An Achievable Rate Region for the Gaussian Interference Channel

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Abstract

An achievable rate region for the Gaussian interference channel is derived using Sato's modified frequency division multiplexing idea [1] and a special case of Han and Kobayashi's rate region (denoted by $\mathcal{G}'$ in [2]). We show that the new inner bound includes $\mathcal{G}'$, Sason's rate region $\mathcal{D}$ [3], as well as the achievable region via TDM/FDM [4], as its subsets. The advantage of this improved inner bound over $\mathcal{G}'$ arises due to its inherent ability to utilize the whole transmit power range on the real line without violating the power constraint. We also provide analysis to examine the conditions for the new achievable region to strictly extend $\mathcal{G}'$.

Index terms — Gaussian interference channel, achievable rate region, sum capacity.

I. Introduction

The study of interference channel (IFC) can be traced back to [5]; its capacity region for the general case, however, remains unknown to this date. A two-user discrete memoryless interference channel is defined by $(\mathcal{X}_1^n \times \mathcal{X}_2^n, p^n(\mathbf{y}_1\mathbf{y}_2|\mathbf{x}_1\mathbf{x}_2), \mathcal{Y}_1^n \times \mathcal{Y}_2^n)$, where $\mathcal{X}_i$ and $\mathcal{Y}_i$, $i = 1, 2$, are the input and output alphabet sets of the i^{th} user, $p^n(\mathbf{y}_1\mathbf{y}_2|\mathbf{x}_1\mathbf{x}_2)$ is the channel transition probability

$$p^n(\mathbf{y}_1\mathbf{y}_2|\mathbf{x}_1\mathbf{x}_2) = \prod_{l=1}^{n} p\left(y_1^{(l)}y_2^{(l)}\,\middle|\,x_1^{(l)}x_2^{(l)}\right)$$

where $\mathbf{x}_i \in \mathcal{X}_i^n$, $\mathbf{y}_i \in \mathcal{Y}_i^n$. A (M_1, M_2, n) code of this channel has respectively two encoding functions $f_i : \{1, 2, \cdots, M_i\} \to \mathcal{X}_i^n$, and two decoding functions $g_i : \mathcal{Y}_i^n \to \{1, 2, \cdots, M_i\}$, $i = 1, 2$. The average error of this channel is defined as

$$p_e^n = \max\{p_{e1}^n, p_{e2}^n\}$$

where

$$\begin{aligned} p_{e1}^n &= \frac{1}{M_1M_2}\sum_{j=1}^{M_1}\sum_{k=1}^{M_2} p_r\left\{g_1(y_1) \neq j|(j,k) \text{ sent}\right\} \\ p_{e2}^n &= \frac{1}{M_1M_2}\sum_{j=1}^{M_1}\sum_{k=1}^{M_2} p_r\left\{g_2(y_2) \neq k|(j,k) \text{ sent}\right\} \end{aligned}$$

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The rate pair (R_1, R_2) is achievable for the IFC if and only if there exists a sequence of codes (M_1, M_2, n) such that

$$M_1 \geq 2^{nR_1}, \quad M_2 \geq 2^{nR_2}, \quad p_e^n \to 0$$

for some sufficiently large n. The capacity region is defined as the closure of all the achievable rates.

In this correspondence, we focus on the Gaussian IFC. Carleial has shown that any two-user Gaussian IFC can be reduced to the standard form [4]

$$\begin{aligned} y_1 &= x_1 + \sqrt{a} x_2 + n_1 \\ y_2 &= \sqrt{b} x_1 + x_2 + n_2 \end{aligned} \tag{1}$$

where x_i, y_i and n_i are the transmit signal, receive signal and Gaussian noise of user i with power constraints $\mathcal{E}(x_i^2) \leq P_i$ and noise variances $\mathcal{E}(n_i^2) = 1$, $i = 1, 2$. For the Gaussian IFC, the capacity region is only known for the very strong interference case [6], [7] ($a \geq P_1 + 1$ and $b \geq P_2 + 1$) and the strong interference case [2], [7] ($1 \leq a < 1 + P_1$ and $1 \leq b < 1 + P_2$).

While the capacity region for the general Gaussian IFC remains an open problem, various attempts have been made to find reasonably tight achievable rate regions [2], [4], [5], [8], [9]. Carleial's work in 1978 [4] was the first to use the superposition coding idea, proposed originally by Cover for studying the broadcast channels [10], to obtain an inner bound on the capacity region. This was later generalized by Han and Kobayashi [2] where a joint decoder was used instead of the sequential decoder in Carleial's original work [4] . The Han and Kobayshi (HK) region (Theorem 3.1 in [2]) remains to be the largest achievable region up to date. However, the original HK region is prohibitively complex to evaluate. For the Gaussian IFC, a subregion of the HK bound has been obtained by imposing the Gaussian input assumption and ignoring the time sharing argument (Eq. (5.9) in [2]). This subregion, denoted by $\mathcal{G}'$, is amenable to numerical evaluation. For a given standard Gaussian IFC (i.e., fixed a and b), we use $\mathcal{G}'(P_1, P_2)$ to represent the explicit dependence of $\mathcal{G}'$ on the power constraints. It was observed in [2], however, that this subregion does not contain the achievable rate region of simple frequency/time division multiplexing (FDM/TDM), denoted herein as $\mathcal{F}$, for the moderate interference case. In particular, the achievable sum rate of $\mathcal{F}$ is strictly larger than that of $\mathcal{G}'$ for the moderate interference case.

More recently, Sason [3] obtained a new achievable rate region, denoted by $\mathcal{D}$, that combines Sato's modified FDM/TDM method [1] and the transmission modes in which one of the user transmits at the maximum achievable rate and the other user transmits at the rate that both users can reliably decode it. Although Sason's bound is still a subset of Han and Kobayashi's achievable rate region, it is much more amenable to numerical evaluation. Furthermore, Sason's bound always includes $\mathcal{F}$, the achievable rate region of FDM/TDM. While the sum rate of Sason't bound is strictly larger than that of $\mathcal{G}'$ for the moderate interference power, the reverse is true, however, for the weak interference case.

The inner bound reported in this work is also based on Sato's modified FDM/TDM method, hence can be considered as a refinement of Sason's inner bound. The difference with that of Sason's work is that $\mathcal{G}'$ is used in each subband, as opposed to the two extreme modes. Denoted by $\mathcal{S}$, this new bound is shown to include all three achievable rate regions: $\mathcal{G}'$, $\mathcal{F}$, and $\mathcal{D}$ as its subsets. Therefore, it provides the largest inner bound for the Gaussian IFC *that can be numerically evaluated*. We provide a geometric interpretation to explain why it improves upon $\mathcal{G}'$. In particular, the proposed inner bound amounts to the convex combination for the weighted sums of the two users' rates in each subband with much relaxed individual power constraint. As such, whether it can improve upon $\mathcal{G}'$ depends on the concavity of $\mathcal{G}'(P_1, P_2)$ in the entire $[0, +\infty) \times [0, +\infty)$ power region and the precise location of the power constraint (P_1, P_2), which will be elaborated in Section IV.

The rest of the correspondence is organized as follows. In Section II, we summarize the three existing rate regions: $\mathcal{G}'$, $\mathcal{D}$, and $\mathcal{F}$. Section III presents the new achievable region and we also show that this new region includes $\mathcal{G}'$, $\mathcal{D}$, and $\mathcal{F}$ as its subsets. We provide detailed analysis in Section IV as to when the new achievable rate region strictly improves upon $\mathcal{G}'$. The analysis is applicable to any achievable rate region by combining any known rate regions with Sato's modified FDM/TDM scheme. Numerical examples are given in Section V and we conclude in Section VI.

II. A BRIEF REVIEW OF THE EXISTING RESULTS

The HK achievable region $\mathcal{G}'$ for the Gaussian IFC is derived by simultaneous superposition coding that divides the messages into common messages, to be decoded by both users, and private messages, to be decoded only by the intended users. We rewrite the rate region $\mathcal{G}'$ below for completeness as the new achievable rate region is also based on $\mathcal{G}'$. Specifically, define

$$\gamma(x) = \frac{1}{2}\log(1+x)$$

we have

$$\mathcal{G}' = \quad \text{Convex closure of} \bigcup_{0\le\beta_1\le 1, 0\le\beta_2\le 1} \left\{ (R_1, R_2) \;\middle|\; \begin{array}{l} R_1 \le \rho_1, R_2 \le \rho_2, R_1 + R_2 \le \rho_{12} \\ 2R_1 + R_2 \le \rho_{10}, R_1 + 2R_2 \le \rho_{20} \end{array} \right\}$$

with

$$\begin{aligned}
\rho_1 &= \sigma_1^\star + \gamma\left(\frac{\beta_1 P_1}{1 + a\beta_2 P_2}\right) \\
\rho_2 &= \sigma_2^\star + \gamma\left(\frac{\beta_2 P_2}{1 + b\beta_1 P_1}\right) \\
\rho_{12} &= \sigma + \gamma\left(\frac{\beta_1 P_1}{1 + a\beta_2 P_2}\right) + \gamma\left(\frac{\beta_2 P_2}{1 + b\beta_1 P_1}\right) \qquad (2) \\
\rho_{10} &= 2\sigma_1^\star + 2\gamma\left(\frac{\beta_1 P_1}{1 + a\beta_2 P_2}\right) + \gamma\left(\frac{\beta_2 P_2}{1 + b\beta_1 P_1}\right) - \left[\sigma_1^\star - \gamma\left(\frac{b\bar{\beta}_1 P_1}{1 + \beta_2 P_2 + b\beta_1 P_1}\right)\right]^+ \\
&\quad + \min\left\{\gamma\left(\frac{\bar{\beta}_2 P_2}{1 + \beta_2 P_2 + b\beta_1 P_1}\right), \gamma\left(\frac{\bar{\beta}_2 P_2}{1 + \beta_2 P_2 + bP_1}\right) + \left[\gamma\left(\frac{b\bar{\beta}_1 P_1}{1 + \beta_2 P_2 + b\beta_1 P_1}\right) - \sigma_1^\star\right]^+, \right. \\
&\qquad \left. \gamma\left(\frac{a\bar{\beta}_2 P_2}{1 + \beta_1 P_1 + a\beta_2 P_2}\right), \gamma\left(\frac{\bar{\beta}_1 P_1 + a\bar{\beta}_2 P_2}{1 + \beta_1 P_1 + a\beta_2 P_2}\right) - \sigma_1^\star\right\} \qquad (3) \\
\rho_{20} &= 2\sigma_2^\star + \gamma\left(\frac{\beta_1 P_1}{1 + a\beta_2 P_2}\right) + 2\gamma\left(\frac{\beta_2 P_2}{1 + b\beta_1 P_1}\right) - \left[\sigma_2^\star - \gamma\left(\frac{a\bar{\beta}_2 P_2}{1 + \beta_1 P_1 + a\beta_2 P_2}\right)\right]^+
\end{aligned}$$

$$+\min\left\{\gamma\left(\frac{\bar{\beta}_1P_1}{1+\beta_1P_1+a\beta_2P_2}\right),\gamma\left(\frac{\bar{\beta}_1P_1}{1+\beta_1P_1+aP_2}\right)+\left[\gamma\left(\frac{a\bar{\beta}_2P_2}{1+\beta_1P_1+a\beta_2P_2}\right)-\sigma_2^\star\right]^+,\right.$$
$$\left.\gamma\left(\frac{b\bar{\beta}_1P_1}{1+\beta_2P_2+b\beta_1P_1}\right),\gamma\left(\frac{\bar{\beta}_2P_2+b\bar{\beta}_1P_1}{1+\beta_2P_2+b\beta_1P_1}\right)-\sigma_2^\star\right\} \tag{4}$$

where $[x]^+=\max\{0,x\}$, $\bar{x}=1-x$, and

$$\sigma_1^\star = \min\left\{\gamma\left(\frac{\bar{\beta}_1P_1}{1+\beta_1P_1+a\beta_2P_2}\right),\gamma\left(\frac{b\bar{\beta}_1P_1}{1+b\beta_1P_1}\right)\right\}$$
$$\sigma_2^\star = \min\left\{\gamma\left(\frac{\bar{\beta}_2P_2}{1+\beta_2P_2+b\beta_1P_1}\right),\gamma\left(\frac{a\bar{\beta}_2P_2}{1+a\beta_2P_2}\right)\right\}$$
$$\sigma = \min\left\{\gamma\left(\frac{\bar{\beta}_1P_1+a\bar{\beta}_2P_2}{1+\beta_1P_1+a\beta_2P_2}\right),\gamma\left(\frac{\bar{\beta}_2P_2+b\bar{\beta}_1P_1}{1+\beta_2P_2+b\beta_1P_1}\right),\gamma\left(\frac{\bar{\beta}_1P_1}{1+\beta_1P_1+a\beta_2P_2}\right)\right.$$
$$\left.+\gamma\left(\frac{\bar{\beta}_2P_2}{1+\beta_2P_2+b\beta_1P_1}\right),\gamma\left(\frac{b\bar{\beta}_1P_1}{1+\beta_2P_2+b\beta_1P_1}\right)+\gamma\left(\frac{a\bar{\beta}_2P_2}{1+\beta_1P_1+a\beta_2P_2}\right)\right\}$$

The region $\mathcal{G}'$ is obtained from the original HK region by assuming Gaussian input signal and ignoring the time sharing variable.

The achievable rate region proposed by Sason [3] is

$$\mathcal{D} = \bigcup_{\alpha,\lambda_1,\lambda_2\in[0,1]}\left\{(R_1,R_2)\,\middle|\,\begin{array}{l} R_1\le\alpha\gamma\left(\frac{\lambda_1P_1}{\alpha}\right)+\bar{\alpha}\min\left\{\gamma\left(\frac{\bar{\lambda}_1P_1}{\bar{\alpha}+a\bar{\lambda}_2P_2}\right),\gamma\left(\frac{b\bar{\lambda}_1P_1}{\bar{\alpha}+\bar{\lambda}_2P_2}\right)\right\}\\ R_2\le\bar{\alpha}\gamma\left(\frac{\bar{\lambda}_2P_1}{\bar{\alpha}}\right)+\alpha\min\left\{\gamma\left(\frac{\lambda_2P_2}{\alpha+b\lambda_1P_1}\right),\gamma\left(\frac{a\lambda_2P_2}{\alpha+\lambda_1P_1}\right)\right\}\end{array}\right\} \tag{5}$$

The region $\mathcal{D}$ is obtained by combining Sato's modified FDM/TDM idea and two transmission modes, i.e., one user transmits at the maximum rate while the other transmits at a rate that both receivers can decode it.

The FDM/TDM achievable region $\mathcal{F}$ is

$$\mathcal{F} = \bigcup_{\alpha\in[0,1]}\left\{(R_1,R_2)\,\middle|\,R_1\le\alpha\gamma\left(1+\frac{P_1}{\alpha}\right);R_2\le\bar{\alpha}\gamma\left(\frac{P_2}{\bar{\alpha}}\right)\right\} \tag{6}$$

Note that $\mathcal{F}\subseteq\mathcal{D}$, as $\mathcal{D}$ reduces to $\mathcal{F}$ by choosing $\lambda_1=1$ and $\lambda_2=0$ in (5).

III. A NEW ACHIEVABLE RATE REGION FOR THE GAUSSIAN IFC

The new achievable rate region is obtained by combining Sato's modified FDM/TDM idea with $\mathcal{G}'$. Specifically, we have,

Theorem 1: For a given standard Gaussian IFC with power constraint (P_1,P_2), if $\mathcal{G}'(P_1,P_2)$ is $\mathcal{G}'$, the HK subregion, then the following rate region, denoted by $\mathcal{S}$, is also achievable.

$$\mathcal{S} = \bigcup_{\alpha\in[0,0.5],\lambda_1,\lambda_2\in[0,1]}\left\{(R_1,R_2)\,\middle|\,\begin{array}{l} R_1\le\alpha R_1^{(1)}+\bar{\alpha}R_1^{(2)},R_2\le\alpha R_2^{(1)}+\bar{\alpha}R_2^{(2)}\\ \left(R_1^{(1)},R_2^{(1)}\right)\in\mathcal{G}'\left(\frac{\lambda_1P_1}{\alpha},\frac{\lambda_2P_2}{\alpha}\right)\\ \left(R_1^{(2)},R_2^{(2)}\right)\in\mathcal{G}'\left(\frac{\bar{\lambda}_1P_1}{\bar{\alpha}},\frac{\bar{\lambda}_2P_2}{\bar{\alpha}}\right)\end{array}\right\} \tag{7}$$

where we define the case for $\alpha=0$ as

$$\mathcal{S}=\mathcal{G}'(P_1,P_2)$$

Proof Divide the total bandwidth into two subbands, one with α and the other $\bar{\alpha} = 1 - \alpha$ fraction of the total bandwidth. The power is allocated into each subband with a factor λ_i. Specifically, the signal model of the first sub-band can be written as

$$\begin{aligned} y_1' &= x_1' + \sqrt{a}x_2' + \sqrt{\alpha}n_1' \\ y_2' &= \sqrt{b}x_1' + x_2' + \sqrt{\alpha}n_2' \end{aligned} \tag{8}$$

where $\mathcal{E}(x_i'^2) \leq \lambda_i P_i$, i.e., with a power constraint $\lambda_i P_i$, and $\mathcal{E}(n_i'^2) = 1$. Denote the achievable rate contributed by this subband as $(\alpha R_1^{(1)}, \alpha R_2^{(1)})$. In the same way, the power constraint in the other subband is $\bar{\lambda}_i P_i$ for the ith user with rate contribution $(\bar{\alpha} R_1^{(2)}, \bar{\alpha} R_2^{(2)})$. The transmit signals of the same user in different bands are independent via independent codeword selection, thus the overall achievable rate is $(\alpha R_1^{(1)} + \bar{\alpha} R_1^{(2)}, \alpha R_2^{(1)} + \bar{\alpha} R_2^{(2)})$. Rewrite (8) in the standard form

$$\begin{aligned} \frac{y_1'}{\sqrt{\alpha}} &= \frac{x_1'}{\sqrt{\alpha}} + \sqrt{a}\frac{x_2'}{\sqrt{\alpha}} + n_1' \triangleq x_1^{(1)} + \sqrt{a}x_2^{(1)} + n_1' \\ \frac{y_2'}{\sqrt{\alpha}} &= \sqrt{b}\frac{x_1'}{\sqrt{\alpha}} + \frac{x_2'}{\sqrt{\alpha}} + n_2' \triangleq \sqrt{b}x_1^{(1)} + x_2^{(1)} + n_2' \end{aligned} \tag{9}$$

Applying $\mathcal{G}'$ to the first subband, its contribution to the total rate is

$$\left\{ \left(\alpha R_1^{(1)}, \alpha R_2^{(1)}\right) \middle| \left(R_1^{(1)}, R_2^{(1)}\right) \in \mathcal{G}'\left(\frac{\lambda_1 P_1}{\alpha}, \frac{\lambda_2 P_2}{\alpha}\right) \right\} \tag{10}$$

In the same way, the achievable rate of the second sub-band by using $\mathcal{G}'$ can be obtained.

Further, since both subbands use $\mathcal{G}'$, we only need to consider $\alpha \in [0, 0.5]$. This proves that the rate in Eq. (7) is achievable. Q.E.D

Notice that the power constraint for the standard Gaussian IFC in Eq. (9) in the first subband is $\left(\frac{\lambda_1 P_1}{\alpha}, \frac{\lambda_2 P_2}{\alpha}\right)$, which can take values on $[0, +\infty) \times [0, +\infty)$ by varying λ_i and α accordingly.

In Theorem 1, if we define

$$\begin{aligned} \mathcal{G}_1' &= \mathcal{G}'\left(\frac{\lambda_1 P_1}{\alpha}, \frac{\lambda_2 P_2}{\alpha}\right) = \mathcal{G}'\left(P_1', P_2'\right) \\ \mathcal{G}_2' &= \mathcal{G}'\left(\frac{\bar{\lambda}_1 P_1}{\bar{\alpha}}, \frac{\bar{\lambda}_2 P_2}{\bar{\alpha}}\right) = \mathcal{G}'\left(P_1'', P_2''\right) \end{aligned}$$

by choosing different α and λ_i, (P_1', P_2') and (P_1'', P_2'') will cover all possible power pairs satisfying

$$\begin{aligned} P_1 = \alpha P_1' + \bar{\alpha} P_1'' \\ P_2 = \alpha P_2' + \bar{\alpha} P_2'' \end{aligned} \tag{11}$$

Thus the convex combination of all rate pairs in $\mathcal{G}_1'$ and $\mathcal{G}_2'$ are achievable. The difference compared with the single mode $\mathcal{G}'$ is that the two modes extend the power constraints in each subband to the whole nonnegative real lines, whereas the original single mode $\mathcal{G}'$ in [2] is limited to the given power constraint (P_1, P_2). This will be the key to understanding when the proposed inner bound is strictly larger than that of $\mathcal{G}'$. Before proceeding, we first present the following obvious result.

Proposition 1: $\mathcal{G}'$, $\mathcal{D}$ and $\mathcal{F}$ are all subsets of proposed achievable region $\mathcal{S}$.

Proof To show $\mathcal{G}' \subseteq \mathcal{S}$, we notice that by setting $\alpha = 0$ the proposed $\mathcal{S}$ reduces to $\mathcal{G}'$. To show $\mathcal{D} \subseteq \mathcal{S}$, we note that the extreme mode in each subband used in Sason's region is a special case of $\mathcal{G}'$. Finally, since $\mathcal{F} \subseteq \mathcal{D}$, it follows directly that $\mathcal{F} \subseteq \mathcal{S}$. Q.E.D.

IV. Discussions

In this section, we analyze the conditions when the proposed rate region strictly extends that of $\mathcal{G}'$. The analysis can be applied in general to any rate region obtained by combining a known achievable rate region with Sato's modified FDM scheme, as to be elaborated later.

A. Conditions for $\mathcal{G}' \subset \mathcal{S}$

Here we consider the conditions that $\mathcal{S}$ strictly enlarges $\mathcal{G}'$. We use the following alternative definition to describe an achievable rate region:

$$\{(R_1, R_2)\,|R_1 + kR_2 \leq c_k, k \in [0, +\infty)\}$$

That is, we find all lines tangent to the boundary of the region and the intersection of the half spaces defined by these lines in the positive quadrant is the corresponding rate region. We denote by k and c_k the slope and intercept, respectively, of all these lines.

We have, for $\mathcal{G}'$,

$$c_k = \max_{(R_1,R_2)\in\mathcal{G}'} \{R_1 + kR_2\} \triangleq c_k^{(\mathcal{G}'(P_1,P_2))} \tag{12}$$

and for $\mathcal{S}$

$$\begin{aligned} c_k &= \max_{(R_1,R_2)\in\mathcal{S}} \{R_1 + kR_2\} \\ &= \max_{\alpha\in[0,0.5]} \left\{ \alpha \max_{(R_1^{(1)},R_2^{(1)})\in\mathcal{G}'_1} \left\{R_1^{(1)} + kR_2^{(1)}\right\} + \bar{\alpha} \max_{(R_1^{(2)},R_2^{(2)})\in\mathcal{G}'_2} \left\{R_1^{(2)} + kR_2^{(2)}\right\} \right\} \\ &= \max_{\alpha\in[0,0.5]} \left\{ \alpha c_k^{(\mathcal{G}'(P'_1,P'_2))} + \bar{\alpha} c_k^{(\mathcal{G}'(P''_1,P''_2))} \right\} \\ &\triangleq c_k^{(\mathcal{S})} \end{aligned} \tag{13}$$

Eq. (13) shows that $c_k^{(\mathcal{S})}$, as a function of (P_1, P_2) is nothing but the convex combination of $c_k^{(\mathcal{G}')}$. Since all the $\left(k, c_k^{(\mathcal{S})}\right)$ pairs fully specify the achievable rate region, $\mathcal{S}$ is equivalent to taking the convex combination on the curved surface of $c_k^{(\mathcal{G}')}$ with power constraints on the entire nonnegative real lines $[0, +\infty) \times [0, +\infty)$ that satisfy Eq. (11). Therefore, $\mathcal{G}' \subset \mathcal{S}$ if and only if there exists a k such that $c_k^{(\mathcal{G}')} < c_k^{(\mathcal{S})}$, which means that $c_k^{(\mathcal{G}')}$ is not a concave function of (P_1, P_2) for this k. It is clear that the same is true if other rate region, instead of $\mathcal{G}'$, is used in combination with Sato's two-mode scheme.

The cases that are of particular interest are $k = 0.5, 1, 2$, since the corresponding $c_k^{(\mathcal{G}')}$ can be obtained from Eq. (3), Eq. (2) and Eq. (4), respectively. In particular, when $k = 1$, c_1 specifies the achievable sum rate which is of the most interest. This case is discussed below.

B. Achievable sum rate

When $k=1$, $c_1^{(\mathcal{G}')}$ and $c_1^{(\mathcal{S})}$, are the respective sum rates for these two achievable rate regions. Based on Eq. (2), we have

$$c_1^{(\mathcal{G}')} = \max_{\beta_1,\beta_2\in[0,1]} \rho_{12}$$

and

$$c_1^{(\mathcal{S})} = \max_{\alpha\in[0,0.5]} \left\{ \alpha c_1^{(\mathcal{G}'(P_1',P_2'))} + \bar{\alpha} c_1^{(\mathcal{G}'(P_1'',P_2''))} \right\} \tag{14}$$

with the constraint in Eq. (11).

The sum capacity of the general Gaussian IFC is still an open problem. It is known only in some special cases, including the strong and very strong IFC [7], the degraded Gaussian IFC [1], and the Gaussian Z channel [3]. We summarize them below

- Degraded channel, $ab=1$

$$C_s = \begin{cases} \gamma(P_1) + \gamma\left(\frac{P_2}{1+bP_1}\right), & a \geq 1 \\ \gamma(P_2) + \gamma\left(\frac{P_1}{1+aP_2}\right), & b \geq 1 \end{cases} \tag{15}$$

- Z channel, $a=0$

$$C_s = \begin{cases} \gamma(P_1) + \gamma\left(\frac{P_2}{1+bP_1}\right), & 0 \leq b \leq 1 \\ \gamma(bP_1 + P_2), & 1 \leq b \leq 1 + P_2 \\ \gamma(P_1) + \gamma(P_2), & b > 1 + P_2 \end{cases} \tag{16}$$

- Very strong IFC, $a \geq 1 + P_1$, $b \geq 1 + P_2$

$$C_s = \gamma(P_1) + \gamma(P_2) \tag{17}$$

- Strong IFC, $1 \leq a < 1 + P_1$, $1 \leq b < 1 + P_1$

$$C_s = \min\left\{\gamma(P_1 + aP_2), \gamma(bP_1 + P_2)\right\} \tag{18}$$

In all above cases, the sum capacity C_s can be achieved by $\mathcal{G}'$. Therefore $c_1^{(\mathcal{S})} = c_1^{(\mathcal{G}')} = C_s$ at the given power constraint, i.e., $c_1^{(\mathcal{S})}$ does not improve upon $c_1^{(\mathcal{G}')}$. We show that in all the cases, $c_1^{(\mathcal{G}')}$ is a concave function of (P_1, P_2) in the whole $(P_1, P_2) \in [0,\infty) \times [0,\infty)$ range.

It is straightforward to check that $c_1^{(\mathcal{G}')} = C_s$ is concave of (P_1, P_2) for Eq. (15) and Eq. (16). For the very strong and strong IFC in Eq. (17) and Eq. (18), it is much more complicated since $c_1^{(\mathcal{G}')} = C_s$ does not hold for the entire $(P_1, P_2) \in [0,\infty) \times [0,\infty)$ range. However, we can proceed to rewrite Eq. (17) and Eq. (18) in a unified way: For a standard Gaussian IFC with $a \geq 1$ and $b \geq 1$,

$$\begin{aligned} C_s &= \begin{cases} \gamma(P_1) + \gamma(P_2), & 0 \leq P_1 \leq a-1,\ 0 \leq P_2 \leq b-1 \\ \min\{\gamma(P_1 + aP_2), \gamma(bP_1 + P_2)\}, & P_1 > a-1,\ P_2 > b-1 \\ \gamma(bP_1 + P_2), & 0 \leq P_1 \leq a-1,\ P_2 > b-1 \\ \gamma(P_1 + aP_2), & P_1 > a-1,\ 0 \leq P_2 \leq b-1 \end{cases} & (19) \\ &= \min\{\gamma(P_1) + \gamma(P_2), \gamma(bP_1 + P_2), \gamma(P_1 + aP_2)\} & (20) \end{aligned}$$

The first and second equations of (19) are very strong and strong IFC. The third and forth equations are actually equivalent to Z channel with $a = 0$ and $b = 0$ respectively, since one of the user can decode and subtract all the interference signals from the other user before decoding its own signals. Eq. (19) reduces to Eq. (20), which is a concave function in $(P_1, P_2) \in [0, \infty) \times [0, \infty)$ range. Therefore, with $c_1^{(\mathcal{G}')} = C_s$, the convex combination operation can not improve upon $c_1^{(\mathcal{G}')}$.

The more interesting cases are those when $c_1^{(\mathcal{S})} > c_1^{(\mathcal{G}')}$. One example is the moderate IFC [4] [2], where FDM outperforms $\mathcal{G}'$ in terms of sum rate. Clearly, since $\mathcal{F} \subseteq \mathcal{S}$, we have $c_1^{(\mathcal{S})} > c_1^{(\mathcal{G}')}$. Summarizing, the general conditions for $c_1^{(\mathcal{S})} > c_1^{(\mathcal{G}')}$ for a given IFC is that, $c_1^{(\mathcal{G}')}$ is not a concave function throughout $[0, +\infty) \times [0, +\infty)$ power range, and the power constraint for this IFC happens to be at the point such that one can find two pairs of power constraints that satisfy Eq. (11) and the convex combination of the corresponding rate pairs improves upon $c_1^{(\mathcal{G}')}$. We will illustrate this with an example in Section V.

C. The relation of $\mathcal{S}$ with $\mathcal{D}$ and $\mathcal{F}$ from a geometric viewpoint

In the previous section, we show that $\mathcal{S}$ can be equivalently obtained by computing $c_k^{(\mathcal{S})}$ via convex combination of all $c_k^{(\mathcal{G}')}$. Each $(\alpha, \lambda_1, \lambda_2)$ triple determine a line segment passing through the point (P_1, P_2) with two extreme points at (P_1', P_2') and (P_1'', P_2''). In the same way, we can give geometric interpretation of $\mathcal{S}$ in comparison with $\mathcal{D}$ and $\mathcal{F}$.

As we have elaborated before, the advantage of Sato's modified FDM scheme is attributed to its inherent convex combination operation of two pairs of rates in two subbands with power contraints in the entire real line. The same argument applies to Sason's rate region. However, comparing Eq. (5) with Eq. (7), one can see that the corresponding rate pairs $\left(R_1^{(1)}, R_2^{(1)}\right)$ and $\left(R_1^{(2)}, R_2^{(2)}\right)$ of $\mathcal{D}$ are chosen from special cases of $\mathcal{G}'$, whose achievable c_k is no larger than that of $\mathcal{G}'$. Therefore, the obtained $c_k^{(\mathcal{D})}$ by taking the convex combination of some $c_k \le c_k^{(\mathcal{G}')}$ will be no larger than $c_k^{(\mathcal{S})}$. That is why $\mathcal{D}$ may be better than $\mathcal{G}'$ but $\mathcal{D} \subseteq \mathcal{S}$ is always true.

In Proposition 1, $\mathcal{S}$ reduces to $\mathcal{F}$ by setting $\lambda_1 = 1$, $\lambda_2 = 0$, and $\lambda_1 = 0$, $\lambda_2 = 1$. Then, $c_k^{(\mathcal{F})}$ is obtained by taking the weighted sum in Eq. (13) with a more restricted power constraint $P_1' = \frac{P_1}{\alpha}$, $P_1'' = 0$ and $P_2' = 0$, $P_2'' = \frac{P_1}{\bar{\alpha}}$ in addition to the constraint in Eq. (11). Therefore, $c_k^{(\mathcal{F})}$ is also a convex combination of $c_k^{(\mathcal{G}')}$, but instead of using the all the points in the $(P_1, P_2) \in [0, +\infty) \times [0, \infty)$ plane like $c_k^{(\mathcal{S})}$, it only uses the points along two lines $P_1 = 0$, $P_2 \in [0, \infty)$ and $P_2 = 0$, $P_1 \in [0, \infty)$. Thus, $\mathcal{F} \subseteq \mathcal{S}$.

V. Numerical Results

Figure 1 is the achievable rate region for a symmetric IFC with $a = b = \frac{1}{3}$ and $P_1 = P_2 = 6$, in which $\mathcal{G}' \subset \mathcal{S} = \mathcal{D}$. This is a moderate interference case [3]. Since the sum rate of $\mathcal{F}$ will outperform that of $\mathcal{G}'$, and $\mathcal{F} \subset \mathcal{S}$, we have $c_1^{(\mathcal{S})} > c_1^{(\mathcal{G}')}$ at the point $(P_1 = 6, P_2 = 6)$, i.e. $c_1^{(\mathcal{G}')}$ is not a concave function. This is shown in Figures 2 and 3. Figure 2 is $c_1^{(\mathcal{G}')}$ as a function of (P_1, P_2), where M is the point of the power constraint $(P_1 = 6, P_2 = 6)$, while $E(P_1'^*, P_2'^*)$ and $F(P_1''^*, P_2''^*)$ are the two extreme points of the line segment passing through M. The power constraints of E and F satisfy (11), and the convex combination of the rates for E and F gives the largest achievable $c_1^{(\mathcal{S})}$. Figure 3 is a two dimensional plot obtained by intersecting $c_1^{(\mathcal{G}')}$ with a plane vertical to $P_1 O P_2$ and passing through the two points E and

F. This intersection is also illustrated in Figure 2. Clearly $c_1^{(\mathcal{S})}$ improves most upon $c_1^{(\mathcal{G}')}$ at M by using E and F. Since E and F are on the line $\overline{OP_1}$ and $\overline{OP_2}$, we have $c_1^{(\mathcal{S})} = c_1^{(\mathcal{F})}$, and hence $c_1^{(\mathcal{S})} = c_1^{(\mathcal{D})}$, as is shown in Figure 1.

Figure 4 gives an example that $\mathcal{G}' \subset \mathcal{S}$ and $\mathcal{D} \subset \mathcal{S}$, i.e., both $\mathcal{G}'$ and $\mathcal{D}$ are strictly subsets of $\mathcal{S}$. It is an asymmetric IFC with $a = 0.25$, b=1.44, $P_1 = 6$ and $P_2 = 2$. In this case $c_2^{(\mathcal{G}')}$ is not concave and $c_2^{(\mathcal{S})} > c_2^{(\mathcal{G}')}$ as is shown in Figures 5 and 6.

VI. Conclusion

A new achievable rate region of general Gaussian IFC is obtained by combining the typical HK region $\mathcal{G}'$ with Sato's modified FDM/TDM idea. This new region is shown to improve upon $\mathcal{G}'$, Sason's rate region, and that of FDM/TDM.

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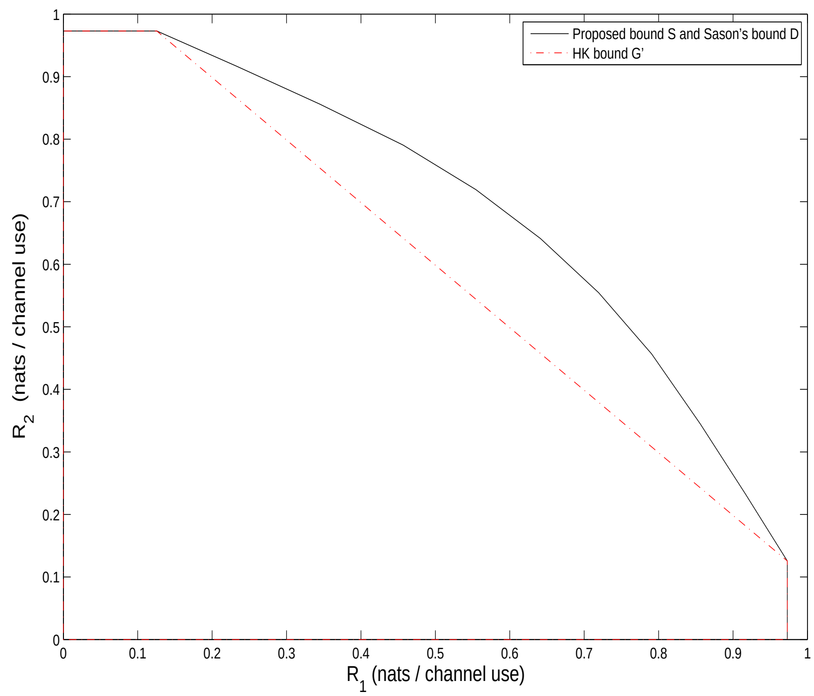

Fig. 1. The achievable rate region for IFC with $a = b = \frac{1}{3}$, and $P_1 = P_2 = 6$.

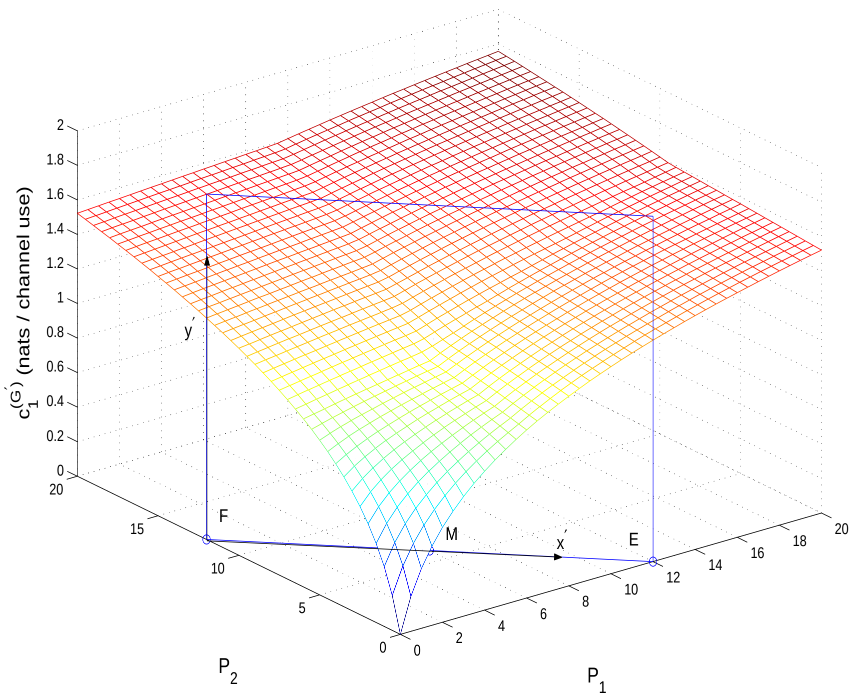

Fig. 2. The rate $c_1^{(\mathcal{G}')} = \max_{(R_1,R_2)\in\mathcal{G}'} \{R_1 + R_2\}$, as a function of (P_1, P_2).

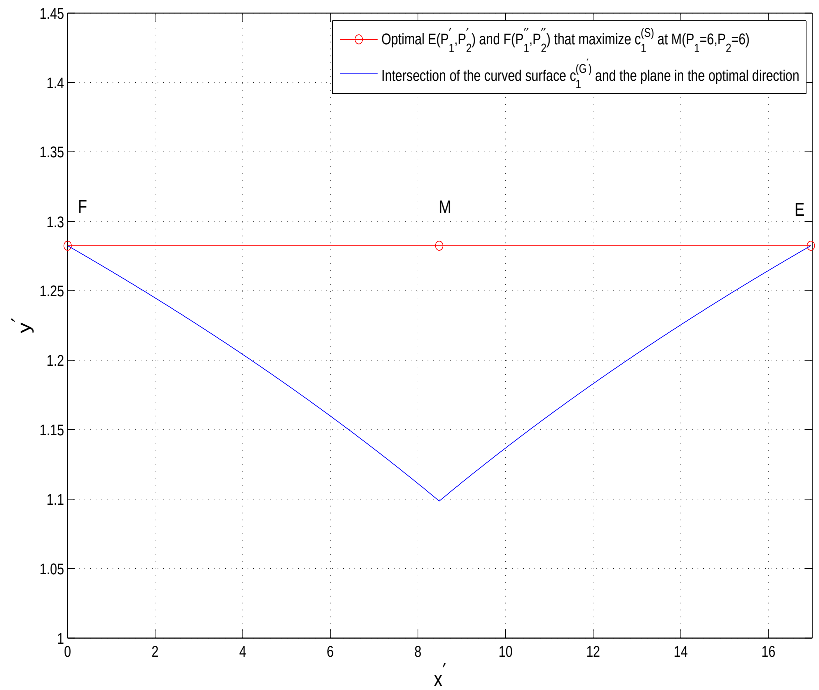

Fig. 3. The intersection of the curved surface $c_1^{(\mathcal{G}')}$ and the plane vertical to P_1OP_2 and passing through optimal combination points E and F in Figure 2.

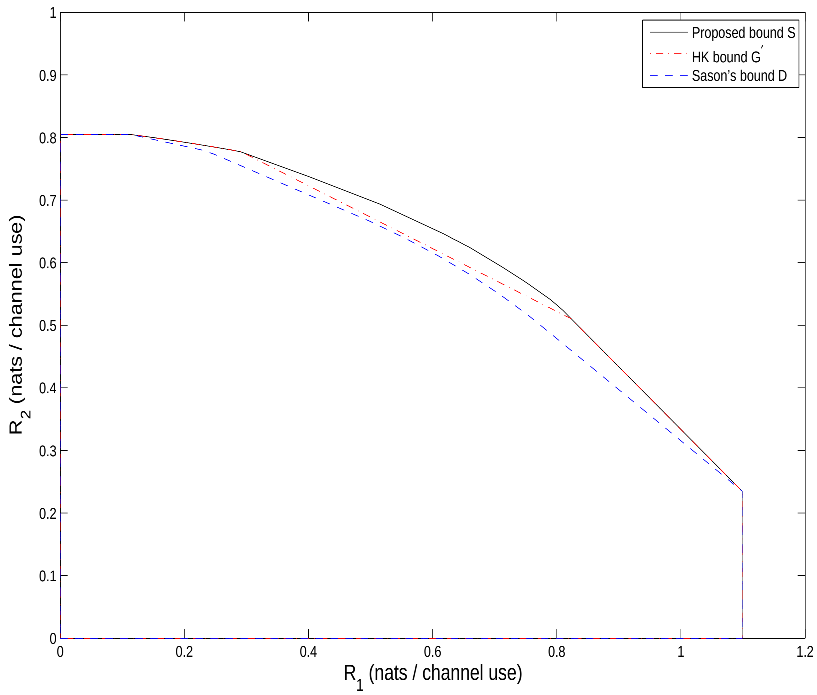

Fig. 4. The achievable rate region for IFC with $a = 0.25$, $b = 1.44$, $P_1 = 6$ and $P_2 = 2$

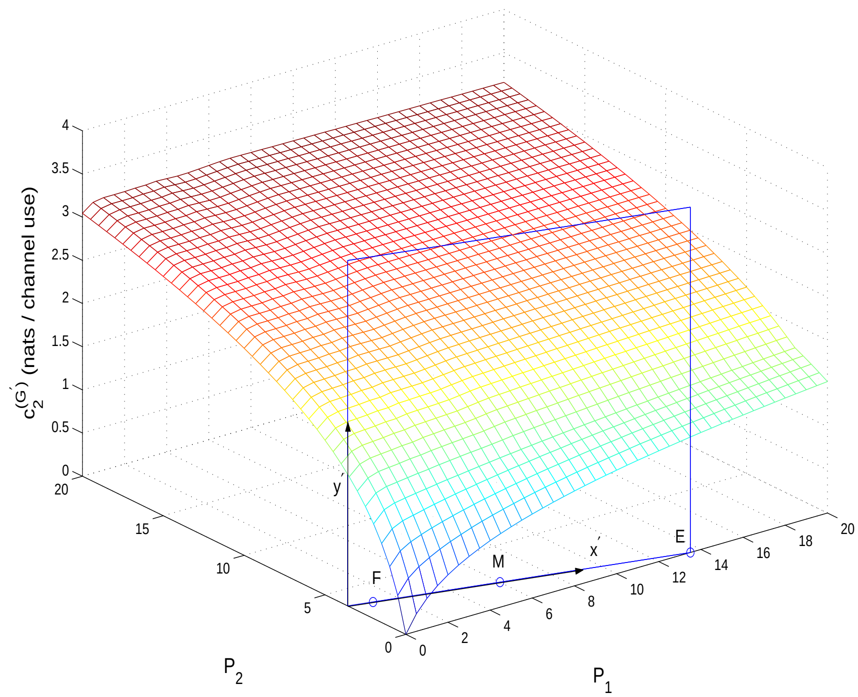

Fig. 5. The rate $c_2^{(\mathcal{G}')} = \max_{(R_1,R_2)\in\mathcal{G}'}\{R_1 + 2R_2\}$, as a function of (P_1, P_2)

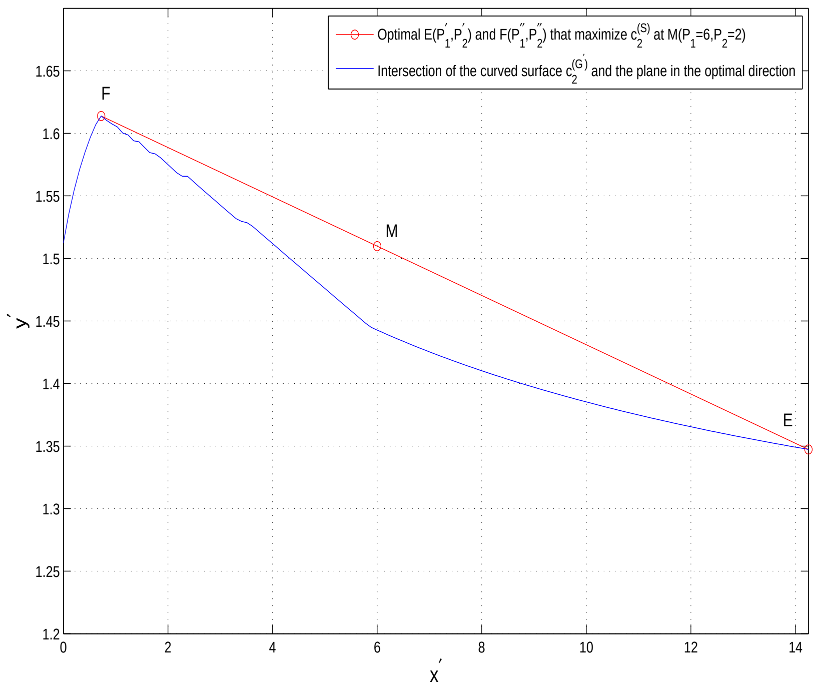

Fig. 6. The intercept of the curved surface $c_2^{(\mathcal{G}')}$ and the plane vertical to P_1OP_2 and passing through optimal combination points E and F in Figure 5