# On the Role of Implication in Formal $Logic^1$

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#### Abstract

Evidence is given that implication (and its special case, negation) carry the logical strength of a system of formal logic. This is done by proving normalization and cut elimination for a system based on combinatory logic or  $\lambda$ -calculus with logical constants for and, or, all, and exists, but with none for either implication or negation. The proof is strictly finitary, showing that this system is very weak. The results can be extended to a "classical" version of the system. They can also be extended to a system with a restricted set of rules for implication: the result is a system of intuitionistic higher-order BCK logic with unrestricted comprehension and without restriction on the rules for disjunction elimination and existential elimination. The result does not extend to the classical version of the BCK logic.

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The main aim of this paper is to provide evidence that implication and implication alone carries the logical strength of a system of formal logic. By this I mean that if implication and its rules are left out of a system of formal logic, the system is very weak. This conclusion requires that negation be treated as a special case of implication by means of the definition of  $\neg A$  as  $A \supset \bot$ . (The intuitionistic property of  $\bot$ , namely that any formula follows from it, adds no logical strength to the system, as we shall see below.) Furthermore, before we are done, we will see that there are some restricted rules for implication that can be assumed without adding logical strength to the system.

As an indication of how the argument will go, consider a standard natural deduction formulation of first-order minimal logic, and drop the rules for implication. The remaining rules are as follows:



where in  $\forall I$  and  $\exists E \ a$  is a free variable which does not occur free in any undischarged assumption and where in  $\land E$  and  $\lor I$ , i = 1 or 2.

Now suppose that we transform deductions by modifying each inference by  $\lor E$  or  $\exists E$  by placing a copy of the deduction of the major (left) premise above each assumption discharged by the rule: i.e., change

$$\begin{array}{cccc} & 1 & 2 \\ & [A_1] & [A_2] \\ D_1 & D_2 & D_3 \\ \underline{A_1 \lor A_2} & C & C \\ \hline C & & \\ \end{array} \lor \mathbf{E} - 1 - 2$$

 $\mathrm{to}$ 

$$\begin{array}{ccccc} & D_{1} & D_{1} \\ & \frac{A_{1} \lor A_{2}}{A_{1}} \ast 1 & \frac{A_{1} \lor A_{2}}{A_{2}} \ast 2 \\ D_{1} & D_{2} & D_{3} \\ \hline & \frac{A_{1} \lor A_{2}}{C} & \frac{C}{C} & \frac{C}{V} \lor E \ast -1 - 2 \end{array}$$

and

$$\begin{array}{c} 1\\ [A(a)]\\ D_1 & D_2(a)\\ \hline (\exists x)A(x) & C\\ \hline C & \exists E-1 \end{array}$$

 $\operatorname{to}$ 

$$\begin{array}{c}
D_{1} \\
(\exists x)A(x) \\
A(a) \\
 & 1 \\
D_{1} \\
D_{2}(a) \\
(\exists x)A(x) \\
C. \\
\end{array} *1$$

These changes could be made systemmatically throughout any deduction, say by proceeding from the top down (so that each transformation is carried out on an inference by  $\forall E$  or  $\exists E$  for which there are no untransformed inferences by either rule above any premise).

Now consider the standard (non-permutative) proof reduction steps. The steps for  $\wedge$ -reductions and  $\forall$ -reductions are unchanged, and are as follows:

 $\wedge$ -reductions

$$\begin{array}{ccc} D_1 & D_2 & \text{reduces to} & D_i \\ A_1 & A_2 & & A_i \end{array}$$

where i = 1 or 2, and

$$\begin{array}{ccc} \forall \text{-reductions} & D_1(a) & \text{reduces to} & D_1(t) \\ & \underline{A(a)} & & A(t) \\ \hline \hline (\forall x)A(x) & \forall \mathbf{I} & & D_2. \\ \hline \hline A(t) & \forall \mathbf{E} & & \\ & D_2 & & \end{array}$$

The steps for  $\lor$ -reductions and  $\exists$ -reductions are transformed respectively into the following:

## $\vee^*$ -reductions

reduces to  

$$D_0$$
  $D_0$   $D_0$   
 $A_i$   $\forall I$   $A_i$   $\forall I$   $A_i$   $D_i$ 

$$\frac{A_1 \lor A_2}{4} * 1 \quad \frac{A_1 \lor A_2}{4} * 2 \qquad \qquad C$$

$$\begin{array}{ccc} A_1 \lor A_2 & C & C \\ \hline C & \\ D_3 \end{array} \lor \mathbf{E} * -1 - 2 \\ \end{array}$$

where i = 1 or 2, and

 $D_0$ 

$$\exists^*$$
-reductions

reduces to  

$$\begin{array}{ccc}
D_0 & D_0 \\
\underline{A(t)} \\
\underline{(\exists x)A(x)} \\
A(a) \\ \ast 1 & D_1(t) \\
\end{array}$$

Note that *Each of these reduction steps shortens the deduction*. This means that any sequence of these reduction steps terminates, and this is proved without any reference to the complexity of the formulas involved. The

result will still hold if instead of first-order quantifiers we have second- or higher-order quantifiers. Furthermore, although the normalization is not complete (in the sense of Prawitz [16], it is sufficient to prove the consistency of the system.

At first this result seems to contradict Gödel's Second Theorem, since the proof is completely finitary but implies the consistency of a system with higher-order quantifiers. However, the absence of implication and negation means that we cannot represent this proof within the system. In fact, the system is so weak that we cannot even represent the famous Aristotelian premise "All men are mortal." Furthermore, this proof fails if implication (or negation) is included in the system. If implication is present, the proof requires a transformation at each implication cut formula that takes

$$\begin{array}{c}
1\\
[A]\\
D_1\\
\hline
D_1\\
\hline
\frac{B}{A \supset B} \supset I - 1 \qquad D_2\\
\hline
B\\
D_3 \supset E
\end{array}$$

to

$$\frac{D_{2}}{\frac{A}{A} * 1}$$

$$\frac{D_{1}}{\frac{B}{A \supset B} \supset I - 1} \qquad D_{2}$$

$$\frac{D_{2}}{\frac{A}{D_{1}} \supset E} \supset E$$

$$\frac{D_{2}}{B} \supset E$$

and these transformations must all be carried out before the reduction process starts. But since new implication cut formulas can be created as part of the reduction process, there is no way all of them can be transformed at that stage. The problem is that the assumption discharged in the creation of the cut formula does not occur in the same rule as the part of the deduction to be duplicated. Hence, this proof cannot be carried out this way if implication (or its special case, negation) occurs with its usual rules.

However, implication (and negation) can be permitted in the system if the introduction rule is restricted so that the assumption discharged by the rule cannot occur more than once (after the transformation that precedes the reduction).

What Gödel's Second Theorem really tells us about this proof is that the system is quite weak, no matter what the order of the quantifiers. This, in turn, suggests the main theme of this paper, that the strength of a system of logic is determined by its rules for implication (and negation). The purpose of this paper is to explore this importance of implication in connection with systems of logic based on combinatory logic or  $\lambda$ -calculus in the tradition of H. B. Curry.

There is previous evidence in Curry's work for the importance of implication in this regard. It is well known that after Kleene and Rosser [11] proved inconsistent the original systems of Curry and Church, Curry was the only one (except for F. B. Fitch) who remained interested in using combinatory logic or  $\lambda$ -calculus as a basis for logic and mathematics. This story is told in [19]. Actually, the paradox of Kleene and Rosser did not apply to Curry's original system in [3] but to its extension in [4]; however, this extension was so much a part of Curry's original objectives that the part of his original system dealing with logical connectives and quantifiers was not very interesting without it. Curry's first assumption was that the cause of the contradiction lay in his postulates for the universal quantifier. Later, however, he derived in [5] a contradiction from the postulates for implication alone.

In [8, Theorem 16C3, p. 441] we claimed to have a proof that if implication (and its special case, negation) is left out of the system, no further restrictions are needed to avoid the contradiction. If the proof had been valid, this result would also have applied to other systems, such as second order logic, higher order logic (type theory), logic with comprehension terms for set theory, etc. But Curry found a gap in that proof in 1975. (The proof was, in fact, my personal responsibility, and originally appeared in [17, Theorem 5C3, p. 130f]. The error is that Stage 2 case ( $\beta$ ) does not go through as claimed in [8, p. 202].) This paper arose from an attempt to give a new proof of the same result.

In Section 1, a system of logic based on combinatory logic or  $\lambda$ -calculus with conjunction, disjunction, and the universal and existential quantifiers is defined, first as an L-system and then as a natural deduction system, and reduction rules are given for the latter formulation. The system is formalized as minimal logic, but the quantifiers are essentially of infinite order. In Section 2. the above sketch of a proof of normalization is carried out in detail. This is done, following a suggestion of G. E. Mints (in private correspondence) by defining in(D), the *index* of a deduction, which is the length (number of formulas) in the tree-form of the deduction obtained from D by carrying out the transformation suggested above. It is then shown that each reduction step reduces the index of the deduction, and so normalization follows. It is then shown that this implies cut-elimination for the L-system. In Section 3, these results are extended to a "classical" version of the system: for the L-system this means allowing more than one formula on the right-hand side of the sequent, but without implication a new way of defining the classical version of the natural deduction formulation is needed. It turns out that this classical system is essentially a version (without implication or negation) of the logic of constant domains. Finally, in Section 4, the system with implication restricted as suggested above is considered. This system turns out to be a variation of intuitionistic higher-order BCK logic with unrestricted comprehension. BCK logic is a logic in which the postulates for implication correspond under the formulas-as-types notion [10] to the types of a system of combinators in which no combinator duplicates an argument or to a system of  $\lambda$ -calculus in which  $\lambda x \cdot M$  is well-formed only when x occurs free at most once in M. This particular BCK logic is unlike other formulations of BCK logic such as that of [20] in that there are no restrictions on the rules for other connectives and quantifiers. This result does not hold for the corresponding classical system. The result is stronger than the similar result of White [20] because in White's system the rules restricting the number of occurrences of discharged assumptions for the rule of implication introduction that characterize BCK logic also apply to disjunction elimination and existential elimination, whereas in the system considered here, the restructions apply only to the rule for implication introduction (and its special case of negation introduction).

The incorrect proof of [8, Theorem 16C3, p. 441] was a minor modification of another proof for a much simpler system with an operator representing equality. That proof is also incorrect for the same reason. Hence, this simpler system is treated in Appendix A.

Because this paper is a part of the program of H. B. Curry, it is written in his language. This means that I am following Curry in using ' $\Lambda$ ' for ' $\wedge$ ', 'V' for ' $\vee$ ', ' $\Pi$ ' for ' $\forall$ ', and ' $\Sigma$ ' for ' $\exists$ ' in the names of rules. Furthermore, I follow Curry in using 'i' and 'e' instead of 'I' and 'E' in the names of natural deduction rules, and I use an asterisk as in '\* $\Lambda$ ' to indicate the L-rule for conjunction on the left. Except where otherwise specified, the other basic definitions and conventions are those of [9] and [8]. In particular, as in [9], conversion will be denoted by ' $=_*$ ' and reduction by ' $\triangleright$ '. The major change is that, for the reasons given in [18, p. 31, footnote 1], I will not use the symbol ' $\vdash$ ' in some places where it is used in [8].

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## 1 The System for Logic without Implication

Let us begin with the system called  $\mathcal{F}_{33}$  in [8, §16C2].

**Definition 1** The system  $\mathcal{F}_{33}^L$  (or, when more precision is needed,  $\mathcal{F}_{33}^{LA}$ ) is formed from the terms of a system of combinatory logic or  $\lambda$ -calculus with the following non-redex constants:  $\Lambda$  (conjunction), V (disjunction),  $\Pi$  (universal quantifier), and  $\Sigma$  (existential quantifier). The provability relation is given by an L-system with the following axioms and rules, where M and M' are sequences of terms:

Axiom Scheme:

×

\*

 $X \Vdash X$ ,

for each term X.

Rules:

\*K 
$$\frac{M \Vdash Z}{M, X \Vdash Z,}$$
  
\*Exp 
$$\frac{M, Y \Vdash Z}{M, X \Vdash Z,}$$
 Exp\* 
$$\frac{M \Vdash Y}{M \Vdash X}$$
  
\* $\land \qquad \frac{M, X, Y \Vdash Z}{M, \Lambda X Y \Vdash Z,}$   $\land^* \qquad \frac{M \Vdash X \qquad M \Vdash Y}{M \Vdash \Lambda XY,}$   
\* $\lor \qquad \frac{M, X \Vdash N \qquad M, Y \Vdash Z}{M, \nabla XY \Vdash Z,}$   $\lor^* \qquad \frac{M \Vdash X_i}{M \Vdash \nabla X_1 X_2,}$   
\* $\Pi \qquad \frac{M, XY \Vdash Z}{M, \Pi X \Vdash Z,}$   $\Pi^* \qquad \frac{M \Vdash Xx}{M \Vdash \Pi X,}$   
\* $\Sigma \qquad \frac{M, Xx \Vdash Z}{M, \Sigma X \Vdash Z,}$   $\Gamma^* \qquad \frac{M \Vdash Xx}{M \Vdash \Pi X,}$   
Cut  $\qquad \frac{M \Vdash X \qquad M, X \Vdash Z}{M \Vdash Z,}$   $\Sigma^* \qquad \frac{M \Vdash XY}{M \Vdash \Sigma X,}$ 

where in  $\Pi^*$  and  $*\Sigma$ , x is a variable which does not occur free in X, M, or Z, where in  $\lor *$ , i = 1 or 2, where in rules  $*Exp^*$ , there is the condition that  $X \triangleright Y$ , and where in \*C, M' is a permutation of M.

**Remark** The main departure from the conventions of [8] is that I am not writing the range of quantification inside the symbol ' $\Vdash$ ' and I am writing ' $\triangleright$ ' for reduction.) By [8, Theorem 12C7, p. 193], the rules

\*Eq 
$$\frac{M, Y \Vdash N}{M, X \vDash N}$$
 Eq\*  $\frac{M \Vdash Y, L}{M \vDash X, L}$ 

where the convention here is that  $X =_* Y$ , are admissible in the system, and, indeed, in any other system of this kind; I shall use this fact throughout the paper without further mention. Of course, in the singular system, rules C<sup>\*</sup>, W<sup>\*</sup>, and K<sup>\*</sup> are not postulated, but Exp<sup>\*</sup> is, and so Eq<sup>\*</sup> is admissible.

**Remark** This is not quite the system  $\mathcal{F}_{33}$  of [8, §16C2], which has, in addition, the non-redex constants Q and and P and the rules  $*Q^*$  of Appendix A and \*P of §4 below. The system presented here will be modified in §4 so that Q can be defined in such a way as to make rules  $*Q^*$  of Appendix A valid, and a natural deduction rule corresponding to \*P will be one of the rules postulated in that modification.

The natural deduction system, which is of more interest to us here, is defined as follows:

**Definition 2** The system  $\mathcal{F}_{33}^T$  (or, to be more precise,  $\mathcal{F}_{33}^{TA}$ ), is defined from the same terms as  $\mathcal{F}_{33}^T$  of Definition 1. Its proof system is a natural deduction system with the following rules:



Here, in  $\Pi$ i and  $\Sigma$ e, x is a variable which does not occur free in X, Z, or any undischarged assumption.

We are interested in the following proof-reduction steps, where in each case the deduction on the left reduces to the one on the right:

A-reductions For i = 1 or i = 2,

$$\begin{array}{cccc}
D_1 & D_2 & & D_i \\
\underline{X_1 & X_2} \\
\hline
\underline{\Lambda X_1 X_2} \\
\hline
\underline{\Lambda Y_1 Y_2} \\
\hline
\underline{Y_i} \\
D_3
\end{array} \land \mathbf{h} \qquad \qquad D_3.$$

V-reductions For i = 1 or i = 2,

### $\Pi$ -reductions

$$D_{1}(x) \qquad D_{1}(Z)$$

$$\frac{Xx}{\Pi X} \prod_{i} \qquad \frac{XZ}{YZ} Eq$$

$$\frac{\Pi Y}{YZ} \Pi e \qquad D_{2}.$$

 $\Sigma$ -reductions

$$\begin{array}{ccccc}
 & D_1 & 1 & & D_1 \\
 & \underline{XU} & [Yx] & & & \underline{XU} \\
 & \underline{\SigmaX} & Eq & D_2(x) & & & \\
 & \underline{YU} & Eq & & \\
 & & & Z \\
 & & & & & D_3
\end{array}$$

Note that we do not have the permutative reduction steps of [16]; I know of no way to extend the normalization proof to cover these rules. This means that the standard proof in [16] that normalization implies cut elimination cannot be applied to this system without modification. Nevertheless, if we modify the definition of "branch", then it is true that in a normal deduction all of the e-rules precede all of the i-rules in a given branch. Here a *branch* is the first part of a thread down to the first minor premise for an inference by Ve or  $\Sigma$ e if there is one. (In §4 below it will mean the first part of a thread down to the first minor premise for an inference by Ve,  $\Sigma$ e, or Pe.) Because of the above property of normal deductions, it is easy to show by the usual methods than an "atomic formula" (in this case, a term in normal form which does not convert to one of the forms  $\Lambda XY$ ,  $\nabla XY$ ,  $\Pi X$ , or  $\Sigma X$ ) cannot be proved without an undischarged assumption

**Remark** It follows that if  $\perp$  is taken to be an abbreviation for  $\sqcap I$  (which is  $(\forall x)x$  in the usual notation), then there is no proof of  $\perp$ ; for if there were, then there would be the following proof of x for any variable x, which is ruled out.

$$\frac{\prod}{Ix}_{x.} \frac{\Pi e}{Eq}$$

Note that this is a property of intuitionistic logic not shared by minimal logic. This is why the name of the system does not refer to minimal logic, but to what Curry in [6, Chapter 5] calls *absolute* logic. A different definition of  $\perp$  would give us a form of minimal logic.

# 2 The Proof of Normalization

We now come to the definition of in(D) for a deduction D as explained in the Introduction, where until the end of the proof of Theorem 1, "deduction" means deduction of  $\mathcal{F}_{33}^{TA}$ . The definition requires first a function of a formula, an specification of occurrences of that formula as an undischarged assumption (e.g., by the numbers normally used to indicate where they are discharged), and a deduction, which tells us how many times that assumption is duplicated in the transformation described in the introduction.

**Definition 3 (Index of an assumption)** If D is a deduction, X a term, and O is a specification of occurrences of X as an undischarged assumption, then the *index of* X with respect to O and D, in(X, O, D), is defined by induction on the length of D as follows:

(a) if  $D \equiv X$ , then in(X, O, D) = 1 if this one occurrence of X is specified by O and in(X, O, D) = 0 otherwise;

(b) if  $D \equiv Z \not\equiv X$ , then in(X, O, D) = 0;

(c) if the last inference in D is by any of the rules Eq, Ae, Vi,  $\Pi e$ ,  $\Pi i$ , or  $\Sigma i$  (i.e., any of the rules with one premise), if D' is the result of deleting the last inference from D, and if O' specifies the occurrences of X in D' specified by O in D, then in(X, O, D) = in(X, O', D');

(d) if D is the deduction

(1) 
$$\begin{aligned} D_1 & D_2 \\ \frac{Y_1 & Y_2}{\Lambda Y_1 Y_2} & \Lambda i \end{aligned}$$

then  $in(X, O, D) = in(X, O_1, D_1) + in(X, O_2, D_2)$ , where  $O_1$  and  $O_2$  specify the occurrences of X in  $D_1$  and  $D_2$  respectively specified by O in D; (e) if D is the deduction

then  $in(X, O, D) = [in(Y_1, O'_1, D_2) + in(Y_2, O'_2, D_3) + 1] \cdot in(X, O_1, D_1) + in(X, O_2, D_2) + in(X, O_3, D_3)$ , where, for i = 1, 2, 3,  $O_i$  specifies the occurrences of X as an undischarged assumption in  $D_i$  specified by O (which, by the assumption about O, do not include any of the assumptions discharged by the rule Ve) and, for  $j = 1, 2, O'_j$  specifies the occurrences of  $Y_j$  in  $D_{j+1}$  discharged by the inference by the rule Ve; and (f) if D is the deduction

(3) 
$$\begin{array}{c} 1\\ [Yx]\\ D_1 \quad D_2(x)\\ \underline{\Sigma Y \quad Z}\\ \overline{Z, \quad} \Sigma e - 1 \end{array}$$

then  $in(X, O, D) = [in(Yx, O', D_2(x)) + 1] \cdot in(X, O_1, D_1) + in(X, O_2, D_2(x))$ , where O' specifies the occurrences of Yx in  $D_2(x)$  discharged by the inference by  $\Sigma$ e and where, for  $i = 1, 2, O_i$  specifies the occurrences of X in  $D_i$  specified by O in D. Note that  $in(X, O, D) \neq 0$  if and only if X occurs as an undischarged assumption in D.

**Definition 4 (Index of a deduction)** The *index* of a deduction D, in(D), is defined by induction on the length of D as follows:

(a) if  $D \equiv X$ , then in(D) = 1;

(b) if the last inference in D is by Rule Eq, and if D' is the result of deleting the last inference from D, then in(D) = in(D');

(c) if the last inference in D is by any of the rules with one premise except Eq, and if the result of deleting the last inference from D is D', then

$$in(D) = in(D') + 1;$$

(d) if D is the deduction (1), then

$$in(D) = in(D_1) + in(D_2) + 1;$$

(e) if D is the deduction (2), then

 $in(D) = [in(Y_1, O_1, D_2) + in(Y_2, O_2, D_3) + 1] \cdot in(D_1) + in(D_2) + in(D_3) + 1,$ 

where, for  $i = 1, 2, O_i$  specifies the occurrences of  $Y_i$  in  $D_{i+1}$  discharged by the inference by Ve; and

(f) if D is the deduction (3), then

$$in(D) = [in(Yx, O, D_2(x)) + 1] \cdot in(D_1) + in(D_2(x)) + 1,$$

where O specifies the occurrences of Yx in  $D_2$  discharged by the inference by  $\Sigma e$ .

**Remark** Note that in (e) and (f) of Definition 4, if there are no inferences by Ve or  $\Sigma e$  in  $D_2$ ,  $D_3$ , or  $D_2(x)$ , then the number by which  $in(D_1)$  is multiplied in the formula for in(D) is one more than the total number of occurrences of the assumptions discharged by the inferences in question. Note also that in(D) does not count the inferences by Rule Eq in D.

**Theorem 1** If D' is obtained from D by a reduction step, then in(D') < in(D).

The proof requires three lemmas.

**Lemma 1** (a) in([Z/x]X, O', [Z/x]D) = in(X, O, D) if O' specifies the occurrences of [Z/x]X in [Z/x]D corresponding to the occurrences of X specified by O in D; and (b) in([Z/x]D) = in(D).

**Proof** By an easy induction on the length of D.

**Lemma 2** Let  $D_1$  and  $D_2$  be the deductions

$$\begin{array}{ccc} D_1 & & X \\ X & & and & D_2 \\ & & & Z. \end{array}$$

 $D_1 \\ X \\ D_2 \\ Z,$ 

Then if D is

we have 
$$in(D) = in(X, O, D_2) \cdot [in(D_1) - 1] + in(D_2)$$
, where O specifies  
the indicated occurrences of X in  $D_2$  (i.e., the occurrences over which  $D_1$  is  
placed to form D).

**Proof** By induction on the length of  $D_2$ .

(a)  $D_2 \equiv X$  (and  $Z \equiv X$ ) and the indicated occurrence of X is specified (so that  $D_1$  is placed over it). Then  $in(D_2) = in(X, O, D_2) = 1$ . Also  $D \equiv D_1$ . Hence,

$$in(D) = in(D_1) = in(D_1) - 1 + 1 = 1 \cdot [in(D_1) - 1] + 1 = in(X, O, D_2) \cdot [in(D_1) - 1] + in(D_2).$$

(b) X is not an undischarged assumption of  $D_2$  or else is an undischarged

assumption which is not specified (so that  $D_1$  is not placed over it to form D). Then  $D \equiv D_2$ ,  $in(X, O, D_2) = 0$ , and

$$in(D) = in(D_2) = 0 + in(D_2) = 0 \cdot [in(D_1) - 1] + in(D_2) = in(X, O, D_2) \cdot [in(D_1) - 1] + in(D_2).$$

(c) The last inference in  $D_2$  (and hence also in D) is by Eq. Let  $D'_2(D')$  be the result of deleting the last inference from  $D_2(D)$ . Then  $in(X, O, D_2) =$  $in(X, O', D'_2)$  where O' specifies the occurrences of X as an undischarged assumption in  $D'_2$  specified by O in  $D_2$ ,  $in(D_2) = in(D'_2)$ , and by the hypothesis of induction,

$$in(D') = in(X_2, O', D'_2) \cdot [in(D_1) - 1] + in(D'_2).$$

Hence,

$$in(D) = in(D') = in(X_2, O', D'_2) \cdot [in(D_1) - 1] + in(D'_2) = in(X_2, O, D_2) \cdot [in(D_1) - 1] + in(D_2).$$

(d) The last inference in  $D_2$  (and hence also in D) is by one of the one-premise rules except for Eq. Let  $D'_2(D')$  be the result of deleting the last inference from  $D_2(D)$ . Then  $in(X, O', D'_2) = in(X, O, D_2)$  where O' is as in Case (c),  $in(D_2) = in(D'_2) + 1$ , and by the induction hypothesis,

$$in(D') = in(X, O', D'_2) \cdot [in(D_1) - 1] + in(D'_2).$$

Hence,

$$in(D) = in(D') + 1$$
  
=  $in(X, O', D'_2) \cdot [in(D_1) - 1] + in(D'_2) + 1$   
=  $in(X, O, D_2) \cdot [in(D_1) - 1] + in(D_2).$ 

(e)  $D_2$  is the deduction

$$\begin{array}{ccc}
X & X \\
D_3 & D_4 \\
\hline
Z_1 & Z_2 \\
\hline
\Lambda Z_1 Z_2.
\end{array} \Lambda i$$

Then D is

$$D_1 \qquad D_1 \\ X \qquad X \\ D_3 \qquad D_4 \\ \frac{Z_1 \qquad Z_2}{\Lambda Z_1 Z_2} \Lambda i$$

Let  $D'_3, D'_4$  be

If, for  $i = 2, 3, 4, O_i$  specifies the occurrences of X in  $D_i$  specified by O in D, then

$$in(X, O_2, D_2) = in(X, O_3, D_3) + in(X, O_4, D_4),$$
  
 $in(D_2) = in(D_3) + in(D_4) + 1,$ 

and, by the induction hypothesis,

$$in(D'_3) = in(X, O_3, D_3) \cdot [in(D_1) - 1] + in(D_3),$$
  
 $in(D'_4) = in(X, O_4, D_4) \cdot [in(D_1) - 1] + in(D_4).$ 

Hence,

$$in(D) = in(D'_3) + in(D'_4) + 1$$
  
=  $in(X, O_3, D_3) \cdot [in(D_1) - 1] + in(D_3)$   
 $+in(X, O_4, D_4) \cdot [in(D_1) - 1] + in(D_4) + 1$   
=  $[in(X, O_3, D_3) + in(X, O_4, D_4)] \cdot [in(D_1) - 1] + in(D_3) + in(D_4) + 1$   
=  $in(X, O_2, D_2) \cdot [in(D_1) - 1] + in(D_2).$ 

(f)  $D_2$  is the deduction

$$\begin{array}{cccc} & 1 & 2 \\ X & [Y_1], X & [Y_2], X \\ D_3 & D_4 & D_5 \\ \hline \frac{\mathsf{V}Y_1Y_2}{Z} & \frac{Z}{Z} & \frac{Z}{Z} \\ \hline \end{array} \mathsf{Ve} - 1 - 2 \end{array}$$

Then D is

Let  $D'_3, D'_4, D'_5$  be

$D_1$	$D_1$	$D_1$
X	$Y_1, X$	$Y_2, X$
$D_3$	$D_4$	$D_5$
$VY_1Y_2,$	Z,	Z.

If, for i = 2, 3, 4, 5,  $O_i$  specifies the occurrences of X in  $D_i$  specified by O in D and, for  $j = 1, 2, O'_j$  specifies the occurrences of  $Y_j$  in  $D_{j+3}$  discharged by the inference by Ve, then

$$in(X, O_2, D_2) = [in(Y_1, O'_1, D_4) + in(Y_2, O'_2, D_5) + 1] \cdot in(X, O_3, D_3) + in(X, O_4, D_4) + in(X, O_5, D_5), in(D_2) = [in(Y_1, O'_1, D_4) + in(Y_2, O'_2, D_5) + 1] \cdot in(D_3) + in(D_4) + in(D_5) + 1,$$

and, by the induction hypothesis,

$$in(D'_i) = in(X, O_i, D_i) \cdot [in(D_1) - 1] + in(D_i), \qquad i = 3, 4, 5.$$

Thus,

$$\begin{split} in(D) &= [in(Y_1, O_1', D_4) + in(Y_2, O_2', D_5) + 1] \cdot in(D_3') + in(D_4') + in(D_5') + 1 \\ &= [in(Y_1, O_1', D_4) + in(Y_2, O_2', D_5) + 1] \\ &\cdot \{in(X, O_3, D_3) \cdot [in(D_1) - 1] + in(D_3)\} \\ &+ in(X, O_4, D_4) \cdot [in(D_1) - 1] + in(D_4) \\ &+ in(X, O_5, D_5) \cdot [in(D_1) - 1] + in(D_5) + 1 \\ &= \{[in(Y_1, O_1', D_4) + in(Y_2, O_2', D_5) + 1] \cdot in(X, O_3, D_3) \\ &+ in(X, O_4, D_4) + in(X, O_5, D_5)\} \\ &\cdot [in(D_1) - 1] + [in(Y_1, O_1', D_4) + in(Y_2, O_2', D_5) + 1] \cdot in(D_3) \\ &+ in(D_4) + in(D_5) + 1 \\ &= in(X, O_2, D_2) \cdot [in(D_1) - 1] + in(D_2). \end{split}$$

(g)  $D_2$  is the deduction

$$\begin{array}{ccc}
 & 1 \\
 & [Yx], X \\
 & D_3 & D_4(x) \\
 & \underline{\Sigma Y} & \underline{Z} \\
 & \underline{Z}. \\
\end{array}$$

Then D is

$$\begin{array}{cccc}
D_1 & 1 & D_1 \\
X & [Yx], X \\
D_3 & D_4(x) \\
\underline{\Sigma Y} & \underline{Z} \\
\underline{Z}.
\end{array}$$

Let  $D'_3, D'_4$  be

$D_1$	$D_1$
X	Yx, X
$D_3$	$D_4(x)$
Σ <i>Υ</i> ,	Z.

If, for  $i = 2, 3, 4, O_i$  specifies the occurrences of X in  $D_i$  specified by O in D and if O' specifies the occurrences of Yx in  $D_4(x)$  discharged by the inference by  $\Sigma e$ , then

$$in(X, O_2, D_2) = [in(Yx, O', D_4(x)) + 1] \cdot in(X, O_3, D_3) + in(X, O_4, D_4(x)),$$
  

$$in(D_2) = [in(Yx, O', D_4(x)) + 1] \cdot in(D_3) + in(D_4(x)) + 1,$$

and, by the induction hypothesis,

$$in(D'_i) = in(X, O_i, D_i) \cdot [in(D_1) - 1] + in(D_i), \qquad i = 3, 4.$$

Hence,

$$\begin{split} in(D) &= [in(Yx,O',D_4(x))+1] \cdot in(D'_3) + in(D'_4) + 1 \\ &= [in(Yx,O',D_4(x))+1] \cdot \{in(X,O_3,D_3) \cdot [in(D_1)-1] + in(D_3)\} \\ &\quad + in(X,O_4,D_4(x)) \cdot [in(D_1)-1] + in(D_4(x)) + 1 \\ &= \{[in(Yx,O',D_4(x))+1] \cdot in(X,O_3,D_3) + in(X,O_4,D_4(x))\} \cdot [in(D_1)-1] \\ &\quad + [in(Yx,O',D_4(x))+1] \cdot in(D_3) + in(D_4(x)) + 1 \\ &= in(X,O_2,D_2) \cdot [in(D_1)-1] + in(D_2). \end{split}$$

	-

As an immediate corollary of Lemma 2, we have the following result:

**Lemma 3** If D and D' are the deductions

$D_1$		$D'_1$
X	and	X
$D_2$		$D_2,$

and if  $in(D_1) < in(D'_1)$ , then in(D) < in(D').

**Proof of Theorem 1** There are four cases, depending on the reduction step.

 $\Lambda$ -reductions. Let D and D' be

$$\begin{array}{cccc}
D_1 & D_2 \\
X_1 & X_2 \\
\hline
\frac{X_1 & X_2}{\Lambda X_1 X_2} \Lambda i & & D_i \\
\hline
\frac{\Lambda X_1 X_2}{\Lambda Y_1 Y_2} Eq & \text{and} & & \frac{X_i}{Y_i} Eq \\
\hline
\frac{\Lambda Y_1 Y_2}{Y_i} \Lambda e & & D_3 \\
\hline
D_3
\end{array}$$

respectively, where by Lemma 3 we may disregard  $D_3$ . Then

$$in(D) = in(D_1) + in(D_2) > in(D_i) = in(D').$$

V-reductions. Let D and D' be

respectively, where, by Lemma 3, we may disregard  $D_4$ . If, for  $i = 1, 2, O'_i$  specifies the occurrences of  $Y_i$  in  $D_{i+1}$  discharged by the inference by Ve, then

$$in(D) = [in(Y_1, O'_1, D_2) + in(Y_2, O'_2, D_3) + 1] \cdot [in(D_1) + 1] + in(D_2) + in(D_3) + 1,$$
  
$$in(D') = in(Y_i, O'_i, D_{i+1}) \cdot [in(D_1) - 1] + in(D_{i+1}),$$

and clearly in(D') < in(D).

 $\Pi$ -reductions. Let D and D' be

$$\begin{array}{cccc}
D_1(x) \\
\frac{Xx}{\Pi X} & \Pi i \\
\frac{\Pi Y}{\Psi Z} & Eq \\
D_2
\end{array}$$
and
$$\begin{array}{cccc}
D_1(Z) \\
D_1(Z) \\
D_1(Z) \\
D_2 & Eq \\
D_2
\end{array}$$

respectively, where, by Lemma 3, we may disregard  $D_2$ . Then

$$in(D) = in(D_1(x)) + 2 = in(D_1(Z)) + 2 > in(D_1(Z)) = in(D').$$

 $\Sigma$ -reductions. Let D and D' be

$$\frac{\begin{array}{cccc}
D_1 & & & D_1 \\
\frac{XU}{\Sigma X} & \Sigma i & [Yx] & & & \frac{XU}{YU} Eq \\
\frac{\Sigma Y}{\Sigma Y} & Eq & D_2(x) & \text{and} & & D_2(U) \\
\hline
& & & & & & & D_2(U) \\
\hline
& & & & & & & & Z \\
& & & & & & & & D_3
\end{array}$$

respectively, where, by Lemma 3, we may disregard  $D_3$ . If O' specifies the occurrences of Yx in  $D_2(x)$  discharged by the inference by  $\Sigma e$  and if O'' specifies the corresponding occurrences of YU in  $D_2(U)$ , then

$$in(D) = [in(Yx, O', D_2(x)) + 1] \cdot [in(D_1) + 1] + in(D_2(x)) + 1$$
  
$$in(D') = in(YU, O'', D_2(U)) \cdot [in(D_1) - 1] + in(D_2(U)),$$

and since  $in(YU, O'', D_2(U)) = in(Yx, O', D_2(x))$  and  $in(D_2(U)) = in(D_2(x))$  by Lemma 1, we clearly have in(D') < in(D).

**Corollary 1.1** Every deduction can be reduced to a normal deduction (a deduction which is irreducible).

**Proof** An easy induction on the index of the deduction.

This normalization result makes it possible to prove cut elimination for the (singular) L-system introduced in §2.

**Theorem 2** The cut elimination theorem holds for  $\mathcal{F}_{33}^L$ .

The proof consists of two lemmas:

Lemma 4 If (4)  $M \Vdash X$ is provable in the  $\mathcal{F}_{33}^L$ , then (5)  $M \vdash X$ holds in  $\mathcal{F}_{33}^T$ .

**Proof** A straightforward induction on the length of the proof of (4). If (4) is the conclusion of \*Exp or Exp\*, then (5) follows by Rule Eq. The other cases are similar to those in [6, Theorems 5D6 and 7C1 (necessity)].

**Lemma 5** If (5) holds in  $\mathcal{F}_{33}^T$ , then there is a cut-free proof of (4) in  $\mathcal{F}_{33}^L$ .

**Proof** By Corollary 1.1, if (5) holds there is a normal deduction of it. Let this normal deduction be D. The proof is an induction on (in(D), ln(D)), where ln(D) is the length of D (the number of nodes in the tree diagram of D) and where the pairs are ordered by the usual lexicographic order, so that (a, b) < (c, d) if a < c or else a = c and b < d. Note that a proper part of a deduction has an index no higher than that of the entire deduction. For the basis of the induction, note that the result is trivial if (5) is is a deduction consisting of a single formula, since (4) an instance of the axiom scheme. For the induction step, we have the following cases:

Case 1. The last inference is an i-inference. Let D' be the result of deleting the last inference (D' may consist of two separate deductions). By applying the corresponding rule on the right to the induction hypothesis (whose index(es) is (are) lower than in(D)), we obtain a cut-free proof of (4).

Case 2. The last inference is by Eq. Similar to Case 1, but now in(D') = in(D) and ln(D') < ln(D), and the corresponding rule on the right is Eq<sup>\*</sup>.

Case 3. The last inference is an e-inference. Then the only inferences which occur in the left branch of D are e-inferences and inferences by Eq. Let the top formula of this left branch be Y (note that it is not discharged in D) and let the first inference be by rule R. If R is  $\Lambda e$ ,  $\Pi e$ , or Eq, then deleting the inference results in a deduction of lower index or equal index and lower length than D, and so we can apply the corresponding rule on the left to the induction hypothesis to obtain a cut-free proof of (4). The only cases left are those in which R is Ve and  $\Sigma e$ .

If R is Ve, then Y is  $VY_1Y_2$  and D is

Now the two deductions

$Y_1$		$Y_2$
$D_1$		$D_2$
Ζ	and	Z
$D_3$		$D_3$
X		X

have lower indexes than does D. Furthermore, although they may not be in normal form (because our normalization procedure does not eliminate all maximum segments but only cut formulas), they can certainly be normalized by Corollary 1.1, and the resulting normal deductions will have still lower indexes. Hence, if M' is all of M except Y, then by the induction hypothesis there are cut-free proofs of

$$M', Y_1 \Vdash X, \qquad \qquad M', Y_2 \Vdash X.$$

By \*V there is a cut-free proof of (4).

The case for  $\Sigma$ e is similar.

**Remark** This proof is somewhat complicated, since it is necessary to obtain some of the effects of the permutative reduction steps without having them as part of the normalization process. It therefore seems worth considering an example. Let us begin with the following deduction of  $\forall YZ, X \vdash X$ :

$$\frac{\frac{\left[Y\right]}{\Lambda YX}X}{\frac{\Lambda YX}{V(\Lambda YX)(\Lambda ZX)}V_{i}}\frac{\left[Z\right]}{\frac{\Lambda ZX}{V(\Lambda YX)(\Lambda ZX)}}X_{i}}\frac{X}{V_{i}}$$

$$\frac{\frac{X}{V(\Lambda YX)(\Lambda ZX)}V_{i}}{\frac{V(\Lambda YX)(\Lambda ZX)}{V(\Lambda ZX)}}V_{i}$$

$$\frac{X}{V_{i}}$$

$$\frac{X}{V_$$

This deduction is normal in the sense of this paper (although not in the sense of Prawitz [16]). If we apply Lemma 5 to this deduction, we are in Case 3, since the last inference is an e-inference. The top of the main branch is VYZ; the rest of the main branch consists of  $V(\Lambda YX)(\Lambda ZX), X$ . The subcase is that for Ve, so we need to look at the following two deductions:

$$\frac{\frac{Y - X}{\Lambda Y X} \Lambda i}{\frac{V(\Lambda Y X)(\Lambda Z X)}{X} V i} \frac{\frac{[\Lambda Y X]}{X} \Lambda e}{X} \frac{\frac{[\Lambda Z X]}{X} \Lambda e}{X} \Lambda e}{X} \frac{V e - 1 - 2}{2}$$

and

$$\frac{\frac{Z}{\Lambda ZX} \Lambda i}{\frac{V(\Lambda YX)(\Lambda ZX)}{X} V i} \frac{[\Lambda YX]}{X} \Lambda e \frac{[\Lambda ZX]}{X} \Lambda e}{X} \Lambda e \frac{[\Lambda ZX]}{X} V e - 1 - 2$$

Neither of these deductions is normalized, but both can be normalized: in each case a V-reduction followed by a  $\Lambda$ -reduction leads to a one-step deduction

X.

Hence, the cut-free proof in  $\mathcal{F}_{33}^L$  is

$$\frac{X \Vdash X}{\mathsf{V}YZ, X \Vdash X.} * \mathsf{K}$$

Note that Lemmas 4 and 5 also imply the equivalence of  $\mathcal{F}_{33}^T$  and  $\mathcal{F}_{33}^L$ .

An examination of the proof of Lemma 5 shows that we can, in fact, prove the following stronger result:

**Corollary 2.1** If (5) holds in  $\mathcal{F}_{33}^T$ , then there is a cut-free proof of (4) in  $\mathcal{F}_{33}^L$  in which rule \*W is not used.

Corollary 2.2 Rule \*W is redundant in  $\mathcal{F}_{33}^L$ .

**Proof** Redefine  $\mathcal{F}_{33}^L$  without this rule. Then Lemma 4 is proved as before, and Lemma 5 holds by Corollary 2.1. Hence, Theorem 2 holds for this modified  $\mathcal{F}_{33}^L$ . It is then possible to derive rule \*W as follows:

$$\frac{\frac{M, X, X \Vdash Z}{M, \forall XX \Vdash Z} * \mathsf{V} \qquad \frac{X \Vdash X}{X \Vdash \forall XX} \mathsf{V} *}{M, X \Vdash Z} \mathsf{Cut}$$

## 3 The Classical Version

Finding a classical version of  $\mathcal{F}_{33}^L$  is easy; take the system with more than one formula on the right of a sequent.

**Definition 5** The system  $\mathcal{F}_{33}^{LC}$  is defined by taking the same terms as in Definition 1 for  $\mathcal{F}_{33}^{LA}$ . The proof system is defined by the following axioms and rules, where M, N, and L are sequences of terms:

Axiom Scheme: 
$$X \Vdash X$$
, for each term  $X$   
Pulse:

Rules:

\*C 
$$\frac{M \Vdash N}{M' \vDash N}$$
, C\*  $\frac{M \Vdash N}{M \vDash N'}$ ,  
\*W  $\frac{M, X, X \vDash N}{M, X \vDash N}$ , W\*  $\frac{M \vDash X, X, L}{M \vDash X, L}$ ,  
\*K  $\frac{M \vDash N}{M, X \vDash N}$ , K\*  $\frac{M \vDash L}{M \vDash X, L}$ ,

\*Exp 
$$\frac{M, Y \Vdash N}{M, X \vDash N}$$
Exp\* 
$$\frac{M \Vdash Y, L}{M \vDash X, L}$$
\* $\Lambda \qquad \frac{M, X, Y \vDash N}{M, \Lambda XY \vDash N}$ 

$$^{*}\Lambda \qquad \frac{M, X, Y \vDash N}{M, \Lambda XY \vDash N}$$

$$^{*}V \qquad \frac{M, X \vDash N}{M, \nabla XY \vDash N}$$

$$^{*}\Pi \qquad \frac{M, XZ \vDash N}{M, \Pi X \vDash N}$$

$$^{*}\Sigma \qquad \frac{M, XX \vDash N}{M, \SigmaX \vDash N}$$
Cut 
$$\frac{M, X \vDash N}{M \vDash N, L}$$

$$^{*}\Sigma \qquad \frac{M, XX \vDash N}{M, \SigmaX \vDash N}$$

$$^{*}\Sigma \qquad \frac{M, XX \vDash N}{M, \SigmaX \vDash N}$$

$$^{*}\Sigma \qquad \frac{M, XX \vDash N}{M, \SigmaX \vDash N}$$

$$^{*}\Sigma \qquad \frac{M, XX \vDash N}{M, \SigmaX, L \leftarrow N}$$

where, in rules  $\Pi^*$  and  $*\Sigma$ , x is a variable which does not occur free in M, X, N, or L, in rules  $*Exp^*$  there is the condition that  $X \triangleright Y$ , and in rules  $*C^*$ , M' and N' are permutations of M and N respectively.

**Remark** This statement of the rules follows a convention from [8] in that for a system that is singular (i.e., with only one term on the right of a sequent), N is to consist of one formula and L is to be void. This makes it possible to state the rules, except for rule V<sup>\*</sup>, for singular and multiple systems together.

However, for the natural deduction system, it is not so easy, since all of the usual natural deduction rules which lead to classical logic when added to intuitionistic logic involve implication or negation. (A rule with implication alone is Pk of [6].) One way to find such a rule with neither implication nor negation is to try to prove that if

$$(6) M \Vdash Y_1, Y_2, \dots, Y_n$$

holds in  $\mathcal{F}_{33}^{LC}$ , then

(7) 
$$M \vdash \mathsf{V}Y_1(\mathsf{V}Y_2(\dots(\mathsf{V}Y_{n-1}Y_n)\dots))$$

holds in  $\mathcal{F}_{33}^{TA}$ . If we are trying an induction on the length of the proof of (6), then the only case that fails is the case in which (6) is the conclusion of  $\Pi^*$ . This reflects the fact that in an L-system for first-order intuitionistic

logic, the rules which must be singular on the right are  $\Pi^*$  and the rules for implication and negation. To complete this case of the proof here, it is sufficient to add the rule

$$\frac{\mathsf{V}(Xx)Y}{\mathsf{V}(\Pi X)Y}$$

where x does not occur free in X, Y, or any undischarged assumption.

Now in first-order predicate calculus, adding this rule to intuitionistic logic leads to the logic of constant domains, and as Lopez-Escobar shows in [13], there is no cut-free, complete, and sound L-system for this logic. But Lopez-Escobar is working in a mixed system in which the rules for implication and negation must be singular on the right but  $\Pi^*$  need not be. Since we are dealing with a system without implication or negation that is not mixed (all rules on the right are multiple), we do not have this problem.

**Remark** One might have thought that cut elimination could be proved for the mixed system of Lopez-Escobar by the methods of [6, Theorem 5D3, pp. 213–215]. But this is not the case. Curry's proof works only for systems with the following property: if the rule for a connective or quantifier on the left fails to be invertible, then the rule for the same connective or quantifier on the right must be singular. Since  $^{*}\Pi$  is not invertible, Curry's proof does not apply here.

**Definition 6** The system  $\mathcal{F}_{33}^{TC}$  is defined by adding rule  $\Pi V$  to  $\mathcal{F}_{33}^{TA}$ .

For this new rule we take the following reduction steps, where the deduction on the left reduces to the one on the right:

#### $\Pi V_1$ -reductions

$$\begin{array}{ccc} D_1 & & D_1 \\ \frac{U}{\nabla UV} \operatorname{Vi} & & \frac{U}{Xx} \operatorname{Eq} \\ \frac{V(Xx)Y}{\nabla(\Pi X)Y} \operatorname{\PiV} & & \frac{\Pi X}{\nabla(\Pi X)Y} \operatorname{Vi} \\ D_2 & & D_2. \end{array}$$

 $\Pi V_2$ -reductions

$$\begin{array}{ccc} D_1 & & D_1 \\ \frac{V}{\nabla UV} \mathsf{Vi} & & \frac{V}{Y} \mathrm{Eq} \\ \frac{V(Xx)Y}{\mathsf{V}(\Pi X)Y} \mathrm{\Pi V} & & D_2. \\ D_2 \end{array}$$

With these reduction rules, we can show that in a normalized deduction no inference by  $\Pi V$  will follow an i-inference. Hence, although  $\Pi V$  may look like an introduction rule, it behaves like an elimination rule. It differs from other e-rules in that because of the eigenvariable it cannot be the top inference in a branch of a deduction.

To extend the definition of index to this system, include  $\Pi V$  in clause (c) of Definition 3 and add the following clause to Definition 4:

(c) if the lst inference of D is by  $\Pi V$ , and if D' is the result of deleting the last inference from D, then in(D) = in(D') + 2.

(This is "in(D') + 2" rather than "in(D') + 1" for technical reasons; see the proof of Theorem 1 below.)

The proofs of Lemmas 1 and 3 for this system are straigntforward. We can prove Lemma 2 and Theorem 1 as follows:

**Proof of Lemma 2** If the last inference in  $D_2$  (and hence also in D) is by  $\Pi V$ , let  $D'_2(D')$  be the result of deleting the last inference from  $D_2(D)$ and let O' specify the occurrences of X in  $D'_2$  specified by O in D. Then  $in(X, O', D'_2) = in(X, O, D_2), in(D_2) = in(D'_2) + 2$ , and by the hypothesis of induction

$$in(D') = in(X, O', D'_2) \cdot [in(D_1) - 1] + in(D'_2).$$

Hence

$$in(D) = in(D') + 2$$
  
=  $in(X, O', D'_2) \cdot [in(D_1) - 1] + in(D'_2) + 2$   
=  $in(X, O, D_2) \cdot [in(D_1) - 1] + in(D_2).$ 

**Proof of Theorem 1**  $\Pi V_1$ -*reductions.* Let D reduce to D' by a  $\Pi V_1$ reduction where by Lemma 3 we may ignore  $D_2$ . Then

$$in(D) = in(D_1) + 3 > in(D_1) + 2 = in(D').$$

 $\Pi V_2$ -reductions. Let D reduce to D' by a  $\Pi V_2$ -reduction where by Lemma 3 we may ignore  $D_2$ . Then

$$in(D) = in(D_1) + 3 > in(D_1) + 1 = in(D').$$

-		
I		
I		

Corollary 1.1 now follows as before.

To extend Theorem 2 to the classical system, note first that rules  $^{*}\Lambda^{*}$ ,  $^{*}V^{*}$ ,  $\Pi^{*}$ , and  $^{*}\Sigma$  are invertible but  $^{*}\Pi$  and  $\Sigma^{*}$  are not; see [6, Theorem 7B4, p. 329]. To restate Lemma 4 for the multiple system, let  $N^{\vee}$  be  $\forall Y_{1}(\forall Y_{2}(\ldots(\forall Y_{n-1}Y_{n})\ldots))$  where N is  $Y_{1}, Y_{2}, \ldots, Y_{n-1}, Y_{n}$ . Then the revised form of the lemma is the following:

 $M \Vdash Y_1, Y_2, \ldots, Y_n$ 

holds in the  $\mathcal{F}_{33}^{LC}$ , then

(7)  $M \vdash \mathsf{V}Y_1(\mathsf{V}Y_2(\dots(\mathsf{V}Y_{n-1}Y_n)\dots))$ 

can be deduced in  $\mathcal{F}_{33}^{TC}$ .

The proof is an easy induction on the length of the proof of (6).

In order to prove Lemma 5 for this system, we need a theorem, but first we need a definition.

- **Definition 7** 1. Suppose A is a formula (term) in the conclusion of an inference by a rule and suppose that B is a formula in a premise for the same inference. Then B is an *immediate ancestor* of A if
  - (a) A is the principal formula for the inference and B is a side formula, or

- (b) A and B are corresonding parameters.
- 2. Suppose A is a formula (term) in the conclusion of an inference by a rule and suppose that B is a formula in a premise for the same inference. Then B is an *immediate parametric ancestor* of A if A and B are corresonding parameters.
- 3. Suppose A is a formula (term) in the conclusion of an inference by a rule and suppose that B is a formula in a premise for the same inference. Then B is an *immediate quasi-parametric ancestor* of A if
  - (a) A is the principal formula for the inference and B is a side formula and the rule is one of  $*W^*$ , or
  - (b) A and B are corresonding parameters.
- 4. Suppose A is a formula (term) in the conclusion of an inference by a rule and suppose that B is a formula in a premise for the same inference. Then B is an *immediate semiparametric ancestor* of A if
  - (a) A is the principal formula for the inference and B is a side formula and the rule is one of  $W^*$  or  $Exp^*$ , or
  - (b) A and B are corresonding parameters.
- 5. A formula B is an ancestor [parametric ancestor, quasi-parametric ancestor, semiparametric ancestor] if there is a sequence of formulas  $A_1, \ldots, A_n$ such that for each i = 2 to  $n, A_i$  is an immediate ancestor [immediate parametric ancestor, immediate quasi-parametric ancestor, immediate semiparametric ancestor] of  $A_{i-1}$ .

**Remark** These definitions are from [8, p. 191]. Note that if *B* is a parametric or quasi-parametric ancestor of *A*, then *A* and *B* are identical, and if *B* is a semiparametric ancestor of *A* then  $A \triangleright B$ .

**Theorem 3** Let D be a normal deduction in  $\mathcal{F}_{33}^{TC}$  of

 $(8) M \vdash X.$ 

Let there be for any K and N a cut-free proof in  $\mathcal{F}_{33}^{LC}$  of

(9)  $K, X \Vdash N.$ 

Then there is a cut-free proof in  $\mathcal{F}_{33}^{LC}$  of

**Proof** We may assume without loss of generality that no variable which occurs free in D [in the proof of (9)] is an eigenvariable in the proof of (9) [in D]. (If necessary, we can change the eigenvariables until this is true.)

The proof is by induction on (in(D), ln(D)) as in Lemma 5. The basis is immediate, since if D consists only of X, then (10) is just (9). For the induction step there are three cases:

Case 1. The last inference in D is by an i-rule. If the rule is  $\Lambda i$ , then X is  $\Lambda X_1 X_2$ , and D is

$$egin{array}{ccc} D_1 & D_2 \ X_1 & X_2 \ \hline \Lambda X_1 X_2. \end{array}$$
 Ai

Now for i = 1, 2, we have that  $in(D_i) < in(D)$ . Also, by (9) and the invertibility of  $*\Lambda$ , there is a cut-free proof of

$$K, X_1, X_2 \Vdash N.$$

By this,  $D_1$ , and the induction hypothesis, there is a cut-free proof of

$$M, K, X_2 \Vdash N.$$

By this,  $D_2$ , and the induction hypothesis, there is a cut-free proof of

$$M, M, K \Vdash N,$$

and (10) follows by \*C and \*W.

If the rule is Vi or  $\Sigma$ i the proof is similar. If the rule is  $\Pi$ i, then X is  $\Pi X_1$  and D is

$$\frac{D_1(x)}{\prod X_{1:}} \prod i$$

Here  $in(D_1(x)) < in(D)$ . Now let  $A_1, A_2, \ldots, A_n$  be a cut-free proof of (9) in which each  $A_i$  for i < n is used exactly once as the premise of an inference. Let  $A_k$  be

$$K_k, U_k \Vdash N_k,$$

where  $U_k$  consists of the semiparametric ancestors of X in  $A_k$  (if Y is in any  $U_k$ , then  $X \triangleright Y$ , and so Y is  $\prod Y_1$  where  $X_1 \triangleright Y_1$ ). For each k let  $A'_k$  be

$$M, K_k \Vdash N_k$$

Then  $A'_n$  is (10), and it is sufficient to prove for each k by a secondary induction on k that there is a cut-free proof of  $A'_k$ . There are the following cases in this secondary induction:

( $\alpha$ )  $U_k$  is void. Then  $A'_k$  follows from  $A_k$  by \*K, and there is a cut-free proof of  $A_k$  because it is part of the proof of (9).

( $\beta$ )  $U_k$  is not void and  $A_k$  is an axiom. Then  $K_k$  is void and  $U_k$  and  $N_k$  each consist of one term,  $\Pi Y_1$ , where  $X_1 \triangleright Y_1$ . If x is a variable which is not free in any of the terms of the context, then  $Y_1 x \Vdash Y_1 x$  is an axiom. By \*Exp we get  $X_1 x \Vdash Y_1 x$ . Then by  $D_1(x)$  and the main induction hypothesis, there is a cut-free proof of  $M \Vdash Y_1 x$ , and  $A'_k$  now follows by  $\Pi^*$ .

 $(\gamma)$   $U_k$  is not void and  $A_k$  is derived from  $A_i, A_j$  by a rule R for which all of the terms in  $U_k$  are parametric. By the induction hypothesis on k there are cut-free proofs of  $A'_i, A'_j$ , and  $A'_k$  now follows by R.

( $\delta$ )  $U_k$  is not void and  $A_k$  is derived from  $A_i$  by a structural rule (\*C, \*K, or \*W) or \*Exp whose principal formula is in  $U_k$ . By the induction hypothesis on k, there is a cut-free proof of  $A'_i$ , and  $A'_k$  is identical to  $A'_i$ .

( $\epsilon$ )  $U_k$  is not void and  $A_k$  is derived by one of the rules \* $\Lambda^*$ , \* $V^*$ , \* $\Pi^*$ , or \* $\Sigma^*$  whose principal formula is in  $U_k$ . The only such rule possible is \* $\Pi$ , and hence there is only one premise, say  $A_i$ . The principal formula is  $\Pi Y_1$ , where  $X_1 \triangleright Y_1$ , and  $A_i$  is

$$K_k, U_i, Y_1Z \Vdash N_k$$

By the argument of  $(\gamma)$  above, there is a cut-free proof of

$$M, K_k, Y_1Z \Vdash N_k.$$

By \*Exp there is a cut-free proof of

$$M, K_k, X_1Z \Vdash N_k.$$

Now substituting Z for x in  $D_1(x)$  gives us a deduction  $D_1(Z)$  of  $X_1Z$  where  $in(D_1(Z)) < in(D)$ . Hence, by the main induction hypothesis, there is a cut-free proof of

$$M, M, K_k \Vdash N_k,$$

and  $A'_k$  follows by \*W and \*C.

Case 2. D ends in an inference by Eq. This case is easy using Eq<sup>\*</sup>.

Case 3. D ends in an e-inference (which may be by  $\Pi V$ ). Then since D is normalized, the left branch consists entirely of such inferences or inferences by Eq. Consider the first inference of the branch (which cannot be by  $\Pi V$ because of the eigenvariable). If it is by  $\Lambda e$  or Eq, then deleting it we get a deduction with a lower index or an equal index but lower length, and we can apply the induction hypothesis and the corresponding rule on the left.

If the first inference is by Ve, then M is  $M_1, VY_1Y_2$  and D is

Now the deductions

$Y_1$		$Y_2$
$D_1$		$D_2$
Z	and	Z
$D_3$		$D_3$
X		X

may not be normalized, but each has index less than in(D), and normalizing them reduces the indexes still more. Hence, by the induction hypothesis, there is a cut-free proof of

$$M_1, Y_i, K \Vdash N, \qquad \qquad i = 1, 2,$$

and then (10) can be obtained by \*V.

If the first inference is by  $\Sigma$ e the proof is similar.

If the first inference is by  $\Pi e$ , then M is  $M_1, \Pi Y$  and D is

$$\frac{\Pi Y}{YZ} \Pi e$$
$$D_1$$
$$X.$$

There are the following subcases:

(i) No eigenvariable of an inference by  $\Pi V$  in the main branch of D occurs free in Z. Then  $D_1$  is a valid deduction and  $in(D_1) < in(D)$ . Hence, we can proceed as in the case for  $\Lambda e$ .

(ii) There is an eigenvariable of an inference by  $\Pi V$  in the main branch of D which occurs free in Z. Then  $D_1$  is not a valid deduction, but D can be written as follows:

$$\frac{\Pi Y}{YZ} \Pi e$$

$$\frac{D_2(x)}{V(Ux)W}$$

$$\frac{V(Ux)W}{V(\Pi U)W} \Pi V$$

$$\frac{D_3}{X,}$$

where there are no inferences by  $\Pi V$  (and hence no eigenvariables) in the main branch of  $D_3$ . Now the deductions

(11) 
$$\begin{aligned} \frac{\Pi Y}{YZ} \Pi e \\ D_2(x) \\ V(Ux)W \end{aligned}$$

and

(12) 
$$\begin{array}{c} \mathsf{V}(\mathsf{\Pi}U)W\\ D_3\\ X \end{array}$$

are valid normal deductions with indexes less than in(D). By (12), (9), the induction hypothesis, and perhaps \*K, there is a cut-free proof of

$$M_1, \mathsf{V}(\mathsf{\Pi} U)W, K \Vdash N.$$

Because \*V is invertible, this implies that there are cut-free proofs of

$$(13) M_1, \Pi U, K \Vdash N$$

- and
- (14)  $M_1, W, K \Vdash N.$

Let  $A_1, A_2, \ldots, A_n$  be a cut-free proof of (13) in which each  $A_i$  for i < n is used exactly once as the premise for an inference. Let  $A_k$  be

$$M'_k, U_k \Vdash N_k,$$

where, as before,  $U_k$  consists of the semiparametric ancestors of  $\Pi U$  that occur in  $A_k$ . Let  $A'_k$  be

$$M'_k, M_1, \Pi Y, K \Vdash N_k, N.$$

Then (10) follows from  $A'_n$  by \*W\* and \*C\*. As before, we prove by a secondary induction on k that there is a cut-free proof of  $A'_k$ , and as before we have the following subcases:

( $\alpha$ )  $U_k$  is void. Then  $A'_k$  follows from  $A_k$  by \*K\*.

( $\beta$ )  $U_k$  is not void and  $A_k$  is an axiom. Then  $M'_k$  is void and  $U_k$  and  $N_k$  both consist of the same term, which is  $\Pi U'$  where  $U \triangleright U'$ . We have the following cut-free proof:

$$\frac{\frac{U'x \Vdash U'x}{M_1, U'x, K \Vdash U'x, N} *K * (14)}{\frac{M_1, W, K \Vdash U'x, N}{M_1, W, K \Vdash U'x, N}} *V$$

$$\frac{\frac{M_1, V(U'x)W, K \Vdash U'x, N}{M_1, V(Ux)W, K \Vdash U'x, N} *Exp$$

Hence, by (11) and the main induction hypothesis, there is a cut-free proof of

 $M_1, \Pi Y, K \Vdash U'x, N.$ 

Applying  $\Pi^*$ , we get a cut-free proof of  $A'_k$ .

 $(\gamma)$   $U_k$  is not void and  $A_k$  follows from  $A_i$ ,  $A_j$  by a rule for which all the terms in  $U_k$  are parametric. Then  $A'_k$  follows from  $A'_i$ ,  $A'_j$  by the same rule.

( $\delta$ )  $U_k$  is not void and  $A_k$  follows from  $A_i$  by a structural rule or \*Exp. Then  $A'_k$  is identical to  $A'_i$ .

( $\epsilon$ )  $U_k$  is not void and  $A_k$  follows from  $A_i$  by \* $\Pi$  and the principal formula is in  $U_k$ . Then the principal formula is  $\Pi U'$  where  $U \triangleright U'$ . Since  $U_k$  is  $U_i, \Pi U', A_i$  is

$$M'_k, U_i, U'V \Vdash N_k$$

By the argument of  $(\gamma)$  above, there is a cut-free proof of  $A_k''$ :

$$M'_k, M_1, \Pi Y, U'V, K \Vdash N_k, N.$$

We now have the following cut-free proof:

$$\frac{A_k''}{M_k', M_1, \Pi Y, UV, K \Vdash N_k, N} * \operatorname{Exp} \frac{(14)}{M_k', M_1, \Pi Y, W, K \Vdash N_k, N} * \operatorname{K*}_{\mathsf{W}_k', M_1, \Pi Y, \mathsf{V}(UV)W, K \Vdash N_k, N.} * \mathsf{V}$$

If we substitute V for x in (11), we get

$$\frac{\Pi Y}{Y([V/x]Z)} \prod_{\substack{D_2(V)\\ \mathsf{V}(UV)W,}}$$

which has an index less than in(D). Thus, by the main induction hypothesis, there is a cut-free proof of

$$M'_k, M_1, \Pi Y, \Pi Y, K \Vdash N_k, N,$$

and  $A'_k$  follows from this by \*W.

#### Remarks

- 1. Note that rule \*W is used in the proof in an essential way. It does not appear that we can prove Corollary 2.2 for the classical system.
- 2. Since the theorem is, in a sense, a special case of the cut-elimination theorem, it is perhaps not surprising that parts of the proof (for  $\Pi$ i in Case 1 and for  $\Pi$ e in Case 3) resemble part of the proof (Stage 1) of proofs of the elimination theorem in [6] and [8].
- 3. This proof cannot be applied to the logic of constant domains because subcases  $(\beta)$  and  $(\eta)$  of Case 3 fail if there is a rule in the system whose right-hand side must be singular.
- 4. This proof is sufficiently complicated that it seems advisable to give an example. Let a, b, c, and d be distinct non-redex constants, and let x and y be distinct variables. Let D be the normal deduction:

$$\frac{ \prod(\lambda y \cdot \Pi(\lambda x \cdot V(ax)(by))) }{ \prod(\lambda x \cdot V(ax)(by)) } \prod_{\text{e, Eq}} \text{Eq} \\ \frac{ \prod(\lambda x \cdot V(ax)(by)) }{ V(ax)(by) } \prod_{\text{e, Eq}} \text{Eq} \\ \frac{ V(ax)(by) }{ V(\Pi a)(by) } \prod_{\text{e, Eq}} \text{Eq}, \prod_{\text{e, Eq}} \text{Eq}$$

and let E be the cut-free proof

$$\frac{\frac{ac \Vdash ac}{ac \Vdash ac, bd}}{\prod a \Vdash ac, bd} \overset{\mathsf{K}*}{*\Pi} \frac{bd \Vdash bd}{bd \Vdash ac, bd} \overset{\mathsf{K}*}{*\mathsf{V}} \frac{V(\Pi a)(bd) \Vdash ac, bd}{*\mathsf{V}} \overset{\mathsf{K}*}{*\mathsf{V}}$$

By Theorem 3, there should be a cut-free proof of

(15) 
$$\Pi(\lambda y . \Pi(\lambda x . V(ax)(by))) \Vdash ac, bd.$$

. . . . . .

Let us see how to obtain this cut-free proof by following the proof of the theorem. Since the last step of D is by  $\Pi$ i, we apply Case 1, and since the last step of E is by  $*\Pi$ , we apply subcase ( $\epsilon$ ). Thus, we need to look at  $D_1(y)$ , which is obtained from D by deleting the last inference,

and also at  $E_1$ , which is obtained from E by deleting the last inference. To apply subcase  $(\epsilon)$ , we apply the theorem to  $D_1(d)$  and  $E_1$ , where  $D_1(d)$  is obtained from  $D_1(y)$  by substituting d for y, and is therefore as follows:

$$\frac{\Pi(\lambda y \cdot \Pi(\lambda x \cdot V(ax)(by)))}{\frac{\Pi(\lambda x \cdot V(ax)(bd))}{\frac{V(ax)(bd)}{V(ax)(bd)}} \Pi e, Eq}$$

To apply the theorem, since this ends in an e-rule, we use Case 3, and since there is no occurrence in d of the eigenvariable of the inference by  $\Pi V$ , we need subcase (i) of the subcase of Case 3 for  $\Pi e$ . This means that we delete the first inference from  $D_1(d)$ , giving us  $D_2$ :

$$\frac{\Pi(\lambda x . V(ax)(bd))}{\frac{V(ax)(bd)}{V(\Pi a)(bd)}} \Pi V$$

We then apply the theorem to this and  $E_1$  to get a cut-free proof of

(16)  $\Pi(\lambda x . \mathsf{V}(ax)(bd) \Vdash ac, bd,$ 

and we then obtain (15) from this by an inference by  $^{*}\Pi$ . Now because x in  $D_2$  is the eigenvariable for the inference by  $\Pi V$ , we apply Case 3, subcase (ii) of the case for  $\Pi e$ . Thus, we break  $D_2$  into two deductions:  $D_3(x)$ , which is

$$\frac{\Pi(\lambda x \cdot \mathsf{V}(ax)(bd))}{\mathsf{V}(ax)(bd),} \, \mathsf{\Pi}\mathrm{e}, \mathrm{Eq}$$

and  $D_4$ , which consists of the single step  $V(\Pi a)(bd)$ . Since  $D_4$  consists of a single step,  $E_1$  is itself the result of applying the theorem to  $D_4$  and  $E_1$ . Next, we note that the use of the invertibility of \*V is unnecessary, since the last inference of  $E_1$  is by \*V. Hence, we look at the cut-free proofs of the premises of this last inference in  $E_1$ . These are  $E_2$ , which is

$$\frac{ac \Vdash ac}{ac \Vdash ac, bd} \mathbf{K}^*$$
$$\overline{\Pi a, \Vdash ac, bd,} * \Pi$$

and  $E_3$ , which is

$$\frac{bd \Vdash bd}{bd \Vdash ac, bd.} \mathbf{K} \ast$$

In particular, we look at  $E_2$ , and since its last inference is by  $^*\Pi$ , we apply subcase ( $\epsilon$ ). This means that we delete the last inference of  $E_2$  in order to get a cut-free proof of  $A'_k$ , and when we put this together with  $E_3$  and apply  $^*V$ , we get  $E_4$ :

$$\frac{\frac{ac \Vdash ac}{ac \Vdash ac, bd} \operatorname{K*} \quad \frac{bd \Vdash bd}{bd \Vdash ac, bd} \operatorname{K*}}{\mathsf{V}(ac)(bd) \Vdash ac, bd.} \operatorname{K*}$$

Now we take  $D_3(c)$ , which is

$$\frac{\Pi(\lambda x \cdot \mathsf{V}(ax)(bd))}{\mathsf{V}(ac)(bd),} \, \mathsf{\Pi}\mathrm{e}, \mathrm{Eq}$$

and apply the theorem to it and  $E_4$ . The result (by Case 3, subcase (i) for the case for  $\Pi e$ ) is the following cut-free proof of (16):

$$\frac{E_4}{\Pi(\lambda x . \mathsf{V}(ax)(bd) \Vdash ac, bd.} * \Pi$$

A cut-free proof of (15) can then be obtained by another inference by  $^{*}\Pi$ .

If in this system we let (9) be the axiom  $X \Vdash X$ , then (10) becomes  $M \Vdash X$ ; hence Lemma 5 holds, and we have the following result:

**Corollary 3.1** Theorem 2 holds for  $\mathcal{F}_{33}^{LC}$ .

Note that as in the case of the intuitionistic system, the proof also implies the equivalence of  $\mathcal{F}_{33}^{TC}$  and  $\mathcal{F}_{33}^{LC}$ .

**Remark** It is possible to deduce the cut-elimination theorem directly from Theorem 3 without using Lemma 5. Thus, suppose we have cut-free proofs of

and

If L is void, then by (18) and Lemma 3.1', we have  $K \vdash X$ . The deduction of this can be normalized, and then by (17) and Theorem 3 there is a cut-free proof of

$$M, K \Vdash N.$$

If L is not void, then by (18) and Lemma 3.1',

 $K \vdash \mathsf{V}XL^{\mathsf{V}}.$ 

By using (17), the axiom  $L^{\sf V} \Vdash L^{\sf V}$ , and  ${}^*{\sf V}$ , we can get a cut-free proof of

$$M, \mathsf{V}XL^{\mathsf{V}} \Vdash N, L^{\mathsf{V}}.$$

Then, by Theorem 3, there is a cut-free proof of

$$M, K \Vdash N, L^{\mathsf{V}},$$

and by the invertibility of  $\mathsf{V}^*$  there is a cut-free proof of

$$M, K \Vdash N, L.$$

## 4 Introducing Implication with Restrictions

By the contradiction Curry found in [5], the theory we have developed so far cannot be applied if implication is added to the system without any restriction. In order to see if there are any restrictions under which the rules for implication can be added, it is worth looking for the place at which the argument of the paper so far breaks down when implication is present in the system.

On the other hand, as we saw in the introduction to the paper, if the discharged assumption of the inference by Pi occurs at most once (after all the duplicating; i.e., occurs at most once and is not duplicated later in the deduction, which means that it has no descendent in the deduction which is a major premise for an inference by Ve or  $\Sigma e$ ), then a P-reduction step will shorten the proof. This suggests that we adopt the following restriction:

**Restriction on Rule** Pi *The discharged assumption is to occur at most once and have no descendant which is a major premise for an inference by* Ve *or*  $\Sigma e$  (down to the inference by Pi).

**Remark** Since we now have implication, we also have negation.  $\neg X$  is defined to be  $\mathsf{P}X(\mathsf{\Pi}\mathsf{I})$ . However, the rule for negation introduction is a special case of the rule for implication introduction, so that in this logic implication and negation do not satisfy all of their usual properties, even in intuitionistic logic.

**Definition 8** The system  $\mathcal{F}_{33}^{TJ}$  is obtained from  $\mathcal{F}_{33}^{TA}$  by adding the rules Pi and Pe, where rule Pi is subject to the Restriction on Rule Pi.

To see that this works, extend Definitions 3 and 4 by treating Pi with the other one-premise rules in case (c) of each definition and treat Pe like Ai in case (d) of each definition. Lemmas 1–3 can be proved as before. We also want the following lemmas:

**Lemma 6** Let D be a deduction, X a term, and O a specification of occurrences of X as an undischarged assumption in D. If the specified occurrence(s) of X satisfy the Restriction on Rule Pi, then  $in(X, O, D) \leq 1$ .

The proof is a straightforward induction on the length of D using Definition 3. Note that if O specifies no occurrences of X in D, then in(X, O, D) =0. Hence, it is sufficient to suppose that O specifies one occurrence of X in D and to prove that in(X, O, D) = 1.

**Lemma 7** Suppose D and D' are deductions such that D reduces to D', and suppose X is an undischarged assumption in both deductions. If X satisfies the Restriction on Rule Pi in D, then it satisfies it in D'.

The proof is a straightforward induction on the number of reduction steps from D to D', with cases according to the reduction step.

Now to complete the proof of Theorem 1, it remains to consider the case for P-reductions. Let D be the first and D' the second deduction in a Preduction and disregard  $D_3$  (as we can because of Lemma 3). It is easy to see that if O specifies the occurrences of X in  $D_1$  discharged by the inference by Pe,

$$in(D) = in(D_1) + in(D_2) + 2,$$
  
 $in(D') = in(X, O, D_1) \cdot [in(D_2) - 1] + in(D_2).$ 

By Lemma 6 and the restriction on Rule Pi,  $in(X, O, D_1) \leq 1$ . It follows that in(D') < in(D).

This proves the following result:

**Theorem 4** Theorem 1 holds for  $\mathcal{F}_{33}^{TJ}$ .

In [8, Corollary 16C3.1, p. 442] a corresponding result was claimed for the L-formulation, where the rules for P are

$$*\mathsf{P} \quad \frac{M \Vdash X \quad M, Y \Vdash Z}{M, \mathsf{P}XY \Vdash Z,} \qquad \mathsf{P}^* \qquad \frac{M, X \Vdash Y}{M \Vdash \mathsf{P}XY,}$$

and where the restriction on  $\mathsf{P}^*$  is that the X in the premise not have any ancestor which is the principal formula of an operational rule on the left. This clearly does not correspond to the restriction on the rule  $\mathsf{P}$  is stated above.

Instead, let us note that Bunder and da Costa give in [2] a natural deduction system with just the above restriction on rule Pi as a formulation of BCK logic. Furthermore, White [20] gives a natural deduction formulation for a variant of higher-order BCK logic with comprehension that differs from the system given here in that his rules for disjunction and existential elimination satisfy the same restriction on discharged assumptions that we have here for Pi, so that his system is weaker than this one. Elsewhere, for example in [15], [12], and [14], BCK logic is identified with an L-system without rules \*W\*. This suggests that we might find an L-formulation equivalent to our natural deduction formulation by leaving out these rules. But there is a technical problem with this: consider the following deduction in  $\mathcal{F}_{33}^{TA}$  of the distributive rule:

The restriction on Pi prevents us from discharging the assumption  $\Lambda X(VYZ)$  because there are three occurrences of it and one of them has a descendant which is a major premise for an inference by Ve. But now consider the equivalent proof in  $\mathcal{F}_{33}^{LA}$ :

$$\frac{\frac{X \Vdash X}{X, Y \vDash X} * \mathbb{K} \quad \frac{Y \Vdash Y}{X, Y \vDash Y} * \mathbb{K}}{\frac{X, Y \vDash \Lambda XY}{X, Y \vDash \Lambda XY} \wedge *} \xrightarrow{\begin{array}{c} X \Vdash X \\ X, Z \vDash X} * \mathbb{K} \quad \frac{Z \vDash Z}{X, Z \vDash Z} * \mathbb{K} \\ \frac{X, Z \vDash \Lambda XZ}{X, Z \vDash \Lambda XZ} \wedge * \\ \frac{X, VYZ \vDash V(\Lambda XY)(\Lambda XZ)}{\Lambda X(VYZ) \vDash V(\Lambda XY)(\Lambda XZ)} * \\ \end{array}$$

There is nothing to prevent the use of  $P^*$  here.

On the other hand, the occurrence of  $\Lambda X(VYZ)$  on the left of the conclusion does have an ancestor which is a principal formula for an inference by \*V. This suggests that we form the L-system by deleting rules \*W\* and adding rules \*P\* with the restriction that no ancestor of X on the left of the premise is the principal formula for an inference by \*V or \* $\Sigma$ . However, this is still not quite right; there is nothing corresponding to the condition in the Restriction on Rule Pi that the discharged assumption occur at most once.

The solution turns out to be to assume rules  $*W^*$  but take as an restriction on  $P^*$  that the left side formula (X in \*P) not have an ancestor which is a principal formula for an inference by \*W. Also, to make all this work properly, it is necessary to modify some of the rules.

**Definition 9** The system  $\mathcal{F}_{33}^{LJ}$  will be singular. (See below for the reason we cannot extend the result to the multiple [classical] version of the calculus.) Its rules are \*C, \*K, \*W, and \*Exp\* of Definition 1, and

*Λ	$\frac{M, X_i \Vdash Z}{M, \Lambda X_1 X_2 \Vdash Z,}$	٨*	$\frac{M_1 \Vdash X_1}{M_1, M_2 \Vdash \Lambda X_1 X_2,}$
*V	$\frac{M_1, X_1 \Vdash Z}{M_1, M_2, V X_1 X_2 \Vdash Z}$	V*	$\frac{M \Vdash X_i}{M \Vdash VX_1 X_2,}$
*P	$\frac{M_1 \Vdash X \qquad M_2, Y \Vdash Z}{M_1, M_2, PXY \Vdash Z,}$	P*	$\frac{M, X \Vdash Y}{M \Vdash PXY,}$
*П	$\frac{M, XY \Vdash Z}{M, \Pi X \Vdash Z,}$	Π*	$\frac{M \Vdash Xx}{M \Vdash \Pi X,}$
*Σ	$\frac{M, Xx \Vdash Z}{M, \Sigma X \Vdash Z,}$	Σ*	$\frac{M \Vdash XZ}{M \Vdash \Sigma X,}$
Cut	t $\frac{M_1 \Vdash X}{M_1, M_2 \Vdash Z}$		

where, in  $\Pi i^*$  and  $^*\Sigma$ , x does not occur free in M, X, or Z; where in  $^*\Lambda$  and  $V^*$ , i = 1 or i = 2; and where there is a restriction on  $P^*$  to be stated below.

To state the restriction on  $\mathsf{P}^*$ , define X to be *side-connected* to Y if X is a parametric constituent for the left premise of  $*\mathsf{P}$  (i.e., is in  $M_1$ ) and Y is the side formula of the right premise, or else X is a parametric constituent for the left premise of Cut (i.e., is in  $M_1$ ) and Y is the side formula of the right premise (i.e., the X in the right premise). Then define X to be *chained*  if (i) X has an ancestor which is a principal formula for an inference by \*W, \*V, or  $*\Sigma$ ; or (ii) X is side-connected to Y and Y is chained.

Restriction on Rule P\* The side formula on the left is not chained.

(The "side formula on the left" of Rule  $P^*$  is the occurrence of X on the left of the premise.)

**Remark** Note that in this system there are no provable sequents with void right-hand sides. This formulation is what Curry in [6] calls an F-formulation, where his F is  $\Pi$ I. What in another system would be a sequent with a void right-hand-side corresponds here to a sequent whose right-hand-side is  $\Pi$ I.

To prove Theorem 2, we need the following modifications of Lemmas 4 and 5.

Lemma 3.1" If (4)  $M \Vdash X$ is provable in  $\mathcal{F}_{33}^{LJ}$ , then there is a deduction of (5)  $M \vdash X$ 

in  $\mathcal{F}_{33}^{TJ}$  such that for each Y in M, if Y satisfies the Restriction on Rule P\* in the proof of (4), then it satisfies the Restriction on Rule Pi in the deduction of (5).

**Proof** A straightforward induction on the proof of (4). In each case of the induction step, it is necessary to check that satisfying the restriction on Rule Pi is preserved by the induction step. In this process, note that \*W corresponds to grouping occurrences of an assumption together for the purpose of discharging them together; the Restriction on Rule Pi precludes this for that rule, but it can be done for Rules Ve and  $\Sigma$ e. One case should illustrate the way to complete the proof from the proof of Lemma 4: the case for  $\Lambda^*$ . Here  $X \equiv \Lambda X_1 X_2$ ,  $M \equiv M_1, M_2$ , and the premises are

$$M_1 \Vdash X_1, \qquad \qquad M_2 \Vdash X_2.$$

By the induction hypothesis, there are deductions

$$\begin{array}{ccc} D_1 & D_2 \\ X_1, & X_2, \end{array}$$

where, for i = 1, 2, the undischarged assumptions of  $D_i$  are in  $M_i$  and where, for each Y in  $M_i$ , if Y satisfies the Restriction on Rule P\* then Y satisfies the restriction on Rule Pi in  $D_i$ . Then

$$rac{D_1}{X_1} rac{D_2}{\Lambda X_1 X_2}$$
 Ai

is a deduction of (5) in which each Y satisfies the Restriction on Rule Pi if it satisfies the restriction on Rule  $P^*$  in (4). (Note that Y satisfies the Restriction on Rule  $P^*$  in (4) if and only if it does in the premise in which it occurs.)

**Lemma 3.2**" If there is a deduction of (5) in  $\mathcal{F}_{33}^{TJ}$ , then there is a cut-free proof of (4) in  $\mathcal{F}_{33}^{LJ}$  in which, for each Y in M, if Y satisfies the Restriction on Rule Pi in the deduction of (5), then it is not chained in the proof of (4).

**Proof** A straightforward modification of the proof of Lemma 5, in which a case for Pi is added in Case 1 and a case for Pe is added in Case 3, and where, in each case, we have to check the condition on not being chained. One subcase each from Cases 1 and 3 should make clear how the rest of the proof goes.

Under Case 1, let us consider the case for Ai. Here  $X \equiv \Lambda X_1 X_2$  and D is

$$rac{D_1}{X_1} rac{D_2}{\Lambda X_1 X_2}$$
 Ai

For i = 1, 2, let  $M_i$  be a sequence consisting of the undischarged assumptions in  $D_i$ . By the hypothesis of induction, there is a cut-free proof of

$$M_1 \Vdash X_1, \qquad \qquad M_2 \Vdash X_2$$

in which, for each Y in one of the  $M_i$  which satisfies the Restriction on Rule Pi in  $D_i$  satisfies the Restriction on Rule \*P. By  $\Lambda$ \*, we get a cut-free proof of

$$M_1, M_2 \Vdash \Lambda X_1 X_2$$

in which each Y in one of the  $M_i$  which satisfies the Restriction on Rule Pi satisfies the Restriction on Rule P<sup>\*</sup>. Now if M cannot be obtained from  $M_1, M_2$  by adding assumptions (for which we can use Rule \*K), it can only be because there is an assumption Y which is in both  $M_1$  and  $M_2$ ; this assumption is an undischarged assumption in both  $D_1$  and  $D_2$ , and is to be considered as one undischarged assumption until it is discharged. In this case, we use \*W; the terms which occur as principal formulas of these inferences by \*W, and which do not satisfy the Restriction on Rule P<sup>\*</sup> in (4), do not satisfy the Restriction on Rule Pi in (5).

Under Case 3, let us consider the case for Pe. Here D is

$$\frac{\begin{array}{cc} D_1 \\ PYZ & Y \\ \hline Z \\ D_2 \\ X. \end{array} \mathsf{Pe}$$

For i = 1, 2, let  $M_i$  be the undischarged assumptions of  $D_i$  other than the indicated occurrence of Z in  $D_2$ . By the hypothesis of induction, there is a cut-free proof of

$$(19) M_1 \Vdash Y, M_2, Z \Vdash X$$

such that, for each Y in  $M_i$ , if Y satisfies the Restriction on Rule in Pi in  $D_i$ , then Y satisfies the Restriction on Rule P\* in whichever of (19) it occurs on the left, and the same is true for Z in the second premise (and in  $D_2$ ). Now by \*P we get

$$(20) M_1, M_2, \mathsf{P}YZ \Vdash X.$$

If M cannot be obtained from  $M_1, M_2$ , PYZ by adding terms (using \*K), then either there is an overlap between  $M_1$  and  $M_2$  or there is an overlap between  $M_i$  and PYZ. In either case, we can use \*W to get (4) from (20). Now suppose W in M satisfies in D the Restriction on Rule Pi. Then W is in  $M_i$  or is PYZ, and it is not one of the formulas in an overlap as described above (since it can occur at most once). Hence, it does not have an ancestor in the steps from (20) to (4) which is the principal formula for an inference by \*W. If it is in  $M_i$ , then it satisfies in  $D_i$  the Restriction on Rule Pi, and hence it satisfies in the proof of (19) the Restriction on Rule  $P^*$ . If W is in  $M_2$ , then it satisfies in the proof of (4) the Restriction on Rule  $P^*$ . If W is in  $M_1$  then it satisfies the Restriction on Rule Pi in  $D_1$ . If it satisfies the Restriction on Rule Pi in D, then the indicated occurrence of Z in  $D_2$ must satisfy this restriction in  $D_2$ , and in this case W will satisfy the Restriction on Rule P<sup>\*</sup> in the cut-free proof of (4). Similarly, if PYZ satisfies the Restriction on Rule Pi in D, then Z satisfies this restriction in  $D_2$ , and therefore PYZ satisfies the Restriction on Rule P<sup>\*</sup> in the cut-free proof of (20) and hence (since there is no overlap) in the cut-free proof of (4).

This proves

**Theorem 5** Theorem 3 holds for  $\mathcal{F}_{33}^{LJ}$ .

This intuitionistic variant of BCK logic,  $\mathcal{F}_{33}^{TJ}$ , is strong enough to prove some interesting results. For example, here are proofs of the implication formulas (PI), (PK), (PB), and (PC):

(PI):

$$\frac{1}{[X]}$$

$$Pi - 1$$

(PK):

$$\frac{\begin{bmatrix} X \\ PYX \end{bmatrix}}{PYX} Pi - v}$$

$$\frac{PX(PYX)}{PX(PYX)} Pi - 1$$

(PB):



(PC):

If we define  ${\sf Q}$  by

$$\mathsf{Q} \equiv \lambda xy . \mathsf{P}(\lambda z . \mathsf{P}(zx)(zy)),$$

then, using the obvious abbreviations, we have

$$\mathsf{Q}XY =_* (\forall z)(zX \supset zY),$$

and we can interapret the system Q of Appendix A. Finally, if we let  $X \sim Y$  be an abbreviation for  $\Lambda(\mathsf{P}XY)(\mathsf{P}YX)$ , then we can prove a comprehension scheme

$$(\exists z)(\forall y)(zy \sim X),$$

where X is any term in which z does not occur free (but y may occur free):

$$\frac{\frac{1}{[(\lambda y \cdot X)y]}}{X} \operatorname{Eq} \qquad \frac{[X]}{(\lambda y \cdot X)y} \operatorname{Eq} \\
\frac{\overline{P((\lambda y \cdot X)y)X}}{(\lambda y \cdot X)yX} \operatorname{Pi} - 1 \qquad \frac{[X]}{(\lambda y \cdot X)y} \operatorname{Eq} \\
\frac{\overline{P((\lambda y \cdot X)y)X}}{(\lambda y \cdot X)y \sim X} \operatorname{Pi} - 2 \\
\frac{\overline{(\lambda y \cdot (\lambda y \cdot X)y \sim X)y}}{(\lambda y \cdot (\lambda y \cdot X)y \sim X)y} \operatorname{Eq} \\
\frac{\overline{P((\lambda y \cdot (\lambda y \cdot X)y \sim X)y})}{(\lambda z \cdot (\lambda y \cdot X)y \sim X)(\lambda y \cdot X)} \operatorname{Eq} \\
\frac{\overline{P((\lambda x \cdot (\lambda y \cdot x)y \sim X)y})}{\Sigma(\lambda z \cdot (\lambda y \cdot x) \sim X))} \operatorname{Eq} \\
\frac{\overline{P((\lambda x \cdot (\lambda y \cdot x)y \sim X)y})}{\Sigma(\lambda z \cdot (\lambda y \cdot x) \sim X))} \operatorname{Eq} \\
\frac{\overline{P((\lambda x \cdot (\lambda y \cdot x)y \sim X)y})}{\Sigma(\lambda z \cdot (\lambda y \cdot x) \sim X))} \operatorname{Eq} \\
\frac{\overline{P((\lambda x \cdot (\lambda y \cdot x)y \sim X)y})}{\Sigma(\lambda z \cdot (\lambda y \cdot x)y \sim X))} \operatorname{Eq} \\
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\frac{\overline{P((\lambda x \cdot x)y)}}{\Sigma(\lambda x \cdot x)} \operatorname{Eq} \\
\frac{\overline{P((\lambda x \cdot x)y)}}{\Sigma(\lambda x \cdot x)} \operatorname{Eq} \\
\frac{\overline{P((\lambda x$$

If we think in terms of formulas-as-types [9, §§14D, 15D] (also known as the Curry-Howard isomorphism), then the implication fragment of this system is equivalent to a system of typed combinatory logic in which the basic combinators are B, I, C, and K, or, equivalently, to a typed  $\lambda$ -calculus in which  $\lambda x \cdot X$  is only defined when x occurs at most once in X. This system has long been known to be consistent; see [7, Theorem 10C3, p. 364]. Since I can be defined in terms of B, C, and K (as, e.g., CKB), this shows something of the origin of the name of BCK logic.

Note that if we define  $\neg$  to be  $\lambda x \cdot \mathsf{P}x \bot$ , where  $\bot$  is  $\sqcap \mathsf{I}$ , then we get properties of intuitionistic negation.

It is well known that if we add

$$(\mathsf{PW}) \qquad \qquad \mathsf{P}(\mathsf{P}X(\mathsf{P}XY))(\mathsf{P}XY)$$

as a new axiom scheme, then the system becomes inconsistent; see [8, §12B3, pp. 180-1]. Bunder shows in [1] that we can obtain a contradiction by adding the scheme

$$(\mathsf{PS}) \qquad \qquad \mathsf{P}(\mathsf{P}X(\mathsf{P}YZ))(\mathsf{P}(\mathsf{P}XY)(\mathsf{P}XZ))$$

or, surprisingly, either of the schemes

(Pc) 
$$P(P(PXY)X)X$$

or

$$(\neg V)$$
  $VX(\neg X).$ 

To see this, let Y be any arbitrary term, and define

$$X \equiv \mathsf{Y}(\lambda x . \mathsf{P} xY),$$

where Y is a fixed-point operator. Then

$$X =_* \mathsf{P} X Y.$$

Hence, we can prove  $\mathsf{P}XY \vdash Y$ :

$$\frac{\mathsf{P}XY}{Y.} \frac{\frac{\mathsf{P}XY}{X}}{\mathsf{Pe}} \mathsf{Eq}$$

This means that if we can prove that PXY follows from a scheme, any arbitrary term can be proved from that scheme and the system is inconsistent.

The proof that  $\mathsf{P}XY$  follows from ( $\mathsf{PS}$ ) is



The proof that it follows from (Pc) is



The proof that it follows from  $(\neg V)$  is



This proves

**Theorem 6** Theorem 1 cannot be extended to the classical version of this variant of BCK logic.

# A The System for Equality

**Definition 10** The terms of system  $Q^T$  a system of combinatory logic or  $\lambda$ -calculus with an atomic constant Q which behaves like a variable with respect to reduction and conversion (i.e., is a non-redex constant in the sense of [9] or a C-indeterminate in the sense of [8]). The provability relation is defined by the following rules:

Eq 
$$\frac{X}{Y}$$
, Condition:  $Y =_* X$ .  
Qe  $\frac{QXY}{ZY}$ , Qi  $\begin{bmatrix} xX \end{bmatrix}$   
 $\frac{xY}{QXY}$ ,

where, in Qi, the variable x does not occur (free) in X, Y, or in any undischarged assumption. This system is called the T-formulation of Q in [8, §12C1].

For this system we define a  $Q\mbox{-}\mathrm{reduction}$  step as one which takes a deduction of the form

(21)  
$$\begin{aligned} & \begin{bmatrix} 1 \\ [xX] \\ D_1(x) \\ \\ \frac{XY}{QXY} Qi - 1 \\ \frac{XY}{QUV} Eq \quad D_2 \\ UV \quad ZU \\ D_3 \\ \end{aligned} Qe$$

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(22)  
$$\begin{array}{c}
D_{2} \\
ZU \\
ZX \\
D_{1}(Z) \\
\frac{ZY}{ZV} \\
D_{3}.
\end{array}$$

Note that since Q is a non-redex constant, if  $QXY =_* QUV$  then  $X =_* U$  and  $Y =_* V$ . The term QXY in (21) is called the *cut formula* of the reduction step.

Now ZU in (22) cannot be a cut formula, because if ZX were a major premise for an inference by Qe in  $D_1(Z)$ , then xX would be such a major premise in  $D_1(x)$  in (21), and this is impossible since x is a variable and Q is a non-redex constant. For the same reason, ZY in (22) is not a cut formula. (This depends on the fact that if a and b are either variables or nonredex constants and if  $aX_1X_2...X_n =_* bY_1Y_2...Y_m$ , then  $a =_* b, n = m$ , and  $X_i =_* Y_i$  for i = 1, ..., m. This follows from [8, p. 143, property C3] and the Church-Rosser Theorem.) It follows from this that if a Q-reduction step is applied to a deduction of the form (21) in which there are no cut formulas in  $D_1(x)$  or in  $D_2$ , then the only cut formulas in (22) are those of  $D_3$ , and the reduction step has reduced the number of cut formulas. Thus, by induction on the number of cut formulas in a deduction, we get the following theorem:

## **Theorem 7** Every deduction in $Q^T$ can be normalized.

It is not hard to see that this kind of normalization corresponds to the normalization of Prawitz [16] for classical logic, and that normal deductions have a structure similar to the normal deductions of Prawitz for the classical predicate calculus. In particular, if a *branch* is taken to be an initial part of a thread which ends in the first minor (right) premise for an inference by Qe if there is one (see [16, p. 41]), then in any branch of a normal deduction, all the inferences by Qe precede all those by Qi. From this it is easy to prove that if  $\vdash QXY$  (with no undischarged assumptions), then  $X =_* Y$ .

A Gentzen L-formulation for this system is given in [8, §12C2, pp. 187ff].

to

**Definition 11** The system  $Q^L$  has the same terms as system  $Q^T$  of Definition 10. It has the following axioms and rules:

Axiom Scheme:  $X \Vdash X$ , for each term X.

Rules:

\*C 
$$\frac{M \Vdash N}{M' \Vdash N}$$
, C\*  $\frac{M \Vdash N}{M \Vdash N'}$ ,

\*W 
$$\frac{M, X, X \Vdash N}{M, X \Vdash N,}$$
 W\*  $\frac{M, \Vdash X, X, L}{M \Vdash X, L,}$ 

\*K 
$$\frac{M \Vdash N}{M, X \vDash N,}$$
 K\*  $\frac{M \Vdash L}{M \vDash X, L,}$ 

\*Exp 
$$\frac{M, Y \Vdash N}{M, X \vDash N}$$
 Exp\*  $\frac{M \Vdash Y, L}{M \vDash X, L}$ 

\*Q 
$$\frac{M \Vdash ZX, L \quad M, ZY \Vdash N, L}{M, QXY \Vdash N, L} \quad Q^* \qquad \frac{M, xX \Vdash xY, L}{M \Vdash QXY, L}$$
Cut 
$$\frac{M, X \Vdash N \qquad M \Vdash X, L}{M \Vdash N, L}$$

The general conventions are that M, N, and L are sequences of terms, that in rules  $C^*$ , M' and N' are permutations of M and N respectively, that in rules  $Exp^*$ ,  $X \triangleright Y$ , the variable x in  $\mathbb{Q}^*$  does not occur (free) in M, L, X, or Y, and, if the system is to be singular (on the right) then N has only one term in it and L is void.

By the standard property of normal natural deduction derivations noted above, it is possible to use Theorem 7 to prove cut elimination for the singular version of this L-system in much the same way that Prawitz proves similar results in [16, Appendix A, §3]. The fact that cut elimination for this L-system can be proved so easily by proving normalization for the corresponding natural deduction system gave me a strong incentive to look for a similar proof for  $\mathcal{F}_{33}$ . Note, by the way, that in this case the singular and multiple formulations are equivalent; see [8, Theorem 12C12 and Corollary 12C12.1, pp. 206-207].

**Remark** An alternative formulation of the L-system, in which rule  $Q^*$  is replaced by the axiom scheme  $\Vdash QXX$  (for any term X), is presented in [8, §12C3, pp. 195ff]. This system is closely related to what is called the A-formulation of Q in the introduction to [8, §12C, p. 186], in which Rule Qi is replaced by the same axiom scheme. The proof reductions take

$$\frac{\begin{array}{c} \text{Axiom} \\ \hline \mathbf{Q}XX \\ \hline \overline{\mathbf{Q}UV} \xrightarrow{\mathbf{Eq}} & \begin{array}{c} D_1 \\ ZU \\ \hline \hline \\ \hline \\ D_2 \end{array} \\ \begin{array}{c} \mathbf{Q} \\ \mathbf{Q} \\$$

to

$$\begin{array}{c}
D_1 \\
ZU \\
\overline{ZV} \\
D_2;
\end{array}$$
Eq

this works because  $QUV =_* QXX$  implies  $U =_* X =_* V$ . Furthermore, each reduction step shortens the deduction. Hence, Theorem 7 holds for the A-formulation, and we can use this fact to obtain a proof of the cut-elimination theorem for the corresponding L-formulation.

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