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Gravitational fields with a non Abelian bidimensional Lie algebra of symmetries

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Abstract

Vacuum gravitational fields invariant for a bidimensional non Abelian Lie algebra of Killing fields, are explicitly described. They are parameterized either by solutions of a transcendental equation (the *tortoise equation*) or by solutions of a linear second order differential equation on the plane. Gravitational fields determined *via* the tortoise equation, are invariant for a 3-dimensional Lie algebra of Killing fields with bidimensional leaves. Global gravitational fields out of local ones are also constructed.

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In the last years a great deal of attention has been devoted to the detection of gravitational waves. However, all the experimental devices, interferometers or resonant antennas, are constructed coherently with results obtained from the non covariant linearized Einstein field equations, in close analogy with that is normally done in Maxwell theory of electromagnetic fields.

Starting from the seventy's, however, new powerful mathematical methods have been invented to deal with nonlinear evolution equations and their exact solutions. One of this methods, namely a suitable generalization of the *Inverse Scattering Transform*, allowed to solve reduced vacuum Einstein field equations and to obtain *solitary waves solutions* [3] (see [14] and references therein).

This paper is the first in a series devoted to the study of gravitational fields g admitting a Lie algebra $\mathcal{G}$ of Killing fields. The case of a non Abelian bidimensional Killing Lie algebra has been only partially studied. Here, this case will be completely analyzed within the following general problem which, as we will see, emerges naturally.

I. the distribution $\mathcal{D}$, generated by the vector fields of $\mathcal{G}$, is bidimensional.

II. the distribution $\mathcal{D}^{\perp}$ orthogonal to $\mathcal{D}$, is integrable and transversal to D.

According to whether $\dim \mathcal{G}$ is 2 or 3, two qualitatively different cases occur.

A bidimensional $\mathcal{G}$, is either Abelian ($\mathcal{A}_2$) or non-Abelian ($\mathcal{G}_2$). A metric g satisfying I and II, with $\mathcal{G} = \mathcal{A}_2$ or $\mathcal{G}_2$, will be called $\mathcal{G}$ *-integrable* .

The study of $\mathcal{A}_2$-integrable Einstein metrics goes back to Einstein and Rosen [5], Rosen [11], Kompaneyets [8], Geroch [6], Belinsky and Khalatnikov [2].

The greater rigidity of $\mathcal{G}_2$-integrable metrics, for which some partial results can be found in [7, 1, 4], allows an exhaustive analysis. It will be shown that they are parameterized by solutions of a linear second order differential equation on the plane which, in its turn, depends linearly on the choice of a $\mathbf{j}$-*harmonic* function (see later). Thus, this class of solutions has a *bilinear structure* and, as such, admits two *superposition laws.*

When dim $\mathcal{G} = 3$, assumption II follows from I and the local structure of this class of Einstein metrics can be explicitly described. Some well known exact solutions [10, 9], *e.g.* Schwarzschild, belong to this class.

Besides the new local $\mathcal{G}_2$-integrable solutions, a procedure to construct new global solutions, suitable for all such $\mathcal{G}$-integrable metrics, will be also described.

The following notation will be adopted

Metric: a non-degenerate symmetric $(0,2)$ tensor field,
$\mathcal{K}il\,(g)$: the Lie algebra of all Killing fields of a metric g,
Killing algebra: a sub-algebra of $\mathcal{K}il\,(g)$.

Semiadapted coordinates. Let g be a metric on the space-time M (a connected smooth manifold) and $\mathcal{G}_2$ one of its Killing algebras whose generators X, Y satisfy

$$[X,Y] = sY, \quad s \in \mathcal{R} \tag{1}$$

The Frobenius distribution $\mathcal{D}$ generated by $\mathcal{G}_2$ is bidimensional and a chart $(x^1,\, x^2,\, x^3,\, x^4)$ exists such that

$$X = \frac{\partial}{\partial x^3}, \quad Y = \left(\exp s x^3\right) \frac{\partial}{\partial x^4}. \tag{2}$$

¿From now on such a chart will be called *semiadapted* (to the Killing fields).

Invariant metrics It can be easily verified [13] that in a semiadapted chart g has the form

$$\begin{aligned} g \;\; = \;\; & g_{ij} dx^i dx^j + 2\left(l_i + s m_i x^4\right) dx^i dx^3 - 2 m_i dx^i dx^4 + \\ & \left(s^2 \lambda \left(x^4\right)^2 - 2 s \mu x^4 + \nu\right) dx^3 dx^3 + \\ & 2\left(\mu - s \lambda x^4\right) dx^3 dx^4 + \lambda dx^4 dx^4, \quad i = 1,2; j = 1,2 \end{aligned}$$

with g_{ij}, m_i, l_i, λ, μ, ν arbitrary functions of $\left(x^1, x^2\right)$.

Killing leaves. Condition II allows to construct semi-adapted charts, with new coordinates $\left(x, y, x^3, x^4\right)$, such that the fields $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, belong to $\mathcal{D}^\perp$. In such a chart, called from now on *adapted*, the components l_i's and m_i's vanish.

We will call *Killing leaf* an integral (bidimensional) submanifold of $\mathcal{D}$ and *orthogonal leaf* an integral (bidimensional) submanifold of $\mathcal{D}^\perp$. Since $\mathcal{D}^\perp$ is transversal to $\mathcal{D}$, the restriction of g to any Killing leaf, S, is non-degenerate. Thus, $(S, g|_S)$ is a homogeneous bidimensional Riemannian manifold. Then, the Gauss curvature $K\,(S)$ of the Killing leaves is constant (depending on the leave). In the chart $(p = x^3|_S,\, q = x^4|_S)$ one has

$$g|_S = \left(s^2 \widetilde{\lambda} q^2 - 2 s \widetilde{\mu} q + \widetilde{\nu}\right) dp^2 + 2\left(\widetilde{\mu} - s \widetilde{\lambda} q\right) dp dq + \widetilde{\lambda} dq^2,$$

where $\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}$, being the restrictions to S of λ, μ, ν, are constants, and

$$K(S) = \widetilde{\lambda} s^2 \left(\widetilde{\mu}^2 - \widetilde{\lambda}\widetilde{\nu}\right)^{-1}. \tag{3}$$

Einstein metrics when $g(Y,Y) \neq 0$. In the considered class of metrics, vacuum Einstein equations, $R_{\mu\nu} = 0$, can be completely solved [13]. If the Killing field Y is not of *light type*, *i.e.* $g(Y,Y) \neq 0$, then in the adapted coordinates (x, y, p, q) the general solution is

$$g = f(dx^2 \pm dy^2) + \beta^2[(s^2k^2q^2 - 2slq + m)dp^2 + 2(l - skq)dpdq + kdq^2] \tag{4}$$

where $f = -\frac{1}{2s^2k} \triangle_\pm \beta^2$, and $\beta(x, y)$ is a solution of the *tortoise equation*

$$\beta + A \ln|\beta - A| = u(x, y),$$

the function u being a solution either of Laplace or d' Alembert equation, $\triangle_\pm u = 0$, $\triangle_\pm = \partial^2_{xx} \pm \partial^2_{yy}$, such that $(\partial_x u)^2 \pm (\partial_y u)^2 \neq 0$. The constants k, l, m are constrained by $km - l^2 = \pm 1$, $k \neq 0$.

Canonical form of metrics when $g(Y,Y) \neq 0$ The gauge freedom of the above solution, allowed by the function u, can be locally eliminated by introducing the coordinates (u, v, p, q), the function $v(x, y)$ being conjugate to $u(x, y)$, *i.e.* $\triangle_\pm v = 0$ and $u_x = v_y, u_y = \mp v_x$. In these coordinates the metric g takes the form (local "Birkhoff's theorem")

$$g = \frac{exp[\frac{u-\beta}{A}]}{2s^2k\beta}(du^2 \pm dv^2) + \beta^2[(s^2k^2q^2 - 2slq + m)dp^2 + 2(l - skq)dpdq + kdq^2]$$

with $\beta(u)$ a solution of $\beta + A \ln|\beta - A| = u$.

Normal form of metrics when $g(Y,Y) \neq 0$. In *geographic coordinates* (ϑ, φ) along Killing leaves one has $g|_S = \beta^2 \left[d\vartheta^2 + \mathcal{F}(\vartheta) d\varphi^2\right]$, where $\mathcal{F}(\vartheta)$ is equal either to $\sinh^2\vartheta$ or $-\cosh^2\vartheta$, depending on the signature of the metric. Thus, in the *normal coordinates*, $\left(r = 2s^2k\beta, \tau = v, \vartheta, \varphi\right)$, the metric takes the form

$$g = \varepsilon_1 \left(\left[1 - \frac{A}{r}\right] d\tau^2 \pm \left[1 - \frac{A}{r}\right]^{-1} dr^2\right) + \varepsilon_2 r^2 \left[d\vartheta^2 + \mathcal{F}(\vartheta) d\varphi^2\right] \tag{5}$$

where $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$ with a choice coherent with the required *signature* 2.

The geometric reason for this form is that, when $g(Y,Y) \neq 0$, a third Killing field exists which together with X and Y constitute a basis of $so(2,1)$. The larger symmetry implies that the geodesic equations describe a *non-commutatively integrable system* [12], and the corresponding geodesic flow projects on the geodesic flow of the metric restricted to the Killing leaves. *The above local form* *does not allow, however, to treat properly the singularities appearing inevitably in global solutions.* The metrics (4), although they all are locally diffeomorphic to (5), play a relevant role in the construction of new global solutions as described later.

Einstein metrics when $g(Y,Y)=0$. If the Killing field Y is of *light type*, then the general solution of vacuum Einstein equations, in the adapted coordinates (x,y,p,q), is given by

$$g = 2f(dx^2+dy^2)+\mu[(w\,(x,y)-2sq)dp^2+2dpdq], \tag{6}$$

where $\mu = D\Phi + B$; $D, B \in \mathcal{R}$, Φ is a non constant harmonic function, $f = \pm\left(\nabla\Phi\right)^2/\sqrt{|\mu|}$, and $w\,(x,y)$ is a solution of

$$\Delta w + \left(\partial_x \ln|\mu|\right)\partial_x w + \left(\partial_y \ln|\mu|\right)\partial_y w = 0.$$

Special solutions are $w=\widetilde{\mu},\;\; w=\ln|\mu|$, where $\widetilde{\mu}$ is the harmonic function conjugate to μ. When μ is not constant, in the coordinates $\xi = \mu+\widetilde{\mu}$, $\eta = \mu-\widetilde{\mu}$, the above equation appears to be the *Darboux equation*

$$(\xi+\eta)\left(\partial^2_{\xi\xi}+\partial^2_{\eta\eta}\right)w + \partial_\xi w + \partial_\eta w = 0.$$

In this case the curvature of Killing leaves vanishes.

The new solutions (6) together with (4) exhaust all local Lorentzian Ricci-flat metrics invariant for a $\mathcal{G}_2$ Lie algebra.

Ricci-flat g with $\dim\mathcal{K}il(g|_S)=3$ & $\dim\mathcal{S}=2$. In view of the construction of global solutions, the previous results suggest to consider with the same approach all metrics having 3-dimensional Killing algebras with bidimensional leaves. A Killing algebra $\mathcal{G}$ of a metric g will be called *normal* if the restrictions of g to Killing leaves S of $\mathcal{G}$ are non-degenerate. Obviously, a normal Killing algebra $\mathcal{G}$ is isomorphic to a subalgebra of $\mathcal{K}il\,(g|_S)$. Thus, when $\dim\mathcal{G}=3$ and the Killing leaves are bidimensional, $\mathcal{G}=\mathcal{K}il\,(g|_S)$. In this situation there are just five options for $\mathcal{K}il\,(g|_S)$ and therefore for $\mathcal{G}$:

$$so\,(2,1)\,,\;\mathcal{K}il\left(dp^2-dq^2\right),\;\; so\,(3)\,,\;\mathcal{K}il\left(dp^2+dq^2\right),\;\; \mathcal{A}_3\,.$$

The method used before allows to describe completely Einstein metrics admitting one of these algebra.

The gravitational fields invariant for $so(2,1)$ and $\mathcal{K}il(dp^2-dq^2)$, which are the only ones possessing non Abelian bidimensional subalgebras, can be found among solutions (4) and (6).

The gravitational fields invariant for $so(3)$ and corresponding to a positive choice of the solution $\beta(u)$ of the tortoise equation have the following local form:

$$g = \triangle_{\pm}\beta^2(dx^2 \pm dy^2) + \beta^2\left[d\vartheta^2+\sin^2\vartheta d\varphi^2\right]. \tag{7}$$

The choice of normal coordinates as in equation (5) and of minus sign gives the Schwarzschild solution with a new insight to the physical meaning of the so called *Regge-Wheeler tortoise coordinate* [16].

The gravitational fields g invariant for $\mathcal{K}il(dp^2+dq^2)$, have the local form

$$\begin{aligned} g &= 2f\left(dx^2-dy^2\right)+\alpha\left(x,y\right)\left[dr^2+r^2d\varphi^2\right], \\ \alpha &\equiv C_1F\left(x+y\right)+C_2G\left(x-y\right)+C, \quad f\equiv F'G'/\sqrt{|\alpha|}, \end{aligned}$$

F and G being arbitrary functions, C, C_1, C_2, arbitrary constants such that α and f are nonvanishing.

The Lie algebra $\mathcal{A}_3$ belongs to the Abelian case of [3].

Global solutions. Any of previous metrics is fixed by a solution of the wave or Laplace equation, and a choice

- of the constant A and one of the branches of a solution of the tortoise equation, if $g(X,Y)\neq 0$.
- of a solution of Darboux equation, if $g(X,Y)=0$.

The metric manifold (M,g) has a bundle structure whose fibers are the Killing leaves and whose base $\mathcal{W}$ is a bidimensional manifold diffeomorphic to the orthogonal leaves. The wave and Laplace equations mentioned above are defined on $\mathcal{W}$. Thus, the problem of the extension of our local solutions is reduced to that of the extension of $\mathcal{W}$. Such an extension carries a geometric structure, the **j** -*complex structure*, that gives an intrinsic sense to the notion of the wave and Laplace equations and clarifies what variety of different geometries is, in fact, obtained.

j-complex structures In full parallel with ordinary complex numbers, **j**-*complex numbers* of the form $z=x+\mathbf{j}y$, with $\mathbf{j}^2=-1,0,1$, can be introduced. Thus, a **j**-*complex analysis* can be developed by defining **j**-*holomorphic* functions as $\mathcal{R}^2_{\mathbf{j}}$ -valued differentiable functions of $z=x+\mathbf{j}y$, where $\mathcal{R}^2_{\mathbf{j}}$ is the algebra of **j**-complex numbers. Just as in the case of ordinary complex numbers, a function $f\left(z\right)=u\left(x,y\right)+\mathbf{j}v\left(x,y\right)$ is **j**-*holomorphic* if and only if the **j**-*Cauchy-Riemann* conditions hold:

$$u_x=v_y, \qquad u_y=\mathbf{j}^2v_x \tag{8}$$

The compatibility conditions of the above system require that both u and v satisfy the **j**-*Laplace equation*, that is

$$-\mathbf{j}^2u_{xx}+u_{yy}=0, \qquad -\mathbf{j}^2v_{xx}+v_{yy}=0. \tag{9}$$

Of course, the **j**-Laplace equation reduces for $\mathbf{j}^2=-1$ to the ordinary Laplace equation, while for $\mathbf{j}^2=1$ to the wave equation.

A bidimensional manifold $\mathcal{W}$ supplied with a **j**-*complex structure* is called a **j**-*complex curve*. Obviously, for $\mathbf{j}^2=-1$ a **j**-complex curve is just a 1-dimensional complex manifold. The case $\mathbf{j}^2=0$ will not be considered.

Thus, any global metric is associated with a pair consisting of a **j** -complex curve $\mathcal{W}$ and a **j** -harmonic function u on it.

Model solutions The pairs $(\mathcal{W}, u)$ and $(\mathcal{W}', u')$, corresponding to two equivalent solutions, are related by an invertible $\mathbf{j}$-holomorphic map $\Phi : (\mathcal{W}, u) \rightarrow (\mathcal{W}', u')$ such that $\Phi^*(u') = u$.

Particularly important are then the *model solutions*, namely those solution for which $(\mathcal{W}, u) = \left(\mathcal{R}^2_{\mathbf{j}}, x\right)$. The pair $\left(\mathcal{R}^2_{\mathbf{j}.}, x\right)$ is *universal* in the sense that any solution characterized by a given pair $(\mathcal{W}, u)$ is the pull-back of a model solution by a $\mathbf{j}$-holomorphic map $\Phi : \mathcal{W} \longrightarrow \mathcal{R}^2_{\mathbf{j}}$ defined uniquely by the relations $\Phi^*(x) = u$ and $\Phi^*(y) = v$, where v is conjugated with u.

It will be now described in detail how to construct global solutions in the case $\dim \mathcal{K}il\,(g_\Sigma) = 3$. The remaining cases can be found in [13].

Let us first consider $so\,(3)$ and $so\,(2,1)$. Denote by (Σ, g_Σ) a homogeneous bidimensional Riemannian manifold, whose Gauss curvature $K\,(g_\Sigma)$, if different from zero, is normalized to ± 1. Let $(\mathcal{W}, u)$ be a pair consisting of a $\mathbf{j}$-complex curve $\mathcal{W}$ and a $\mathbf{j}$-harmonic function u on $\mathcal{W}$. The bundle structure $\pi_1 : M \rightarrow \mathcal{W}$ canonically splits in the product $\mathcal{W} \times \Sigma$. Denote by $\pi_2 : M \rightarrow \Sigma$ the also natural projection of $M = \mathcal{W} \times \Sigma$ on Σ. Then, the above data determine the following Ricci-flat manifold (M, g) with

$$g = \pi_1^*\left(g_{[u]}\right) + \pi_1^*\left(\beta^2\right)\pi_2^*\left(g_\Sigma\right) \tag{10}$$

where $\beta\,(u)$ is implicitly determined by the tortoise equation, and

$$g_{[u]} = \pm\frac{(\beta - A)}{\beta}\left(du^2 - \mathbf{j}^2 dv^2\right). \tag{11}$$

In the case of normal Killing algebras isomorphic to $\mathcal{K}il\left(dx^2 \pm dy^2\right)$ it is sufficient to consider Ricci-flat manifolds M of the form

$$M = \mathcal{W} \times \Sigma, \qquad g = \pi_1^*\left(g_{[u]}\right) + \pi_1^*\,(u)\,\pi_2^*\left(g_\Sigma\right) \tag{12}$$

where (Σ, g_Σ) is a flat bidimensional manifold and

$$g_{[u]} = \pm\frac{1}{\sqrt{u}}\left(du^2 - \mathbf{j}^2 dv^2\right). \tag{13}$$

Examples

Algebraic solutions Let $\mathcal{W}$ be an algebraic curve over $\mathcal{C}$, understood as a $\mathbf{j}$-complex curve with $\mathbf{j}^2 = -1$. With a given meromorphic function Φ on $\mathcal{W}$ a pair $(\mathcal{W}_\Phi, u)$ is associated, where $\mathcal{W}_\Phi$ is $\mathcal{W}$ deprived of the poles of Φ and u the real part of Φ. A solution (metric) constructed over such a pair will be called algebraic. Algebraic metrics are generally singular [15], *e.g.* they are degenerate along the fiber $\pi^{-1}\,(a)$ if at $a \in \mathcal{W}$ $d_a u = 0$.

A star "outside" the universe The Schwarzschild solution describes a star generating a space around itself. It is an $so\,(3)$-invariant solution of the vacuum Einstein equations. Its $so\,(2,1)$ analogue shows a star generating the space only on "*one side of itself*". More exactly, the fact that the space in the Schwarzschild universe is formed by a 1-parameter family of concentric spheres allows one to give a sense to the adverb *around*. In the $so\,(2,1)$ case the space is formed by a 1-parameter family of *concentric* hyperboloids. The adjective *concentric* means that the curves orthogonal to hyperboloids are geodesics and metrically converge to a singular point. This explains in what sense this singular point generates the space only on "*one side of itself*".

The next example shows how the introduction of **j**-complex structures allows to manipulate already known Einstein metrics to get new ones also with singularities.

The "square root" of the Schwarzschild universe It is an Einstein metric induced by the **j**-*quadratic map:* $z \longrightarrow \frac{\mathbf{j}}{2}z^2$, with $\mathbf{j}^2 = 1$, from a model of the Schwarzschild type. The pair $(\mathcal{W}, u)$ is now $\left(\mathcal{R}^2_{\mathbf{j}}, xy\right)$ and the local expression of the metric is

$$g = \frac{e^{\frac{\beta + xy}{A}}}{2\beta}\left(x^2 - y^2\right)\left(dy^2 - dx^2\right) + \beta^2\left[d\vartheta^2 + \mathcal{F}\left(\vartheta\right)d\varphi^2\right],$$

where $\mathcal{F}$, depending on the Gauss curvature K of the Killing leaves, is one of the functions $\sin^2\vartheta$, $\sinh^2\vartheta$, $-\cosh^2\vartheta$. The metric is degenerate along the lines $x = \pm y$. The Einstein manifolds so obtained consist of four regions soldered along the degeneracy lines and could be interpreted [13] as "parallel" universes generated by "parallel" stars. By repeating this procedure one discover *foam-like universes.*

A detailed discussion of these and other new solutions will appear in a forthcoming paper.

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