We may remark that Goguen’s terminology might be made more explicit, albeit more cumbersome, by referring to “hidden *monadic* many sorted algebras”, since the unary aspect is important as well as the hidden aspect. Indeed one might better refer to them as many sorted *object* algebras. However we will stick to Goguen’s terminology here.

In [3], Goguen defines the hidden many sorted equational institution to have as signatures the usual many sorted signatures with the sorts divided into “hidden” and “visible” ones and a restriction to unary operations on hidden sorts. The sentences and models are the same as for many sorted institutions, but the satisfaction relation is different. A model now satisfies an equation if the left hand side gives the same result as the right hand side in all visible contexts. Here a *visible context* means a term with one free variable and a visible result sort. We shall call this *behavioural satisfaction*. The definition of morphism for hidden many sorted signatures has some restrictions. We shall make all this precise later.

Given a hidden many sorted algebra, A , it behaviourally satisfies a set of sentences just if a certain quotient algebra satisfies the sentences in the usual sense. This quotient algebra identifies elements which are behaviourally indistinguishable, that is there is no visible context which discriminate between them.

Consider the following example of a many sorted signature. The comments in parentheses are just for motivation.

Sorts

Nat	(Naturals)
NzNat	(Nonzero Naturals)
R	

Operations

<code>_+_</code> , <code>_*_</code>	<code>: Nat Nat -> Nat</code>	(addition and multiplication)
<code>_+_</code> , <code>_*_</code>	<code>: NzNat NzNat -> NzNat</code>	(addition and multiplication)
<code>_div_</code>	<code>: Nat NzNat -> Nat</code>	(division ignoring remainder)
<code>natof</code>	<code>: NzNat -> Nat</code>	(injection)
<code>in</code>	<code>: NzNat -> R</code>	(injection)
<code>_*_</code>	<code>: R Nat -> R</code>	
<code>_/_</code>	<code>: R NzNat -> R</code>	
<code>wh</code>	<code>: R -> Nat</code>	(whole part)

Based on this example, we can define a hidden many sorted signature, by defining the sort \mathbf{R} to be hidden, \mathbf{Nat} and \mathbf{NzNat} to be visible. The interpretation of the visible sorts is fixed thus $|\mathbf{Nat}| = \omega$, the set of natural numbers, $|\mathbf{NzNat}| = \omega_+$, the set of nonzero natural numbers and the interpretation of the operations on these is the standard one.

There are many hidden sorted algebras for this signature. A particular familiar one is obtained by interpreting the hidden sort \mathbf{R} as $|\mathbf{R}| = \omega \times \omega_+$. Using n, n', \dots for elements of ω , d, d', \dots for elements of ω_+ and r, r', \dots for elements of $|\mathbf{R}|$, we define the operations

$$\begin{aligned} \mathbf{in}(n) &= (n, 1) \\ (n, d) * n' &= (n * n', d) \\ (n, d) / d' &= (n, d * d') \\ \mathbf{wh}(n, d) &= n \text{ div } d \end{aligned}$$

We think of n as the numerator and d as the denominator. Now r and r' are *behaviourally equivalent* iff any term of sort \mathbf{Nat} involving r and the corresponding term involving r' have the same value. (As it happens, our signature has no terms of sort \mathbf{NzNat} , the other visible sort.)

For example, $(3, 2)$ is behaviourally equivalent to $(6, 4)$ since $\mathbf{wh}(3, 2) = 1 = \mathbf{wh}(6, 4)$, and $\mathbf{wh}[(3, 2) \times 10] = 15 = \mathbf{wh}[(6, 4) \times 10]$, and so on for all terms. But $(3, 2)$ is not behaviourally equivalent to $(7, 4)$ since $\mathbf{wh}[(3, 2) \times 10] = 15 \neq 17 = \mathbf{wh}[(7, 4) \times 10]$. It is not hard to see that (n, d) is behaviourally equivalent to (n', d') iff the rationals n/d and n'/d' are equal, that is, $n \times d' = n' \times d$. This algebra is not quite the nonnegative rationals, since it has many (n, d) pairs corresponding to each rational.

Another hidden sorted algebra could be defined by taking $|\mathbf{R}|$ to be \mathbb{Q}^{nn} , the nonnegative rationals with the operations \mathbf{in} , $*$, $/$, and \mathbf{wh} interpreted in the usual way. Now, r and r' are behaviourally equivalent if and only if they are equal. This algebra with $|\mathbf{R}| = \mathbb{Q}^{nn}$ is actually a quotient of the algebra above with $|\mathbf{R}| = \omega \times \omega_+$. This quotienting will play a central role in our investigation, since a hidden many sorted algebra gives rise by quotienting to a many sorted algebra which in a sense represents its behaviour.

A more object oriented example may give more computational motivation:

```
Sorts
  int
  counter [hidden]

Operations
  _+_, _-_ : int int -> int
```

```

0,1 : -> int
incr,decr : counter -> counter
init : -> counter
read : counter -> int

```

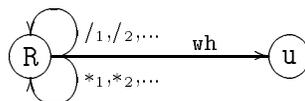
Now we can have an algebra where `|counter|` is pairs of natural numbers (the number of increments and the number of decrements) or an algebra where `|counter|` is integers (the difference between these numbers). The latter is a quotient of the former.

1.1.1 Behaviour algebras

In order to describe the notion of behaviourally indistinguishable concisely we may convert a hidden many sorted algebra into another kind of algebra which is simpler and has a concise formulation of “behaviour”. This is what we shall call a *behaviour algebra*; it somewhat reminiscent of the Lawvere notion of algebra for an algebraic theory. Like a Lawvere algebra, it is a functor from a category to **Set**, but here the category has a distinguished object and this object has a fixed interpretation. The distinguished object will represent the totality of visible sorts. We are able to show that hidden many sorted algebras convert to behaviour algebras and back again and that there are terminal behaviour algebras. The morphism to the terminal object yields the required behavioural quotient. We may say that the category algebras explicate our notion of *object*.

From a hidden sorted signature we can derive a category playing the role of a signature for behaviour algebras. The derivation should be intuitively natural; it will be spelled out formally later. We first convert the signature to a graph, whose nodes are the hidden sorts plus a distinguished node representing all the visible sorts. Each operation on a hidden sort gives a family of edges in the graph. Now the paths on this graph, or possibly a quotient of them, yield a category. So here is the graph corresponding to the hidden sorted signature in the “rationals” example presented above.

- Nodes: \mathbb{R} and \mathbf{u}
- Edges: $*_n, /_d : \mathbb{R} \rightarrow \mathbb{R}$ for each $n \in \omega, d \in \omega_+$, and $\mathbf{wh} : \mathbb{R} \rightarrow \mathbf{u}$.



Thus the *binary* operator $* : \mathbb{R} \text{ Nat} \rightarrow \mathbb{R}$ becomes a family of *unary* operators $*_n : \mathbb{R} \rightarrow \mathbb{R}$, taking one argument of hidden sort.

For the signature of the behaviour algebras we will take the path category of this graph. A typical arrow is $*_3;*_4;/_2;\mathbf{wh} : \mathbf{R} \rightarrow \mathbf{u}$. We can think of each path ending in the visible sort \mathbf{u} as a possible experiment. An interpretation of the graph nodes as sets and the edges as functions extends to an interpretation the path category, assigning to each path the composition of the functions for the edges making up the path; we call this an algebra of the path category. It is what we called a behaviour algebra above.

In the case of the rational numbers example, as an interpretation for \mathbf{u} we take $|\mathbf{u}| = \omega$ (strictly, in our construction given below, $|\mathbf{u}| = \omega + \omega_+$, but, since there are no terms of sort \mathbf{NzNat} , we can simplify this to ω and avoid a boring injection of ω into $\omega + \omega_+$). We can think of $|\mathbf{u}|$ as the set of answers. Corresponding to the first hidden sorted algebra given above we have the behaviour algebra given as follows:

- $|\mathbf{R}| = \omega \times \omega_+$
- $*_{n'}(n, d) = (n*n', d)$
- $/_{d'}(n, d) = (n, d*d')$
- $\mathbf{wh}(n, d) = n \text{ div } d$

The interpretation of a path to $|\mathbf{u}|$ (an experiment) is a function; when this function is applied to a value in the interpretation of \mathbf{R} it gives an answer to the experiment. It turns out that the collection of all functions from experiments to answers is itself a behaviour algebra, indeed the terminal behaviour algebra.

Two elements of the hidden sort $|\mathbf{R}|$ are indistinguishable by observation if all experiments give the same answer. We can now quotient the algebra by this observation equivalence, obtaining in fact the behaviour algebra corresponding to the second hidden sorted algebra given above, that interprets \mathbf{R} as the nonnegative rationals. We obtain this quotient by factorising the unique morphism to the terminal algebra.

1.1.2 Making a hidden version of an institution

We use this to develop a way of taking an institution equipped with some extra data and producing an object version of the institution (or as Goguen calls it a “hidden” version). The extra data shows how the models of the institution can be viewed as representing behaviour algebras. So you might take the institution of many sorted algebras and produce the institution of hidden many sorted algebras. Or you might start with order sorted algebras and produce hidden order sorted algebras.

Here is the idea in outline. We take an institution \mathcal{I} plus the extra data and produce an institution \mathcal{H} , thus

- We have a notion of \mathcal{H} signature, and each \mathcal{H} signature should have a corresponding \mathcal{I} signature

- The models of the \mathcal{H} signature are a subcategory of the models of the \mathcal{I} signature
- For each model M of an \mathcal{H} signature Σ we can derive a behaviour algebra.
- We compute a quotient of this behaviour algebra by equating elements which have the same observable behaviour
- We convert this quotient algebra back into an \mathcal{H} model $\beta_{\Sigma}(M)$.
- The satisfaction relation in \mathcal{H} between the model M and any sentence e is defined as the satisfaction in \mathcal{I} between $\beta_{\Sigma}(M)$ and e .

All this should work smoothly when you change signatures so that we get appropriate functors and natural transformations, so that we get an institution \mathcal{H} equipped with an institution morphism to \mathcal{I} .

1.1.3 The order sorted institution

Goguen also wishes to define a hidden version of the order sorted institution. We use the machinery outlined above to accomplish this. The translation from an order sorted signature to a category again uses the path category, but because of the inclusions induced by the ordering on sorts certain paths have to be identified. It is for this reason that we preferred to define a behaviour algebra as a functor rather than taking an algebra to be a morphism from a graph to **Set**; this simpler notion would have sufficed to translate many sorted algebras, but not order sorted ones.

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Rod Burstall wishes to dedicate his work presented here to the memory of his wife Sissi and her unfailing love.

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1.2 Preliminaries

1.2.1 Institutions

An adequate formalisation of logic as used in Computing Science must achieve a delicate balance between syntax and semantics. Tarski’s semantic definition of truth for first order logic [12] is a traditional reconciliation of these two aspects of logics, based on the notion of *satisfaction* as a binary relation between models and sentences. Some such notion is needed for the very basic notions of *soundness* and *completeness* of logical systems, because these notions depend in an essential way upon the relationship between provability (which is syntactic) and satisfaction (which is semantic, i.e., concerns “truth” in Tarski’s sense). These notions, in turn, are basic to classical treatments of the adequacy of rules of deduction for logical systems; soundness and completeness with respect to an intuitively plausible class of models give us far greater confidence in a set of rules of deduction, and make their range of applicability more precise.

In a series of papers beginning in 1979, Goguen and Burstall developed institutions to formalise the intuitive notion of a logical system; the most recent and complete exposition is [5]. This approach allows us to discuss the crucial relationship between theories and models without commitment to either side at the expense of the other. Institutions are much more abstract than Tarski’s model theory, and they also add another basic ingredient, namely signatures and the possibility of translating sentences and models from one signature to another. A special case of this translation may be familiar from first order model theory: if $\Sigma \rightarrow \Sigma'$ is an inclusion of first order signatures, and if M is a Σ' -model, then we can form $M|_{\Sigma}$, called the *reduct* of M to Σ . Similarly, if e is a Σ -sentence, then we can always view it as a Σ' -sentence (but there is no standard notation for this). The key axiom, called the Satisfaction Condition, says that *truth is invariant under change of notation*, which is surely a very basic intuition for traditional logic.

Definition 1 An **institution** consists of

1. a category $Sign$, whose objects are called **signatures**,
2. a functor $Sen : Sign \rightarrow \mathbf{Set}$, giving for each signature a set whose elements are called **sentences** over that signature,
3. a functor $MOD : Sign^{op} \rightarrow \mathbf{Cat}$ giving for each signature Σ a category whose objects are called Σ -**models**, and whose arrows are called Σ -**(model) morphisms**, and
4. a relation $\models_{\Sigma} \subseteq |MOD(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |Sign|$, called Σ -**satisfaction**,

such that for each morphism $\phi : \Sigma \rightarrow \Sigma'$ in *Sign*, the **Satisfaction Condition**

$$M' \models_{\Sigma'} Sen(\phi)(e) \text{ iff } MOD(\phi)(M') \models_{\Sigma} e$$

holds for each $M' \in |MOD(\Sigma')|$ and $e \in Sen(\Sigma)$. \square

The notion of institution morphism is especially relevant for comparing different logical systems and eventually for transferring results from one logical system to another [5], or from one theorem prover built on the top of one particular logical system to another theorem prover built on the top of another logical system. Here is its original definition as given by [5]:

Definition 2 Let \mathcal{I} and \mathcal{I}' be institutions. Then an **institution morphism** $\Phi : \mathcal{I} \rightarrow \mathcal{I}'$ consists of

1. a functor $\Phi : \text{SIGN} \rightarrow \text{SIGN}'$,
2. a natural transformation $\alpha : \Phi; Sen' \Rightarrow Sen$, and
3. a natural transformation $\beta : MOD \Rightarrow \Phi; MOD'$

such that the following **Satisfaction Condition** holds

$$M \models_{\Sigma} \alpha_{\Sigma}(e') \text{ iff } \beta_{\Sigma}(M) \models'_{\Phi(\Sigma)} e'$$

for any Σ -model M from \mathcal{I} and any $\Phi(\Sigma)$ -sentence e' from \mathcal{I}' . \square

1.2.2 Many sorted algebra

An important institution in Computing Science is that of many sorted equational logic, which played a major role in the theory of algebraic specifications [2], semantics of imperative programming languages [10], and theorem proving [4]:

Definition 3 [4] A **many sorted (algebraic) signature** is a pair (S, Σ) , where S is a set of sorts and Σ is a family of sets of operator names, indexed by $S^* \times S$. A **morphism of many sorted signatures** $\phi : (S, \Sigma) \rightarrow (S', \Sigma')$ consists of a map $S \rightarrow S'$ of sorts and an $S^* \times S$ -indexed family of maps $\phi_{w,s} : \Sigma_{w,s} \rightarrow \Sigma'_{\phi w, \phi s}$.

A **many sorted algebra** A over the signature (S, Σ) is an S -sorted set $\langle A_s \mid s \in S \rangle$ together with an interpretation of operator names as functions $\langle A_{\sigma} : A_w \rightarrow A_s \mid \sigma \in \Sigma_{w,s} \rangle$ (by A_w we mean the cartesian products of all A_v for all elements v of w). A **homomorphism** $h : A \rightarrow A'$ of many sorted algebras over the signature (S, Σ) is an S -sorted map $A \rightarrow A'$ such that for all σ in $\Sigma_{w,s}$ and all $a = a_1 \dots a_n$ in A_w , $h_s(A_{\sigma}a) = A'_{\sigma}(h_{s_1}a_1 \dots h_{s_n}a_n)$. \square

Definition 4 [4] An **equation** e for the many sorted signature (S, Σ) is a triple $\langle X, l, r \rangle$, where X is an S -sorted set, where l and r are terms over X having the same sort $s \in S$. A many sorted algebra A over (S, Σ) **satisfies** the equation

$\langle X, l, r \rangle$ iff for all valuations $v : X \rightarrow A$, $v^\#(l) = v^\#(r)$ (where $v^\#$ is the unique extension of v to a homomorphism from the term algebra over X to the algebra A). In this case we write $A \models_\Sigma \langle X, l, r \rangle$. \square

[5] describes the way equations are translated and models are “reduced” along many sorted signature morphisms and proves the Satisfaction Condition for many sorted equational logic.

1.3 Behaviour algebras and observations

Definition 5 A **behaviour signature** is a category with distinguished object u such that there are no arrows whose source is the distinguished object except the identity; we let a morphism of behaviour signatures be a functor preserving the distinguished object and such that distinguished objects are the only objects mapped to distinguished objects. Let \mathbf{Cat}_u denote the category of behaviour signatures.

We write a behaviour signature as a pair (C, u) where C is a category and u is its distinguished object. We use h and h' for objects different from u , and we use e for arrows. \square

Definition 6 Let \mathcal{U} be a set. By a **behaviour algebra** over (C, u, \mathcal{U}) we mean a functor $A : C \rightarrow \mathbf{Set}$ such that $Au = \mathcal{U}$. A homomorphism of these algebras is a natural transformation such that its component at u is the identity function on \mathcal{U} . These algebras and their homomorphisms form a category, $\mathbf{BALG}(C, u, \mathcal{U})$. \square

From now on we fix a behaviour signature (C, u) and a set \mathcal{U} ; also, we call a behaviour algebra over these simply an algebra. We now construct a terminal object, \mathbf{B} for this category:

1. for each object h different from u , define $\mathbf{B}h$ to be the set of functions which take arrows $h \rightarrow u$ to an element of \mathcal{U} , that is $\mathcal{U}^{C[h, u]}$, and
2. for an arrow $e \in C[h, h']$, define $\mathbf{B}e$ to be $\mathcal{U}^{C[e, u]}$, the function $\psi : \mathcal{U}^{C[h, u]} \rightarrow \mathcal{U}^{C[h', u]}$ given by $(\psi f)e' = f(e; e')$ where $f \in \mathcal{U}^{C[h, u]}$ and $e' \in C[h', u]$.

Note that the definition of \mathbf{B} on arrows $e \in C[h, u]$ makes sense under the canonical identification of \mathcal{U} with $\mathcal{U}^{C[u, u]}$ (since $C[u, u]$ is a singleton set because no arrows go out of u except the identity).

Proposition 7 \mathbf{B} is the terminal algebra over (C, u, \mathcal{U}) .

Proof: Let A be any algebra over (C, u, \mathcal{U}) . Fix any object h . If $m : A \rightarrow \mathbf{B}$ is a homomorphism, then for any $e \in C[h, h']$ it satisfies $m_h; \mathbf{B}e = Ae; m_{h'}$ and hence for any $p \in C[h, u]$, $m_h; \mathbf{B}p = Ap; m_u$. But m_u is the identity, so this is $m_h; \mathbf{B}p = Ap$.

Observe that for any arrow $e \in C[h, u]$, $\mathbf{B}e$ is the projection $\mathcal{U}^{C[h, u]} \rightarrow \mathcal{U}$ on the e -th component. By the universal property of products there is a unique function $m_h: Ah \rightarrow \mathbf{B}h$ such that $m_h; \mathbf{B}e = Ae$ for any arrow $e \in C[h, u]$. Thus, there is at most one algebra homomorphism $m: A \rightarrow \mathbf{B}$.

All we still have to prove is that the unique function m defined above is indeed an algebra homomorphism, that is, $m_h; \mathbf{B}e = Ae; m_{h'}$ for any objects h and h' and for any arrow $e \in C[h, h']$. Pick an arbitrary arrow $e' \in C[h', u]$. Then:

$$\begin{aligned}
m_h; \mathbf{B}e; \mathbf{B}e' & \\
&= m_h; \mathbf{B}(e; e') \\
&= A(e; e') \quad (\text{by definition of } m) \\
&= Ae; Ae' \\
&= Ae; m_{h'}; \mathbf{B}e' \quad (\text{by definition of } m)
\end{aligned}$$

By the universal property of the product $\mathcal{U}^{C[h', u]}$ we get the desired equality $m_h; \mathbf{B}e = Ae; m_{h'}$. \square

We now turn to the definition of quotient behaviour algebra using a rather abstract formulation, the notion of *image factorisation system* as defined in [1].

Fact 8 The category of algebras over (C, u, \mathcal{U}) has a canonical image factorisation system $(\mathcal{E}_C, \mathcal{M}_C)$ with $\mathcal{E}_C = \{e \text{ morphism of behaviour algebras} \mid e_h \text{ surjective for any } h\}$ and $\mathcal{M}_C = \{m \text{ morphism of behaviour algebras} \mid m_h \text{ injective for any } h\}$.

Proof: Each component of a morphism of behaviour algebras factors in **Set**, then we use the Diagonal Fill-in Property for image factorisation systems to define the image behaviour algebra on arrows.

More specifically, suppose $f: A \rightarrow \mathbf{B}$ is a homomorphism of algebras and $k: s \rightarrow t$ in C , so that $A_k: A_s \rightarrow A_t$ and $\mathbf{B}_k: \mathbf{B}_s \rightarrow \mathbf{B}_t$. Then we factorise $f_s: A_s \rightarrow \mathbf{B}_s$ to get an intermediate I_s , similarly for t . We then use the fill in property to get $I_k: I_s \rightarrow I_t$. This gives the factorisation of f with image algebra I .

\square

Any morphism $\phi: (C, u) \rightarrow (C', u)$ of behaviour signatures determines a functor $\text{BALG}(\phi): \text{BALG}(C', u, \mathcal{U}) \rightarrow \text{BALG}(C, u, \mathcal{U})$ mapping a behaviour algebra A' to $\phi; A'$ and any morphism f' of category algebras to $\phi f'$ (i.e., the vertical composition between ϕ as a functor and f' as a natural transformation).

Corollary 9 For any morphism $\phi: (C, u) \rightarrow (C', u)$ of behaviour signatures, $\text{BALG}(\phi)$ is a morphism of factorisation systems $(\mathcal{E}_{C'}, \mathcal{M}_{C'}) \rightarrow (\mathcal{E}_C, \mathcal{M}_C)$, i.e., $\text{BALG}(\phi)\mathcal{E}_{C'} \subseteq \mathcal{E}_C$ and $\text{BALG}(\phi)\mathcal{M}_{C'} \subseteq \mathcal{M}_C$. \square

Lemma 10 Let $\phi: (C, u) \rightarrow (C', u)$ be a morphism of behaviour signatures with $\phi[h, u]: C[h, u] \rightarrow C'[\phi h, u]$ surjective for any h . Let \mathbf{B} be the terminal behaviour algebra over (C, u, \mathcal{U}) and \mathbf{B}' be the terminal algebra over (C', u, \mathcal{U}) . Then the unique homomorphism of category algebras $m: \text{BALG}(\phi)\mathbf{B}' \rightarrow \mathbf{B}$ is injective in all components, i.e., it belongs to \mathcal{M}_C .

Proof: Fix h object of C . $(\text{BALG}(\phi)\mathbf{B}')h = \mathbf{B}'(\phi h) = \mathcal{U}^{C'[\phi h, u]}$. For any $f: C'[\phi h, u] \rightarrow \mathcal{U}$, $m_h(f) = \phi[h, u]; f$. Since surjective functions are epis, $m_h(f) = m_h(g)$ implies $f = g$ which proves the injectivity of m_h . \square

1.4 Hidden many sorted signatures and models

In [3], Goguen introduces the institution of hidden many sorted equational logic as a logical support for an algebraic semantics for object oriented programming. His definition of hidden many sorted signatures emphasize on the *monadic* nature of observations. Furthermore, by forcing the Satisfaction Condition to hold, Goguen obtained the right notion of hidden many sorted signature. The restrictions imposed on morphisms of hidden many sorted signatures correspond exactly to the methodological principle of encapsulation from the practice of object oriented programming (see [3] for further details).

Here are his definitions for hidden many sorted signatures, morphisms of signatures and models:

Definition 11 Fix a set V of visible sorts, a many sorted signature (V, Ψ) and an algebra \mathcal{U} over (V, Ψ) . A **hidden many sorted signature** (H, V, Σ) is a many sorted signature $(H \cup V, \Sigma)$ with H the set of hidden sorts, V the set of visible sorts and satisfying the following conditions:

- (S1) if $\sigma \in \Sigma_{w,v}$ with $w \in V^*$ and $v \in V$, then $\sigma \in \Psi_{w,v}$; and
- (S2) if $\sigma \in \Sigma_{w,v}$ then at most one element of w lies in H .

A **hidden many sorted signature morphism** $\phi: (H, V, \Sigma) \rightarrow (H', V, \Sigma')$, is a many sorted signature morphism $\phi: (H \cup V, \Sigma) \rightarrow (H' \cup V, \Sigma')$ such that:

- (M1) $\phi v = v$ for all $v \in V$ and $\phi \sigma = \sigma$ for all $\sigma \in \Psi$,
- (M2) $\phi(H) \subseteq H'$, and
- (M3) if $\sigma' \in \Sigma'_{w',s'}$ and some sort in w' lies in $\phi(H)$, then $\sigma' = \phi(\sigma)$ for some $\sigma \in \Sigma$.

A **hidden many sorted model** over (H, V, Σ) is a many sorted algebra M over $(H \cup V, \Sigma)$ such that $M \upharpoonright_{\Psi} = \mathcal{U}$ and a **homomorphism of hidden many sorted models** $h: M \rightarrow M'$ is a homomorphism of many sorted algebras such that $h \upharpoonright_{\Psi} = 1_{\mathcal{U}}$. \square

Consider a hidden many sorted signature (H, V, Σ) , with H as the set of hidden sorts, V as the set of visible sorts and \mathcal{U}_v the fixed sets of “data values” for the visible sorts $v \in V$. Let now v and v' denote strings of visible sorts, i.e., elements of V^* . For each $v \in V^*$ we denote by \mathcal{U}_v the cartesian product of all sets of data values corresponding to the elements of v .

The hidden many sorted signature (H, V, Σ) canonically determines a pointed graph (G, u) in the following way:

1. the nodes are all elements of H plus a distinguished node u ,
2. for any nodes h, n with $h \in H$,
 $G[h, n]$ is the set $\{\langle a, \sigma, a' \rangle : v, v' \in V^*, \sigma \in \Sigma_{vhv', n}, a \in \mathcal{U}_v, a' \in \mathcal{U}_{v'}\}$, and
3. $G[u, u]$ is empty.

Let $C = G^*$ be the path category of G , i.e., the category freely generated by the graph G . Notice that the category C does not have any arrows out of u except the identity.

We will now define the behaviour algebras underlying hidden many sorted models. Since there is no danger of confusion, we denote by \mathcal{U} be the disjoint union of the sets \mathcal{U}_v for all $v \in V$.

Definition 12 There is a forgetful functor δ from the category of hidden many sorted models, $\text{HALG}(H, V, \Sigma)$, to the category of behaviour algebras, $\text{BALG}(C, u, \mathcal{U})$, defined by:

1. for any hidden many sorted model A , $\delta(A)h = A_h$ for $h \in H$ and each edge $\langle a, \sigma, a' \rangle \in G[h, n]$ is interpreted as $A_\sigma(a, -, a') : A_h \rightarrow A_n$; now since C is the path category of G , this extends uniquely to an interpretation of all path, i.e., all morphisms of C , and
2. any homomorphism of hidden many sorted models $m : A \rightarrow B$ is mapped into the homomorphism of behaviour algebras $\delta(m)$ with $\delta(m)_h = m_h$ for $h \in H$.

□

The forgetful functor from hidden many sorted models to behaviour algebras has a lifting property which enables us to take advantage of working at the level of behaviour algebras rather than at the level of hidden many sorted models.

Lemma 13 Let A be a hidden many sorted model and $m : \delta(A) \rightarrow B$ be a homomorphism of behaviour algebras. There is a unique homomorphism of hidden many sorted models $m^\sharp : A \rightarrow B^\sharp$ such that $\delta(m^\sharp) = m$.

Proof: Because $\delta(m^\sharp)$ should be m , for any $h \in H$ we take B_h^\sharp to be Bh and m_h^\sharp to be m_h . If $\sigma \in \Sigma_{vhv', h'}$, then, for all $a \in \mathcal{U}_v, a' \in \mathcal{U}_{v'}, b \in Bh, B_\sigma^\sharp(a, b, a')$ is $(B\langle a, \sigma, a' \rangle)(b)$. If $\sigma \in \Sigma_{v, h}, v \in V^*, h \in H$, then, for all $a \in \mathcal{U}_v, B_\sigma^\sharp(a)$ is

defined as $m_h(A_\sigma(a))$. These define a hidden sorted model B^\sharp and a homomorphism $m^\sharp: A \rightarrow B^\sharp$. Notice that m^\sharp is indeed a homomorphism of hidden sorted models because of the naturality of m and of the definition of the interpretations of the operations $\sigma \in \Sigma_{v,h}$, $v \in V^*$, $h \in H$, in B^\sharp . \square

Fact 14 Any morphism of hidden many sorted signatures $\phi: (H, V, \Sigma) \rightarrow (H', V', \Sigma')$ determines a full morphism of behaviour signatures $\phi^*: (C, u) \rightarrow (C', u)$.

Proof: Let (G, u) and (G', u) be the pointed graphs determined by (H, V, Σ) and (H', V', Σ') respectively. The morphism of pointed graphs determined by ϕ maps edges $\langle a, \sigma, a' \rangle$ to edges $\langle a, \phi(\sigma), a' \rangle$. It is full because any edge $\langle a, \sigma', a' \rangle$ in G' is an image of an edge $\langle a, \sigma, a' \rangle$ in G , where $\phi(\sigma) = \sigma'$. Its unique extension to a functor ϕ^* between the path categories of G and G' is thus full. \square

1.5 Hiding sorts in institutions

This section is devoted to the main result of this paper. The following theorem gives an abstract model theoretical construction of a “behavioural” (or “hidden”) institution over any institution satisfying some mild and natural conditions. This method of constructing a behavioural satisfaction relation on the top of an ordinary satisfaction relation is totally independent of the form of the sentences and it contrasts with the more syntactical way of defining the behavioural satisfaction in the particular case of equational logic by using the concept of *context* (see [11] or [3]).

Let $\mathcal{I} = (\text{SIGN}, \text{MOD}, \text{Sen}, \models)$ be an institution and suppose that the following data is given:

- a subcategory of “hidden sorted” signatures $\text{HSIGN} \hookrightarrow \text{SIGN}$, and
- a subfunctor of “hidden sorted” models $\text{HMOD} \subseteq (\text{HSIGN} \hookrightarrow \text{SIGN}; \text{MOD})$.

Theorem 15 Given a functor $\text{BMOD}: \text{HSIGN} \rightarrow \mathbf{Cat}^{op}$ and a natural transformation $\delta: \text{HMOD} \rightarrow \text{BMOD}$ such that

- for each $\Sigma \in |\text{HSIGN}|$, $\text{BMOD}(\Sigma)$ has a terminal object \mathbf{B}_Σ and an image factorisation system $(\mathcal{E}_\Sigma, \mathcal{M}_\Sigma)$,
- for each $\sigma: \Sigma \rightarrow \Sigma'$ in HSIGN , the unique map $\text{BMOD}(\sigma)(\mathbf{B}_{\Sigma'}) \rightarrow \mathbf{B}_\Sigma$ is in \mathcal{M}_Σ and $\text{BMOD}(\sigma)$ preserves the image factorisation systems, and
- $\delta_\Sigma: M/\text{HMOD}(\Sigma) \simeq \delta_\Sigma(M)/\text{BMOD}(\Sigma)$ is a natural isomorphism of slice categories

then these canonically determine a “hidden sorted” institution

$$\mathbf{H}(\mathcal{I}) = (\text{HSIGN}, \text{HMOD}, \text{Sen} \upharpoonright_{\text{HSIGN}}, \models^b)$$

and a morphism of institutions $\langle \text{HSIGN} \hookrightarrow \text{SIGN}, \beta, 1_{\text{Sen}} \upharpoonright_{\text{HSIGN}} \rangle: \mathbf{H}(\mathcal{I}) \rightarrow \mathcal{I}$.

Proof: We first define the natural transformation $\beta: \mathbf{HMOD} \rightarrow \mathbf{MOD} \mid \mathbf{HSIGN}$ translating the models. Fix a signature $\Sigma \in \mid \mathbf{HSIGN} \mid$.

For any model $M \in \mid \mathbf{HMOD}(\Sigma) \mid$, let the ‘observation map’ obs_{Σ}^M be the unique arrow $\delta_{\Sigma}(M) \rightarrow \mathbf{B}_{\Sigma}$. Let $obs_{\Sigma}^M = e_M; m_M$ be its image factorisation and consider $e_M^{\sharp}: M \rightarrow \beta_{\Sigma}(M)$ the unique map of hidden sorted models such that $\delta_{\Sigma}(e_M^{\sharp}) = e_M$. For the definition of β_{Σ} on model morphisms, consider a morphism of models $f: M \rightarrow N$. By the Diagonal Fill-in Property of image factorisation systems, $\delta_{\Sigma}(f)$ induces a canonical map f' such that $e_M; f' = \delta_{\Sigma}(f); e_N$ and $f'; m_N = m_M$. Define $\beta_{\Sigma}(f)$ to be f'^{\sharp} , i.e., the unique model morphism such that $\delta_{\Sigma}(f'^{\sharp}) = f'$. The functoriality of β_{Σ} follows from the uniqueness in the Diagonal Fill-in Property.

$$\begin{array}{ccc}
 & \xrightarrow{obs_{\Sigma}^M} & \\
 \delta_{\Sigma}(M) & \xrightarrow{e_M} \delta_{\Sigma}(\beta_{\Sigma}M) \xrightarrow{m_M} & \mathbf{B}_{\Sigma} \\
 \delta_{\Sigma}(f) \downarrow & \delta_{\Sigma}(\beta_{\Sigma}f) \downarrow & \nearrow m_N \\
 \delta_{\Sigma}(N) & \xrightarrow{e_N} \delta_{\Sigma}(\beta_{\Sigma}N) &
 \end{array}$$

For proving the naturality of β we pick any morphism of hidden sorted signatures $\phi \in \mathbf{HSIGN}(\Sigma, \Sigma')$. Let M' be any Σ' -model.

Let $obs_{\Sigma'}^{M'} = e_{M'}; m_{M'}$ be the image factorisation of the observation map of M' . Because $\mathbf{BMOD}(\phi)$ preserves the factorisation system, $\mathbf{BMOD}(\phi)(e_{M'}); \mathbf{BMOD}(\phi)(m_{M'})$ is an image factorisation for $\mathbf{BMOD}(\phi)(obs_{\Sigma'}^{M'})$. $\mathbf{BMOD}(\phi)(\delta_{\Sigma'}(M')) = \delta_{\Sigma}(M' \mid_{\phi})$ by the naturality of δ . We know that the unique map $m: \mathbf{BMOD}(\phi)\mathbf{B}_{\Sigma'} \rightarrow \mathbf{B}_{\Sigma}$ is in \mathcal{M}_{Σ} . This shows that $\mathbf{BMOD}(\phi)(e_{M'}); (\mathbf{BMOD}(\phi)(m_{M'}); m)$ is an image factorisation for $\mathbf{BMOD}(\phi)(obs_{\Sigma'}^{M'}); m$ which is equal to $obs_{\Sigma}^{M' \mid_{\phi}}$ by the universal property of the terminal object. Then $\mathbf{BMOD}(\phi)(e_{M'}) = e_{M' \mid_{\phi}}$ which proves that $\beta_{\Sigma'}^{M'} \mid_{\phi} = \beta_{\Sigma}(M' \mid_{\phi})$. The naturality of β on model morphisms follows in a similar way.

The ‘behavioural satisfaction’ relation is the only possible choice for a satisfaction relation in $\mathbf{H}(\mathcal{I})$ which makes $\langle \mathbf{HSIGN} \hookrightarrow \mathbf{SIGN}, \beta, 1_{Sen \mid \mathbf{HSIGN}} \rangle: \mathbf{H}(\mathcal{I}) \rightarrow \mathcal{I}$ into a morphism of institutions. More precisely, given a hidden sorted signature Σ , a hidden sorted model $M \in \mid \mathbf{HMOD}(\Sigma) \mid$ and a Σ -sentence e ,

$$M \models_{\Sigma}^b e \text{ iff } \beta_{\Sigma}(M) \models_{\Sigma} e .$$

All we still have to prove is that the behavioural satisfaction \models^b verifies the Satisfaction Condition for the institution $\mathbf{H}(\mathcal{I})$.

Let $\phi \in \mathbf{HSIGN}(\Sigma, \Sigma')$ be any morphism of hidden sorted signatures and consider any hidden sorted model $M' \in \mid \mathbf{MOD}(\Sigma') \mid$ and any Σ -sentence e . Then $M' \models_{\Sigma'}^b (Sen \phi)(e)$ iff $\beta_{\Sigma'}(M') \models_{\Sigma'} (Sen \phi)(e)$ iff (by the Satisfaction Condition in \mathcal{I}) $\beta_{\Sigma'}(M') \mid_{\phi} \models_{\Sigma} e$. But $\beta_{\Sigma'}(M') \mid_{\phi} = \beta_{\Sigma}(M' \mid_{\phi})$ by the naturality of β . It follows

that $M' \models_{\Sigma'}^b (Sen\phi)(e)$ iff $\beta_{\Sigma}(M' |_{\phi}) \models_{\Sigma} e$ iff $M' |_{\phi} \models_{\Sigma}^b e$. \square

Turning to the example of hidden many sorted logic we have:

Corollary 16 Hidden many sorted equational logic with the behavioural satisfaction of equations by algebras is an institution. Moreover, there is a canonical morphism from the institution of hidden many sorted equational logic to the institution of many sorted equational logic.

Proof: Let \mathcal{I} of the previous theorem be the institution of many sorted equational logic. Consider **HSIGN** to be the category of hidden sorted signatures for a fixed signature and a fixed algebra of data values, and let \mathcal{U} be the disjoint union of the carriers of the algebra of data values. Let **HMOD** be the functor constructing the category of hidden many sorted models for any hidden many sorted signature.

Consider the forgetful functor $\mathcal{B} : \mathbf{HSIGN} \rightarrow \mathbf{Cat}_u$ mapping hidden many sorted signatures to their underlying behaviour signatures (see Section 1.4) and morphisms of hidden many sorted signatures to morphisms of behaviour signatures. Consider the functor $\mathbf{BALG} : \mathbf{Cat}_u \rightarrow \mathbf{Cat}^{op}$ mapping any behaviour signature (C, u) to the category of behaviour algebras over (C, u, \mathcal{U}) . The composition $\mathcal{B}; \mathbf{BALG}$ gives us the functor **BMOD** and Definition 12 parameterised by signatures defines a natural transformation $\delta : \mathbf{HMOD} \rightarrow \mathcal{B}; \mathbf{BALG}$.

The existence of terminal objects for the categories $\mathbf{BALG}(\mathcal{B}(H, V, \Sigma))$, where (H, V, Σ) is a hidden sorted many sorted signature, is assured by Proposition 7.

For any morphism of hidden sorted many sorted signatures ϕ , $\mathbf{BALG}(\mathcal{B}(\phi))$ maps the terminal behaviour algebra over $(\mathcal{B}(H', V, \Sigma'), \mathcal{U})$ to a subalgebra of the terminal behaviour algebra over $(\mathcal{B}(H, V, \Sigma), \mathcal{U})$ by Lemma 10 and Fact 14.

Lemma 13 proves that the components of δ satisfy the lifting property expressed by the isomorphism of slice categories in the statement of the previous theorem.

Corollary 9 proves that the reduct functors between categories of behaviour algebras preserve the image factorisation systems. \square

1.6 Hidden order sorted logics

In this section we build the institution of hidden order sorted equational logic where the satisfaction relation between order sorted algebras and order sorted equations is behavioural. The notion of order-sortedness we use is due to Goguen and Meseguer [9] and it is the basis for the OBJ family languages, including the object oriented language FOOPS [8]. A comparison between different types of order-sortedness could be found in the survey [6] (where the Goguen-Meseguer approach has been referred as *overloaded* order sorted algebra).

Here is the definition of hidden order sorted signatures, signature morphisms and models, by adapting Definition 11 to the order sorted case:

Definition 17 Fix a set V of visible sorts, an order sorted signature (V, \leq, Ψ) and an order sorted algebra U over (V, \leq, Ψ) . A **hidden order sorted signature** (H, V, \leq, Σ) is an order sorted signature $(H \cup V, \leq, \Sigma)$ with (H, \leq) the partially ordered set of hidden sorts, (V, \leq) the partially ordered set of visible sorts such that no visible sort is related to any hidden sort and such that (H, V, Σ) is a hidden many sorted signature.

A **hidden order sorted signature morphism** $\phi: (H, V, \leq, \Sigma) \rightarrow (H', V, \leq, \Sigma')$ is both a morphism of order sorted signatures $(H \cup V, \leq, \Sigma) \rightarrow (H' \cup V, \leq, \Sigma')$ and a morphism of hidden many sorted signature $(H, V, \Sigma) \rightarrow (H', V, \Sigma')$ such that for any hidden sorts h, h' , $\phi h < \phi h'$ implies $h \leq h'$.

A **hidden order sorted model** over (H, V, \leq, Σ) is an order sorted algebra M over $(H \cup V, \leq, \Sigma)$ such that $M|_{\Psi} = U$ and a homomorphism of hidden sorted models $h: M \rightarrow M'$ is a homomorphism of order sorted algebras such that $h|_{\Psi} = 1_U$. \square

Let (H, V, \leq, Σ) be a hidden order sorted signature. Then the hidden many sorted signature (H, V, Σ) determines a pointed graph (G_0, u) as described in Section 1.4. Then (H, V, \leq, Σ) determines another pointed graph (G, u) thus:

- the nodes of G are the same as those of G_0 together with a new node 1,
- $G[1, 1]$ is empty,
- for each $h \in H$, $G[1, h]$ is $\{\perp_h\}$, and
- if $h < h'$ then $G[h, h'] = G_0[h, h'] \cup \{i_{h, h'}\}$ and $G[h', h] = G_0[h', h] \cup \{r_{h', h}\}$.

The intuitive understanding of these new edges is that \perp_h stands for the undefined element of sort h , $i_{h, h'}$ stands for the inclusion between subsorts and $r_{h', h}$ stands for the right inverse to $i_{h, h'}$.

Now, we form a category $C = G^*/Q$, where Q is the congruence generated by the following identities:

- $i_{h, h'}; r_{h', h} = 1_h$ for all pairs h, h' with $h < h'$,
- $i_{h, h'}; i_{h', h''} = i_{h, h''}$ and $r_{h'', h'}; r_{h', h} = r_{h'', h}$ for all triples h, h', h'' with $h < h' < h''$,
- for all nodes n, n' and $f \in G^*[n, n']$, $\perp_n; f = \perp_{n'}$, and
- for all $\sigma \in \Sigma_{vhv', h'} \cap \Sigma_{v_0 h_0 v'_0, h'_0}$ with $h'_0 < h'$ and $a \in \mathcal{U}_{v_0}$, $a' \in \mathcal{U}_{v'_0}$, $i_{h_0, h}; \langle a, \sigma, a' \rangle = \langle a, \sigma, a' \rangle; i_{h'_0, h'}$.

Let \mathcal{U}_v be a fixed set of data values for the visible sort v and let \mathcal{U} be the set obtained by adding a new element \perp_u to the colimit of all these sets. There is a forgetful functor

$\delta: \text{HALG}(H, V, \leq, \Sigma) \rightarrow \text{BALG}(C, u, \mathcal{U})$ defined as follows for a hidden order sorted model A :

- $\delta(A)h = A_h \cup \{\perp_h\}$,
- $\delta(A)1$ is a singleton set and \perp_h evaluates its element as \perp_h ,

- for all pairs h, h' with $h < h'$, $i_{h,h'}$ is interpreted as the expansion of the inclusion $A_h \subseteq A_{h'}$ mapping \perp_h to $\perp_{h'}$ and $r_{h',h}$ is interpreted as its right inverse which maps all elements of $A_{h'}$ that are not in A_h to \perp_h , and
- the interpretation of edges $\langle a, \sigma, a' \rangle$ is the same as in Definition 12.

The definition of δ on homomorphisms of hidden order sorted models m expands each component m_h by preserving \perp_h . By similarity with Lemma 13, we have the following lifting property:

Fact 18 For any hidden order sorted model M and any behaviour algebra morphism $f: \delta(M) \rightarrow N$ there is a unique morphism of hidden order sorted models $f^\sharp: M \rightarrow N^\sharp$ such that $\delta(f^\sharp) = f$. \square

Corollary 19 Hidden order sorted equational logic is an institution. Moreover, there is a canonical morphism of institutions from hidden order sorted equational logic to order sorted equational logic.

Proof: Let \mathcal{I} of the previous theorem be the institution of order sorted equational logic. Consider **HSIGN** to be the category of hidden order sorted signatures for a fixed signature and a fixed algebra of data values, and let \mathcal{U} be the colimit of the carriers of the algebra of data values. Let **HMOD** be the functor constructing the category of hidden order sorted models for any hidden order sorted signature.

Consider the forgetful functor $\mathcal{B}: \mathbf{HSIGN} \rightarrow \mathbf{Cat}_u$ mapping hidden order sorted signatures to their underlying behaviour signatures as described above and mapping morphisms of hidden order sorted signatures to morphisms of behaviour signatures in the canonical way. Consider the functor $\mathbf{BALG}: \mathbf{Cat}_u \rightarrow \mathbf{Cat}^{op}$ mapping any behaviour signature (C, u) to the category of behaviour algebras over (C, u, \mathcal{U}) . The definition of forgetful functors from categories of hidden order sorted models to behaviour algebras described above gives a natural transformation $\delta: \mathbf{HMOD} \rightarrow \mathcal{B}; \mathbf{BALG}$.

The existence of terminal objects for the categories $\mathbf{BALG}(\mathcal{B}(H, V, \leq, \Sigma))$, where (H, V, \leq, Σ) is a hidden order sorted signature, is assured by Proposition 7.

For any morphism of hidden order sorted signatures ϕ , $\mathbf{BALG}(\mathcal{B}(\phi))$ maps the terminal behaviour algebra over $(\mathcal{B}(H', V', \Sigma'), \mathcal{U})$ to a subalgebra of the terminal category algebra over $(\mathcal{B}(H, V, \Sigma), \mathcal{U})$ by Lemma 10 and Fact 14.

Lemma 13 proves that the components of δ satisfy the lifting property expressed by the isomorphism of slice categories in the statement of the previous theorem.

The reduct functors between categories of behaviour algebras preserve image factorisation systems by Corollary 9. \square

1.7 Conclusions

We showed how abstract model theory could be used to define a behavioural satisfaction relation on top of any satisfaction relation. An important consequence of

this construction is that it is totally independent of syntax (i.e., the shape of the sentences of the institution involved) and it could be therefore applied to institutions like Horn clause logics, first order logics, modal logics etc.

As a corollary we have proved that hidden order sorted equational logic is an institution, this institution underlying the algebraic foundations of object oriented programming.

An obvious step forward would be a proof theory for the behavioural satisfaction in connection with the quotienting of hidden sorted models through the image factorisation of the canonical morphisms to the terminal behaviour algebra.

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