The Relativistic Composite-Velocity Reciprocity Principle

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ABSTRACT *Gyrogroup theory [A.A. Ungar, Thomas precession: its underlying gyrogroup axioms and their use in hyperbolic geometry and relativistic physics, Found. Phys. 27 (1997), pp. 881-951] enables the study of the algebra of Einstein's addition to be guided by analogies shared with the algebra of vector addition. The capability of gyrogroup theory to capture analogies is demonstrated in this article by exposing the Relativistic Composite-Velocity Reciprocity Principle. The breakdown of commutativity in the Einstein velocity addition* $\oplus$ *of relativistically admissible velocities seemingly gives rise to a corresponding breakdown of the relativistic composite-velocity reciprocity principle, since seemingly (i) on one hand the velocity reciprocal to the composite velocity* $\mathbf{u}\oplus\mathbf{v}$ *is* $-(\mathbf{u}\oplus\mathbf{v})$ *and (ii) on the other hand it is* $(-\mathbf{v})\oplus(-\mathbf{u})$. *But, (iii)* $-(\mathbf{u}\oplus\mathbf{v}) \neq (-\mathbf{v})\oplus(-\mathbf{u})$. *We remove the confusion in (i), (ii) and (iii) by employing the gyrocommutative gyrogroup structure of Einstein's addition and, subsequently, present the relativistic composite-velocity reciprocity principle with the Thomas rotation that it involves.*

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1 INTRODUCTION

Einstein's addition of relativistically admissible velocities, $\oplus$, is neither commutative nor associative. It seems, on first glance, that the breakdown of commutativity in Einstein's addition results in a violation of the relativistic composite-velocity reciprocity principle that is explained below. We will see, however, how the Thomas precession comes to the rescue.

The relativistic velocity reciprocity principle asserts that

(1) if the velocity of an inertial frame of reference Σ'' relative to another inertial frame of reference Σ' is $\mathbf{v}$ then, reciprocally,

(2) the velocity of Σ' relative to Σ'' is $-\mathbf{v}$, Fig 1.1.

An interesting study of the relativistic velocity reciprocity principle is found in [1]. For composite velocities $\mathbf{u}\oplus\mathbf{v}$, Fig. 1.1, the relativistic velocity principle seemingly implies that

(1) if the velocity of Σ'' relative to Σ is $\mathbf{u}\oplus\mathbf{v}$ then, reciprocally,

(2) the velocity of Σ relative to Σ'' is simultaneously

$$-(\mathbf{u}\oplus\mathbf{v}) \tag{1.1}$$

and

$$(-\mathbf{v})\oplus(-\mathbf{u}) = -(\mathbf{v}\oplus\mathbf{u}) \tag{1.2}$$

Contradictingly, however, in general

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u} \tag{1.3}$$

The resulting composite-velocity reciprocity paradox is closely related to the Mocanu paradox according to which the velocity of Σ'' relative to Σ in Fig. 1.1 is simultaneously $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ while, in general, $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ are distinct. The Mocanu paradox [8] raises the problem: which one is the correct velocity of Σ'' relative to Σ in Fig. 1.1? Is it $\mathbf{u}\oplus\mathbf{v}$ or $\mathbf{v}\oplus\mathbf{u}$?

The obscured Thomas precession soared into prominence in 1988 [7] following the discovery of the gyroassociative law of Einstein's addition, to which it gives rise. The gyroassociative law of Einstein's addition, in turn, gives rise to the relativistic composite-velocity reciprocity principle that we present in Section 4. In Section 2 we describe the Thomas precession generated by two relativistically admissible velocities, and indicate the central role that it plays in the algebra of Einstein's addition, paying special attention to the gyrocommutative and the gyroassociative laws, (2.5)-(2.7). In Section 3 we are led to the notion of the gyrogroup, which captures analogies shared by Einstein's addition and vector addition.

Since the gyroassociative law of Einstein's addition is unheard of in mainstream literature on relativity physics, the problem of composite-velocity reciprocity in the study of relativistic velocities causes the confusion described in (1.1)-(1.3). We therefore employ the gyroassociative law of Einstein's addition in Section 4 for the exposition of the truly, paradox free, relativistic composite-velocity reciprocity principle, (4.5)-(4.8), the presentation of which is the goal of this article.

2 THE THOMAS PRECESSION

Relativistically admissible velocities are elements of the open ball $\mathbb{R}^3_c$ of the Euclidean three-space $\mathbb{R}^3$,

$$\mathbb{R}^3_c = \left\{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\right\} \tag{2.1}$$

with radius c, that represents the vacuum speed of light. The Einstein velocity addition is a binary operation $\oplus$ in the ball $\mathbb{R}^3_c$ given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u}+\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\times(\mathbf{u}\times\mathbf{v}))\right\} \tag{2.2}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$. Here $\cdot$ and $\times$ represent the usual dot and cross products in $\mathbb{R}^3$, and $\gamma_{\mathbf{u}}$ is the *Lorentz factor* given by

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}} \tag{2.3}$$

The Thomas precession is illustrated in Fig. 1.1. Let Σ, Σ' and Σ'' be three inertial frames of reference which were coincident at time $t = 0$. The three inertial frames Σ, Σ' and Σ'', equipped with spacetime coordinates (only two space coordinates are shown in the Figure for clarity), are in relative motion with relative velocities $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$. Observers at rest relative to Σ (relative to Σ') agree with observers at rest relative to Σ' (relative to Σ'') that their space coordinates are parallel. Yet, observers at rest relative to Σ agree with observers at rest relative to Σ'' that their space coordinates are in relative rotation. This relative rotation generated by $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, which does not exist in classical mechanics, is the relativistic rotation known as the Thomas precession, denoted $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$. The rotation angle ϵ of the Thomas precession $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ generated by $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3_c$ is shown in Figure 1.1.

Paradoxically, the velocity of Σ'' relative to Σ is described simultaneously by the two vectors $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ of $\mathbb{R}^3_c$, which are distinct whenever $\mathbf{u}$ and $\mathbf{v}$ are non-parallel. This paradox, called the Mocanu paradox [8], is resolved below.

Following the description of the Thomas precession in Fig. 1.1, if

- Σ'' moves relative to Σ' with velocity $\mathbf{v} \in \mathbb{R}^3_c$ *without* rotation.

and

- Σ' moves relative to Σ with velocity $\mathbf{u} \in \mathbb{R}^3_c$ *without* rotation,

then

- Σ'' moves relative to Σ'
  - with velocity $\mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^3_c$ *preceded* by the Thomas rotation $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$

or, equivalently,

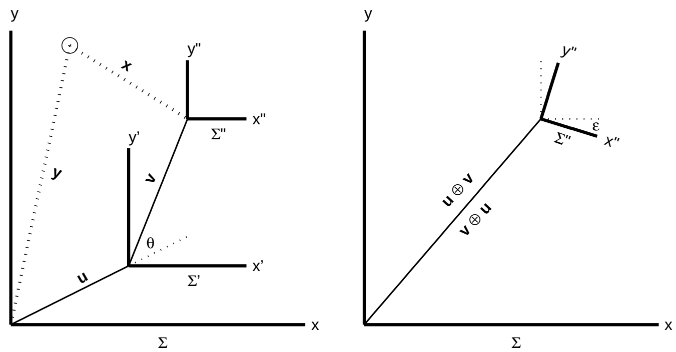

Figure 1.1. The Thomas precession (also known as the Thomas rotation or the Thomas gyration) $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ generated by two relativistically admissible velocities $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$. The rotation angle ϵ of the precession is a function of $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and θ. The angles θ and ε have opposite signs. Paradoxically, "the" velocity of frame Σ'' relative to frame Σ is both $\mathbf{u}\oplus\mathbf{v}$ and $\mathbf{v}\oplus\mathbf{u}$ which are, in general, distinct due to the presence of the Thomas precession that they generate. For the discussion of velocity reciprocity in Section 4 an object is shown, moving uniformly with velocity $\mathbf{y}$ (velocity $\mathbf{x}$) relative to Σ (to Σ'').

■ – with velocity $\mathbf{v}\oplus\mathbf{u}\in\mathbb{R}^3_c$ *followed* by the Thomas rotation $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$

thus resolving the paradox.

Fig. 1.1 illustrates the noncommutativity of Einstein's velocity addition law (2.2). This noncommutativity, as counterintuitive as it may seem, is allowed in special relativity since relative velocities and relative orientations between frames are coupled. The paradoxical question of whether the "correct" velocity of Σ'' relative to Σ in Fig. 1.1 is given by $\mathbf{u}\oplus\mathbf{v}$ or by $\mathbf{v}\oplus\mathbf{u}$ makes no sense in special relativity because relations between frames cannot be determined by relative velocities alone. They are determined by both relative velocities and relative orientations which, unlike their Galilean counterparts, are woven together and cannot be decoupled unless the Thomas precession is invoked [8].

Thomas precession $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ has rich mathematical structure [11]; it possesses, for instance, the following useful algebraic rules,

$$\begin{aligned}\mathrm{gyr}[\mathbf{u},\mathbf{v}] &= (\mathrm{gyr}[\mathbf{v},\mathbf{u}])^{-1}\\ \mathrm{gyr}[-\mathbf{u},-\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}]\\ \mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] = \mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}]\end{aligned}\tag{2.4}$$

The application of any Thomas precession $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, to any $\mathbf{x}\in\mathbb{R}^3_c$ is expressible in terms of Einstein's addition by the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} = -(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{x})\}\tag{2.5}$$

and it "repairs" the breakdown of commutativity and associativity in Einstein's addition, giving rise to the gyrocommutative law

$$\mathbf{u}\oplus\mathbf{v} = \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) \tag{2.6}$$

and to the gyroassociative law (left and right)

$$\begin{aligned} \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{x}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} \\ (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{x} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{x}) \end{aligned} \tag{2.7}$$

of Einstein's addition.

The gyrocommutative law (2.6), though not recognized as such, is well known and appeared in early literature on special relativity [6]. The Thomas precession $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ in (2.6) can be characterized as the unique rotation of R^3 about its origin which takes the vector $\mathbf{v}\oplus\mathbf{u} \in \mathbb{R}^3_c$ to the vector $\mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^3_c$ by a rotation about an axis perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$. In contrast, the gyroassociative law (2.7) is a relatively recent discovery made in 1988 [7], signaling the birth of gyrogroup theory. In that theory, the abstract Thomas precession is called *Thomas gyration*, suggesting the prefix *gyro* that we use to emphasize analogies [11].

A groupoid is a non-empty space with a binary operation. The groupoid $(\mathbb{R}^3_c, \oplus)$, called the Einstein gyrogroup, thus shares remarkable analogies with the group $(\mathbb{R}^3, +)$ of all Newtonian velocities. In general, Gyrogroups are grouplike objects modeled on Einstein's addition with its Thomas precession. Guided by analogies with group theory, gyrogroups are defined in Section 3 below, and their theory is studied in [11] and [12]. The Einstein gyrocommutative gyrogroup $(\mathbb{R}^3_c, \oplus)$ is not an isolated specimen; both gyrocommutative and non-gyrocommutative gyrogroups abound in group theory, as demonstrated by Foguel and Ungar in [2] and [3]. Some gyrocommutative gyrogroups can be extended to gyrovector spaces, and the latter form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry [11][12][13][14][15].

3 THE ABSTRACT GYROGROUP

As indicated in (2.4)-(2.7), Einstein's addition has rich algebraic structure and, therefore, merits extension by abstraction. Motivated by the definition of a group, the key features of Einstein's addition are abstracted and placed in the following formal definition of a gyrogroup.

Definition 3.1 (Gyrogroups). *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $0\oplus a = a$ *Left Identity*

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each a in G there is an element $\ominus a$ in G, called a left inverse of a, satisfying

(G2) $\ominus a \oplus a = 0$ *Left Inverse*

Moreover, for any $a, b, z \in G$ there exists a unique element $\mathrm{gyr}[a, b]z \in G$ such that

(G3) $a \oplus (b \oplus z) = (a \oplus b) \oplus \mathrm{gyr}[a, b]z$ *Left Gyroassociative Law*

If $gyr[a, b]$ denotes the map $gyr[a, b] : G \to G$ given by $z \mapsto gyr[a, b]z$ then

(G4) $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$ *Gyroautomorphism*

and $gyr[a, b]$ is called the Thomas gyration, or the gyroautomorphism of G, generated by $a, b \in G$. The operation $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the gyrooperation of G. Finally, the gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ satisfies

(G5) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$ *Left Loop Property*

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups. The definition of gyrocommutativity in gyrogroups follows.

Definition 3.2 (Gyrocommutative Gyrogroups). *The gyrogroup $(G, \oplus)$ is gyrocommutative if for all $a, b \in G$*

(G6) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$ *Gyrocommutative Law*

Theorem 3.3 ([11]). *A gyrogroup $(G, \oplus)$ is gyrocommutative if and only if*

$$\ominus(a \oplus b) = \ominus a \ominus b$$

Instructively, an equivalent definition of a gyrogroups is presented in Theorem 3.4 below.

Theorem 3.4 (Gyrogroups: A Second, Equivalent Definition). *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms and properties. In G there exists a unique element, 0, called the identity, satisfying*

(g1) $0 \oplus a = a \oplus 0 = a$ *Identity*

for all $a \in G$*. For each* a *in* G *there exists a unique inverse* $\ominus a$ *in* G*, satisfying*

(g2) $\ominus a \oplus a = a \ominus a = 0$ *Inverse*

where we use the notation $a \ominus b = a \oplus (\ominus b)$*,* $a, b \in G$*. Moreover, if for any* $a, b \in G$ *the self-map* $\mathrm{gyr}[a, b]$ *of* G *is given by the equation*

$$\mathrm{gyr}[a, b]z = \ominus(a \oplus b) \oplus (a \oplus (b \oplus z)) \tag{3.1}$$

for all $z \in G$*, then the following hold for all* $a, b, c \in G$*:*

(g3)	$\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$	*Gyroautomorphism Property*
(g4a)	$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$	*Left Gyroassociative Law*
(g4b)	$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$	*Right Gyroassociative Law*
(g5a)	$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$	*Left Loop Property*
(g5b)	$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$	*Right Loop Property*
(g6)	$\ominus(a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a)$	*Gyrosum Inversion Law*
(g7)	$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a]$	*Gyroautomorphism Inversion*

A gyrogroup is gyrocommutative if it satisfies

(g8) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$ *Gyrocommutative Law*

The groupoid $(\mathbb{R}^3_c, \oplus)$, consisting of the c-ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$ and Einstein's addition $\oplus$, is a gyrocommutative gyrogroup called an Einstein gyrogroup; for other interesting gyrocommutative gyrogroups see [13]. In the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus)$ we have the identity $\ominus a = -a$ for all $a \in \mathbb{R}^3_c$, that is, the gyrogroup inversion is identical with vector inversion. The Einstein gyrogroup is, in fact, the first known gyrogroup, discovered in 1988 [7]. Gyrogroups, both gyrocommutative and non-gyrocommutative, abound in group theory as indicated in [3] and [2]. Some gyrocommutative gyrogroups support scalar multiplication, turning them into gyrovector spaces. These, in turn, form the setting for hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry [11][12].

The gyrocommutative gyrogroup structure of Einstein's velocity addition allows the relativistic composite-velocity reciprocity principle to be placed in a context that makes it (i) obvious and (ii) compatible with the well-known Lorentz transformation reciprocity principle, as we show in Section 5.

4 THE RELATIVISTIC COMPOSITE-VELOCITY RECIPROCITY PRINCIPLE

Illustrated by the left part of Fig. 1.1,

(1) a Σ-observer (that is, an observer at rest relative to Σ) observes an object moving uniformly with relative velocity $\mathbf{y}$. He relates the velocity $\mathbf{y}$

of the moving object to its velocity $\mathbf{x}$ as seen by a Σ''-observer by the equation

$$\mathbf{y} = \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{x}) \tag{4.1}$$

and, reciprocally,

(2) a Σ''-observer relates the velocity $\mathbf{x}$ of a moving object that he observes to its velocity $\mathbf{y}$ as seen by a Σ-observer by the equation

$$\mathbf{x} = -\mathbf{v}\oplus(-\mathbf{u}\oplus\mathbf{y}) \tag{4.2}$$

The application of the left gyroassociative law to the reciprocal viewpoints in (4.1) and (4.2) exposes the relativistic composite-velocity reciprocity principle: Viewpoint (4.1) gives

$$\mathbf{y} = \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{x}) = (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{x} \tag{4.3}$$

and its reciprocal viewpoint (4.2) gives, by Theorem 3.3 and by means of (2.4),

$$\begin{aligned}\mathbf{x} = -\mathbf{v}\oplus(-\mathbf{u}\oplus\mathbf{y}) &= (-\mathbf{v}\oplus(-\mathbf{u}))\oplus\mathrm{gyr}[-\mathbf{v},-\mathbf{u}]\mathbf{y} \\ &= -(\mathbf{v}\oplus\mathbf{u})\oplus(\mathrm{gyr}[\mathbf{v},\mathbf{u}])]\mathbf{y} \\ &= -(\mathbf{v}\oplus\mathbf{u})\oplus(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1}]\mathbf{y}\end{aligned} \tag{4.4}$$

The emerging relativistic composite-velocity reciprocity principle in (4.3) and (4.4) is now clear:

The Relativistic Composite-Velocity Reciprocity Principle

(1) If the Σ''-observer sees a spinning object moving uniformly with relative velocity $\mathbf{x}$ without relative rotation, Fig. 1.1, then the Σ-observer sees the same spinning object moving uniformly, boosted with relative velocity

$$\mathbf{u}\oplus\mathbf{v} \tag{4.5}$$

and with relative rotation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}] \tag{4.6}$$

as evidenced from (4.3) (the relative rotation of the moving spinning object is recognized by the orientation of its spin-axis); and reciprocally,

(2) If the Σ-observer sees a spinning object moving uniformly with relative velocity $\mathbf{y}$ without relative rotation, Fig. 1.1, then the Σ''-observer sees the same spinning object moving uniformly, boosted with reciprocal relative velocity

$$-(\mathbf{v}\oplus\mathbf{u}) \tag{4.7}$$

and with reciprocal relative rotation

$$(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1} \tag{4.8}$$

as evidenced from (4.4).

The two relativistically reciprocal velocities (4.5) and (4.7) are not negative to each other. Rather, they are related by the gyrocommutative law of Einstein's addition,

$$\begin{aligned} -(\mathbf{v}\oplus\mathbf{u}) &= \mathrm{gyr}[\mathbf{v},\mathbf{u}](\mathbf{u}\oplus\mathbf{v}) \\ &= -(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1}(\mathbf{u}\oplus\mathbf{v}) \end{aligned} \tag{4.9}$$

a relationship which is consistent with hyperbolic geometry [11]. The results of hyperbolic geometry found applications in physics upon Einstein's introduction of the special theory of relativity in 1905, as was pointed out soon later by Varičak in 1908 [16] [17] whose work has been cited by Pauli [4].

5 THE LORENTZ GROUP AND THE COMPOSITE-VELOCITY RECIPROCITY PRINCIPLE

In this section we extend the analogies shared by relativistic and prerelativistic velocities to analogies shared by the Lorentz and by the Galilean transformation group. The extension enables us to show that our relativistic composite-velocity reciprocity principle in (4.5)-(4.8) is compatible with the well-known reciprocity principle in the Lorentz transformations between two inertial frames.

The relativistic composite-velocity reciprocity principle can, most convincingly, be demonstrated in terms of the Lorentz group and analogies that it shares with the Galilei group. The Galilei transformation group G is commonly parametrized, $G(\mathbf{v},V)$, by a velocity parameter $\mathbf{v}\in\mathbb{R}^3$ and an orientation parameter $V\in SO(3)$ so that

(i) The application of a Galilean transformation $G(\mathbf{v},V)$ to spacetime coordinates $(t,\mathbf{x})^t$ (exponent t assigns transposition), $t\in\mathbb{R}$, $\mathbf{x}\in\mathbb{R}^3$, is given by

$$G(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x}\end{pmatrix} = \begin{pmatrix} t \\ \mathbf{v}t+V\mathbf{x}\end{pmatrix} \tag{5.1}$$

and so that

(ii) The composition of two successive Galilei transformations $G(\mathbf{v},V)$ followed by $G(\mathbf{u},U)$ is, again, a Galilei transformation,

$$G(\mathbf{u},U)G(\mathbf{v},V) = G(\mathbf{u}+U\mathbf{v},UV) \tag{5.2}$$

for $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$ and $U,V\in SO(3)$.

The parameter composition

$$(\mathbf{u},U)(\mathbf{v},V)=(\mathbf{u}+U\mathbf{v},UV) \tag{5.3}$$

in (5.2) is known in group theory as a *semidirect product* [5], a product of two groups that generates a group of pairs.

By analogy, the Lorentz group L can also be parametrized, $L(\mathbf{v},V)$, by a velocity parameter $\mathbf{v}\in\mathbb{R}^3_c$ and an orientation parameter $V\in SO(3)$. The parametrization $L(\mathbf{v},V)$ of L can be found in old texts on relativity theory [6]. But, seemingly having no use, this parametrization has been abandoned in modern texts. Sharing analogies with $G(\mathbf{v},V)$, the parametrization $L(\mathbf{v},V)$ of L has the following properties:

(i) The application of a Lorentz transformation $L(\mathbf{v},V)$ to spacetime coordinates $(t,\mathbf{x})^t$, $t\in\mathbb{R}$, $\mathbf{x}\in\mathbb{R}^3$, is given by

$$L(\mathbf{v},V)\begin{pmatrix}t\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}\gamma_{\mathbf{v}}(t+\frac{1}{c^2}\mathbf{v}\cdot V\mathbf{x})\\ \gamma_{\mathbf{v}}\mathbf{v}t+V\mathbf{x}+\frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot V\mathbf{x})\mathbf{v}\end{pmatrix} \tag{5.4}$$

and

(ii) The composition of two successive Lorentz transformations $L(\mathbf{v},V)$ followed by $L(\mathbf{u},U)$ is, again, a Lorentz transformation,

$$L(\mathbf{u},U)L(\mathbf{v},V)=L(\mathbf{u}\oplus U\mathbf{v},\mathrm{gyr}[\mathbf{u},U\mathbf{v}]UV) \tag{5.5}$$

for $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$ and $U,V\in SO(3)$,

as shown in [11]. The Lorentz transformation composition law (5.5) is a novel result (Eq. (99) in [9]) that we employ in this section to demonstrate that our relativistic composite-velocity reciprocity principle in (4.5)-(4.8) is compatible with the well-known reciprocity principle to which the Lorentz transformation group gives rise.

The parameter composition

$$(\mathbf{u},U)(\mathbf{v},V)=(\mathbf{u}\oplus U\mathbf{v},\mathrm{gyr}[\mathbf{u},U\mathbf{v}]UV) \tag{5.6}$$

in (5.5) is known in gyrogroup theory as a *gyrosemidirect product* [11], a product of a gyrogroup and a group that generates a group of pairs.

The analogies shared by the Galilei and the Lorentz transformation composition law in (5.2) and (5.5) are clear. These, as well as other analogies, are exposed in [9] and [10]. It is hoped that following the exposition of these analogies, the parametrization $L(\mathbf{v},V)$ of the Lorentz group L, being fully analogous to the parametrization $(G(\mathbf{v}, V)$ of the Galilei group G, will find its way back to mainstream literature.

Let the spacetime coordinates of the moving object in Fig. 1.1, as measured by the Σ-observer and by the Σ''-observer, be respectively $(t_{\Sigma}, \mathbf{y})$ and $(t_{\Sigma''}, \mathbf{x})$. They are related by composite Lorentz transformations,

$$\begin{aligned} \begin{pmatrix} t_{\Sigma} \\ \mathbf{y} \end{pmatrix} &= L(\mathbf{u}, I)L(\mathbf{v}, I) \begin{pmatrix} t_{\Sigma''} \\ \mathbf{x} \end{pmatrix} \\ &= L(\mathbf{u}\oplus\mathbf{v}, \mathrm{gyr}[\mathbf{u}, \mathbf{v}]) \begin{pmatrix} t_{\Sigma''} \\ \mathbf{x} \end{pmatrix} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \begin{pmatrix} t_{\Sigma''} \\ \mathbf{x} \end{pmatrix} &= L(-\mathbf{v}, I)L(-\mathbf{u}, I) \begin{pmatrix} t_{\Sigma} \\ \mathbf{y} \end{pmatrix} \\ &= L(-\mathbf{v}\oplus(-\mathbf{u}), \mathrm{gyr}[-\mathbf{v}, -\mathbf{u}]) \begin{pmatrix} t_{\Sigma} \\ \mathbf{y} \end{pmatrix} \\ &= L(-(\mathbf{v}\oplus\mathbf{u}), (\mathrm{gyr}[\mathbf{u}, \mathbf{v}])^{-1}) \begin{pmatrix} t_{\Sigma} \\ \mathbf{y} \end{pmatrix} \end{aligned} \tag{5.8}$$

where $I \in SO(3)$ is the identity rotation, that is, the identity transformation of $\mathbb{R}^3$.

The two Lorentz transformations in (5.7) and (5.8) correspond to the two viewpoints in (4.1) and (4.2) and, accordingly, they are parametrized by the velocity-orientation parameters in the two viewpoint (4.1) and (4.2), which are $\mathbf{u}\oplus\mathbf{v}$ in (4.5), and $-(\mathbf{v}\oplus\mathbf{u})$ in (4.7).

Moreover, the two Lorentz transformations

$$\begin{gathered} L(\mathbf{u}\oplus\mathbf{v}, \mathrm{gyr}[\mathbf{u}, \mathbf{v}]) \\ L(-(\mathbf{v}\oplus\mathbf{u}), (\mathrm{gyr}[\mathbf{u}, \mathbf{v}])^{-1}) \end{gathered} \tag{5.9}$$

in (5.7) and (5.8) are inverse to each other. This is verified (i) directly, by employing the Lorentz transformation composition law (5.5) and well-known gyrogroup identities from [11], or (ii) indirectly, by means of their decompositions in (5.7) and (5.8).

The fact that the two Lorentz transformations (5.9) in (5.7) and (5.8) are inverse to each other demonstrates that the relativistic composite-velocity reciprocity principle in (4.5)-(4.8) is compatible with the well-known Lorentz transformation reciprocity principle. The latter asserts that if the Lorentz transformation from an inertial frame S to an inertial frame T is $L(\mathbf{v}, V)$, $\mathbf{v} \in \mathbb{R}^3_c$, $V \in SO(3)$, then the Lorentz transformation from T to S is the inverse Lorentz transformation $(L(\mathbf{v}, V))^{-1} = L(-V^{-1}\mathbf{v}, V^{-1})$.

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