A Liquid Model Analogue
for
Black Hole Thermodynamics

David Hochberg[1] and Juan Pérez-Mercader[2, 3]

Laboratorio de Astrofísica Espacial y Física Fundamental
Apartado 50727
28080 Madrid, Spain

February 7, 2008

Abstract

We are able to characterize a 2–dimensional classical fluid sharing some of the same thermodynamic state functions as the Schwarzschild black hole. This phenomenological correspondence between black holes and fluids is established by means of the model liquid's pair-correlation function and the two-body atomic interaction potential. These latter two functions are calculated exactly in terms of the black hole internal (quasilocal) energy and the isothermal compressibility. We find the existence of a "screening" like effect for the components of the liquid.

[1]Electronic address: hochberg@laeff.esa.es
[2]Also at: IMAFF, CSIC, Madrid.
[3]Electronic address: mercader@laeff.esa.es

arXiv:gr-qc/9609043v1 16 Sep 1996

1 Introduction.

The derivation of the thermal properties of a black hole is typically carried out in the context of the Euclidean path integral approach, as initiated by Gibbons and Hawking [1]. In this language, which is manifestly geometrical, the black hole partition function is identified with the Euclidean path integral, the integration over four-metrics playing the role of the configurational sum, and the hamiltonian, a function of the metric and curvature, taken to be that of the gravitational system.

A black hole of mass M has a temperature $T_\infty = \hbar/(8\pi M)$ measured at large distances from the hole. Following and extending this approach, York demonstrated that the canonical ensemble with elements of radius r and temperature $T(r)$ for hot gravity with black holes is well-defined [4]. That is, one treats a collection of spherical cavities of radius r with a temperature T at r. These cavities may contain either, no black hole, or one of two physically distinct black holes depending on the value of the product rT. In the case when the two distinct solutions pertains, only one of them will correspond to a thermodynamically stable black hole. This ensemble resolves a number of difficulties in assessing the physical significance of the classical black hole action in its contribution to the Euclidean path integral.

However, the reasons for considering the implantation of a black hole in a finite cavity, or "box", go well beyond the resolution of these initial difficulties and in fact the spatial cut-off provided by the cavity has been recognized as being crucial for making sense of black hole thermodynamics in general, quite independent of the path integral approach. For example, when one comes to consider the back-reaction of the radiation on the spacetime geometry of the black hole, the system comprised of black hole plus radiation must be put into a finite cavity, lest the radiation in a spatial region that is too large collapses onto the hole, thereby producing a larger one [3]. Related to this (but much more general) is the fact that the usual thermodynamic limit in which one invokes the infinite system volume limit does not exist for equilibrium, self-gravitating systems at a finite temperature. This follows since the system is unstable to gravitational collapse, or recollapse, if a black hole is already present. This, in practice, presents no problem since physically, one only requires that the system can in *principle* become sufficiently large so that fluctuations become negligible. This peculiarity of gravitational thermodynamics will play an important role in the present paper.

While the Euclidean path integral approach is well-defined and allows one to obtain the same value of the entropy as required by the thermodynamic analogy, the Bekenstein-Hawking entropy, it does not shed any light on the so-called dynamic origin of the entropy, nor does it explain the "microphysics" giving rise to the macroscopic thermodynamic functions, such as the internal energy, heat capacity, the equation-of-state, etc., that characterize the black hole. This state of affairs has spawned numerous efforts to understand the dynamical, or statistical mechanical, origins of black hole thermodynamics with particular emphasis paid to an explanation of the dynamical origin of black hole entropy as for example in Refs. [2] and [5]-[8].

In contrast to on-going efforts devoted to identifying the black hole's fundamental degrees of freedom, we wish to take a model-oriented approach and promote a phenomenological *analogy* between black holes and liquids. The analogy will be established at the level of thermodynamics. In the present paper, we seek what might be termed an effective *atomic* picture of black hole thermodynamics. By this we mean that it may be possible to reproduce (some) black hole thermodynamics in terms of microscopic properties of an interacting fluid or gas. The components of this analogue fluid are massive point particles interacting mutually via a pairwise additive potential. If such a correspondence is possible, we will have effected a mapping between the inherently geometrical degrees of freedom part and parcel of the Euclidean approach, and the "atomic" variables actually appearing in standard partition functions for fluids. The geometric quantities include metric, curvature, manifolds and boundaries. The so-called atomic quantities include the particles, their momenta and positions, and their interaction potentials. The correspondence is established via the black hole's thermodynamic functions, as derived from the standard Euclidean path integral approach, using these as given input. The task is then to characterize a liquid or (dense) gas whose microscopic properties (as encoded for example by the potential, pressure, pair-correlation function) can be "tuned", or adjusted suitably, so as to reproduce mathematically the same set of black hole state functions. If the program so described is successful, then one has in effect, replaced integration over metrics by an integration over a multi-particle classical phase space while the gravitational action is replaced by a particle hamiltonian, containing a (non-relativistic) kinetic energy and potential energy term.

In the next Section we review and comment on the essential features of black hole thermodynamics as derived from the Euclidean path integral in

the saddle point approximation. The black hole energy, entropy, equation of state and compressibility are displayed and their qualitative features are revealed through various limits and graphical representations. The way in which we establish a connection between liquids and black holes is taken up in Section III. The key in building a "liquid" model is provided by the fundamental equations employed in the study of the atomic dynamics of simple liquids and the atomic picture of the thermodynamics of the liquid state. These fundamental relations equate the macroscopic (thermodynamic) to the microscopic (internal structure, potential energy) properties of fluids. These relations are derived as rigorous consequences of statistical mechanics applied to fluids and (dense) gases. A particular type of fluid is singled out the moment we identify the macroscopic variables of the fluid with those of the black hole. The points of contact between the black hole and fluid are set up via their respective internal energies and compressibilities. The analog fluid is identified to the extent that we can write down its pair-potential and two-body correlation function. The, necessarily, bounded spatial extent of the black hole ensemble is crucial in allowing us to solve for the liquid's microscopic parameters exactly. These are calculated in closed form as well as presented graphically.

The ultimate purpose of establishing such a mapping is the double benefit to be gained in being able to relate black hole physics to the molecular dynamics of fluids. Recent work of a similar spirit includes the possible correspondence between black holes and quantized vortices in superfluids [9] and a connection between fluid surface tension and black hole entropy [10]. A summary is given in Section III. Absolute units ($G = c = \hbar = k_B = 1$) are used throughout except where restoration of conventional units may be helpful.

2   Black hole thermodynamics in brief.

In deriving gravitational thermodynamics from an Euclidean path integral

$$Z(\beta) = \int d\mu[g, \phi] \, e^{-I[g,\phi]} = e^{-\beta F}, \qquad (1)$$

one expects the dominant contribution to the canonical partition function to come from those metrics g and matter fields ϕ that are near a background

4

metric $g^{(0)}$ and background field $\phi^{(0)}$, respectively. These background fields are obtained from solutions of the classical field equations. The classical contribution to Z is obtained by evaluating Z at the point $(g^{(0)}, \phi^{(0)})$ in which case one obtains the familiar relation

$$\beta F = I[g^{(0)}, \phi^{(0)}] = I[g^{(0)}], \tag{2}$$

where we have taken $\phi^{(0)} = 0$ in the last equality. This provides the free energy F of the gravitational system in the saddle-point approximation. The action I is the first-order Euclidean Einstein action including a subtraction term necessary to avoid runaway solutions. The action appropriate to a black hole in a spherical cavity of radius r is given by [4]

$$I = I_1 - I_{subtract}, \tag{3}$$

where

$$I_1 = -\frac{1}{16\pi} \int_{2M}^{r} \int_0^{\beta_*} d^4x \sqrt{g}\, R + \frac{1}{8\pi} \oint_{S^1 \times S^2} d^3x \sqrt{\gamma}\, Tr(\mathcal{K}) \tag{4}$$

The action receives in general both volume and boundary contributions. The Euclidean four-space metric is

$$g_{\mu\nu} = \text{diag}\left((1 - \frac{2M}{r}), (1 - \frac{2M}{r})^{-1}, r^2, r^2 \sin^2\theta\right). \tag{5}$$

For this metric, the volume contribution vanishes identically. The boundary at $r = const.$ is the product of $S^1 \times S^2$ of the periodically identified Euclidean time with the two-sphere of area $A = 4\pi r^2$. The period of Euclidean time, identified with the S^1-coordinate, is $\beta_* = 8\pi M$. The trace of the boundary extrinsic curvature is denoted by $Tr(\mathcal{K})$ and γ is the induced 3-metric on the boundary. Finally, $I_{subtract}$ is I_1 evaluated on a flat spacetime having the same boundary $S^1 \times S^2$.

It is important to remember that for the canonical ensemble the mass parameter M appearing in these formulae is not simply a constant but is instead a specific function of the cavity radius $r \geq 0$ and the cavity wall temperature $T(r) \geq 0$ [3, 4]. This can be verified by inverting the expression for the blue-shifted temperature in equation (11) below and solving for $M = M(r, T)$. The relation so obtained is a cubic equation in M. When $rT <$

$\sqrt{27}/8\pi = (rT)_{min} \approx 0.207$, there are no real solutions of this equation. On the other hand, when $rT \geq (rT)_{min}$, there exist two real and non–negative branches given by

$$M_2(r, T) = \frac{r}{6}\left[1 + 2\cos(\frac{\alpha}{3})\right], \tag{6}$$

$$M_1(r, T) = \frac{r}{6}\left[1 - 2\cos(\frac{\alpha + \pi}{3})\right], \tag{7}$$

$$\cos(\alpha) = 1 - \frac{27}{32\pi^2 r^2 T^2}, \tag{8}$$

$$0 \leq \alpha \leq \pi. \tag{9}$$

This shows that the Schwarzschild mass is in fact double-valued in the canonical ensemble. One has that $M_2 \geq M_1$, with equality holding at $rT = (rT)_{min}$. The heavier mass branch, M_2, is the thermodynamically stable solution because it leads to the lowest free energy, Eq. (2), and is the one we shall be considering in the remainder of this work.

Calculating the action I from (3) and (4) yields ($I_{subtract} = -\beta r$)

$$I = 12\pi M^2 - 8\pi Mr + \beta r, \tag{10}$$

where

$$\beta = T^{-1}(r) = 8\pi M \left(1 - \frac{2M}{r}\right)^{1/2}, \tag{11}$$

is the inverse local temperature and is the proper length of the S^1 component of the boundary. Employing I and the saddle-point approximation $\beta F = I$, it is a straightforward exercise to calculate the thermodynamic state functions associated with a black hole in the canonical ensemble. In so doing, it is useful to note the following two identities

$$\left(\frac{\partial M}{\partial r}\right)_\beta = -\frac{\frac{M^2}{r^2}}{(1 - \frac{3M}{r})}, \tag{12}$$

$$\left(\frac{\partial M}{\partial \beta}\right)_A = \frac{1}{8\pi}\frac{(1 - \frac{2M}{r})^{1/2}}{(1 - \frac{3M}{r})}, \tag{13}$$

which may be deduced from the expression for the inverse local temperature (11). The black hole's internal, or thermal, energy is

$$E = -\left(\frac{\partial \ln Z}{\partial \beta}\right)_A = \left(\frac{\partial I}{\partial \beta}\right)_A = r - r\left(1 - \frac{2M}{r}\right)^{1/2}. \tag{14}$$

The entropy S is

$$S = \beta \left(\frac{\partial I}{\partial \beta} \right)_A - I = 4\pi M^2, \tag{15}$$

while the surface pressure σ is

$$\sigma = - \left(\frac{\partial F}{\partial A} \right)_T = \frac{1}{8\pi r} \left[\frac{\left(1 - \frac{M}{r} \right)}{\left(1 - \frac{2M}{r} \right)^{1/2}} - 1 \right]. \tag{16}$$

Another quantity of special interest in what is to follow, is the black hole isothermal compressibility, $\kappa_T(A)$, which again can be calculated using the standard prescription,

$$\kappa_T(A) = -\frac{1}{A} \left(\frac{\partial A}{\partial \sigma} \right)_T = 16\pi r \left(\frac{r}{M} \right)^3 \left(1 - \frac{3M}{r} \right) \left(1 - \frac{2M}{r} \right)^{3/2} /$$

$$\left\{ 1 + \left(\frac{r}{M} \right)^3 \left(\frac{3M}{r} - 1 \right) \left[\left(1 - \frac{2M}{r} \right)^{3/2} - 1 + \frac{3M}{r} - \frac{3M^2}{r^2} \right] \right\}. \tag{17}$$

Although at face value these functions appear to have a complicated dependence on r and T, they are actually quite simple, owing to their dependence on the slowly varying ratio M/r. To gain some insight into the behavior of these functions, it is useful to examine some limiting cases, namely when $(i)\, rT \rightarrow \infty$ and when $(ii)\, rT \rightarrow (rT)_{min}$. The limit (i) is understood to mean either that $r \rightarrow \infty$ or $T \rightarrow \infty$, or both, simultaneously. The second limit (ii) actually defines a hyperbola in the $r - T$ plane along which the two independent limits $r \rightarrow 0\, (T \rightarrow \infty)$ or $r \rightarrow \infty\, (T \rightarrow 0)$ can be taken. The mass function takes the form $M(r, T) \approx \frac{r}{2}(1 - \frac{1}{4\pi^2 r^2 T^2})$, and $M(r, T) = \frac{r}{3}$, respectively. Physically, these limits indicate that for all allowed values of r and T, the cavity wall radius always lies between the black hole's event horizon $(r = 2M)$ and the unstable circular photon orbit $(r = 3M)$. The behavior of the black hole internal energy with respect to these limits is

$$(i) \qquad E \rightarrow r - \frac{1}{(4\pi T)}, \tag{18}$$

and

$$(ii) \qquad E = \frac{\sqrt{3} - 1}{\sqrt{3}} r. \tag{19}$$

7

E is essentially a positive linear function of r, depending only very weakly on the temperature for large values of rT. For $rT = (rT)_{min}$, E is strictly linear in r, or inversely proportional to T, since of course, in this latter limit, $r \sim 1/T$. Note that on this hyperbola, $E \to 0$ for $r \to 0$ (or $T \to \infty$). The equation for the surface pressure is also an equation of state for the black hole pressure since it is expressed as a function of the cavity radius r, which gives a measure of the system size, and the boundary temperature T: $\sigma = \sigma(r, T)$. Using the limiting forms of the mass function, one can show that the asymptotic limit of the surface pressure is

$$(i) \qquad \sigma \to \frac{T}{4} - \frac{1}{8\pi r}, \qquad (20)$$

for $rT \to \infty$, so that the pressure increases with the temperature, depending only very weakly on the system size. When evaluated along the limit hyperbola, one obtains

$$(ii) \qquad \sigma = \left(\frac{2\sqrt{3}}{3} - 1 \right) \frac{1}{8\pi r}; \qquad (21)$$

in other words, in this regime, the pressure increases as the cavity radius (or area) decreases. Because of the reciprocal relation between r and T along the hyperbola, this is equivalent to increasing pressure with increasing temperature. Such qualitative behavior is familiar from the ideal gas. Finally, the limiting forms of the isothermal compressibility are

$$(i) \qquad \kappa_T(A) \to -16\pi r < 0, \qquad (22)$$

and

$$(ii) \qquad \kappa_T(A) = 0. \qquad (23)$$

These two latter limits deserve some special comment. First, note that the black hole isothermal compressibility is generally negative. This is an unfamiliar property in regards to conventional thermodynamic systems. Indeed, standard textbook arguments prove that $\kappa_T \geq 0$, irrespective of the nature of the substance comprising the system. However, a key step in those proofs assumes that quantities such as the temperature and pressure are *intensive*, that is, independent of the size of the system and constant throughout its interior. Such is most emphatically *not* the case for gravitating systems such

as black holes, where in fact, the temperature and pressure are not intensive quantities but are instead scale dependent. An equilibrium self-gravitating object does not have a spatially constant temperature. This is a consequence of the principle of equivalence which implies that temperature is red- or blue-shifted in the same manner as the frecuency of photons in a gravitational field. Secondly, for values $rT = (rT)_{min}$, the compressibility vanishes identically. This qualitative behavior is familiar from the classical picture of a solid at $T = 0$ (no density fluctuations $\implies$ zero compressibility).

In the Figures 1-3, we have plotted E, σ and κ_T in the $r - T$ plane subject to the condition $rT \geq (rT)_{min}$. As indicated in Fig. 1, the black hole energy is a positive increasing function in r and is fairly insensitive to changes in the temperature for values of $T \geq 0.4$. The flat region at zero level corresponds to the locus of excluded points satisfying $rT < (rT)_{min}$ and is therefore not to be considered as part of the graph as such. The cavity wall surface pressure, shown in Fig. 2, is a positive increasing function of T and varies slowly in r for $r \geq 0.5$. The flat null region represents the same locus of points as in the prior graph. Finally, the black hole isothermal compressibility is a *negative definite function* for all $rT > (rT)_{min}$, decreasing for increasing r and relatively constant with respect to changes in the temperature.

Other functions that may be calculated include the specific heats at constant area and at constant pressure, respectively, as well as the adiabatic compressibility, but these are of no direct interest for the present consideration.

Finally, to complete this brief overview of black hole thermodynamics, we need to identify the effective spatial dimension of the system. It will not have escaped the reader's attention that the above functions have been defined and calculated in terms of the cavity wall area A, rather than in terms of the cavity volume. Spatial volume is not well defined in the presence of a black hole, whereas the area is, and it is the latter which provides the correct means for measuring the size of the system [4]. That the wall area is the proper extensive variable to use is confirmed by considering the black hole's thermodynamic identity. The explicit calculation of the following partial derivatives

$$\left(\frac{\partial E}{\partial S} \right)_A = \frac{1}{8\pi M} \left(\frac{\partial E}{\partial M} \right)_A = \frac{1}{8\pi M} \left(1 - \frac{2M}{r} \right)^{-1/2} \equiv T, \qquad (24)$$

and

$$\left(\frac{\partial E}{\partial A}\right)_S = \frac{1}{8\pi r}\left(\frac{\partial E}{\partial r}\right)_M = \frac{1}{8\pi r}\left[1 - \frac{\left(1 - \frac{M}{r}\right)}{\left(1 - \frac{2M}{r}\right)^{1/2}}\right] \equiv -\sigma. \qquad (25)$$

proves that

$$dE = T\,dS - \sigma\,dA \qquad (26)$$

is an exact differential. In other words, the energy when expressed in terms of its proper independent variables, $E = E(S, A)$, is integrable. We remark that all the above functions may be considered as functions either of the cavity radius or the wall area, since obviously, $r = \sqrt{A/(4\pi)}$ and $dA = 8\pi r dr$.

3   A Liquid Model for Black Holes.

The starting point for attempting to model black hole thermodynamics in terms of liquids is the statistical mechanical treatment of fluids. There, it is known how to relate the various macroscopic thermodynamic properties of liquids (energy, pressure, temperature) to the internal, microscopic features such as the interaction/intermolecular potential and the pair-correlation function. This latter function provides a measure of the local structure of the fluid or gas. The typical approach to the study of the liquid state starts with (a perhaps imperfect) knowledge of the interatomic force law and the measured short range order (obtained experimentally via X-ray or neutron scattering experiments). One then attempts to infer the macroscopic or thermodynamic behavior of the liquid on the basis of this microscopic information. By marked contrast, here we shall turn the reasoning around and solve for the local "structure" and the "interatomic" potential of an analog (and possible fictitious!) fluid from explicit knowledge of black hole thermodynamics. Two very important ingredients which will allow this inverted procedure to be carried out in closed form are (i) the fact that the thermal ensemble of black holes is spatially bounded and (ii) the fact that we know the spatial dimension of this ensemble to be $d = 2$.

The model fluid we shall deduce will be described in classical terms. Let us use our knowledge of the spatial dimensionality of the black hole ensemble at the outset. To make a thermodynamic correspondence between black hole and fluid means we seek a two-dimensional fluid whose partition function

over the $2N$-dimensional phase space is given by (restoring the dependence on $\hbar$)

$$Z_N(\beta) = \frac{1}{(2\pi\hbar)^{2N} N!} \int d\{p_i\} \int d\{r_i\} \, e^{-\mathcal{E}/kT} \qquad (27)$$

and where the total, nonrelativistic energy of a system of N interacting point particles of mass m in $d = 2$ is

$$\mathcal{E}(\{p_i\}, \{r_i\}) = \sum_{i \neq j}^{2N} \frac{p_i^2}{2m} + U(\{r_i\}). \qquad (28)$$

Here, $U(\{r_i\})$ is the potential energy of the particle system which we assume to be pairwise additive:

$$U = \frac{1}{2} \sum_{i \neq j}^{N} \phi(|\mathbf{r}_i - \mathbf{r}_j|). \qquad (29)$$

This is always a reasonable assumption provided the fluid constituents have no internal structure that couples to the potential. From the theory of liquids, it is well known that an equation of state for the isothermal compressibility κ_T and the internal energy E can be calculated in terms of the pair-potential ϕ and the pair-correlation function g [11]- [13], to whit (restoring the dependence on k_B),

$$(a): \qquad \rho \, k_B T \kappa_T = \rho \int_{system} d^2\tilde{\mathbf{r}} \, [g(\tilde{r}) - 1] + 1, \qquad (30)$$

and

$$(b): \qquad E = N k_B T + \frac{1}{2} N \rho \int_{system} d^2\tilde{\mathbf{r}} \, \phi(\tilde{r}) g(\tilde{r}), \qquad (31)$$

where $\rho = \frac{N}{A}$ is the two-dimensional particle density, T is the fluid temperature, and the radial distribution function g is defined via

$$\rho \, g(r) = \frac{1}{N} \langle \sum_{i \neq j} \delta(\mathbf{r} + \mathbf{r}_i - \mathbf{r}_j) \rangle, \qquad (32)$$

the angular brackets denote the average computed using the grand canonical ensemble.

Before we go on to use these relations, we remark that the equation of state (a) is exact while the expression for the internal energy (b) makes use of

11

the pairwise summability of the total potential energy. They are valid for any single-component, monatomic system in thermodynamic equilibrium (gas, liquid or solid) whose energy is expressible in the form (28) with a pairwise additive potential (29). Although these expressions are derived primarily for their application to the liquid state, they can also be applied to the study of solids. The only modification would be that g and ϕ depend on the full vector coordinate $\mathbf{r}$ (magnitude and direction). For liquids, however, the results are isotropic so it is enough to write $g(r)$ and $\phi(r)$. For the present consideration, modelling a liquid which is capable of reproducing certain aspects of black hole thermodynamics is carried out once we identify the κ_T and E in (a) and (b) with those of the black hole.

The idea of representing a black hole at finite temperature as a thermal fluid is novel and therefore deserves careful explanation. A black hole is but one example of a thermodynamic system having, among other things, a well-defined temperature, energy and compressibility. On the other hand, any equilibrium many-body system with hamiltonian given in Eq.(28) and Eq.(29) has a well-defined energy and compressibility which can be calculated in terms of an associated g and ϕ. When we *formally* identify the κ_T and E appearing there with those belonging to the black hole, we are simply demanding that these particular thermodynamic functions be reproducible in terms of the internal variables of a certain classical many-body system. This is not to say that these variables actually represent the true degrees of freedom of a black hole. The identification must be carried out in a consistent way. First, the temperature T appearing in (30) and (31) is the uniform cavity wall temperature. Since (a) and (b) are to decribe a liquid, the temperature of that liquid must be identified with this temperature: $T_{liquid} = T$. Note that the temperature of the liquid is *intensive*. That is, the temperature of the cavity wall of the black hole ensemble is identified with the temperature of the bulk fluid. Next, the density ρ of the fluid is simply the number of "atoms" per unit area of fluid. For the black hole, both E and κ_T depend explicitly on the cavity radius r, reflecting the fact that the black hole ensemble is spatially finite. This means that the integrations in (a) and (b) are to be carried out over a fluid of bounded spatial extent. The integrations over the *system* are bounded. Since the integrands are functions only of r and T, we can therefore write

$$\int_{system} d^2\mathbf{r} = \int_0^r r \, dr \int_0^{2\pi} d\theta. \tag{33}$$

It is natural to take the length scale of the liquid coincident with that of the cavity containing the black hole; any other choice would introduce a second, and arbitrary, length scale into the problem. Taking E and κ_T from (14) and (17) as input, the relations (a) and (b) yield two equations in the two unknowns g and ϕ. We can easily solve for these microscopic functions in terms of the macroscopic functions and their first derivatives. To do so, we make use of (33) and differentiate both sides of the relations in (a) and (b) with respect to r. The results of this operation are that

$$\phi(r,T)g(r,T) = \frac{4r}{N^2}\left[\left(\frac{\partial E}{\partial r}\right)_T + \frac{2}{r}\left(E - Nk_BT\right)\right], \tag{34}$$

and

$$g(r,T) - 1 = \frac{2r}{N}\left[k_BT\left(\frac{\partial \rho\kappa_T}{\partial r}\right)_T + \frac{2}{r}\left(\rho k_BT\kappa_T - 1\right)\right]. \tag{35}$$

By explicit construction, these give the pair correlation function and the inter-particle potential of the model fluid whose energy and isothermal compressibility are identical with those of the black hole. Moreover, these two functions depend on the two <u>independent</u> variables r and T. Since we can vary them independently, we have actually obtained ϕ and g as functions of their arguments for all $T \geq 0$ and $r \geq 0$, subject only to the restraint that the product rT always be greater than or equal to $(rT)_{min}$. By way of a trivial but illustrative example, consider the ideal gas in two-dimensions whose equation of state is $pA = Nk_BT$. Then $E = Nk_BT$ and $\kappa_T = -\frac{1}{A}(\frac{\partial p}{\partial A})_T = 1/(\rho k_BT)$. Inserting these into the above relations immediately yields $g(r) = 1$ and $\phi(r) = 0$, which is also a solution of the pair of equations (30) and (31). As is to be expected, the ideal gas has no structure (it is uniform: homogeneous and isotropic) and lacks interatomic interactions (by definition). Therefore, any deviation in either g and or ϕ with respect to these limits may be considered as deviations from an ideal gas.

It is of interest to consider the limiting forms of the pair correlation function and potential energy for the black hole; these may be deduced easily from the associated limits calculated above for E and κ_T, Eqs.(18,19) and

Eqs.(22,23). When $rT >> (rT)_{min}$, the pair correlation function goes as

$$(i) \qquad g(r,T) \sim 1 - \frac{8k_BT}{r} - \frac{4}{N}. \qquad (36)$$

In particular, for fixed temperature, $g(r,T) \to 1 - \delta$, as $r \to \infty$ where $\delta = 4/N$ is small for N large. In normal simple liquids, $g(r)$ has the asymptotic limit $g(r) \to 1$ (compare to the ideal gas limit) and deviations from this value represent molecular correlations (or anti-correlations). When evaluated along the boundary hyperbola $rT = (rT)_{min}$ we get,

$$(ii) \qquad g(r,T) = 1 - \frac{4}{N}, \qquad (37)$$

a constant independent of r and T. The corresponding limits for the two-body potential energy may be worked out and yield

$$(i) \qquad \phi(r,T) \sim \frac{4r}{N^2} \left(3 - \frac{2Nk_BT}{r} \right) \Big/ \left(1 - \frac{8k_BT}{r} - \frac{4}{N} \right). \qquad (38)$$

For fixed temperature, $\phi \sim r$. When $rT = (rT)_{min}$,

$$(ii) \qquad \phi(r,T) = \frac{4r}{N^2} \left[(3 - \sqrt{3}) - \frac{2Nk_BT}{r} \right] \Big/ \left(1 - \frac{4}{N} \right). \qquad (39)$$

The black hole pair correlation function is calculated and presented in Fig. 4. For fixed T and small r, this function is negative, then increases, becoming positive and approaches unity from below as $r \to \infty$. This behavior is also revealed in the one dimensional plot of $g(r,T)$ for the value $T = 0.5$ in Fig. 5. What can we make of this behavior in g and what physical interpretation can it admit? For this, let us turn to the meaning of $g(r)$. Imagine we select a particular particle of the fluid, whose average density is ρ, and fix our origin at that point. Then, the number of particles dN contained within the (two-dimensional) spherical shell of thickness dr centered at $r = 0$ is

$$dN = \rho \, g(r)2\pi r \, dr. \qquad (40)$$

Here we see that g gives a measure of the deviation from perfect homogeneity ($g = 1$). Evidently, $g < 0 \iff dN < 0$ in that shell. On the other

hand, a negative value for dN is the signature for the phenomenon of charge-screening, i.e., it indicates the presence of *holes* in the neighborhood of our reference particle at $r = 0$. Thus, it would appear that the analogue fluid which could model some aspects of black hole thermodynamics should have something to do with a charged fluid or plasma. These latter systems are defined as a collection of identical point charges, of equal mass m and charge e, embedded in a uniform background of charge (the dielectric) obeying classical statistical mechanics. If one adds a given charge to the plasma, the plasma density is locally depleted so as to neutralize the impurity charge. This is the well-known phenomenon of charge screening. The depletion shows up as an underdensity of particles (or an overdensity of holes), and is reflected in a $g < 0$ near the origin, that is, where the impurity charge is located. Calculations of the pair-correlation function for degenerate electron plasmas at metallic densities yield functions exhibiting the same general qualititative features as those in Figures 4 and 5 [12]. In addition, screening is known to be a characteristic property of interactions like the electromagnetic interaction, where there exist two species of charges; they have the property that renormalization effects induce corrections that make the *effective* charge decrease with distance. This, in itself, is not surprising because as is well known there exists an analog of a black–hole with an electromagnetic membrane; an analogy that in the literature is called the "membrane paradigm" [14]. What we find here is a different manifestation of this analogy, this time through the dynamical and statistical properties of the liquid.

The weakly temperature dependent potential is scaled by N^2 and plotted in Figure 6. Again, recall the physical part of the graph consists of those points r and T satisfying $rT > (RT)_{min}$. The potential is seen to be a positive increasing function of r. The limit calculated above in Eq.(38) shows the growth is essentially linear. Apart from the "glitch" near $T \approx 0.2$ the potential is practically independent of the temperature.

4   Conclusions.

It is worth emphasizing that the analogue fluid selected to account for the black hole compressibility and internal energy was "engineered" at the fluid's atomic level. As there is no corresponding "atomic" level for the black hole, the bridge between the thermodynamics of the black hole and liquid is es-

tablished via thermodynamic state functions. Surprisingly, only two state functions are needed in order to specify completely the "atomic potential" and local structure of the analogue fluid. However, as we have seen, we can only be sure that the fluid will reproduce the correct compressibility and internal energy. That is, only partial aspects of black hole thermodynamics will be reproducible, since, evidently, there exist other state functions that characterize a black hole, namely, its entropy, pressure, specific heats, etc. which must be calculable by a more complete "microscopic" description of the black hole (see below). The analogy with the liquid leads to a screening effect that can be understood in terms of a connection with the membrane paradigm.

The partial rendering of black hole thermodynamics in terms of atomic fluid elements achieved here points to the possibility of directly effecting a mapping between the black hole variables (mass M and cavity radius r, or cavity wall temperature $T(r)$) and the internal variables of an analogue model which might serve to reproduce all of black hole thermodynamics. Evidently, this would amount to a formal correspondence at the level of the degrees-of-freedom and so bypass the need to call into play the macroscopic state functions. A concrete example of such a mapping between two entirely distinct systems is that established recently between a Newtonian cosmology of pointlike galaxies and spins in a three-dimensional Ising model [15]. The degree-of-freedom mapping problem is well worth pursuing as intriguing deep connections between gravitation, thermodynamics and information theory have been hinted at recently [16]. Another hint is supplied by Wheeler's depiction of the Bekenstein bit number as a "covering" of the event horizon by a binary string code representing the information contained in the black hole [17, 18]. It may well be possible to go beyond these provocative hints and actually establish a rigorous connection between black holes, computation, information theory and complexity. We hope to report on these developments in a separate paper.

References

[1] G.W. Gibbons and S.W. Hawking, Phys. Rev. D**15**, 2752 (1977); see also S.W. Hawking in *General Relativity: An Einstein Centenary Survey*,

edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

[2] J.W. York, Jr., Phys. Rev. D**28**, 2929 (1983).

[3] J.W. York, Jr., Phys. Rev. D**31**, 775 (1985).

[4] J.W. York, Jr., Phys. Rev. D**33**, 2092 (1986).

[5] W.H. Zurek and K.S. Thorne, Phys. Rev. Lett. **54**, 2172 (1985).

[6] G. t'Hooft, Nucl. Phys. B**256**, 727 (1985).

[7] V. Frolov and I. Novikov, Phys. Rev. D**48**, 4545 (1993).

[8] A.O. Barvinsky, V.P. Frolov and A.I. Zelnikov, Phys. Rev. D**51**, 1741 (1995).

[9] G.E. Volovik, Helsinki University of Technology report LTL-95006, gr-qc/9510001 (October 1995).

[10] D.J.E. Callaway, Rockerfeller University report RU 94-48, cond-mat/9601111, (January 1996).

[11] David L. Goodstein, *States of Matter*, (Dover, New York, 1985), Chap 4.

[12] N.H. March and M.P. Tosi, *Atomic Dynamics in Liquids* (Dover, New York, 1991).

[13] A. Ishihara, *Condensed Matter Physics* (Oxford University Press, Oxford, 1991), Chap 1.

[14] Thorne, K and R. Price, *The Membrane Paradigm* (Yale University Press, New haven 1989).

[15] J. Pérez-Mercader, T. Goldman, D. Hochberg and R. Laflamme, to appear in Phys. Lett. **A**, (1996)

[16] J.D. Brown and J.W. York, Jr., in *Physical Origins of Time Asymmetry*, eds. J.J. Halliwell, J. Pérez-Mercader and W.H. Zurek (Cambridge University Press, Cambridge, 1994).

[17] J.A. Wheeler, *A Journey into Gravity and Spacetime*, (Freeman, New York, 1990), Chap. 12.

[18] J.A. Wheeler, in *Physical Origins of Time Asymmetry*, eds. J.J. Halliwell, J. Pérez-Mercader and W.H. Zurek (Cambridge University Press, Cambridge, 1994).

Figure Captions.

Figure 1. The black hole energy, E, as a function of the cavity radius r and cavity wall temperature T. It is seen that E grows with cavity radius and with temperature. The flat portion of the graph (in this and in the following figures) is not part of the physical quantity shown in the graph; it corresponds to the region $rT < (rT)_{min}$ in which the black hole mass is no longer real and positive.

Figure 2. The black hole surface pressure, σ, as a function of the cavity radius r and cavity wall temperature T. It is seen that σ grows sharply at first with cavity radius to then stabilize its growth, while its growth with temperature is unimpeded. The flat portion of the graph (as in all figures in this paper figures) is not part of the physical quantity shown in the graph; it corresponds to the region $rT < (rT)_{min}$ in which the black hole mass is no longer real and positive.

Figure 3. The black hole isothermal compressibility, κ_T, as a function of the cavity radius r and cavity wall temperature T. It is seen that κ_T decreases with cavity radius and with temperature. The flat portion of the graph (as in all figures in this paper figures) is not part of the physical quantity shown in the graph; it corresponds to the region $rT < (rT)_{min}$ in which the black hole mass is no longer real and positive.

Figure 4. The black hole pair correlation function, g, inferred in the liquid model of the black hole as a function of the cavity radius r and cavity wall temperature T. It is seen that for $r > .2$, g grows with cavity radius but decreases with temperature. The flat portion of the graph (in this and in the following figures) is not part of the physical quantity shown in the graph; it corresponds to the region $rT < (rT)_{min}$ in which the black hole mass is no longer real and positive. The spikes are *numerical artifacts* of the plotting routine used to plot the figure and which could not render the deep trough from which the physical region emerges.

Figure 5. Two dimensional representation of the black hole pair correlation function, g, as a function of the cavity radius r but for a fixed value of the cavity wall temperature $T = 0.5$. This is plotted here to illustrate the

existence of the trough which was not clearly seen in the three dimensional plot of Figure 4. Notice that for large r the pair correlation function tends to the constant value $+1$.

Figure 6. The black hole "liquid" two–particle potential, ϕ, as a function of the cavity radius r and cavity wall temperature T. It is seen that apart from the sudden rise near $T = 0.2$, the potential is a rather boring function of cavity radius and temperature. The flat portion of the graph (in this and in the previous figures) is not part of the physical quantity shown in the graph; it corresponds to the region $rT < (rT)_{min}$ in which the black hole mass is no longer real and positive.