Coupling of quantum angular momenta: an insight into analogic/discrete and local/global models of computation

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Abstract In the past few years there has been a tumultuous activity aimed at introducing novel conceptual schemes for quantum computing. The approach proposed in (Marzuoli A and Rasetti M 2002, 2005a) relies on the (re)coupling theory of SU(2) angular momenta and can be viewed as a generalization to arbitrary values of the spin variables of the usual quantum–circuit model based on 'qubits' and Boolean gates. Computational states belong to finite–dimensional Hilbert spaces labelled by both discrete and continuous parameters, and unitary gates may depend on quantum numbers ranging over finite sets of values as well as continuous (angular) variables. Such a framework is an ideal playground to discuss discrete (digital) and analogic computational processes, together with their relationships occuring when a consistent semiclassical limit takes place on discrete quantum gates. When working with purely discrete unitary gates, the simulator is naturally modelled as families of quantum finite states–machines which in turn represent discrete versions of topological quantum computation models.
We argue that our model embodies a sort of unifying paradigm for computing inspired by Nature and, even more ambitiously, a universal setting in which suitably encoded quantum symbolic manipulations of combinatorial, topological and algebraic problems might find their 'natural' computational reference model.

Key words:
analogic and discrete models of computation; binary coupling of quantum angular momenta; quantum automata; quantum complexity theory; Racah–Wigner algebra; $SU(2)$ recoupling theory; topological quantum computation.

Introduction

> " ...Of course, we might get useful ideas from studying how the brain works, but we must remember that automobiles do not have legs like cheetas nor do airplanes flap their wings! We do not need to study the neurologic minutiae of living things to produce useful technologies; but even wrong theories may help in designing machines. Anyway, you can see that computer science has more than just technical interest." (Feynman, 1996)

Richard Feynman was interested in computer science mainly in the last few years of his life, and the recent blossoming of quantum information theory and computing makes his 'Simulating physics with computers' (Feynman, 1982) a prescient paper in that field of research.
Leaving aside Feynman's somehow iconoclastic quotation, let us go over John von Neumann's deep vision of computing inspired by the complex living organism par excellence, the brain. In his lecture at the Hixon Symposium 'Cerebral mechanisms in behaviour' published in (von Neumann, 1951), he discussed first of all the 'dicotomy of the problem': the dialectic between the task of modelling of elementary computational units, on the one hand, and analysing the interconnections among such units on the other. In his words:

> " The first part of the problem is at present the dominant one in physiology. It is closely connected with the most difficult chapters of organic chemistry and of physical chemistry, and may in due course be greately helped by quantum mechanics. (...)
> The second part, on the other hand, is the one which is likely to attract those of us who have the background and the tastes of a mathematician or a logician. With this attitude, we will be inclined to remove the first part of the problem by the process of axiomatization, and concentrate on the second one."

Of course computer models constrained by neurobiological data can help reveal how the physical properties of networks of neurons can be used to encode information at both the levels introduced by von Neumann. Moreover, in the emerging field of computational neuroscience people are currently addressing both top–down and bottom–up approaches (Churchland and Sejnowski, 1992) and the borderline between the concepts of 'elementary computational unit' and 'networks made of units' could be destined to fade out

at some sufficiently small scale.
Taking by now for granted von Neumann's axiomatic procedure –namely treating units as 'black boxes', the inner structure of which need not to be be disclosed– his analysis goes on by illustrating the analogy and digital principles on which computational processes can be based. Coming to the point, he concludes that:

> " When the central nervous system is examined, elements of both procedures, digital and analogic, are discernible. (...) It is well known that there are various composite functional sequences in the organism which have to go through a variety of steps from the original stimulus to the ultimate effect –some of the steps being neural, that is, digital, and others humoral, that is analogic. These digital and analogical portions in such a chain may alternately multiply."

In the following we shall see how a mixed (analogic and discrete) model of 'quantum' simulator arises naturally from an advanced branch of quantum theory of angular momentum. Such a 'spin network' model, together with its semiclassical counterpart, might indeed represent a sort of unifying paradigm embracing analog/discrete, microscopic/macroscopic, local/global features of computing processes inspired by Nature on the one hand, and a powerful implementation of quantum symbolic manipulation, on the other.

Mixed quantum computing: the spin network simulator

The theory of binary coupling of $N = n + 1$ $SU(2)$ angular momenta represents the generalization to an arbitrary N of the coupling of two angular momentum operators $\mathbf{J}_1, \mathbf{J}_2$ which involves Clebsch–Gordan (or Wigner) coefficients in their role of unitary transformations between uncoupled and coupled basis vectors, $|j_1\, m_1 > \otimes |j_2\, m_2 >$ and $|j_1\, j_2; JM >$ respectively. The quantum numbers j_1, j_2 associated with $\mathbf{J}_1, \mathbf{J}_2$ label irreducible representations of $SU(2)$ ranging over $\{0, 1/2, 1, 3/2, \ldots\}$; m_1, m_2 are the magnetic quantum numbers, $-j_i \leq m_i \leq j_i$ in integer steps; J is the spin quantum number of the total angular momentum operator $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ whose magnetic quantum number is $M = m_1 + m_2$, $-J \leq M \leq J$. Here units are chosen for which $\hbar = 1$ and we refer to (Varshalovich et al. , 1988) for a complete

account on the theory of angular momentum in quantum physics. On the other hand, $SU(2)$ 'recoupling' theory –which deals with relationships between distinct binary coupling schemes of N angular momentum operators– is a generalization to any N of the simplest case of three operators $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ which calls into play unitary transformations known as Racah coefficients or $6j$ symbols. A full fledged review on this advanced topic in the general framework of Racah–Wigner algebra can be found in (Biedenharn and Louck, 1981).

The architecture of the 'spin network' simulator proposed in (Marzuoli and Rasetti, 2002) and worked out in (Marzuoli and Rasetti, 2005a) relies extensively on recoupling theory and can be better summarized by resorting to a combinatorial setting proposed by the same authors in (Marzuoli and Rasetti, 2005b).
The computational space is there modelled as an $SU(2)$–fiber space structure over a discrete base space V

$$(V,\, \mathbb{C}^{2J+1},\, SU(2)^J)_n \tag{1}$$

which encodes all possible computational Hilbert spaces as well as unitary gates for any fixed number $N = n + 1$ of incoming angular momenta.

• The base space $V \doteq \{v(\mathfrak{b})\}$ represents the vertex set of a regular, 3–valent graph $\mathfrak{G}_n(V, E)$ whose cardinality is $|V| = (2n)!/n!$. There exists a one–to–one correspondence

$$\{v(\mathfrak{b})\} \longleftrightarrow \{\mathcal{H}_n^J(\mathfrak{b})\} \tag{2}$$

between the vertices of $\mathfrak{G}_n(V, E)$ and the computational Hilbert spaces of the simulator.

The label $\mathfrak{b}$ above has the following meaning –on which we shall extensively return later on: for any given pair $(n, \mathbf{J})$, all binary coupling schemes of the $n + 1$ angular momenta $\{\mathbf{J}_\ell\}$, identified by the quantum numbers $j_1, \ldots, j_{n+1}$ plus $k_1, \ldots, k_{n-1}$ (corresponding to the $n - 1$ intermediate angular momenta $\{\mathbf{K}_i\}$) and by the brackets defining the binary couplings, provide the 'alphabet' in which quantum information is encoded (the rules and constraints of bracketing are instead part of the 'syntax' of the resulting coding language). The Hilbert spaces $\mathcal{H}_n^J(k_1, \ldots, k_{n-1})$ thus generated, each $(2J+1)$-dimensional, are spanned by complete orthonormal sets of states with quantum number label set $\mathfrak{B}$ such as, *e.g.* for $n = 3$, $\{\big(\big(j_1\big(j_2 j_3\big)_{k_1}\big)_{k_2} j_4\big)_j\,,$ $\big(\big(j_1 j_2\big)_{k_1'}\big(j_3 j_4\big)_{k_2'}\big)_j\}$.

More precisely, for a given value of n, $\mathcal{H}_n^J(\mathfrak{b})$ is the simultaneous eigenspace of the squares of $2(n+1)$ Hermitean, mutually commuting angular momentum operators $\mathbf{J}_1,\ \mathbf{J}_2,\ \mathbf{J}_3, \ldots, \mathbf{J}_{n+1}$ with fixed sum $\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \ldots + \mathbf{J}_{n+1} \ = \ \mathbf{J}$, of the intermediate angular momentum operators $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$, $\ldots$, $\mathbf{K}_{n-1}$ and of the operator J_z (the projection of the total angular momentum $\mathbf{J}$ along the quantization axis). The associated quantum numbers are $j_1, j_2, \ldots, j_{n+1}$; J; $k_1, k_2, \ldots,\ k_{n-1}$ and M, where $-J \leq M \leq$ in integer steps. If $\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2} \otimes \cdots \otimes \mathcal{H}^{j_n} \otimes \mathcal{H}^{j_{n+1}}$ denotes the factorized Hilbert space, namely the $(n+1)$–fold tensor product of the individual eigenspaces of the $(\mathbf{J}_\ell)^2$'s, the operators $\mathbf{K}_i$'s represent intermediate angular momenta generated, through Clebsch–Gordan series, whenever a pair of $\mathbf{J}_\ell$'s are coupled. As an example, by coupling sequentially the $\mathbf{J}_\ell$'s according to the scheme $(\cdots((\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3) + \cdots + \mathbf{J}_{n+1}) = \mathbf{J}$ – which generates $(\mathbf{J}_1 + \mathbf{J}_2) = \mathbf{K}_1$, $(\mathbf{K}_1 + \mathbf{J}_3) = \mathbf{K}_2$, and so on – we should get a binary bracketing structure of the type $(\cdots(((\mathcal{H}^{j_1} \otimes \mathcal{H}^{j_2})_{k_1} \otimes \mathcal{H}^{j_3})_{k_2} \otimes \cdots \otimes \mathcal{H}^{j_{n+1}})_{k_{n-1}})_J$, where for completeness we add an overall bracket labelled by the quantum number of the total angular momentum J. Note that, as far as j_ℓ's quantum numbers are involved, any value belonging to $\{0, 1/2, 1, 3/2, \ldots\}$ is allowed, while the ranges of the k_i's are suitably constrained by Clebsch–Gordan decompositions (*e.g.* if $(\mathbf{J}_1 + \mathbf{J}_2) = \mathbf{K}_1 \Rightarrow |j_1 - j_2| \leq k_1 \leq j_1 + j_2$). We denote a binary coupled basis of $(n+1)$ angular momenta in the JM–representation and the corresponding Hilbert space introduced in (2) as

$$\{\, |\, [j_1,\, j_2,\, j_3, \ldots, j_{n+1}]^{\mathfrak{b}}\, ;\, k_1^{\mathfrak{b}},\, k_2^{\mathfrak{b}}, \ldots, k_{n-1}^{\mathfrak{b}}\, ;\, JM\, \rangle,\ -J \leq M \leq J\}$$

$$= \ \mathcal{H}_n^J\,(\mathfrak{b}) \ \doteq \ \text{span}\ \{\, |\, \mathfrak{b}\, ;\, JM\, \rangle_n\, \}\,, \tag{3}$$

where the string inside $[j_1,\, j_2,\, j_3, \ldots, j_{n+1}]^{\mathfrak{b}}$ is not necessarily an ordered one, $\mathfrak{b} \in \mathfrak{B}$ indicates the current binary bracketing structure and the k_i's are uniquely associated with the chain of pairwise couplings selected by $\mathfrak{b}$.

• For a given value of J each $\mathcal{H}_n^J(\mathfrak{b})$ has dimension $(2J+1)$ over $\mathbb{C}$ and thus there exists one isomorphism

$$\mathcal{H}_n^J(\mathfrak{b}) \quad \cong_{\mathfrak{b}} \quad \mathbb{C}^{2J+1} \tag{4}$$

for each admissible binary coupling scheme $\mathfrak{b}$ of $(n+1)$ incoming spins. The vector space $\mathbb{C}^{2J+1}$ is naturally interpreted as the typical fiber attached to each vertex $v(\mathfrak{b}) \in V$ of the fiber space structure (1) through the isomorphism (4). In other words, Hilbert spaces corresponding to different

bracketing schemes, although isomorphic, are not identical since they actually correspond to (partially) different complete sets of physical observables, namely for instance $\{\mathbf{J}_1^2,\, \mathbf{J}_2^2,\, \mathbf{J}_{12}^2,\, \mathbf{J}_3^2,\, \mathbf{J}^2,\, J_z\}$ and $\{\mathbf{J}_1^2,\, \mathbf{J}_2^2,\, \mathbf{J}_3^2,\, \mathbf{J}_{23}^2,\, \mathbf{J}^2,\, J_z\}$ respectively (in particular, $\mathbf{J}_{12}^2$ and $\mathbf{J}_{23}^2$ cannot be measured simultaneously). On the mathematical side this remark reflects the fact that the tensor product $\otimes$ is not an associative operation.

• For what concerns unitary operations acting on the computational Hilbert spaces (3), we examine first unitary transformations associated with recoupling coefficients ($3nj$ symbols) of $SU(2)$ (j–gates in the present quantum computing context). As shown in (Biedenharn and Louck, 1981) any such coefficient can be splitted into 'elementary' j–gates, namely Racah and phase transforms. A Racah transform applied to a basis vector is defined formally as $\mathcal{R} \;: |\ldots(\,(a\,b)_d\,c)_f\ldots; JM\rangle \mapsto |\ldots(a\,(b\,c)_e\,)_f\ldots; JM\rangle$, where Latin letters $a, b, c, \ldots$ are used here to denote generic, both incoming (j_ℓ's in the previous notation) and intermediate (k_i's) spin quantum numbers. Its explicit expression reads

$$|(a\,(b\,c)_e\,)_f\,; M\rangle$$

$$= \sum_d (-1)^{a+b+c+f}\,[(2d+1)(2e+1)]^{1/2} \left\{ \begin{array}{ccc} a & b & d \\ c & f & e \end{array} \right\} |(\,(a\,b)_d\,c)_f\,; M\rangle, \quad (5)$$

where there appears the $6j$ symbol of $SU(2)$ and f plays the role of the total angular momentum quantum number. Note that, according to the Wigner–Eckart theorem, the quantum number M (as well as the angular part of wave functions) is not altered by such transformations, and that the same happens with $3nj$ symbols. On the other hand, the effect of a phase transform amounts to introducing a suitable phase whenever two spin labels are swapped

$$|\ldots(a\,b)_c\ldots; JM\rangle \;=\; (-1)^{a+b-c}\,|\ldots(b\,a)_c\ldots; JM\rangle. \quad (6)$$

These unitary operations are combinatorially encoded into the edge set $E = \{e\}$ of the graph $\mathfrak{G}_n(V, E)$: E is just the subset of the Cartesian product $(V \times V)$ selected by the action of these elementary j–gates. More precisely, an (undirected) arc between two vertices $v(\mathfrak{b})$ and $v(\mathfrak{b}')$

$$e\,(\mathfrak{b}, \mathfrak{b}') \;\doteq\; (v(\mathfrak{b}),\, v(\mathfrak{b}')) \;\in\; (V \times V) \quad (7)$$

exists if, and only if, the underlying Hilbert spaces are related to each other by an elementary unitary operation (5) or (6). Note also that elements in E can be considered as mappings $(V \times \mathbb{C}^{2J+1})_n \longrightarrow (V \times \mathbb{C}^{2J+1})_n$

$$(v(\mathfrak{b}),\, \mathcal{H}_n^J(\mathfrak{b})\,) \mapsto (v(\mathfrak{b}'),\, \mathcal{H}_n^J(\mathfrak{b}')\,) \tag{8}$$

connecting each given decorated vertex to one of its nearest vertices and thus define a 'transport prescription in the horizontal sections' belonging to the total space $(V \times \mathbb{C}^{2J+1})_n$ of the fiber space (1). The crucial feature that characterizes the graph $\mathfrak{G}_n(V, E)$ arises from compatibility conditions satisfied by $6j$ symbols in (5), *cfr.* (Varshalovich et al. , 1988). The Racah (triangular) identity, the Biedenharn–Elliott (pentagon) identity and the orthogonality conditions for $6j$ symbols ensure indeed that any simple path in $\mathfrak{G}_n(V, E)$ with fixed endpoints can be freely deformed into any other, providing identical quantum transition amplitudes at the kinematical level.

• To complete the description of the structure $(V,\, \mathbb{C}^{2J+1},\, SU(2)^J)_n$ we call into play M–gates which act on the angular dependence of vectors in $\mathcal{H}_n^J(\mathfrak{b})$ by rotating them. By expliciting such dependence according to

$$\mathcal{H}_n^J(\mathfrak{b}) \doteq \text{span}\,\{\, |\mathfrak{b}; \theta,\, \phi;\, JM\rangle_n \,\}, \tag{9}$$

we write the action of a rotation on a basis vector as

$$|\mathfrak{b}; \theta',\, \phi';\, M'J\,\rangle_n \;=\; \sum_{M=-J}^{J} D^J_{MM'}\,(\alpha\beta\gamma)\,|\mathfrak{b}; \theta,\, \phi;\, JM\,\rangle_n\,, \tag{10}$$

where (θ, ϕ) and (θ', ϕ') are polar angles in the original and rotated coordinate systems, respectively. $D^J_{MM'}\,(\alpha\beta\gamma)$ are Wigner rotation matrices in the JM representation (expressed in terms of Euler angles $(\alpha\beta\gamma)$) which form a group under composition (Varshalovich et al. ,1988). The shorthand notation $SU(2)^J$ employed in (1) actually refers to the group of W–rotations, which in turn can be interpreted as actions of the automorphism group of the fiber $\mathbb{C}^{2J+1}$. Since rotations in the JM representation do not alter the binary bracketing structure of vectors in computational Hilbert spaces we can interpret W–rotation operators as 'transport prescriptions along the fiber'.

The framework outlined above should have made it manifest that we can switch, at will and independently

i) j–gates –represented by $3nj$ recoupling coefficients between distinct binary coupled schemes of $(n+1)$ incoming angular momenta– depending only on discrete parameters, the spin quantum numbers;

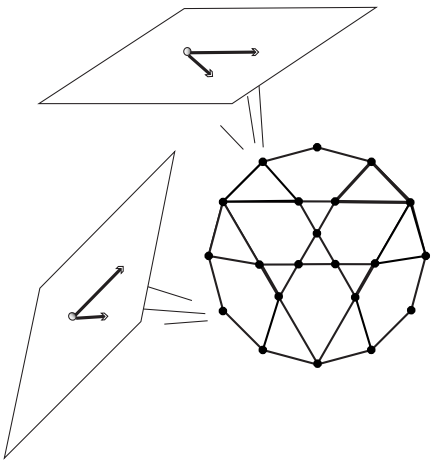

Figure 1: The fiber space structure of the spin network computational space $(V, \mathbb{C}^{2J+1}, SU(2)^J)_n$ for $(n+1) = 4$ incoming angular momenta. The vertices of the graph are in one–to–one correspondence with binary coupled Hilbert spaces: all of them have dimension $(2J+1)$, for some fixed J, and are visualized by 'blowing up' each vertex into a plane. M–gates can be thought of as 'rotations' applied to vectors (depicted as arrows) lying inside any such space. On the other hand, each edge of the graph represents here one Racah transform (phase transforms are not taken into account for simplicity) and each connected path in the graph corresponds to a particular $3nj$ recoupling coefficient (j–gate).

ii) M–gates –represented by Wigner rotation matrices– depending essentially on continuous (angular) parameters, as summarized pictorially in Fig. 1.
These features, which rely on the discreteness of the base space V and on the 'triviality' of transport laws in the total space $(V \times \mathbb{C}^{2J+1})_n$, make the computational space of the spin network simulator an ideal arena for implementing and keeping under control 'mixed' computational processes at the quantum level.

Before entering into more details on this particular issue, let us point out that our model of quantum simulator actually complies with a variety of computing schemes, ranging from (analogical and/or discrete, digital) circuit-type models and finite state–automata up to discretized versions of 'topological' quantum computation (Marzuoli and Rasetti, 2005a). Inside each of these classes we may also consider different types of quantum algorithms (associated with 'programs') which of course must depend on the

particular 'encoding scheme' adopted for the problem we choose to treat. In this respect, as we shall see in the next section, problems from low dimensional topology, geometry, group theory and graph theory (rather than from number theory) are particularly suitable to be addressed in this 'quantum–combinatorial' framework.

Looking for the time being at the simulator as a (not Boolean) quantum circuit model (an all–purpose machine able to implement in principle any computations, so we do not need to specify here a problem nor an encoding scheme) we have to choose a program, one particular input state and a set of (accepted) output states. A program is a collection of step–by–step transition rules (gates), namely a family of 'elementary unitary operations' (Racah, phase trasform or yet Wigner rotation operators for fixed values of the Euler angles) and we assume that it takes one unit of the intrinsic discrete time variable to perform anyone of them. In the combinatorial setting described above such prescriptions amount to select a family of 'directed paths' in the fiber space structure $(V,\, \mathbb{C}^{2J+1},\, SU(2)^J)_n$ (*cfr.* Fig. 1), all starting from the same input state and ending in an admissible output state. A single path in this family is associated with a particular algorithm supported by the given program. By a directed path $\mathcal{P}$ with fixed endpoints we mean a (time) ordered sequence

$$|\mathfrak{v}_{\rm in}\rangle_n \equiv |\mathfrak{v}_0\rangle_n \to |\mathfrak{v}_1\rangle_n \to \cdots \to |\mathfrak{v}_s\rangle_n \to \cdots \to |\mathfrak{v}_L\rangle_n \equiv |\mathfrak{v}_{\rm out}\rangle_n\ , \qquad (11)$$

where we use the shorthand notation $|\mathfrak{v}_s\rangle_n$ for computational states and $s = 0, 1, 2, \ldots, L(\mathcal{P})$ is the lexicographical labelling of the states along the path. $L(\mathcal{P})$ is the length of the path $\mathcal{P}$ and $L(\mathcal{P}) \cdot \tau \doteq T$ is the time required to perform the process in terms of the discrete time unit τ.

A circuit–type computation consists in evaluating the expectation value of the unitary operator $\mathfrak{U}_{\mathcal{P}}$ associated with the path $\mathcal{P}$, namely

$$\langle \mathfrak{v}_{\rm out}\,|\,\mathfrak{U}_{\mathcal{P}}\,|\,\mathfrak{v}_{\rm in}\rangle_n. \qquad (12)$$

By taking advantage of the possibility of decomposing $\mathfrak{U}_{\mathcal{P}}$ uniquely into an ordered sequence of elementary gates, (12) becomes

$$\langle \mathfrak{v}_{\rm out}\,|\,\mathfrak{U}_{\mathcal{P}}\,|\,\mathfrak{v}_{\rm in}\rangle_n \;=\; \lfloor \prod_{s=0}^{L-1} \langle \mathfrak{v}_{s+1}\,|\,\mathcal{U}_{s,s+1}\,|\,\mathfrak{v}_s\rangle_n \rfloor_{\mathcal{P}} \qquad (13)$$

with $L \equiv L(\mathcal{P})$ for short. The symbol $\lfloor\ \rfloor_{\mathcal{P}}$ denotes the ordered product along the path $\mathcal{P}$ and each elementary operation is rewritten as $\mathcal{U}_{s,s+1}$

($s = 0, 1, 2, \ldots L(\mathcal{P})$) to stress its 'one–step' character with respect to such a circuit–type computation.

It is worth noticing that actual computation –namely the choice of families of directed paths in the simulator's computational space $(V, \mathbb{C}^{2J+1}, SU(2)^J)_n$– breaks the invariance with respect to 'intrinsic time–translations' which holds instead at the purely kinematical level (we have to specify the ordering in (11) and (13)). Moreover, different types of evolutions can be grouped into 'computing classes' according to the nature of the gates that a particular program has to employ (see Section 4.2 of (Marzuoli and Rasetti, 2005a) for more details). Of course, a computing class that alternates (a finite number of) j and M–gates is the most general one and, as pointed out before, its kind of behavior is exactly what we need to implement mixed quantum computation. In the pictorial representation given above, this would amount to 'move' a vector in a nearest vertex along the graph and then rotate it inside each space, alternatively.

Looking now in particular at the analogic mode –namely just acting with sequences of rotations inside one space– we argue that the spin network simulator plays the role of 'universal' quantum analog machine, despite von Neumann's early claim that " 'universal' quantum analogical machines do not make sense" (he refers of course to classical devices which are based on a variety of physical mechanisms, not sharing unifying principles, *cfr.* (von Neumann, 1951)). More precisely, such analogic processes belong to M–computing classes containing programs which employ only (finite sequences of) M–gates in their associated directed paths and it would not be difficult to recognize that such kind of computation, when suitably applied to N $\frac{1}{2}$–spins, reproduces the standard Boolean quantum circuit.

On the other hand, a j–computing class includes programs which employ only j-gates at each computational step, namely the only allowed 'moves' are along the edges of the computational graph of Fig. 1. This class is particularly interesting since it shares many features with 'discretized' topological quantum field theories (TQFTs), the so–called state sum models, related in turn with $SU(2)$ Chern–Simons TQFT (*cfr.* the next section). Here the combinatorial structure becomes prominent owing to the existence of a one–to–one correspondence between allowed elementary operations (Racah and phase tranforms) and the edge set E of the graph $\mathfrak{G}_n(V, E)$. Inside this class the selection of (families of) directed paths proceeds as in the most general case illustrated above, but we realize that dynamical processes break as well the combinatorial invariance which holds at the kinematical level (where

paths with fixed endpoints can be freely deformed one into another due to the algebraic identities satisfied by $SU(2)$ $6j$ symbols, which imply that the corresponding quantum amplitudes are equal). When working in such purely discrete modes, the spin network complies with Feynman's requirements for an 'universal' (discrete) simulator (Feynman, 1982) as discussed extensively in (Marzuoli and Rasetti, 2002).

Improving quantum complexity: the 'quantum field computer'

The tremendous efforts spent in the last few years on quantum information processing were motivated and fostered by the single important result constituted by Shor's algorithm. However, they were at the same time frustrated by the fact that the progresses toward successful physical implementation had not been paralled by the discovery of other algorithms, definitely demonstrating the superiority of quantum *vs.* classical computation.

The 1998 pioneering paper by Michael Freedman (from which we borrow the title of this section) opened the possibility of greatly improving standard quantum computing –inspired by the behavior of 'quantum mechanical' physical systems– moving to 'quantum field' theory. His program is outlined in the abstract of (Freedman, 1998):

> The central problem in computer science is the conjecture that two complexity classes, **P** (...) and **NP** (...), are distinct in the standard Turing model of computation: $\mathbf{P} \neq \mathbf{NP}$. As a generality, we propose that each physical theory supports computational models whose power is limited by the physical theory. It is well known that classical physics supports a multitude of implementation of the Turing machine. Non–Abelian topological quantum field theories exhibit the mathematical features to support a model capable of solving all #**P** problems, a computationally intractable class, in polynomial time. Specifically, Witten, in (Witten, 1989), has identified expectation values in a certain $SU(2)$–field theory with values of the Jones polynomial of knots (Jones, 1987) that are #**P**–hard (Jaeger et al. ,1990). This suggest that some physical system whose effective Lagrangian contains a non–Abelian topological term might be manipulated to serve as an analog computer capable of solving **NP** or even #**P**–hard problems in poly-

> nomial time. Defining such a system and addressing the accuracy issues inherent in preparation and measurement is a major unsolved problem.

In a series of papers (*cfr.* (Freedman et al. , 2002) and references therein) this intriguing idea has been worked out in details, both on the theoretical side and in view of actual physical implementation by means of anyonic systems. It is somehow disappointing that these authors provide a proof according to which topological quantum computation based on modular functors of $SU(2)$ Chern–Simons theory is polynomially–reducible to the standard quantum circuit model employing qubits and Boolean elementary quantum gates. This would mean that, after all, the 'quantum field computer' is not more powerful than a quantum Turing machine, rendering unjustified the effort of going through such a conceptually difficult framework. We argue that there is a way out of this dead end, relying on the observation that only a restricted sector of the underlying topological field theory has been involved in showing reducibility, that is to say, just a few degrees of freedom have been actually switched on, hiding the effective performances of this model of computation.

Two comments are in order here. In the quantum approach, the exponentially better efficiency of quantum with respect to classical information manipulation is to a large extent due to the presence of entanglement. One may therefore wonder which is the mechanism that in quantum field computers, in Freedman's sense, promotes the efficiency of quantum vs. classical. It should be observed first of all that the notion of entanglement is strictly speaking proper to first quantization; it refers to the non-separability in quantum mechanics in certain conditions of multi–component superposition states. As such, the notion cannot be exported to the second quantized formalism of (topological) quantum field theory. However, it is well known that in the latter one the basic mechanisms for dynamical evolution is the existence of many degenerate vacua, among which the system can tunnel. The corresponding correlations are generated and carried by soliton–like excitations. In the scheme of (Marzuoli and Rasetti, 2000), based on the recoupling of angular momenta, this feature is implicitly reflected in the property that the single Hilbert spaces entering the tensor product giving life to the global space of states, are all mutually isomorphic, yet different spaces (and hence different recouplings) encode different information, and manipulating information means just transforming a coupling scheme into another.

In Section 6 of (Marzuoli and Rasetti, 2005a) we proved that quantum circuital computational classes of j–type –modelled on the spin network graph $\mathfrak{G}_n(V, E)$ as explained at the end of the previous section– are indeed 'discretized' versions of Freedman and collaborators' 'functors'. In particular, for each fixed n, we can embed paths on the spin network into a (2+1)–dimensional handlebody presentation of the differentiable manifold which support the modular Chern–Simons functor; combinatorial operations on the graph correspond to suitable topological moves on the 'pant decomposition' of 3–manifolds known as Dehn twists. Such discrete counterparts of the topological setting share a number of interesting features:
i) they solve the open problem concerning 'localization' of modular functors;
ii) they can be naturally interpreted as families of 'finite states' automata, in contrast with the somehow disturbing 'analogic' character of the quantum field computer;
iii) the underlying discretized quantum theory belongs to the class of $SU(2)$ 'state sum models' introduced in (Turaev and Viro, 1992) and used extensively also in quantum gravity models (*cfr.* Sections 5 and 6 of (Marzuoli and Rasetti 2005a) and references quoted therein).

For what concerns the issue of quantum complexity, which we would like to focus on for the rest of this section, the key remark is **ii)** above. In the spin network computational space $\mathfrak{G}_n(V, E)$ (for a fixed n) we recognize first of all a finite 'input alphabeth' whose 'letters' are the spin quantum numbers of the incoming angular momenta $\{j_1, j_2, \ldots, j_{n+1}\}$ plus n pairs of brackets $\{(,)\}$ (or, equivalently, n intermediate spin quantum numbers $\{k_1, k_2, \ldots, k_{n-1}\}$, recall (3)). Each of the computational Hilbert spaces $\{\mathcal{H}_n^J\ (\mathfrak{b})\}$ is finite–dimensional according to (4) and unitary operators, associated with $3nj$ symbols and decomposable according to (5) and (6), play the role of of transition functions depending on finite sets of discrete (spin) variables. The inherently step–by–step character of transition functions is associated with the existence of an intrinsic discrete–time variable, denoted by τ in the previous section. These features make the spin network the ideal candidate for a 'general purpose', finite–states and discrete–time machine able to accept any quantum language compatible with the algebra of $SU(2)$ angular momenta on the one hand, and as powerful as the quantum field computer on the other. This last characteristic, in particular, will allow us to address also problems that share a 'global' nature, such as calculating topological invariants of knots and links.

Once discussed the general conceptual scheme, we pass to illustrate our

guiding idea with respect to actual implementation of algorithms. The exponential efficiency that quantum algorithms may achieve *vs.* classical ones might prove especially relevant in addressing problems in which the space of solutions is not only endowed with a numerical representation but is itself characterized by some additional 'combinatorial' structure, definable in terms of a grammar and a syntax and thus suitable to be encoded naturally in the spin network computational framework. There are a number of problems that are not easily formulated in numerical ('digital') terms and that are quite often intractable in classical complexity theory (*cfr.* (Garey and Johnson, 1979)). In combinatorial and algebraic topology typical issues are: the construction of presentations of the fundamental group (or the first homology group) of compact 3–manifolds decomposed as handlebodies; the study of equivalence classes of knots/links in the three–sphere, related in turn to the classification of hyperbolic 3–manifolds; the enumeration of inequivalent triangulations of D–dimensional compact manifolds. As for group theory: the word problem, the coniugacy problem, the isomorphism problem for both finite and finitely presented groups. Finally, a huge number of problems arise in graph theory (the Hamiltonian circuit problem, just to mention one) and we find it intriguing that the the graph underlying the spin network simulator –known as Twist–Rotation graph– turns out to be automatically encoded into the computational quantum space (*cfr.* Appendix A of (Marzuoli and Rasetti, 2005a)).

Just to give an insight into the 'local' and 'global' nature that these types of problems may exhibit, let us turn to the Artin braid group, the representation theory of which enters heavily into many physical applications, ranging from statistical mechanics to (topological) quantum field theories. Historically, three fundamental decision problems were formulated by Max Dehn in 1911 for any finitely presented group G:

• word problem: does there exist an algorithm to determine, for any arbitrary word w in the generators of G, whether or not $w =$ identity in G?

• coniugacy problem: does there exist an algorithm to decide whether any pair of words in the generators of G are conjugate to each other?

• isomorphism problem: given an arbitrary pair of finite presentations in some set of generators, does there exist an algorithm to decide whether the groups they present are isomorphic?

Following the development of the classical theory of algorithms (recursive functions and Turing machine) it is reasonable to expect that Dehn's problems might be recursively solvable or, at least, that the 'local' ones (the word

and the coniugacy problems) be so. It turns out, instead, that not only these problems, but a host of local and global decision problems sharing a combinatorial flavor are unsolvable within such scheme. As for the braid group, an efficient classical algorithm has been found recently for the restricted word problem (deciding whether two words are equal or not), but the best known algorithm for the coniugacy problem is exponential–time with respect to the length of the input word (we refer the reader to (Birman and Brendle, 2004) for an exhaustive review on braid group). The search for quantum algorithms to solve efficiently this kind of problems is a major challenge for improving quantum computation.

Concluding remarks and outlook

Since the spin network quantum circuit can support both analogic and discrete computing processes, in the spirit of von Neumann's quotations reported in the introduction we may ask whether –for example– we can carry out simulations of molecular dynamics in biochemical systems. Typically, organic molecules, by their own, are complex quantum systems, certainly not well modelled on two–level systems as should happen if we keep on using quantum computers handling with qubits. Moreover, any such system displays several 'sources' of angular momentum: electrons' spins, electrons' orbital angular momentum, nuclear spins and also angular momenta associated with rotations of the nuclear axis (or axes) with respect to the laboratory reference frame. Except for electrons' individual spins, all the other sources may have quantum numbers different from 1/2 (and possibly quite large). Experimentally, some types of interactions occuring in such systems are well modelled on two–body interactions, and thus 'binary' coupled states (3) can indeed describe in a quite accurate way the molecule wave function in the reference frame of its center of mass. Different experimental settings do correspond, on the other hand, to different binary coupled schemes, and transitions between pairs of such couplings are described in terms of $3nj$ coefficients, for suitable n (Vincenzo Aquilanti, private communication). On the other hand, the action of an external magnetic field forces angular momenta to align along its direction, and such phenomenon should be well simulated, in some experimental circumstances, by the action of Wigner rotation operators defined in (10).

Having recognized the possibility of simulating (classes of) discrete and

analogic processes at truly quantum scales, let us have a look at a peculiar feature of our model of computation, namely the chance of relating 'quantum'–discrete modes to 'analogic'–classical ones. Recall that purely discrete transition functions at the quantum level are basically implemented by Racah trasforms of the type (5). Going trough the semiclassical limit, where all the angular momentum quantum numbers are $\gg 1$ in $\hbar$ units (or, equivalently, formally letting $\hbar \to 0$), the $6j$ symbol becomes a function of some (suitably defined) angles, and thus acquires a continuous character (*cfr.* (Ponzano and Regge, 1968) and Section 5 of (Marzuoli and Rasetti, 2005a)): accordingly, quantum transition probabilities are turned into classical ones. For what concerns states on which such asymptotic gates act, the inerhently quantum Hilbert space structure (3) is obviuosly lost in approaching the classical limit, but nevertheless we may think of ensembles of 'macroscopic' particles characterized by classical angular momenta and endowed with classical, many–body interactions modelled on two–body ones. Since the asymptotic formula by Ponzano and Regge fits with experimental data already for quantum numbers of the order of few $\hbar$, we argue that some molecular systems might be accurately simulated in this 'analogic' setting. Coming back to Feynman's quotation from (Feynman, 1996), and turning the argument around, we can say that scientists do indeed get useful ideas from studying (neurobiological) molecular systems whose theoretical background is getting more and more well founded. Thus, the hint here is to take these theories as 'true', and rather focus on the search for different, more appropriate models of computation, eventually going beyond the Turing paradigm.

As a final remark, and turning to the issue of quantum complexity, we are currently addressing quantum automaton–type computations for evaluating polynomials of knots and links, including Jones' polynomial (Jones, 1987). This is achieve by encoding into a 'braided' version of the spin network graph $\mathfrak{G}_n(V, E)$ links presented as closures of braids, and preliminary results seem to be encouraging (Garnerone et al., 2005 and 2006).

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