Lie Groups of Conformal Motions acting on Null Orbits

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Abstract

Space-times admitting a 3-dimensional Lie group of conformal motions C_3 acting on null orbits are studied. Coordinate expressions for the metric and the conformal Killing vectors (CKV) are provided (irrespectively of the matter content) and then all possible perfect fluid solutions are found, although none of these verify the weak and dominant energy conditions over the whole space-time manifold.

In this letter we shall consider space-times (M, g) admitting a maximal three-parameter conformal group C_3 containing an Abelian two-parameter subgroup of isometries G_2 whose orbits S_2 are spacelike, diffeomorphic to $\mathbb{R}^2$ and admit orthogonal two-surfaces. Furthermore, we shall assume that the C_3 acts transitively on null orbits N_3, thus complementing a previous paper [1] in which the case of null conformal orbits was explicitly excluded. In particular, in this letter we shall provide the coordinate expressions for the metric and the CKV for each Lie algebra structure and give *all* possible perfect fluid solutions.

A few remarks concerning Lie groups acting on null orbits are in order here. In most cases the study of null orbits has been restricted to isometries only. It is a well known fact that groups G_r, $r \geq 4$, acting on N_3 have at least one subgroup G_3 acting on N_3, N_2 or S_2 [2]. In the case in which the subgroup G_3 acts on S_2, the space-time is a LRS model, and the G_r admits either a different subgroup G_3 acting on N_3 or a null Killing vector (KV) [3]. The case G_3 acting on N_2 was studied by Barnes [4]; the group G_3 is then of Bianchi type II and perfect fluid solutions are excluded since the metric leads to a Ricci tensor whose Segre type is not that of a perfect fluid. Another case that has been considered in the literature is that of a G_3 acting on N_3 in which $R_{ab}k^a k^b = 0$, and this condition excludes perfect fluid sources with $\mu + p \neq 0$. It

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is also known that perfect fluid solutions cannot admit a non-twisting ($w = 0$) null KV except when $\mu + p = 0$. The algebraically special perfect fluid solutions with a twisting null KV are treated by Wainwright [5], and they admit an Abelian group G_2. Space-times admitting a null CKV have been studied recently by Tupper [6]. He has found that, for perfect fluid and null radiation, non-conformally flat space-times admitting a null CKV are algebraically special; furthermore, if one assumes the CKV to be proper (non-homothetic) then the only possibilities are those solutions in which the line element admits a multiply transitive group of isometries G_3 acting on two-spaces of constant curvature.

One might get the impression that space-times admitting a three-dimensional Lie group of conformal motions C_3 acting on null orbits (i.e., the case under consideration here) might not admit any perfect fluid solutions, since the line element of these space-times is, by the theorem of Defrise-Carter [7], conformally related to one admitting a G_3 acting on null orbits and such space-times, as we have pointed out above, do not admit perfect fluid solutions. However, we will show that this is not the case. Indeed, a conformal scaling changes the algebraic structure of the Ricci tensor. Nevertheless, we find that there are only a few perfect fluid solutions possible.

The classification of all possible Lie algebra structures for $\mathcal{C}_3$ was given in [1] where coordinates were adapted so that the line element associated with the metric g can be written as

$$ds^2 = e^{2F}\{-dt^2 + dr^2 + Q[H^{-1}(dy + W dz)^2 + H dz^2]\} \ , \tag{1}$$

where F, Q, H and W are all functions of t and r alone. (The precise hypotheses leading to this classification were given explicitly in Ref. [1].)

If the conformal algebra $\mathcal{C}_3$ belongs to the family A (i.e., the commutator between the CKV and each KV is a KV), it was shown in [1] that, for null conformal orbits, one can always bring the CKV, X, to the form

$$X = \partial_t + \partial_r + X^y(y,z)\partial_y + X^z(y,z)\partial_z \ , \tag{2}$$

where $X^y(y,z)$ and $X^z(y,z)$ are linear functions of their arguments to be determined from the commutation relations between X and the KVs. Considering now the conformal Killing equations for the CKV (2) and the metric (1), for each possible group type, one obtains the following forms for X and the metric functions F, Q, H, and W appearing in (1) as follows:

$$(I) \qquad Q = q(t-r), \quad H = h(t-r), \quad W = w(t-r),$$
$$X = \partial_t + \partial_r. \tag{3}$$

$$(II) \qquad Q = q(t-r), \quad H = h(t-r), \quad W = w(t-r) - \frac{t+r}{2},$$
$$X = \partial_t + \partial_r + z\partial_y. \tag{4}$$

$$(III) \qquad Q = e^{-\frac{t+r}{2}}q(t-r), \quad H = e^{\frac{t+r}{2}}h(t-r), \quad W = e^{\frac{t+r}{2}}w(t-r),$$

$$X = \partial_t + \partial_r + y\partial_y. \tag{5}$$

$$(IV) \qquad Q = e^{-(t+r)} q(t-r), \quad H = h(t-r), \quad W = w(t-r) - \frac{t+r}{2},$$

$$X = \partial_t + \partial_r + (y+z)\partial_y + z\partial_z. \tag{6}$$

$$(V) \qquad Q = e^{-(t+r)} q(t-r), \quad H = h(t-r), \quad W = w(t-r),$$

$$X = \partial_t + \partial_r + y\partial_y + z\partial_z. \tag{7}$$

$$(VI) \qquad Q = e^{-(1+p)\frac{t+r}{2}} q(t-r), \quad H = e^{(1-p)\frac{t+r}{2}} h(t-r), \quad W = e^{(1-p)\frac{t+r}{2}} w(t-r),$$

$$X = \partial_t + \partial_r + y\partial_y + pz\partial_z \quad (p \neq 0,1). \tag{8}$$

$$(VII) \qquad Q = e^{-p\frac{t+r}{2}} q(t-r), \quad c = c(t-r), \quad g = g(t-r),$$

$$H = \frac{\frac{\sqrt{4-p^2}}{2}}{\sqrt{1+c^2+g^2+c\cos(\sqrt{4-p^2}\frac{t+r}{2}) + g\sin(\sqrt{4-p^2}\frac{t+r}{2})}},$$

$$W = \frac{p}{2} + \frac{\frac{\sqrt{4-p^2}}{2}[c\sin(\sqrt{4-p^2}\frac{t+r}{2}) - g\cos(\sqrt{4-p^2}\frac{t+r}{2})]}{\sqrt{1+c^2+g^2+c\cos(\sqrt{4-p^2}\frac{t+r}{2}) + g\sin(\sqrt{4-p^2}\frac{t+r}{2})}},$$

$$X = \partial_t + \partial_r - z\partial_y + (y+pz)\partial_z \quad (p^2 < 4). \tag{9}$$

In all of these cases $F = F(t,r)$ and the conformal factor Ψ is given by

$$\Psi = F_{,t} + F_{,r} \ . \tag{10}$$

Note that these results are completely independent of the Einstein field equations and therefore of the assumed energy-momentum tensor. Furthermore, it is easy to prove that family B (i.e., the case in which the commutator between the CKV and at least one KV is a proper CKV) cannot admit CKV acting on null orbits (the proof can be found in [8]).

Let us now study possible perfect fluid solutions. For a maximal C_3, with a proper CKV, all possible solutions have been found. We will summarize the results obtained for the different metrics (the details can be obtained from Ref. [8]). For type I (i.e., the case in which X is a null CKV), we find that the space-time always admits a further KV tangent to the Killing orbits, and the metric then admits a multiply transitive group G_3 of isometries. This result is consistent with Tupper's analysis [6]. For types II and IV, either X is not a proper CKV or it does not correspond to a perfect fluid solution (i.e., wrong Segre type). For types V and VII it can be shown that either C_3 is not maximal or X is not a proper CKV (see [8] for details). Therefore, perfect fluid solutions under the previous hypotheses can only occur for the types III and VI.

Type VI (including type III for $p=0$):

We make the coordinate transformation $u = t+r$ and $v = t-r$, so that we have $h = h(v)$ and $q = q(v)$. The field equations yield

$$W = 0 \ , \tag{11}$$

$$F = f(x) + \frac{1}{2}\frac{1+p}{1-p}\ln h - \frac{1}{2}\ln q \ , \quad x \equiv u - \frac{2}{1-p}\ln h \ , \tag{12}$$

$$0 = \left\{ \frac{q_{,v}h_{,v}}{qh} + \frac{h_{,vv}}{h} \right\} \Sigma_0 + \left(\frac{h_{,v}}{h} \right)^2 \Sigma_1 \ , \tag{13}$$

where

$$\begin{aligned} \Sigma_0 \ &\equiv \ -1 + p^4 + 4f_{,x} - 4pf_{,x} + 4p^2 f_{,x} - 4p^3 f_{,x} + 8f_{,x}^2 - 8p^2 f_{,x}^2 \\ &- \ 32f_{,x}^3 + 32pf_{,x}^3 - 8f_{,xx} + 8p^2 f_{,xx} + 32f_{,xx}f_{,x} - 32pf_{,xx}f_{,x} \ , \end{aligned} \tag{14}$$

$$\begin{aligned} \Sigma_1 \ &\equiv \ 2 + 2p + 2p^2 + 2p^3 - 16f_{,x} - 8pf_{,x} - 16p^2 f_{,x} - 8p^3 f_{,x} + 32f_{,x}^2 + 16pf_{,x}^2 \\ &+ \ 48p^2 f_{,x}^2 - 64pf_{,x}^3 - 16f_{,xx} + 16pf_{,xx} - 32pf_{,xx} + 64pf_{,xx}f_{,x} \ , \end{aligned} \tag{15}$$

and $h_{,v} = 0$ is excluded since the solution does not correspond to a perfect fluid. Therefore, two possibilities arise:

$$\begin{aligned} &\text{i)} \qquad \Sigma_0 = 0, \quad \Sigma_1 = 0 \ , \\ &\text{ii)} \qquad \frac{q_{,v}h_{,v}}{qh} + \frac{h_{,vv}}{h} = a\left(\frac{h_{,v}}{h}\right)^2 \qquad (a = \text{const}) \ . \end{aligned}$$

In the first case $f_{,x}$ must be a constant, and therefore the CKV is not proper. In the second case we have that

$$\frac{q_{,v}}{q} = a\frac{h_{,v}}{h} - \frac{h_{,vv}}{h_{,v}} \ , \tag{16}$$

which can be integrated to give

$$q = \frac{h^a}{h_{,v}} \ , \tag{17}$$

and equation (13) reduces to:

$$1 = \frac{f_{,xx}[f_{,x}32(ap - a - 2p) + 8(2 - p^2 a - 2p + 4p^2 + a)]}{[4f_{,x} - p - 1][f_{,x}^2 8(ap - a - 2p) + f_{,x}8(p^2+1) + a - ap + ap^2 - ap^3 - 2 - 2p^2]} \ . \tag{18}$$

It is convenient to further divide the analysis into three sub-cases.

Sub-case (a): $a = 2p/(p-1)$. Equation (18) can be readily integrated to give

$$f = \frac{p+1}{4}x - \frac{(1-p)^2}{p^2+1}\frac{1}{2}\ln|x| + c \ , \quad c = \text{const} \ . \tag{19}$$

We notice that for $p = -1$ there exists a third KV of the form

$$\zeta = \left(\frac{1}{2} + \frac{1}{2}\frac{h}{h_{,v}}\right)\partial_t + \left(\frac{1}{2} - \frac{1}{2}\frac{h}{h_{,v}}\right)\partial_r + y\partial_y - z\partial_z \ . \tag{20}$$

Sub-case (b): $a = 2/(1-p)$. When $p = -1$ the solution is a particular case of sub-case (a). The remaining cases may now be integrated giving:

$$f = -\ln|1 - e^{-(1+p)x/4}| + c \ , \quad c = \text{const} \ . \tag{21}$$

We note that in this sub-case there exists a further KV

$$\zeta = \left(\frac{1}{2} + \frac{1-p}{4}\frac{h}{h_{,v}}\right)\partial_t + \left(\frac{1}{2} - \frac{1-p}{4}\frac{h}{h_{,v}}\right)\partial_r + \frac{1-p}{2}y\partial_y - \frac{1-p}{2}z\partial_z \ , \tag{22}$$

which violates our requirement of a maximal three-dimensional conformal group C_3.

Sub-case (c): finally we consider the possibility $a \neq 2p/(p-1)$ and $a \neq 2/(1-p)$. The solution of (18) is then given implicitly by

$$x = \gamma_1 \ln|f_{,x} - \beta_0| + \gamma_2 \ln|f_{,x} - \beta_+| + \gamma_3 \ln|f_{,x} - \beta_-| \ , \tag{23}$$

where

$$\beta_0 = \frac{p+1}{4} \ , \quad \beta_\pm = \frac{-2(p^2+1) \pm \sqrt{2(p^2+1)(1-p)^2(a^2-2a+2)}}{4(ap-a-2p)} \ , \tag{24}$$

and γ_i, $i = 1, 2, 3$, are constants satisfying $\gamma_1 + \gamma_2 + \gamma_3 = 0$.

A careful analysis of the energy conditions shows that for all cases (i.e., for all values of the parameters a and p) the solutions can only satisfy the energy conditions over certain open domains of the manifold (see [8]).

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