Surface Matching with Large Deformations and Arbitrary Topology: a Geodesic Distance Evolution Scheme on a 3-Manifold

Etienne G. Huot[1], Hussein. M. Yahia[1], Isaac Cohen[2], and Isabelle L. Herlin[1]

[1] INRIA Air Project
BP 105, 78153 Le Chesnay Cedex
France.
Etienne.Huot@inria.fr, Hussein.Yahia@inria.fr, Isabelle.Herlin@inria.fr

[2] Institute for Robotics and Intelligent Systems
University of Southern California
Los Angeles, California 90089-0273, USA.
icohen@iris.usc.edu

Abstract. A general formulation for geodesic distance propagation of surfaces is presented. Starting from a surface lying on a 3-manifold in $\mathbb{R}^4$, we set up a partial differential equation governing the propagation of surfaces at equal geodesic distance (on the 3-manifold) from the given original surface. This propagation scheme generalizes a result of Kimmel *et al.* [11] and provides a way to compute distance maps on manifolds. Moreover, the propagation equation is generalized to any number of dimensions. Using an eulerian formulation with level-sets, it gives stable numerical algorithms for computing distance maps. This theory is used to present a new method for surface matching which generalizes a curve matching method [5]. Matching paths are obtained as the orbits of the vector field defined as the sum of two distance maps' gradient values. This surface matching technique applies to the case of large deformation and topological changes.

1 Introduction and previous work

The theory of front propagation has received a particular attention in the past few years [14, 16, 8]. It sheds light on deriving new methods for computing geodesic paths on surfaces and manifolds [11, 13]; this framework is particularly well suited for answering image processing questions (for instance active contours, deformable templates [10, 12, 3], matching structures [5]). The latter problem of matching structures is very important in computer vision. A general matching formulation can be stated as: given two structures $\mathcal{S}$ and $\mathcal{D}$ define a function χ which associates to any point of $\mathcal{S}$ a corresponding point on $\mathcal{D}$. Introducing structures properties such as their geometry or the underlying image representation allows to characterize a unique matching function χ. Relevant geometrical properties are selected on the basis of their ability to characterize

a description of the structures which is invariant to the considered deformation. In the case of rigid or small elastic deformations high curvature points [4, 15] or semi-differential invariants [19] can be considered as an invariant description of the structure. Matching features in 3D images is also an important task in 3D medical image analysis [9, 18].
In this paper we set up a generalization of a curve evolution scheme introduced by Kimmel *et al.* in [11]. We derive a general partial differential equation governing the front propagation for a family of surfaces lying on a 3-manifold (or "hypersurface") in $\mathbb{R}^4$. Considering an eulerian formulation for the projection of the surface onto an hyperplane unveils a stable method for computing distance maps on a 3-manifold. As a direct application of the theorical framework presented in this work, we generalize to the case of surfaces, a matching technique introduced by Cohen *et al.* for curve matching. A brief outline of the method is described in the following. Given two surfaces we characterize the similarity between these two structures by defining a hypersurface $W \subset \mathbb{R}^4$ based on the computation of two geodesic distance maps derived from the two surfaces. These geodesic distance maps are then combined and allow us to define the paths matching the structures as the paths lying on W which maximize a similarity criterion.
This new approach to surface matching problem leads to algorithms able to handle large deformations and change of topology using the geodesic distance map as the key feature for defining a similarity criterion that can be based on distance solely or integrating curvature information. The proposed method is described throughout the paper and we proceed step by step towards this general surface matching approach. First, we set up the geodesic surface evolution method on a 3D hypersurface W embedded in $\mathbb{R}^4$. This theory makes use of the Hodge "star" $*$ operator, a notion briefly reviewed in an appropriate subsection. Another subsection is devoted to the level-set formulation of that geodesic surface evolution scheme, allowing for the computation of distance maps on the 3-manifold W. The level-set evolution equation takes the form of Hamilton-Jacobi partial differential equations, for which J. Sethian [16] has introduced stable and robusts numerical resolution schemes described in another section. Then, the surface matching method is introduced, with the computation of paths between the source and destination surfaces $\mathcal{S}$ and $\mathcal{D}$ which minimize a cost function whose graph is W. Results are displayed in a specifc section. Lastly we contemplate a generalization of the method to higher dimensional manifolds. Then we conclude and sketch some perspectives.

2 A geodesic distance evolution rule for propagating surfaces on a 3-manifold

Let W be a 3-manifold (or hypersurface) in $\mathbb{R}^4$. We suppose W compact and pathwise connected.[1] From these assumptions, one can derive, using an easy application of the Hopf-Rinow-De Rham theorem (see [17]) that given any two points M_0 and $M_1 \in W$, there is a unique path $\gamma : [0,1] \longrightarrow W$ connecting M_0 and M_1 ($\gamma(0) = M_0$, $\gamma(1) = M_1$) whose length minimizes the lengths of all paths between the two points. The length of γ is called the geodesic distance between M_0 and M_1, and will be denoted $d_W(M_0, M_1)$ in the sequel. Moreover, the path γ is necessarily a geodesic curve on W, i.e. a curve such that the second derivative $\frac{\mathrm{d}^2\gamma}{\mathrm{d}u^2}$ is always perpendicular to W. Let $\mathcal{Y} \subset W$ be a surface (2-manifold) "drawn" on W. We consider the surfaces $\Xi_t \subset W$ whose points are located at geodesic distance t from $\mathcal{Y}$:

$$\Xi_t = \{M \in W \mid d_W(M, \mathcal{Y}) = t\} \ . \tag{1}$$

We are interested in defining a partial differential equation governing the evolution of the surface Ξ_t as the parameter t evolves. For this purpose, we will need a notion of cross-product in 4-space, and a method of deriving simple formulae about such a cross-product. Generalizing the cross-product can be done in different ways. Here, we choose the Hodge $*$ operator, because it provides direct formulae needed in the demonstration of intermediate propositions.

2.1 Exterior algebras and the Hodge * operator

The theory is only briefly recalled here. The reader is referred to [1] for a complete presentation of the subject. Let E be a finite dimensional vector space over $\mathbb{R}$, and let $(e_1, e_2, \cdots, e_n)$ be a basis of E, with $n = \dim E$. For any integer p, $0 \leq p \leq n$, one can construct new vector spaces, usually denoted $\Lambda^p(E)$, such that:

- by convention, $\Lambda^0(E) = \mathbb{R}$ and $\Lambda^1(E) = E$.
- $\Lambda^p(E)$ is the set of formal sums $\sum_{(i_1, i_2, \cdots i_p)} a_{i_1, i_2, \cdots i_p} u_{i_1} \wedge u_{i_2} \wedge \cdots u_{i_p}$, for multi-indexes $(i_1, i_2, \cdots i_p)$ and real coefficients $a_{i_1, i_2, \cdots i_p}$, the u_{i_j} being ordinary vectors of E.

The "wedge" products $u_{i_1} \wedge u_{i_2} \wedge \cdots u_{i_p}$ are supposed to be multilinear in the variables $u_{i_1}, u_{i_2}, \cdots, u_{i_p}$ and alternating, that is to say $u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_p} = 0$ if one of the vectors in this wedge product is equal to another vector. From these conditions, one can derive that for every p, $0 \leq p \leq n$, $\Lambda^p(E)$ is finite

[1] These assumptions do not restrict the validity of the theory presented in this work, as the manifold W will appear as the graph of a cost function, which automatically satisfies these requirements in practice.

dimensional, with dimension $\binom{n}{p} = \frac{n!}{p!(n-p)!}$. A basis of $\Lambda^p(E)$ is given by the family of vectors $(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p})$ with $1 \leq i_1 \leq i_2 \leq \cdots i_p \leq n$. For instance, if E is 4-dimensional space $\mathbb{R}^4$ with standard basis (e_1, e_2, e_3, e_4), the standard basis of $\Lambda^3(\mathbb{R}^4)$ is $(e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_3 \wedge e_4)$, and the standard basis of $\Lambda^4(\mathbb{R}^4)$ is $(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$.
Now suppose a dot product $< \bullet, \bullet >$ is defined in E. A general dot product, usually denoted $< \bullet, \bullet >_p$ can be defined in $\Lambda^p(E)$ by:

$$< u_1 \wedge u_2 \wedge \cdots \wedge u_p, w_1 \wedge w_2 \cdots \wedge w_p >_p = \det(< u_i, w_j >) \qquad (2)$$

In equation 2 the quantities $< u_i, w_j >$ inside the determinant are ordinary dot products in E. Since $\binom{n}{p} = \binom{n}{n-p}$, the two vector spaces $\Lambda^p(E)$ and $\Lambda^{(n-p)}(E)$ are isomorphic. The Hodge $*$ operator provides a standard isomorphism between these two vector spaces. It is defined in the following manner. Let λ and μ be two elements of $\Lambda^p(E)$ and $\Lambda^{(n-p)}(E)$ respectively. The image of λ by the Hodge operator, denoted $*\lambda$, belongs to $\Lambda^{(n-p)}(E)$ and is characterized by the equality:

$$\lambda \wedge \mu = < *\lambda, \mu >_{n-p} e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad (3)$$

Let us now see how all of this operates in practice. Suppose that E is the standard 3-space $\mathbb{R}^3$, and that u and v are two vectors in $\mathbb{R}^3$. Using elementary calculus and equation 3, one can easily show that $*(u \wedge v)$ is the standard cross-product in $\mathbb{R}^3$. If w is another third vector in $\mathbb{R}^3$, $*(u \wedge v \wedge w)$ is simply the determinant of the three vectors u, v, w, also known as their triple product. Now take $E = \mathbb{R}^4$ (it is the case that interests us in this work), and choose three linearly independant vectors u, v and w in $\mathbb{R}^4$. It is easy to prove that the associated image $*(u \wedge v \wedge w)$ satisfies the following properties:

- it is a vector in $\mathbb{R}^4$ perpendicular to u, v and w.
- The basis $(u, v, w, *(u \wedge v \wedge w))$ is positively oriented.
- It has components $\left(-\begin{vmatrix} u_2 & u_3 & u_4 \\ v_2 & v_3 & v_4 \\ w_2 & w_3 & w_4 \end{vmatrix}, \begin{vmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ w_1 & w_3 & w_4 \end{vmatrix}, -\begin{vmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ w_1 & w_2 & w_4 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right)$ in the standard basis of $\Lambda^3(\mathbb{R}^4)$ ($u = u_1 e_1 + u_2 e_2 + u_3 e_3 + u_4 e_4$ and similarly for v,w).
- Its squared norm : $\| *(u \wedge v \wedge w) \|^2$ is equal to $\begin{vmatrix} < u, u > & < u, v > & < u, w > \\ < v, u > & < v, v > & < v, w > \\ < w, u > & < w, v > & < w, w > \end{vmatrix}$.

Note that the last equality about the norm $\| *(u \wedge v \wedge w) \|^2$ generalizes the usual formula about the norm of the ordinary cross-product in 3-space.
With this notion of cross-product in 4-space given by the Hodge $*$ operator, we can now derive the geodesic distance evolution scheme for the family of surfaces Ξ_t. This is presented in the following subsection.

2.2 The geodesic distance evolution equation

We consider a local *orthogonal* parametrization $\alpha(u,v,t)$ of the surface Ξ_t, i.e. a parametrization $\alpha(u,v,t)$ such that $<\boldsymbol{\tau^u},\boldsymbol{\tau^v}>=0$, with $\boldsymbol{\tau^u}=\dfrac{\alpha_u}{\|\alpha_u\|}$ and $\boldsymbol{\tau^v}=\dfrac{\alpha_v}{\|\alpha_v\|}$. It is a known result that an orthogonal parametrization can always be found (see [6] p. 183):

Lemma 1 *Around any point on a surface one can always find a local orthogonal parametrization.*

So let $\alpha(u,v,t)$ be a local orthogonal parametrization of a family of surfaces in W, and let $\boldsymbol{\tau^u}=\dfrac{\alpha_u}{\|\alpha_u\|}$, $\boldsymbol{\tau^v}=\dfrac{\alpha_v}{\|\alpha_v\|}$ denote the two unitary tangent vectors determined by α and $\boldsymbol{N}$ the normal vector to W. We suppose that the family $\alpha(u,v,t)$ satisfies the following partial differential equation:

$$\frac{\partial\alpha}{\partial t}=*\left(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v}\right) \tag{4}$$

We want to prove that, for each t, $\alpha(u,v,t)$ is a local parametrization of Ξ_t.
Let $\gamma(t)$ be the curve in W defined by: $\gamma(t)=\alpha(u,v,t)|_{u=u_0,v=v_0,\text{ fixed}}$. Then:

Lemma 2 *For any u_0,v_0, the curve $\gamma(t)$ is a geodesic in W.*

Proof. We prove this lemma by showing that γ_{tt} is perpendicular to $\boldsymbol{\tau^u}$, $\boldsymbol{\tau^v}$, and $*(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v})$, $\boldsymbol{N}$ being the normal to W. The only possibility that will remain will be then: γ_{tt} is colinear to $\boldsymbol{N}$ which means that γ is a geodesic.
First note that

$$\gamma_t=\frac{d\gamma}{dt}=\frac{\partial\alpha}{\partial t}=*\left(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v}\right)\ ,$$

so

$$\begin{aligned}||\gamma_t||^2 &= ||*\left(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v}\right)||^2\\ &=\begin{vmatrix}\langle\boldsymbol{N},\boldsymbol{N}\rangle & \langle\boldsymbol{N},\boldsymbol{\tau^u}\rangle & \langle\boldsymbol{N},\boldsymbol{\tau^v}\rangle\\ \langle\boldsymbol{\tau^u},\boldsymbol{N}\rangle & \langle\boldsymbol{\tau^u},\boldsymbol{\tau^u}\rangle & \langle\boldsymbol{\tau^u},\boldsymbol{\tau^v}\rangle\\ \langle\boldsymbol{\tau^v},\boldsymbol{N}\rangle & \langle\boldsymbol{\tau^v},\boldsymbol{\tau^u}\rangle & \langle\boldsymbol{\tau^v},\boldsymbol{\tau^v}\rangle\end{vmatrix}\\ &=1-\langle\boldsymbol{N},\boldsymbol{\tau^u}\rangle^2-\langle\boldsymbol{N},\boldsymbol{\tau^v}\rangle^2=1\end{aligned}$$

since the parametrization is orthogonal. This proves:

$$\langle\gamma_{tt},\gamma_t\rangle=\left\langle\frac{d^2\gamma}{dt^2},\frac{d\gamma}{dt}\right\rangle=\left\langle\frac{d}{dt}\left(*\left(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v}\right)\right),*\left(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v}\right)\right\rangle=0. \tag{5}$$

Hence, γ_{tt} is perpendicular to $*(\boldsymbol{N}\wedge\boldsymbol{\tau^u}\wedge\boldsymbol{\tau^v})$.
Now, using methods similar to the one used in the 2D case (see proof of Lemma 1 in [11]), one can show that

$$\langle\gamma_{tt},\boldsymbol{\tau^u}\rangle=0, \tag{6}$$

and similarly

$$\langle \gamma_{tt}, \boldsymbol{\tau^v} \rangle = 0. \tag{7}$$

The curve $\gamma(t)$ is a geodesic in W.

Lemma 3 *The evolution of the family of surfaces Ξ_t is given by the equation*

$$\frac{\partial \alpha}{\partial t} = *\,(\boldsymbol{N} \wedge \boldsymbol{\tau^u} \wedge \boldsymbol{\tau^v}) \tag{8}$$

The proof of this lemma is not given here, as it is a simple adaptation, using the the general Gauss Lemma [17], of the proof given in [11] (Lemma 2).
The two preceding lemma demonstrate that, using the Hodge $*$ operator, it is possible to derive a geodesic distance evolution scheme for a family of surfaces described by local orthogonal parametrizations. In the next subsection, we compute the normal speed of the projection of Ξ_t onto the (x, y, z) hyperplane in $\mathbb{R}^4$.

2.3 Normal evolution of the projection of Ξ_t onto the (x, y, z) hyperplane

To generalize the 2D case, we now make the assumption that W is a graph hypersurface, that is to say $W = \{(x, y, z, w(x, y, z))\}$ for a function $w : \mathbb{R}^3 \longrightarrow \mathbb{R}$. Let $\pi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the canonical projection onto the (x, y, z) hyperplane in $\mathbb{R}^4$ and let $\mathcal{S}(t)$ be the projection of the image of $\alpha(u, v, t)$ (that is to say, Ξ_t) onto that hyperplane:

$$\mathcal{S}(t) = \pi \circ \alpha.$$

We denote by p, q, r the following quantities: $p = \dfrac{\partial w}{\partial x}$, $q = \dfrac{\partial w}{\partial y}$ and $r = \dfrac{\partial w}{\partial z}$.
Starting from a result mentionned in [7], we admit that the trace of a propagating surface may be determined only by its normal velocity, as the other components of the velocity influence only the local parametrization. Our goal is then to compute the projected velocity of the evolving surface $V = \langle \pi \circ \alpha_t, \boldsymbol{n} \rangle$, $\boldsymbol{n} = (n_1, n_2, n_3)$ being the normal to the projected surface $\pi \circ \alpha(u, v, t)$. We state the following result:

Lemma 4 *The projected surface $\mathcal{S}(t)$ satisfies the normal propagation rule:*

$$\frac{\partial \mathcal{S}}{\partial t} = V \boldsymbol{n} \tag{9}$$

with

$$\begin{aligned} V &= \sqrt{\begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} M(\varphi_x, \varphi_y, \varphi_z)} \\ &= \sqrt{a n_1^2 + b n_2^2 + c n_3^2 - d n_1 n_2 - e n_1 n_3 - f n_2 n_3} \end{aligned} \tag{10}$$

M being the matrix of a quadratic form:

$$M = \frac{1}{1+p^2+q^2+r^2} \times \begin{pmatrix} 1+q^2+r^2 & -pq & -pr \\ -pq & 1+p^2+r^2 & -qr \\ -pr & -qr & 1+p^2+q^2 \end{pmatrix}$$

($\varphi_x = \dfrac{\partial \varphi}{\partial x}$, etc.)

Sketch of the proof.
The tangent vectors $\boldsymbol{\tau^u} = \dfrac{\alpha_u}{\| \alpha_u \|}$ and $\boldsymbol{\tau^v} = \dfrac{\alpha_v}{\| \alpha_v \|}$ are given by their coordinates:

$$\boldsymbol{\tau^u} = \frac{(x_u, y_u, z_u, w_u)}{\sqrt{x_u^2+y_u^2+z_u^2+w_u^2}} \quad \text{and} \quad \boldsymbol{\tau^v} = \frac{(x_v, y_v, z_v, w_v)}{\sqrt{x_v^2+y_v^2+z_v^2+w_v^2}}$$

so that one finds easily:

$$\boldsymbol{N} = \frac{(-p, -q, -r, 1)}{\sqrt{1+p^2+q^2+r^2}}.$$

The normal velocity of the surface's projection onto the (x, y, z) hyperplane is:

$$V = \langle \pi \circ \alpha_t, \boldsymbol{n} \rangle$$

with $\pi \circ \alpha_t$ having coordinates (obtained from the Hodge operator, see section 2.1):

$$\frac{1}{K} \times \left| \begin{array}{l} q z_u w_v - q w_u z_v - r y_u w_v + r y_v w_u + y_v z_u \\ -q z_u w_v + q w_u z_v + r y_u w_v + y_u z_v - r y_v w_u - y_v z_u \\ p y_u z_v - p y_v z_u - q x_u z_v + r x_u y_v + q x_v z_u - r x_v y_u \end{array} \right.$$

with

$$K = \sqrt{1+p^2+q^2+r^2}\sqrt{x_u^2+y_u^2+z_u^2}\sqrt{x_v^2+y_v^2+z_v^2}$$

Using the fact that the parametrization $\alpha(u, v)$ is orthogonal:

$$x_u x_v + y_u y_v + z_u z_v + w_u w_v = 0$$

it is possible to simplify the scalar product and get

$$V = \sqrt{\begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix} M (\varphi_x, \varphi_y, \varphi_z)}$$

$$= \sqrt{a n_1^2 + b n_2^2 + c n_3^2 - d n_1 n_2 - e n_1 n_3 - f n_2 n_3}$$

which completes the proof.

Having found the normal equation evolution of the projected surface, we now proceed to set up an Eulerian formulation for $\mathcal{S}(t)$, by writing that projected surface as a level-set $\varphi^{-1}(0)$. This idea was introduced by Osher and Sethian [14] for crystal growth modelling. Its major advantage is the ability to handle topological changes and singularities while insuring stability and accuracy. Moreover, this level-set formulation gives us the ability to compute distance maps on the graph of the cost function. We derive such a formulation in the next subsection.

2.4 Level-set formulation

Given a function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that its zero level-set tracks the projected surface $\mathcal{S}(t) = \varphi^{-1}(0)$, we determine, in the following lemma[2] , the equation governing the evolution of φ:

Lemma 5 *The function φ follows the following propagation equation:*

$$\frac{\partial \varphi}{\partial t} = \sqrt{a\frac{\partial \varphi}{\partial x}^2 + b\frac{\partial \varphi}{\partial y}^2 + c\frac{\partial \varphi}{\partial z}^2 - d\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y} - e\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial z} - f\frac{\partial \varphi}{\partial y}\frac{\partial \varphi}{\partial z}} \qquad (11)$$

the coefficients a, b, c, d, e and f having the same values as in equation 10: $a = \dfrac{1+q^2+r^2}{1+p^2+q^2+r^2}$, $b = \dfrac{1+p^2+r^2}{1+p^2+q^2+r^2}$, $c = \dfrac{1+p^2+q^2}{1+p^2+q^2+r^2}$, $d = \dfrac{2pq}{1+p^2+q^2+r^2}$, $e = \dfrac{2pr}{1+p^2+q^2+r^2}$, $f = \dfrac{2qr}{1+p^2+q^2+r^2}$.

Proof. To derive equation 11, one simply uses equation 10 together with the chain rule:

$$\frac{\partial \varphi}{\partial t} = \; < \boldsymbol{\nabla}\varphi, \frac{\partial \mathcal{S}}{\partial t} > \qquad (12)$$

and the fact that the normal $\boldsymbol{n}$ is given by the gradient vector $\boldsymbol{n} = \dfrac{\boldsymbol{\nabla}\varphi}{\parallel \boldsymbol{\nabla}\varphi \parallel}$.

As mentionned in the curve evolution process described in [11] such an Eulerian formulation leads to numerical resolution schemes able to handle problems caused by a time varying coordinate system (u, v, t): curvature singularities and topological changes [16]. We describe, in the following subsection, the numerical resolution method used to solve equation 11. Then we use this numerical algorithm to build distance maps.

2.5 Numerical resolution

The numerical implementation is a generalization of the finite difference approximation described in [11, 14] for Hamilton-Jacobi type equations. It consists in

[2] It is important to note that the function φ depends not only of $(x, y, z) \in \mathbb{R}^3$, but also of the parameter t. To simplify the notations, we do not write explicitely that dependence on t, but it is important to keep it in mind.

an explicit temporal scheme where spatial derivatives are approximated by finite differences using the *minmod* function. The derivative estimates can be bounded, and the variations of the solution can therefore be controlled. The *minmod* function is defined by:

$$\text{minmod}(a,b) = \begin{cases} \text{sign}(a)\min(|a|,|b|) & \text{if } a < b \\ 0 & \text{if } a \geq b. \end{cases}$$

Spatial derivatives are estimated on a discrete three-dimensional grid by:

$$\begin{aligned} \varphi_x(ih_x, jh_y, kh_z) &= \text{minmod}(D_x^+\varphi(ih_x, jh_y, kh_z), D_x^-\varphi(ih_x, jh_y, kh_z)) \\ \varphi_y(ih_x, jh_y, kh_z) &= \text{minmod}(D_y^+\varphi(ih_x, jh_y, kh_z), D_y^-\varphi(ih_x, jh_y, kh_z)) \\ \varphi_z(ih_x, jh_y, kh_z) &= \text{minmod}(D_z^+\varphi(ih_x, jh_y, kh_z), D_z^-\varphi(ih_x, jh_y, kh_z)) \end{aligned}$$

with h_x, h_y and h_z being the spatial discretization steps. D_x^- and D_x^+ (respectively D_y^-, D_y^+ and D_z^-, D_z^+) are the left and right derivatives in the x (resp. y and z) direction. They are defined by:

$$D_x^-\varphi(ih_x, jh_y, kh_z) = \frac{\varphi(((i+1)h_x, jh_y, kh_z) - \varphi(ih_x, jh_y, kh_z)}{h_x}$$

and

$$D_x^+\varphi(ih_x, jh_y, kh_z) = \frac{\varphi(ih_x, jh_y, kh_z) - \varphi((i-1)h_x, jh_y, kh_z)}{h_x}$$

To compute the squared derivatives, one uses [11]:

$$\begin{aligned} \varphi_x^2(ih_x, jh_y, kh_z) &= (\max(D_x^+\varphi(ih_x, jh_y, kh_z), -D_x^-\varphi(ih_x, jh_y, kh_z), 0))^2 \\ \varphi_y^2(ih_x, jh_y, kh_z) &= (\max(D_y^+\varphi(ih_x, jh_y, kh_z), -D_y^-\varphi(ih_x, jh_y, kh_z), 0))^2 \\ \varphi_z^2(ih_x, jh_y, kh_z) &= (\max(D_z^+\varphi(ih_x, jh_y, kh_z), -D_z^-\varphi(ih_x, jh_y, kh_z), 0))^2. \end{aligned}$$

With these spatial derivative estimations, we can now write the discrete numerical scheme for solving equation 11. The function φ is computed at the locations (ih_x, jh_y, kh_z) of a three-dimensional grid using a recurrence sequence denoted $\varphi(ih_x, jh_y, kh_z)^\tau$ and defined by the relations[3]:

$$\begin{aligned} \varphi^{\tau+1} = \varphi^\tau + [\; & a\left(\max\left(D_x^+\varphi^\tau, -D_x^-\varphi^\tau, 0\right)\right)^2 \\ & + b\left(\max\left(D_y^+\varphi^\tau, -D_y^-\varphi^\tau, 0\right)\right)^2 \\ & + c\left(\max\left(D_z^+\varphi^\tau, -D_z^-\varphi^\tau, 0\right)\right)^2 \\ & - d\ \text{minmod}\left(D_x^+\varphi^\tau, D_x^-\varphi^\tau\right) \text{minmod}\left(D_y^+\varphi^\tau, D_y^-\varphi^\tau\right) \\ & - e\ \text{minmod}\left(D_x^+\varphi^\tau, D_x^-\varphi^\tau\right) \text{minmod}\left(D_z^+\varphi^\tau, D_z^-\varphi^\tau\right) \\ & - f\ \text{minmod}\left(D_y^+\varphi^\tau, D_y^-\varphi^\tau\right) \text{minmod}\left(D_z^+\varphi^\tau, D_z^-\varphi^\tau\right)]^{\frac{1}{2}}\ \Delta\tau \end{aligned} \tag{13}$$

This explicit scheme is conditionally stable, and the convergence to a stationary solution is achieved when the time step $\Delta\tau$ and the spatial steps h_x, h_y and h_z satisfy the Courant-Friedrish-Lewy condition:

$$\Delta\tau \leq \frac{1}{\min(h_x, h_y, h_z)} \tag{14}$$

[3] To simplify the notations, φ^τ stands for $\varphi(ih_x, jh_y, kh_z)^\tau$ and similarly for $\varphi^{\tau+1}$

In practice the spatial resolution is given by the image and the τ resolution is determined to satisfy 14. Using this numerical scheme, we can now use the computed function φ to generate distance maps on a manifold. This is explained in the following subsection.

2.6 Distance maps on a 3-manifold

In order to use equation 13 for computing the geodesic distance map of the surface described by a parametrization $\alpha(u, v, 0)$ on the 3-manifold W, we have to define an initial estimate φ_0 such that the initial surface is represented through a level-set of φ_0. This initial estimate can be obtained in several ways according to the data. We use a Euclidean distance map [2] in such a way that:

$$\varphi_0(x, y, z) = \begin{cases} -d(x, y, z) & \text{if } (x, y, z) \text{ is inside } \varphi_0^{-1}(0) \\ 0 & \text{if } (x, y, z) \in \varphi_0^{-1}(0) \\ +d(x, y, z) & \text{if } (x, y, z) \text{ is outside } \varphi_0^{-1}(0) \end{cases} \tag{15}$$

Given a graph hypersurface W and the initial estimate φ_0 on this hypersurface, equation 13 characterizes the distance map of the area which boundary is defined by $\varphi_0^{-1}(0)$.

The tools presented in this section can now be used to perform a surface matching process.

3 A general surface matching process

We now use the previous theory to build a general matching process between two arbitrary surfaces $\mathcal{S}$ and $\mathcal{D}$ in $\mathbb{R}^3$. The matching process consists in computing the paths on a graph hypersurface such that these paths minimize a cost function. This approach is interesting as it does not rely on a parametrization of the two surfaces $\mathcal{S}$ and $\mathcal{D}$: they are only represented as 0-level-sets of two functions φ_0 and ψ_0. Moreover the process can take into account large deformations between the two surfaces and even a topological change. The two functions φ_0 and ψ_0 are computed from the initial data using the rule presented in 15. Then given these two initial functions φ_0 and ψ_0, the numerical process presented in equation 13 is used to generate two distance maps on a graph surface W:

$$D_{\mathcal{S}} = \{(x, y, z, \varphi(x, y, z))\}$$

and

$$D_{\mathcal{D}} = \{(x, y, z, \psi(x, y, z))\}.$$

The hypersurface graph W is chosen in order to incorporate a geometric criterion of similarity. It is chosen such that the two surfaces $\varphi_0^{-1}(0)$ and $\psi_0^{-1}(0)$ are level-sets of W. At this point one may choose to incorporate only distance information or curvature in the definition of W. A possible choice for W using only distance is given by:

$$W = (x, y, z, w(x, y, z)) = (x, y, z, \min(|\varphi_0|, |\psi_0|)) \tag{16}$$

To take into account curvature, we use a matching criterion ρ, function of the euclidean distance d and of the relative difference between the mean curvatures $\Delta\kappa = \kappa_{\mathcal{S}} - \kappa_{\mathcal{D}}$, where $\kappa_{\mathcal{S}}$ and $\kappa_{\mathcal{D}}$ are respectively the mean curvatures of $\mathcal{S}$ and $\mathcal{D}$. The ρ function is defined in such a way that the influence of mean curvature decreases as the euclidean distance d increases:

$$\rho(\Delta\kappa, d) = 1 - \frac{\Delta\kappa^2}{1 + d^2\Delta\kappa^2/\sigma}$$

σ being a scale parameter defining the neighbourhood around the two init ial surfaces where mean curvature is taken into account. This leads to:

$$W = (x, y, z, \min(|\varphi_0|\rho(\Delta\kappa, \varphi_0), |\psi_0|\rho(\Delta\kappa, \psi_0))) \tag{17}$$

The mean curvature is easily computed from the level-set representation of the surfaces:

$$\kappa = \frac{1}{(\varphi_x^2 + \varphi_y^2 + \varphi_z^2)^{3/2}} \times$$

$$(\varphi_{yy}+\varphi_{zz})\varphi_x^2+(\varphi_{xx}+\varphi_{zz})\varphi_y^2+(\varphi_{xx}+\varphi_{yy})\varphi_z^2-2\varphi_x\varphi_y\varphi_{xy}-2\varphi_x\varphi_z\varphi_{xz}-2\varphi_y\varphi_z\varphi_{yz}$$

Once $D_{\mathcal{S}}$ and $D_{\mathcal{D}}$ are computed, we obtain the matching function between $\mathcal{S}$ and $\mathcal{D}$ by determining the paths on the graph surface W starting at $\mathcal{S}$ and ending on $\mathcal{D}$ which minimize a cost function. This cost function is:

$$f(x, y, z) = \varphi(x, y, z) + \psi(x, y, z) \tag{18}$$

due the following lemma, which is generalization of the proposition found in [11]:

Lemma 6 *All the minimal paths $s \longrightarrow \gamma(s)$ between $\mathcal{S}$ and $\mathcal{D}$ on W minimize, for any value of parameter s, the sum*

$$d_W(\gamma(s), \mathcal{S}) + d_W(\gamma(s), \mathcal{D}).$$

This cost function ultimately justify the process of building distance maps on W. For each point $M_{\mathcal{S}} \in \mathcal{S}$ on the first surface, we determine a path ending at an unknown point $M_{\mathcal{D}} \in \mathcal{D}$ on the second surface and minimizing the cost function $f(x, y, z)$. The cost $C(\gamma)$ of a path $s \longrightarrow \gamma(s)$ is defined to be:

$$C(\gamma) = \int_{M_{\mathcal{S}}}^{M_{\mathcal{D}}} f(x, y, z)\mathrm{d}s \tag{19}$$

Hence the minimal paths $s \longrightarrow \gamma(s)$ (where s is the arc-length) are determined as the orbits of the vector field

$$-\nabla f = -(\nabla\varphi + \nabla\psi)$$

and they are computed using a Runge-Kutta. We give some examples in the next section.

4 Results

We apply the matching process described in the previous section on a first example illustrating the ability of the model to cope with a topological change and large deformation. The initial surface $\mathcal{S}$ is given by two spheres and the destination surface $\mathcal{D}$ is the shape of an ellipsoid containing the two spheres. On figure 1 we show the computation of the matching paths, starting from the ellipsoid. The graph of the cost function W used in this example is given incorporates only the distance. This example clearly proves the ability of matching dissimilar surfaces with distinct topologies.

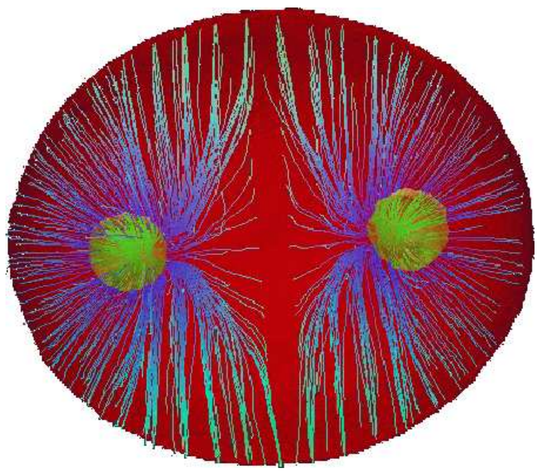

Fig. 1. Matching paths between a surface made of two spheres and an ellipsoid. The paths start from the ellipsoid.

We show, in another example displayed on figure 2, the use of a graph function W incorporating distance function only, as in equation 16. The initial surface $\mathcal{S}$ is given by a digital elevation model, and the destination surface is computed by applying a geophysical deformation model to $\mathcal{S}$. This geophysical model is used to model possible deformations observed in volcanic regions.

5 Generalization to higher dimensions

We sketch in this section a work in progress, consisting in generalizing the theory introduced in this research to higher dimensions. We first note that the Hodge $*$ operator can be used to define a notion of "cross product" of $n-1$ vectors in $\mathbb{R}^n$. Hence we can guess a possible generalization to the geodesic distance evolution scheme 4 as

$$\frac{\partial \alpha}{\partial t} = *\left(\boldsymbol{N} \wedge \boldsymbol{\tau}^{\boldsymbol{u_1}} \wedge \boldsymbol{\tau}^{\boldsymbol{u_2}} \wedge \cdots \wedge \boldsymbol{\tau}^{\boldsymbol{u_{n-2}}}\right) \tag{20}$$

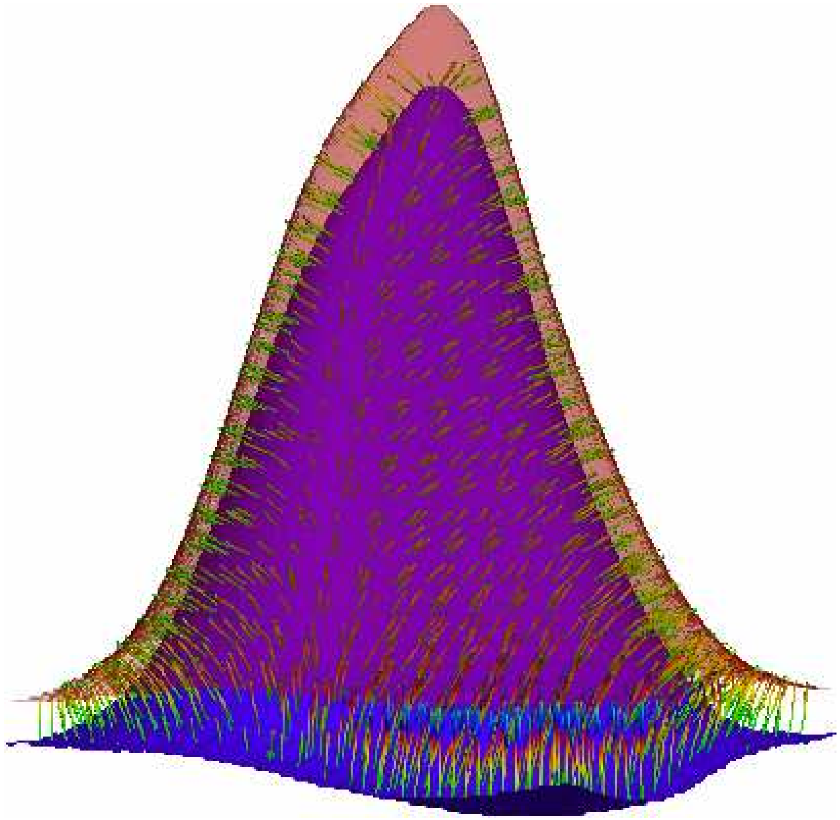

Fig. 2. Matching paths between a surface obtained from a digital elevation model and a destination surface computed by a geophysical model. The graph of the cost function incorporates curvature information. The destination surface is represented with transparency.

where $\alpha(u_1, u_2, \ldots, u_{n-2}, t)$ is an orthogonal parametrization of a family of $n-2$-manifolds embedded in a hypersurface W of $\mathbb{R}^n$. Equation 20 represents the geodesic distance evolution of a family of $n-2$-manifolds in W. (The proof is similar to the one presented in section 3). One can then generalize the theory presented in the previous sections, and set up an Eulerian formulation for the projected manifolds $\pi \circ \alpha(u_1, u_2, \ldots, u_{n-2}, t)$, where π is the projection onto the $(x_1, x_2, \cdots, x_{n-1})$ hyperplane in $\mathbb{R}^n$. To achieve this, the $n-1$-manifold W is here again written in the form of the graph of a function $w : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$. One can then introduce the quantities $p_i = \dfrac{\partial w}{\partial x_i}$, with $1 \leq i \leq n-1$, and the projected manifolds $\pi \circ \alpha(u_1, u_2, \ldots, u_{n-2}, t)$ are written in the form $\varphi^{-1}(0)$, for a function $\varphi : \mathbb{R}^{n-1} \oplus \mathbb{R} \longrightarrow \mathbb{R}$. Equation 11 becomes:

$$\frac{\partial \varphi}{\partial t} = \sqrt{U^t M U}$$

M being the matrix of a quadratic form:

$$M = \frac{1}{1 + \sum_{i=1}^{n-1} p_i^2} \begin{pmatrix} 1 + \sum_{j \neq 1} p_j^2 & -p_1 p_2 & \cdots & -p_1 p_{n-1} \\ -p_2 p_1 & 1 + \sum_{j \neq 2} p_j^2 & \cdots & -p_2 p_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ -p_{n-1} p_1 & -p_{n-1} p_2 & \cdots & 1 + \sum_{j \neq n-1} p_j^2 \end{pmatrix} \tag{21}$$

and U the column vector whose components are $\frac{\partial \varphi}{\partial x_i}$, $1 \leq i \leq n-1$:

$$U = \begin{pmatrix} \frac{\partial \varphi}{\partial x_1} \\ \frac{\partial \varphi}{\partial x_2} \\ \vdots \\ \frac{\partial \varphi}{\partial x_{n-1}} \end{pmatrix}$$

The numerical algorithm introduced in section 2.5 leads to a simple generalization, and permits the computations of distance maps on a n-dimensional grid. The matching process described in section 3 can then be extended in this n-dimensional context.

6 Conclusion

This paper presents a general formulation for the propagation of fronts in any dimension. A special emphasis is given to the case of propagating surfaces on a 3-manifold embedded in $\mathbb{R}^4$. This theory unveils a general method for computing distance maps on manifolds. The algorithms used discards the drawbackks of curvature singularities and topological changes for the projected manifolds. An application to the problem of surface matching is presented. The matching algorithm makes use of distance maps to compute optimal paths on a cost manifold. The optimal paths minimize a cost criterion which can incorporate various geometrical properties such as distance and curvature. We give two examples of cost surface, but model is general enough to include various matching criteria, each leading to a particular cost function.

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