Modeling and Temporal Evolution of a Family of Curves

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Abstract

The vector spline approximation framework is applied to the problem of spatial and temporal modeling of deformable structures in an image sequence. We show that this vector field framework is well adapted to the mathematical modeling of a family of mutually interacting curves associated to deformable structures. The tools presented in this study are applied to satellite oceanographic images, where deformable structures are vortices and temperature fronts. The curves are obtained as integral paths (or orbits) of a suitable vector field described in this work. Motion analysis is obtained with the optical flow and the mathematical notion of the transformation of a vector field by a diffeomorphism.

1 Introduction

There are many situations where deformable structures interact in an image sequence. Whenever these structures are represented with the help of deformable curves, this results in a family of mutually interacting deformable curves. To take an exemple that will serve as our main application, satellite image sequences measuring the Sea Surface Temperature (SST), often display deformable structures such as vortices, temperature fronts, evolving in a complex manner in time. Our problem is then to provide a mathematical tool for modelling the temporal evolution of a family of curves. We propose to set up a vector field approximation framework to provide both spatial and temporal modelling of such curves. More precisely, we want to produce a vector field from image data in such a way that the orbits of the field correspond to the boundaries of general structures. This is applied to examples such as vortices or temperature fronts but it can also be very helpful for modeling and motion tracking of structures undergoing large deformation. Indeed, classical grid deformation and surface template techniques [3] may fail to produce a satisfactory approximation of a structure like a moving vortex for which "small deformation" hypothesis does not apply. Trying to approximate the boundary of such a structure as the orbit of a vector field has many advantages:

- a vector field is a mathematical entity encoding the natural relationships between its orbits. This can be particularily helpful for the temporal evolution of temperature fronts, which interact eachother.
- The data that is used for the initialization of the vector field may come from edge detection algorithms. Such data is often made of linear and disconnected agregations of pixels. The vector field is a single and neat mathematical entity allowing the modeling through well defined mathematical curves of complex and not always well defined boundaries. This is of particular importance. Indeed, the data produced by classical edge detection algorithms does not take the form of a well defined mathematical curve. Using such data as the initialization for computing a vector field is not a serious handicap because the orbits are computed from the field by numerical integration, so the discrepancies of these initial data does not infer considerably on the quality of the orbits.
- Vector field modeling is particularily well suited in the analysis of motion when information derived for motion computation takes also the form of a vector field, e.g.: the optical flow vector field.

If a spatial structure is modeled with the help of a vector field, we will show in this study that the mathematical notion of the transformation of a vector field under a diffeomorphism [7] can be very useful for temporal motion analysis, specially if motion information is given in the form of another vector field, as it is the case with the optical flow. Indeed the spatial model and the temporal evolution model are both given in the form of vector fields.

In this work we are interested in a particular class of vector fields: the div-curl spline vector fields. Spline vector fields are gaining increasing attention in image processing [6, 2, 5]. They can be used for solving approximation problems and deformation estimation. In this study we use this formalism in a vector field approximation framework: this type of vector field approximation has already been used in the context of wind estimation in meteorology [2]. The numerical schemes involved in are easy to implement and robust. We also show that, used in conjunction with the optical flow, it can be a very valuable tool for motion analysis.

From these considerations, we want to attempt the mathematical modeling of deformable structures' boundaries as orbits of vector fields. In this study we come up with methods for the spatial modeling of deformable structures with vector spline tools, and the use of such a modeling for the temporal analysis of motion.

This study is organized as follows. In section 2 the general framework of spline vector fields is presented. Section 3 is devoted to the solution method in dimension 2 and first result is presented on synthetic data. Results are described in section 4, and in section 5 we address the problem of temporal evolution. The paper ends with conclusion and perspectives.

2 The optimization problem in the general case

In the more general setting, the div-curl vector spline approximation problem takes the following form: in an n-dimensional space $I\!R^n$, let be given N data points $(X_i)_{1\leq i\leq N}$ and N vectors $(\overrightarrow{V_i})_{1\leq i\leq N}$. These vectors may correspond to measured displacements, or they can be measured velocities. In this study they will correspond to yet another interpretation: in an image sequence, we will determine contour points at specific locations $(X_i)_{1\leq i\leq N}$ and the vectors $(\overrightarrow{V_i})_{1\leq i\leq N}$ will correspond to tangent information at these points. We seek a global vector field V approximating the vectors $(\overrightarrow{V_i})_{1\leq i\leq N}$. The vector spline method corresponds to finding a C^m vector field V, which is a C^m map

$$V \;:\; I\!R^n \longrightarrow I\!R^n \tag{1}$$

in such a way that the following energy is minimized:

$$\begin{aligned}\mathcal{E} \;=\; & \frac{1}{N}\sum_{i=1}^{N} \|\overrightarrow{V_i} - V(X_i)\|^2 + \\ & \int_{I\!R^n} \alpha\|\nabla^m(\text{div } V)\|^2 + (1-\alpha)\|\nabla^m(\text{curl } V)\|^2\end{aligned} \tag{2}$$

i.e. the vector V approximates the data given by X_i, $\overrightarrow{V_i}$, and satisfies regularity conditions about the variations of its divergence and curl. The regularization term about the variations of div and curl discards pathological behaviours of the solution V, and can be compared to the strain regularizing term in the classical spline curve approximation theory [1]. Parameter α is used to weight the div and curl reglarizing terms of energy $\mathcal{E}$. The general solution of equation 2 can be written using the kernel Φ of the following PDE:

$$\Delta^m[\alpha(\nabla^t\cdot\nabla) + (1-\alpha)(\text{ curl }^t\cdot\text{ curl })]\Phi = \delta I_n \tag{3}$$

(m: order of continuity, n: dimension of the space, α : weighing parameter; in dimension 2 curl $= (-\partial_y\ ,\ \partial_x)$, δI_n is an n-dimensional Dirac impulse at the origin and the superscrip t denotes transposition). The general solution V of the minimization problem 2 takes the form:

$$V(X) = \sum_{i=1}^{N} a_i\Phi(X - X_i) + p_m(X) \tag{4}$$

where $p_m(X)$ is a polynomial with degree at most m.

Once the vector field V is given, its orbits are spatial curves $\gamma(s)$, parameterized by s. They are uniquely defined by an initial position M_0 and the vector field V in the following way:

$$\begin{aligned}\gamma(0) &= M_0 \\ \frac{d\gamma}{ds}(s) &= V(\gamma(s))\end{aligned} \tag{5}$$

3 The optimization problem in 2D

We use the vector spline framework with a first order condition ($m = 1$) and, of course the dimension of the problem is $n = 2$. In this case, the solution of the vector spline approximation is given by the following result [2]:

Theorem 3.1 *Let*

$$\begin{aligned}g(X) = \; & \sum_{i=1}^{N} a_i\Big(\frac{1}{\alpha}\frac{\partial^2}{\partial x^2}K(X-X_i) + \\ & \frac{1}{(1-\alpha)}\frac{\partial^2}{\partial y^2}K(X-X_i)\Big) + \\ & \sum_{i=1}^{N} b_i\Big(\frac{1}{\alpha} - \frac{1}{(1-\alpha)}\Big)\frac{\partial^2}{\partial x\partial y}K(X-X_i)\end{aligned}$$

with a_i and $b_i \in I\!R$ and

$$K(X) = -\frac{1}{2^7\pi}\|X\|^4 \log\|X\| \,.$$

The solution

$$V_\alpha = (u_\alpha, v_\alpha)$$

of the vector spline minimization problem admits a unique expression depending of $X = (x, y)$:

$$\begin{aligned}u_\alpha(X) = &\; g(X) + p(X) \\ v_\alpha(X) = &\; g(X) + q(X)\,,\end{aligned}$$

where $p(X)$ and $q(X)$ are degree 1 polynomials. The coefficients are obtained by solving a linear system.

Let us begin to investigate the results of the minimization process on synthetic data. In figure 1 is shown an instance of the N vectors $\overrightarrow{V_i}$, disposed along a rectangular vortex-like structure. A rectangular shape is symmetric and anisotropic. In figure 2 the result of the approximation process using $\alpha = 0.1$ is shown. Note that the shape of the flow records appropriately the anisotropy of the initial rectangular shape. If the initial vectors were to be lying on a circle, the resulting vector field would be circular. The result seems satisfactory, as it correctly interpolates the original data and generates a singular point (a zero of the vector field) at the center of the picture.

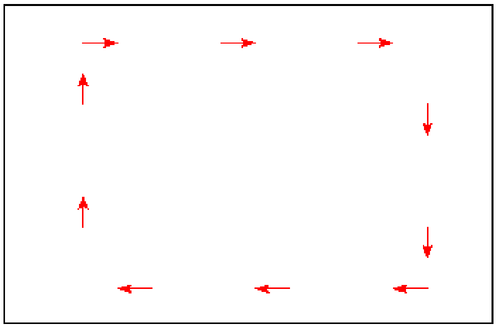

Figure 1: Initial value of N vectors data, disposed along a closed rectangular shape.

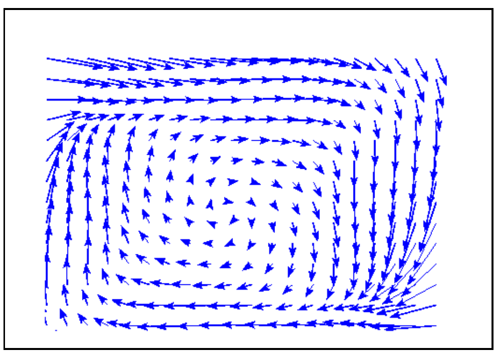

Figure 2: Computation of the interpolating vector field using the initial data of figure 1. Parameter $\alpha = 0.1$.

4 Results

We use a SST (Sea Surface Temperature) NOAA AVHRR image sequence of the sea. It is a sequence of 31 images, each image being of size 146×120 pixels. The sequence displays, as shown in figure 3 a vortex, or gyre, turning clockwise, and also temperature fronts. Our goal is to be able to model the boundary of the gyre, and follow a particule of water along its trajectory path. We first use contour extraction (local extrema of the gradient norm and hysteresis thresholding) [8] to

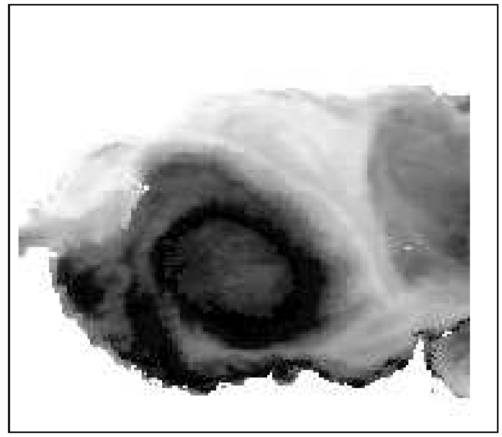

Figure 3: NOAA-AVHRR Sea Surface Temperature over the Alboran Sea. Image acquired on 24 august 1996. False colors.

generate a contour image. We show, in figure 4, the result of contour extraction on image 3. It is a set of linear structures corresponding to local extrema of the gradient norm. From this data we want to generate an initial set of vectors on which the vector spline approximation is computed, in such a way that the linear structures will become the orbits of this spline vector field. Denoting $\mathcal{S}$

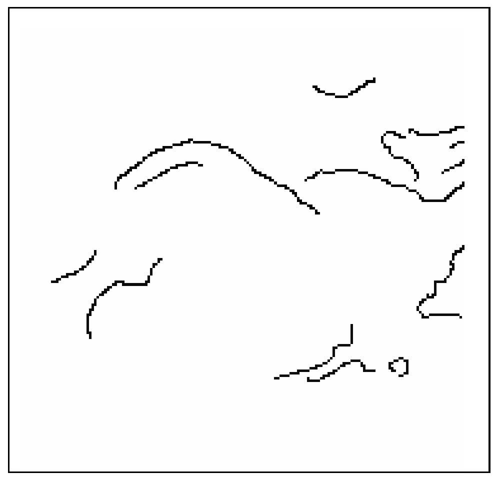

Figure 4: Result of contour extraction.

the set of pixels belonging to the contours, we generate the direction of a tangent vector at a pixel $p \in \mathcal{S}$, by taking the averaged sum of the directions of the segments linking pairs of connected pixels in the neighborhood of p. Hence the tangent direction at pixel p using the set $\mathcal{M}$ of N neighbours is given by θ_p^N:

$$\theta_p^N = \frac{1}{2N} \sum_{j \in \mathcal{M}} \operatorname{atan}\left(\frac{y_{j+1} - y_j}{x_{j+1} - x_j}\right) .$$

In this equation $j+1$ refers to the pixel next to j. In that way, each vector $\overrightarrow{V_i}$ based at p is given the orientation of the computed angle θ_p^N.

But there is a problem at this point: these sets of pixels belonging to $\mathcal{S}$ are not oriented, and the orientation of tangent vectors plays of course a crucial role in the geometry of the approximating field. To overcome this problem, we use a variation of the classical optical flow based on volume conservation, and described in [4]. This kind of optical flow is well adapted to the study of deformable structures. The equation of this kind of optical flow is:

$$\nabla I \cdot W + \frac{\partial I}{\partial t} + I \cdot \text{div}\ (W) = 0 \qquad (6)$$

where I denotes the intensity function of the image and W the optical flow field. Each one of the two possible tangent vectors is projected over the motion vector at that pixel and we keep the vector that gives the highest result (in norm). Each vector $\overrightarrow{V_i}$ is divided by its norm, so we take unitary initial vectors. The result of such a computation is shown in figure 5: at specific location of some contour points, the correct direction of the tangent is computed from the optical flow. We now apply the

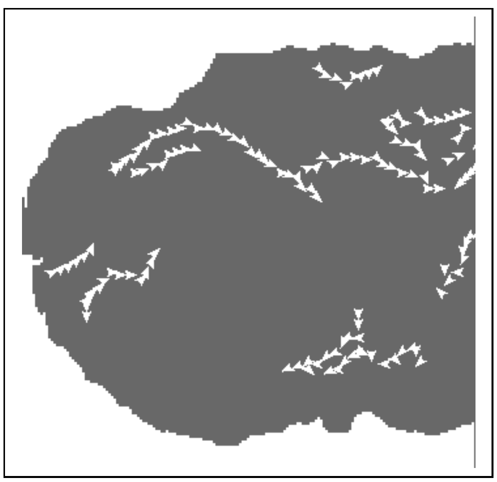

Figure 5: Unitary vectors tangent at contour points. The correct direction comes from the optical flow.

method described in the previous section to the inital data described here. We show, in figure 6 the result of the spline vector field approximation method applied with the result of figure 4. The value of the parameter α of equation 2 is $\alpha = 0.1$. Increasing the value of α would not change the shape of the vector field in regions where it has a noticeable vortex structure. In figure 7 the orbit of a pixel is displayed, showing that the resulting field correspond to the structure present in the image. Lastly, we show in figure 8 the orbit of another pixel, computed on a different frame in the image sequence. In the next section, we addres the problem of motion analysis.

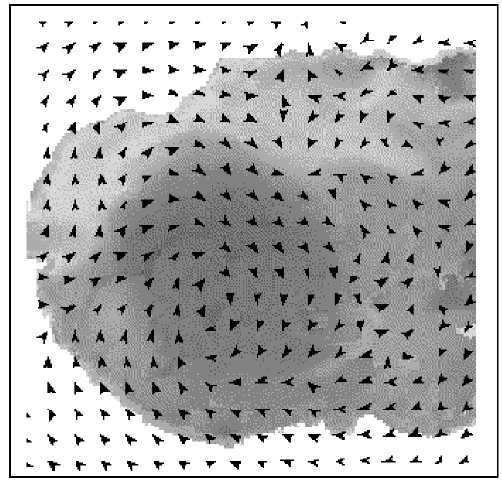

Figure 6: Result of the spatial vector field computation.

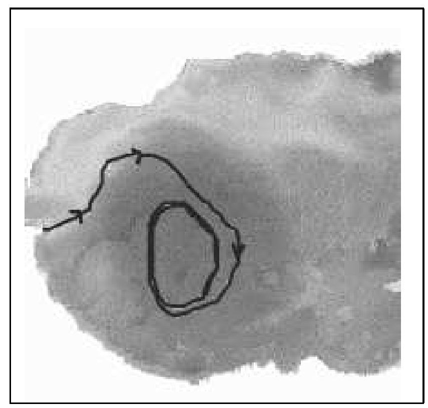

Figure 7: Computation of the orbit of a pixel.

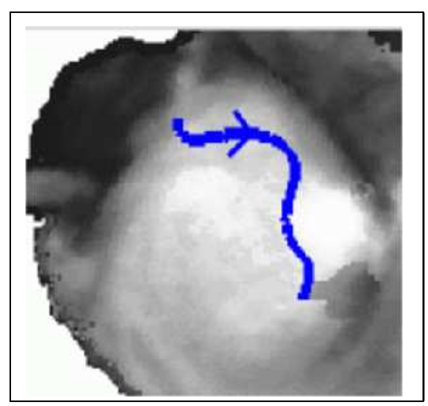

Figure 8: Computation of the orbit of a pixel on another frame in the sequence.

5 Motion analysis

In this section, we use the vector field modeling tool described in the previous sections to perform motion analysis of deformable structures in an image sequence. More precisely, we want to characterize the temporal evolution of vector field V's orbits under the motion encoded by the optical flow. For that matter, we are going to compute a new vector field, denoted X, such that the orbits of X correspond precisely to the temporal evolution of the orbits of V. The temporal evolution is defined by the orbits of the optical flow, so we are using the mathematical notion of the transformation of a vector field by a diffeomorphism [7]. The diffeomorphism, which represents motion, is defined by the optical flow vector field. For clarity of notation, we denote by $\gamma(M_0, s)$ the orbit of the vector field V that goes through M_0 at $s = 0$. The optical flow field W defines orbits, which are temporal curves $\varphi(M_0, t)$ such that

$$\begin{aligned} \varphi(M_0, 0) &= M_0 \\ \frac{d\varphi}{dt}(t) &= W(\varphi(M_0, t)) \end{aligned} \tag{7}$$

The orbits of the optical flow field W define, for each time value t, a diffeomorphism of the image plane, denoted F_t, by the following equation:

$$F_t(M) = \varphi(M, t) \tag{8}$$

In other words, to obtain the image of a pixel M under F_t, one computes the orbit of the optical flow field W going through M, at $t = 0$, and compute the image of this orbit corresponding to parameter value t. The problem of motion modeling can be addressed in the following way: one seeks a spatial vector field X, whose orbits are denoted $\psi(M, s)$ such that:

$$F_t(\gamma(M_0, s)) = \psi(F_t(M_0), s) \tag{9}$$

Hence, if a deformable structure is representec by a spatial orbit of the vector field V, its deformed image at time t is given by the corresponding orbit of the spatial field X. The computation of the orbits of the field X is then equivalent to the computation of the deformable structure under the apparent motion. Such a vector field X is the image of the vector field V under the diffeomorphism F_t. To compute its expression, note that by using differentiation w.r.t. spatial parameter s:

$$\frac{d\psi(F_t(M), s)}{ds} = X(\psi(F_t(M, s)) \tag{10}$$

and that

$$\frac{d\psi(F_t(\gamma(M_0, s)))}{ds} = \nabla F_t(\gamma(M_0, s)) \cdot V(\gamma(M_0, s)) \tag{11}$$

so that the vector field X is given by the following equation:

$$X(M) = \nabla F_t({F_t}^{-1}(M)) \cdot V({F_t}^{-1}(M)) \tag{12}$$

where ∇F_t refers to the spatial derivative of diffeomorphism F_t. Equation 12 involves the computation of the inverse of diffeomorphism F_t. But a general property of vector fields make this computation easy. One has:

$${F_t}^{-1}(M) = \varphi(M, -t) \tag{13}$$

Given an image sequence, the previous formalism allows to perform motion modelling in the following way:

- Perform the vector spline approximation on each image of the sequence as described in the previous sections.
- Compute the optical flow for each image of the sequence.
- Compute the image of the spatial field under the diffeomorphism given by the optical flow.

Such a computation results in a smooth temporal evolution of deformable structures represented by the vector spline approximation.

6 Conclusion

Div-curl vector splines provide a very general tool for the spatial and temporal modeling of deformable structures in image analysis. In this work they are used to provide the spatial modeling of a family of curves and their temporal evolution. The spatial modeling takes advantage of the regularizing properties of spline vector fields. Constraints imposed on the variations of the curl translate into very useful properties for the resulting spline vector field: such a field does not "curl" too much, and its integral paths can serve as the basis for the spatial modeling of complex structures such as vortices or temperature fronts. The vector field encodes in a single mathematical entity the complex behavior of its integral paths. Such a modeling is well suited to large deformation. Moreover, this vector field modeling takes full advantage of the optical flow, which is another vector field, and is particularily well suited for motion. By combining the spatial vector field approximation and the temporal optical flow field through the unified notion of the transformed of a vector field by a diffeomorphism, it is possible to provide a coherent modeling tool for a family of deformable curves.

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References

[1] L. Alt. Parametrization for data approximation. In *Curves and Surfaces, P.J. Laurent and A. Le Méhauté and L.L. Schumaker (eds.)*, pages 1–4, Academic Press, Boston, 1991.

[2] L. Amodei and M.N. Benbourhim. A vector spline approximation. *Journal of Approximation Theory*, 67:51–79, 1991.

[3] S. Basu, N. Olivier, and A. Pentland. 3d modeling and tracking of human lip motions. In *ICCV'98*, pages 337–343, Bombay, January 1998.

[4] J.P. Berroir and D. Béréziat. Use of image processing and remote sensing techniques to improve input data quality and collection. In *International Conference on Air Pollution Modelling and Simulation APMS'98*. Elsevier, 1998.

[5] F. L. Bookstein. Principal warps: Thin-plate splines and the decomposition of deformations. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(6):567–585, 1989.

[6] F. Chen and D. Suter. Image coordinate transformation based on div-curl vector splines. In *ICPR'98*, pages 518–520, Brisbane, August 1998.

[7] R. W. R. Darling. *Differential forms and Connections*. Cambridge University Press, 1994.

[8] R. Deriche. Optimal edge detection using recursive filtering. *International Journal of Computer Vision*, pages 167–187, 1987.