Exponential Quintessence and the End of Acceleration

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Abstract

Recent observations indicate that the universe's expansion has been accelerating of late. But recent theoretical work has highlighted the difficulty of squaring acceleration with the underlying assumptions of string theory, disfavoring most models of quintessence. because they predict eternal acceleration. We show that one of the simplest and most motivated quintessence models described by an exponential potential can produce the acceleration needed to explain the data while also predicting only a finite period of acceleration, consistent with theoretical paradigms. This model is no more tuned than the canonical tracking quintessence models.

1 Introduction

Over the last several years, new and better sources of astrophysical data have allowed us to measure the universe's dynamics to an unprecedented degree. The concensus emerging from this data is that we are living in an (approximately) flat and expanding universe, but one in which that expansion is either currently, or has in the recent past been, accelerating [1]. Such an expansion is quite unexpected, violating our Newtonian intuition and requiring some form of "universal repulsion" to counteract the decelerating efforts of normal matter and radiation.

Modelling the energy content of the universe as a perfect fluid, Einstein's equations (in a Friedmann-Robertson-Walker universe) yield

$$a(t) \propto t^{\frac{2}{3(1+w)}}$$

where $a(t)$ is the metric scale factor and w defines the pressure equation of state of the fluid: $p = w\rho$. In order for $\ddot{a} > 0$[1] (*i.e.*, acceleration) one needs $w < -\frac{1}{3}$. Normal matter and radiation ($w = 0, \frac{1}{3}$ respectively) only lead to deceleration, so that we are forced to conclude that some new, and unknown, energy source has come to dominate the energy density of the universe in the recent past. We will refer to this new source as the "dark energy" henceforth.) Recent fits favor a universe with roughly 2/3 of its energy in the form of dark energy, and 1/3 in the form of matter, most of the latter being dark matter [1].

The simplest of all sources for the dark energy is a cosmological constant, Λ, since $w_\Lambda = -1$. But such an explanation is not without difficulties. First, the observed energy density today is roughly $\rho \simeq 10^{-11} \, \mathrm{eV}^4$ whilst a cosmological constant would naturally be expected to have a density 10^{120} times as great; values of the size observed would seem to indicate a large fine-tuning in the underlying field theory. Second, a cosmological constant suffers from a severe temporal fine-tuning. While matter and radiation energy densities fall as a^{-3} and a^{-4}, ρ_Λ is constant. Why, then, should we happen to live at that one peculiar time in the history of the universe at which $\rho_m \approx \rho_\Lambda$? (The question may be even worse, for one could argue that the proximity of matter-Λ equality to matter-radiation equality implies a near triple coincidence between matter, radiation and Λ, an event that one would not expect to occur now nor at any other time in the life of the universe [2].) Finally, it has been emphasized recently that universes that accelerate without end have no well-defined asymptotic states from which a physical S-matrix can be built; thus it would seem that endless acceleration is in contradiction with the axioms of string theory [3]. A cosmological constant, unfortunately, gives just such an eternal acceleration.

A large class of models have arisen which attempt to replace the cosmological constant with something dynamical, in particular, a scalar field called quintessence [4].

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[1]Dots will represent time derivatives, primes derivatives with respect to fields.

For a scalar field ϕ, the equation of state satisfies

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{1}$$

assuming $\nabla\phi = 0$. Thus a scalar field can be used to generate any w with $-1 \le w \le 1$. But most importantly, the value of w need not remain constant as the universe expands, thus allowing for the possibility that the value of ϕ today is dynamically determined in such a way as to explain the small size of ρ_ϕ and why we live so close to the beginning of the ϕ-dominated era.

Quintessence models, however, fail to fully solve the problems associated with the cosmological constant. For one thing, they generally contain one free parameter which can be taken as their current energy density, or alternatively, the time at which matter and quintessence meet; this is little different than just setting the cosmological constant by hand [5]. More recently, it has been observed that most models cannot solve the last of our three problems either. That is, once the quintessence field has come to dominate the universe, a period of acceleration/inflation begins which has no obvious ending and thus does not allow for a causal definition of asymptotic particle/string states.

In this paper, we will revive one of the more interesting, though discarded, models of quintessence and show that it can generate a period of acceleration which only lasts for a finite time, and that this model is no more contrived or tuned than any of the standard "tracking" quintessence models in the literature [4]. This is the so-called exponential model, also called the "scaling" model for reasons that will soon be clear.

2   Exponential Quintessence

The potential studied in this paper is one of the simplest and most motivated of the various quintessence potentials put forward:

$$V = \hat{V} e^{-\lambda\phi/M} \tag{2}$$

where λ is an unknown coefficient of $\mathcal{O}(1)$ and $M = (8\pi G)^{-1/2}$ is the reduced Planck mass. This potential and its cosmological behavior has been studied already by a number of authors [6]. The behavior of ϕ can be studied by dividing the energy density of the universe into two components: a "background" density, ρ_b, of either matter or radiation, and ϕ itself, where $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \rho_{\mathrm{crit}} - \rho_b$ (we will assume $\Omega \equiv \frac{\rho_{\mathrm{tot}}}{\rho_{\mathrm{crit}}} = \Omega_b + \Omega_\phi = 1$ throughout this work). Assuming no interactions between ϕ and ordinary matter or any self-interactions (consistent with [5]), then the dynamics of ϕ are completely determined by the usual cosmological equations:

$$H^2 = \frac{1}{3M^2}(\rho_b + \rho_\phi), \tag{3}$$

$$\dot{\rho}_b + 3H\rho_b(1 + w_b) = 0, \tag{4}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{5}$$

where H is the Hubble constant and $w_b = 0$ or $\frac{1}{3}$ for a background of matter or radiation. Previous authors had discovered a remarkable attractor solution for ρ_ϕ:

$$\Omega_\phi = \frac{\rho_\phi}{\rho_{\text{crit}}} = \frac{3}{\lambda^2}(1 + w_b). \tag{6}$$

That is, there exists an attractive fixed point trajectory on which the ratio of ρ_ϕ and ρ_b is constant at all times. Thus the energy density of ϕ at the attractor solution is determined completely by λ and is independent of $\hat{V}$. This also implies that $w_\phi = w_b$ at all points along the attractor. One rather remarkable implication of this is that w_ϕ spontaneously changes from $\frac{1}{3}$ to 0 at the time of matter-radiation equality.

A full stability analysis of this model can be performed [6] and the above solution is indeed found to be an attractor for a wide range of initial conditions under the assumption that $\lambda^2 > 3(1 + w_b)$. For smaller λ^2, this solution disappears. (It is clear that such a solution would be inconsistent with our initial assumption of a flat universe.)

Despite its mathematical charm, it appears that the attractor solution cannot describe dark energy as we observe it. It has two fundamental flaws. First, in a universe composed of only matter, radiation and ϕ, it seems to be impossible to generate $w_\phi < 0$ along the attractor. Thus acceleration never occurs. The second problem is that in order for ϕ to dominate ρ today, it must have also dominated ρ at all previous times along the attractor. This is grossly inconsistent with big-bang nucleosynthesis (BBN) constraints; recent analyses [7] obtain a limit $\Omega_\phi < 0.05$ at the time of BBN. Attempts to resuscitate exponential quintessence include altering the potential to include a polynomial prefactor

$$V \propto [A + (\phi - B)^\alpha]\, e^{-\lambda\phi/M} \tag{7}$$

which acts as a potential barrier to ϕ [8]. One fixes A, B and α such that the scaling evolution of ϕ was halted in the recent past, pushing $\dot{\phi} \to 0$ and thus $w_\phi \to -1$. This solves both of the above problems but at the price of introducing free parameters which can again be traded for the value of ρ_ϕ today.

We would like to suggest that the solution to the problems inherent in the exponential quintessence are not nearly so difficult and can be addressed without adding any new pieces to the theory while maintaining the same level of naturalness as exhibited in tracker quintessence models and in the model of Ref. [8]. We begin by dividing the parameter space of λ into three regions:

Case I: $\lambda^2 < 3$

For very small λ there are no attractor solutions for ϕ either during matter- or radiation-domination, and thus ρ_ϕ is strongly dependent in initial conditions. This may not rule out this region of parameters, but it will not give the attractive behavior we will be demonstrating and so we do not consider it further.

Case II: $3 < \lambda^2 < 4$

Here there is no attractor for ϕ during radiation domination, but there is an attractor during matter domination. Thus the universe can be at the attractor solution currently without having to live on it at all times in the past. We will consider this case in detail below.

Case III: $\lambda^2 > 4$

This is the usual case with attractors at all epochs; this case can also be perfectly consistent with observations as will be discussed below.

It is instructive to begin our discussion with Case II from above. Here $3 < \lambda^2 < 4$, so that there is no scaling solution during radiation domination, but there is one during matter domination. Thus when radiation dominates the universe, the behavior of ϕ is completely controlled by initial conditions. Generically there are two broad classes of behavior it could follow. If it has a large initial energy density with $V \gg \dot{\phi}^2$, it will tend to increase, quickly dominating the universe and leading to inflation without end; this case is uninteresting to us. On the other hand, if it has a smaller initial energy density or $\dot{\phi}^2 \gg V$, then it will tend to fall off rapidly, as a^{-6}, typical for kinetic energy. Thus at early times $w_\phi = 1$.

What halts this rapid drop in quintessence energy? When $\frac{1}{2}\dot{\phi}^2 \approx V$, the potential energy begins to dominate the behavior of ϕ. This leads to a stabilization of the energy density at some constant value, that is, $w_\phi = -1$. Following this constant trajectory, the quintessence energy density then approaches the background energy density and surpasses it. If the meeting occurs during matter domination, the attractor solution then kicks in and the quintessence begins to behave as ordinary matter, $w_\phi = 0$.

This brief history of the quintessence field can most easily be seen in Figs. 1 and 2. In Fig. 1 we have plotted the energy densities of matter, radiation, and quintessence as a function of the scale parameter. The axes are log-log and have been left unscaled to emphasize the point that this behavior is independent of the overall time and energy scales. Fig. 2 shows the behavior of w_ϕ as a function of $\log(a)$. In both figures, we begin at the left during radiation domination. The quintessence field has an initial condition that its kinetic energy is much greater than its potential, the latter being of order that observed today. The quintessence field redshifts away its energy as a^{-6} until $\frac{1}{2}\dot{\phi}^2 \approx V$ and then it changes slope quickly to that of a cosmological constant. ρ_ϕ remains constant, passing through matter's curve during the period of matter domination. Once ρ_ϕ surpasses ρ_m, the universe begins accelerating.

However, acceleration only lasts a finite time. Because $\lambda^2 > 3$, there exists an attractor solution for ϕ, in this case with $\rho_\phi > \rho_m$. Once the attractor is met, ϕ turns abruptly to follow it, changing its equation of state to that of matter, $w = 0$, with Ω_ϕ given by Eq. (6). And being dominated by a pressure-less field, the universe discontinues its acceleration. From that point on the universe acts as though it were matter-dominated, with the ratio of ρ_m/ρ_ϕ fixed perpetually. And string theories once again have well-defined asymptotic states.
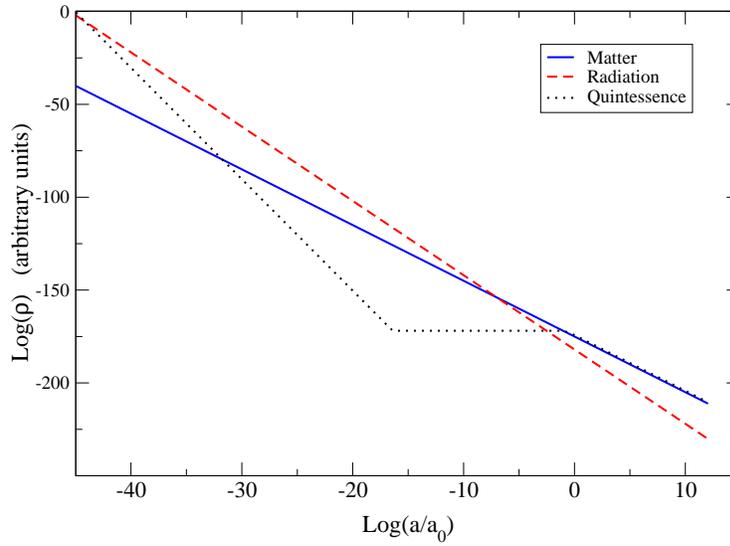
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Figure 1: Scaling of ρ_r, ρ_m and ρ_ϕ as a function of scale factor a. The current epoch is at $a = a_0$.
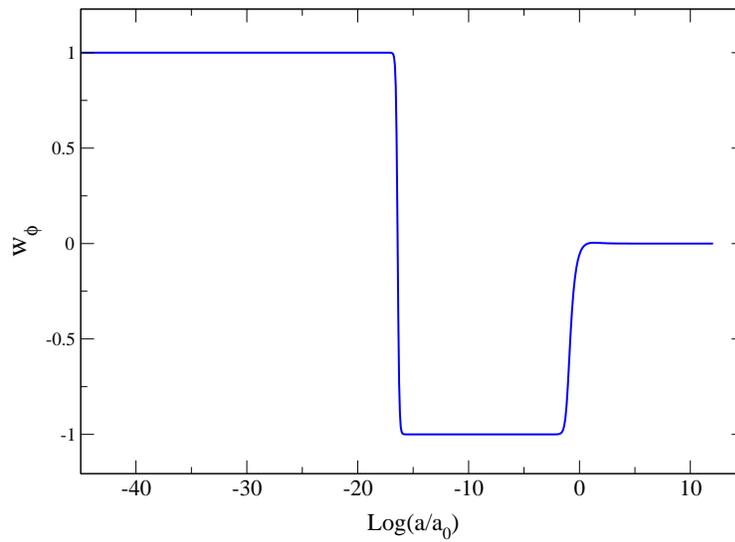

Figure 2: Equation of state of exponential quintessence from Fig. 1.

Notice how this model avoided both of the usual criticisms of exponential models. First, we were able to generate $w < -\frac{1}{3}$ *because we were not yet on the attractor at the time of matter-quintessence equality!* And again for the same reason, we were able to avoid any bounds coming from BBN. In fact, it is clear from the figure that during BBN, the quintessence energy density was many orders of magnitude too small to affect the expansion rate. The role of the attractor has changed completely from the usual case in the literature. Previously one used the attractor to wipe out any dependence on initial conditions at times long before the present. Here one uses the attractor to generate an end to the acceleration.

Let us step back for a moment to discuss naturalness. All models of quintessence of which we are aware require (at least) one parameter to be tuned in order to get matter-quintessence equality at the present time. This model will require exactly that same single input; that is, we will need to input the value of the energy density at which matter and quintessence meet:

$$V_0 = \hat{V} e^{-\lambda \phi_0 / M} \tag{8}$$

Since the values of $\hat{V}$ and ϕ_0 can be traded for each other, some combination of the two must be set to give the observed dark energy density today. Again, this is no worse than any other quintessence model.

Of course, there is now some dependence on initial conditions. For one thing, we require that at some "initial" time, $\dot{\phi}^2 \gg V$. This is simple to imagine, especially given the small value of V needed to reproduce the current dark energy density. We have solved this model for a large number of initial values of $\dot{\phi}$ and find very little dependence in the final matter-quintessence equality point. For example, changing $\dot{\phi}_{\text{initial}}$ from M^2 to $10^{-10} M^2$ resulted in a change of only 10^6 in the energy density at matter-quintessence equality. Thus the dependence on initial conditions is present but not particularly strong, apart from the usual need to tune $\hat{V}$ and/or ϕ_{initial} to reproduce $V = V_0$ today.

We can learn something of the details of the model by carefully measuring Ω_ϕ today. For $3 < \lambda^2 < 4$, the attractor value for Ω_ϕ falls in the range:

$$\frac{3}{4} < \Omega_\phi < 1 \quad \Longrightarrow \quad 0 < \frac{\rho_m}{\rho_\phi} < \frac{1}{3}. \qquad \text{(Case II)} \tag{9}$$

Thus if the observational data settles on $\Omega_{\phi,0} < \frac{3}{4}$, we could conclude that we are still living during the accelerating phase of the universe's expansion. If however the data settles on $\Omega_{\phi,0} > \frac{3}{4}$, it is possible that we have already exited the accelerating phase. Though we would still be dominated by ϕ, the dynamics of the universe as a whole would mimic matter-domination.

How is Case III going to be different? In fact, it is not very different at all. As long as $V \approx V_0$ at some initial time, ϕ will not reach its attractor solution until the present. This is true for any $\lambda^2 > 3$. The only difference between these cases is the

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limit on Ω_ϕ:

$$0 < \Omega_\phi < \frac{3}{4} \quad \Longrightarrow \quad \frac{1}{3} < \frac{\rho_m}{\rho_\phi} < \infty. \qquad \text{(Case III)} \qquad (10)$$

Unfortunately this means that a measurement of $\Omega_\phi < \frac{3}{4}$ will not by itself tell us whether we have exited acceleration yet. Observational signals for differentiating this model from others and for determining the value of λ will be the subject of a future work.

3   Summary

The exponential model of quintessence has been often overlooked in the discussion of viable quintessence models because its attractor solution does not provide acceleration and is inconsistent with BBN constraints. However, we have shown here that viable quintessence models can be built from the exponential potential, but using the attractor solution as a means for *exiting* acceleration, not the converse. One of the key observations of this work is that the period of acceleration can be generated as the quintessence field falls onto its fixed point trajectory. However this behavior is transitory, ending as soon as the fixed point is reached. This model then can explain acceleration, the large ratio of dark energy to matter in the present universe, and still remain consistent with string theory because its acceleration is short-lived.

Finally, we explained that this model is no more fine-tuned than other quintessence models, such as tracking models. There is essentially one free parameter, which can be taken to be $\hat{V}$, which sets the current value of ρ_ϕ and thus the time of matter-quintessence equality. Then for a very wide range of initial conditions on $\dot{\phi}$, $\rho_\phi = \text{constant}$ at the time of matter-quintessence equality, leading to the required acceleration. In short, this model reproduces all the successes of standard quintessence while remaining consistent with the underlying assumptions of string theory.

Acknowledgements

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