

# The Complete Catalog of 3-Regular, Diameter-3 Planar Graphs

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## Abstract

The largest known 3-regular planar graph with diameter 3 has 12 vertices. We consider the problem of determining whether there is a larger graph with these properties. We find all nonisomorphic 3-regular, diameter-3 planar graphs, thus solving the problem completely. There are none with more than 12 vertices.

## An Upper Bound

A graph with maximum degree  $\Delta$  and diameter  $D$  is called a  $(\Delta, D)$ -graph. It is easily seen ([9], p. 171) that the order of a  $(\Delta, D)$ -graph is bounded above by the *Moore bound*, which is given by

$$1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1} = \begin{cases} \frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2} & \text{if } \Delta \neq 2, \\ 2D + 1 & \text{if } \Delta = 2. \end{cases}$$

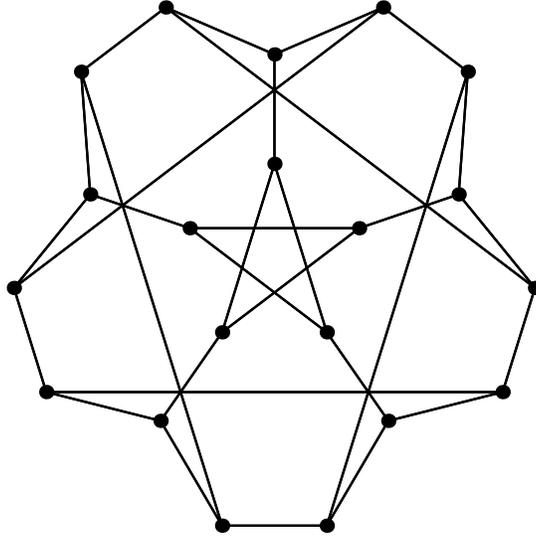


Figure 1: The regular (3,3)-graph on 20 vertices (it is unique up to isomorphism).

For  $D \geq 2$  and  $\Delta \geq 3$ , this bound is attained only if  $D = 2$  and  $\Delta = 3, 7$ , and (perhaps) 57 [3, 14, 23]. Now, except for the case of  $C_4$  (the cycle on four vertices), the number of vertices in a  $(\Delta, D)$ -graph never misses the Moore bound by one [4, 16]. Therefore, a (3,3)-graph can have at most 20 vertices (since the Moore bound is 22). Such an optimal graph has been found [15] (see Figure 1).

These results imply that we need only consider graphs on at most 20 vertices.

## Conventions

In order to solve the problem, we consider only 3-regular graphs. That is, every vertex has degree 3. Hereafter, we suppress the understood prefix *regular* and refer simply to (3,3)-graphs. Note that a  $\Delta$ -regular graph with  $\Delta$  odd must have an even number of vertices. We use the terms *3-regular* and *cubic* interchangeably (some authors also use the term *trivalent*).

Throughout this paper, we consider only unlabeled graphs. Hence, isomorphic graphs are considered to be the same. Hereafter, we suppress the understood prefix *nonisomorphic*.

## Theory

Since we are interested only in graphs with diameter 3, we need only consider connected graphs (finite diameter  $\iff$  connected). A complete catalog of connected cubic graphs on at most 20 vertices has been generated [27]. Since this catalog includes information about the diameters, one could simply examine those with diameter 3 and determine planarity. While this exhaustive search method would yield the same results, the author found the (nonisomorphic) planar (regular) (3,3)-graphs on at most 12 vertices (indeed, the only ones that exist) by using a strengthening of Jackson's Theorem [24] given by Hilbig [22].

Before we develop these theorems, we need some definitions:

A *bridge* (or *cut-edge* or *isthmus*) in a connected graph  $G$  is an edge of  $G$  whose removal disconnects  $G$ . A connected graph with no bridges is said to be *bridgeless* (or *2-edge-connected* or *isthmus-free*).

An  $n$ -*factor* of a graph  $G$  is an  $n$ -regular spanning subgraph of  $G$ . Some authors give a 1-factor the special name *perfect matching*.

**Lemma 1** *Every cubic graph with diameter 3 is bridgeless.*

PROOF: Let  $G$  be a cubic, diameter-3 graph with  $n$  vertices. Assume  $G$  has a bridge  $(A, B)$ . Then, since  $G$  is cubic, the graph in Figure 2 must appear as a subgraph of  $G$ . Call this subgraph  $H$ . By definition of bridge, removing edge  $(A, B)$  leaves two connected components of  $G$ , say  $H_1$  containing  $A$  and  $H_2$  containing  $B$ . That is, no vertex in  $H_1$  (except  $A$ ) is adjacent to any vertex in  $H_2$  (except  $B$ ). So in order to maintain diameter 3, all vertices in  $H_1$  must be at most one edge away from (i.e., adjacent to) vertex  $A$ . Similarly, all vertices in  $H_2$  must be adjacent to vertex  $B$ . But vertices  $A$  and  $B$  are already full, so  $H$  can have no more vertices. We can add at most two more edges to  $H$ , namely  $(C, D)$  and  $(E, F)$ , giving seven edges. But a cubic graph

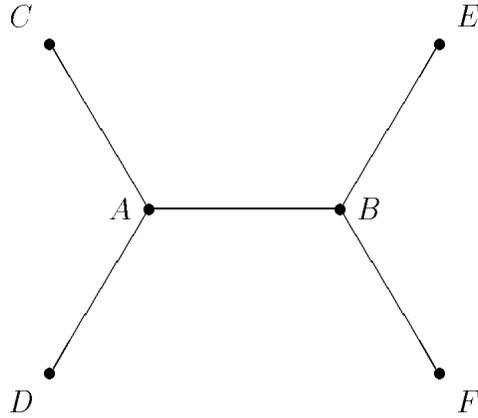


Figure 2: A subgraph of any cubic graph with a bridge.

on six vertices must have nine edges. This contradiction shows that  $G$  cannot have a bridge.  $\square$

We will improve a well-known result of Petersen [28].

**Theorem 1 (Petersen, 1891)** *Every bridgeless cubic graph is the sum of a 1-factor and a 2-factor.*

The following proposition is given without proof because it is a trivial exercise (see [25], p. 361).

**Proposition 1** *Every 2-edge-connected (that is, bridgeless) cubic graph is also 2-connected.*

In fact, this result holds more generally (see [20], p. 55).

**Theorem 2** *The connectivity and edge-connectivity are equal in every cubic graph.*

The following theorem due to Jackson [24] shows that, under additional hypotheses, we can replace the words “a 2-factor” with “ $C_n$ ” in the conclusion of Petersen’s Theorem.

**Theorem 3 (Jackson, 1980)** *Every 2-connected,  $d$ -regular graph on at most  $3d$  vertices is Hamiltonian.*

Zhu, Liu, and Yu [39] improved Jackson's Theorem. A simpler proof of their theorem is given in [10].

**Theorem 4 (Zhu, Liu, Yu, 1985)** *Except for the Petersen graph, every 2-connected,  $d$ -regular graph on at most  $3d + 1$  vertices is Hamiltonian.*

Hilbig [22] strengthened Jackson's Theorem even further.

**Theorem 5 (Hilbig, 1986)** *Except for the Petersen graph and the graph obtained from it by expanding one vertex to a triangle (that is, deleting the vertex, adding a triangle, and joining the three former neighbors of the vertex to the three vertices of the triangle by a matching), every 2-connected,  $d$ -regular graph on at most  $3d + 3$  vertices ( $d \geq 3$ ) is Hamiltonian.*

Since both of these exceptions are nonplanar (being contractible to  $K_5$ ), they do not concern us.

## Exhaustive Search Methods

The conditions of the problem allow us to apply Hilbig's Theorem in constructing the planar (3,3)-graphs when  $n \leq 12$ . We start with the Hamiltonian cycle on  $n$  vertices. Then we add a 1-factor of  $K_n$ . It is well-known that the number of 1-factors of  $K_n$  ( $n$  even) is given by

$$\frac{n!}{2^{n/2} \left(\frac{n}{2}\right)!} = (n-1)!! = (n-1)(n-3) \cdots (3)(1).$$

Now we don't need to consider every 1-factor of  $K_n$ . Instead, we avoid exactly those 1-factors that contain an edge of the Hamiltonian cycle, since adding such a 1-factor to the Hamiltonian cycle would result in a cubic multigraph (not a simple cubic graph). In other words, we consider every 1-factor of  $K_n \setminus C_n$ . This construction clearly gives us a (simple) cubic graph, and by

Vertices	1-Factors of $K_n$	Cases	Diameter 3	Planar	Isomorphism Classes
4	3	1	0	0	0
6	15	4	0	0	0
8	105	31	18	18	3
10	945	293	268	90	6
12	10395	3326	1580	24	2

Table 1: Results of the Search Using Hilbig's Theorem.

Hilbig's Theorem, we can build any (3,3)-graph on at most 12 vertices in this manner. We construct these 1-factors recursively and form the resulting cubic graphs. Then we compute the diameters, keeping only those graphs with diameter 3. From these (3,3)-graphs, we keep only those that are planar. Finally, we determine the isomorphism classes of the planar (3,3)-graphs (see Table 1). The computations were performed using the Combinatorica [35] standard package in *Mathematica* [37]. Unfortunately, this method applies only when  $n \leq 12$ .

We mention that there would be 44189 cases to check for  $n = 14$ , but Hilbig's Theorem does not apply. In general, the number of cases is the same as the answer to the following counting problem, which is similar to Exercise 28 on p. 112 in [8].

*Counting Problem:* Suppose  $n$  people ( $n$  even) are seated around a circular table. They leave the table for a break and then return. In how many ways can the people sit down in pairs so that no one is paired with either neighbor from the first seating? Disregard the order within each pair, and disregard the order of the pairs themselves.

Interestingly, the number of cases for  $n$  vertices ( $n$  even) is also the same as the number of labeled Hamiltonian cycles in the  $m$ -octahedron (with  $m = n/2$ ), which is the complete  $m$ -partite graph  $K_{2,2,\dots,2}$  ( $m$  pairs of opposite vertices with edges connecting each vertex to every other vertex except its opposite). Singmaster [34] notes that such a Hamiltonian cycle can be viewed as a way of seating  $m$  couples around a circular table so that no man is next to his wife. His solution uses the Inclusion-Exclusion Principle.

The number of cases is given by

$$\frac{\sum_{k=0}^{n/2} (-1)^k \binom{n/2}{k} \left[ \frac{n}{n-k} \right] 2^k (n-k)!}{2^{n/2} \left( \frac{n}{2} \right)!}.$$

Now most cubic graphs on at most 20 vertices are Hamiltonian (see Table 2 and also [32, 33]), so Hilbig's theorem does not help much if one considers only unlabeled graphs. However, without a canonical labeling algorithm [26] (as was used in [27]), one saves much computation by using Hilbig's Theorem.

For example, suppose we did not use Hilbig's Theorem for the  $n = 12$  case and used Petersen's Theorem instead. Then we must consider all 2-regular graphs on 12 vertices (not just  $C_{12}$ ). Up to isomorphism, there are nine of these, one for each partition of 12 with minimum no smaller than 3 (the length of the shortest cycle  $C_3$ ). So when  $n = 12$ , using Hilbig's Theorem reduces the number of cases by a factor of nine (if one considers labeled graphs).

The results for the graphs on 14 or more vertices were obtained by searching the catalog constructed by McKay and Royle [27]. The results for the graphs on fewer than 14 vertices (given in the final column of Table 1) were confirmed by a similar search.

## Results

$n = 4$

Clearly, this is the smallest case since we disallow loops or multiple edges. The only 3-regular graph on 4 vertices is the complete graph  $K_4$ . It is planar but, being complete, has diameter 1 (not 3). Hence no graph on 4 vertices satisfies the required properties.

$n = 6$

Since a graph is 2-regular  $\iff$  it is a union of cycles, there are two 2-regular graphs on 6 vertices, namely  $C_6$  and  $C_3 \cup C_3$ . Now every  $k$ -regular

graph on  $n$  vertices is the complement of an  $(n - 1 - k)$ -regular graph on  $n$  vertices. So there are two 3-regular graphs on 6 vertices:  $C_3 \times K_2$  and  $K_{3,3}$  (the complements of  $C_6$  and  $C_3 \cup C_3$ , respectively).  $C_3 \times K_2$  is a prism graph and thus is planar, while  $K_{3,3}$  is manifestly nonplanar. Both have diameter 2 (not 3), as can be easily checked. Hence no graph on 6 vertices satisfies the required properties.

$n = 8$

All three (3,3)-graphs on 8 vertices are planar (see Figures 3-5). Note that all graphs are displayed to clearly exhibit a Hamiltonian cycle, as guaranteed by Hilbig's Theorem. This choice of embedding also emphasizes the method of construction. Furthermore, one can easily see that each graph is planar by simply placing the appropriate chords outside the finite region bordered by the Hamiltonian cycle. Expanding a vertex to a triangle, as described in Hilbig's Theorem, preserves planarity and 3-regularity, and increases the diameter by at most one. The graph in Figure 3 is the unique smallest cubic *matchstick graph*, that is, the unique minimal-order cubic graph that admits a planar embedding where each edge has the same geometric length. The graph in Figure 4 can be obtained from  $K_4$  by expanding two vertices to triangles. The graph in Figure 5 is isomorphic to the *cube graph*  $Q_3 \cong K_2 \times K_2 \times K_2$ , which can also be considered as the prism graph  $C_4 \times K_2$ .

$n = 10$

There are six planar (3,3)-graphs on 10 vertices (see Figures 6-11). The graph in Figure 6 can be obtained from the graph in Figure 3 by expanding an appropriate vertex to a triangle. The graph in Figure 7 can be obtained from the prism  $C_3 \times K_2$  by expanding two adjacent vertices, one from each base, to triangles. Similarly, the graph in Figure 8 can be obtained from the prism  $C_3 \times K_2$  by expanding two nonadjacent vertices, one from each base, to triangles. The graph in Figure 9 can be obtained from  $Q_3$  by expanding a vertex to a triangle. The graph in Figure 10 can be obtained from  $K_4$  by expanding all but one of the vertices to triangles.

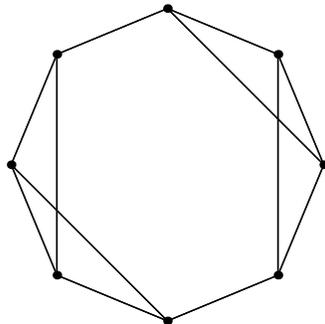


Figure 3: The unique smallest cubic matchstick graph, which is a planar  $(3,3)$ -graph on 8 vertices.

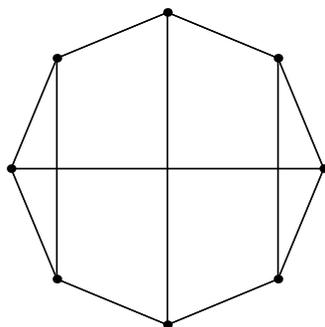


Figure 4: A planar  $(3,3)$ -graph on 8 vertices, which can be obtained from  $K_4$  by expanding two vertices to triangles.

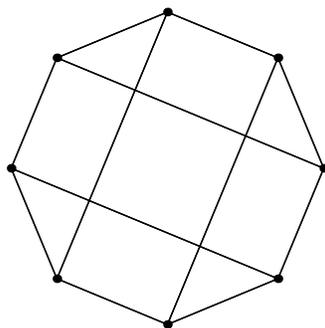


Figure 5: The cube graph  $Q_3$ , which is a planar  $(3,3)$ -graph on 8 vertices.

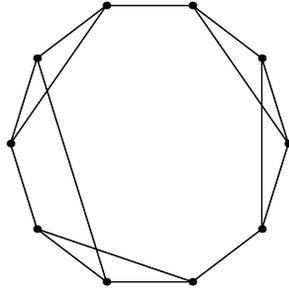


Figure 6: A planar (3,3)-graph on 10 vertices, which can be obtained from the graph in Figure 3 by expanding an appropriate vertex to a triangle.

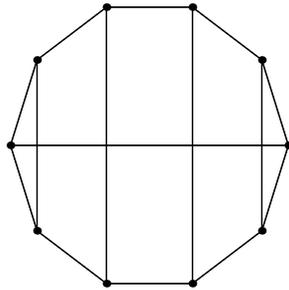


Figure 7: A planar (3,3)-graph on 10 vertices, which can be obtained from the prism  $C_3 \times K_2$  by expanding two adjacent vertices, one from each base, to triangles.

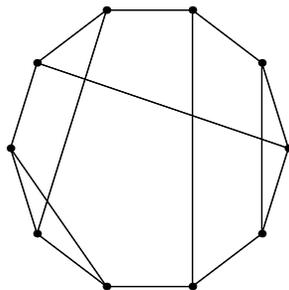


Figure 8: A planar (3,3)-graph on 10 vertices, which can be obtained from the prism  $C_3 \times K_2$  by expanding two nonadjacent vertices, one from each base, to triangles.

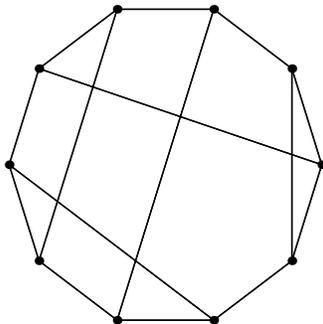


Figure 9: A planar (3,3)-graph on 10 vertices, which can be obtained from  $Q_3$  by expanding a vertex to a triangle.

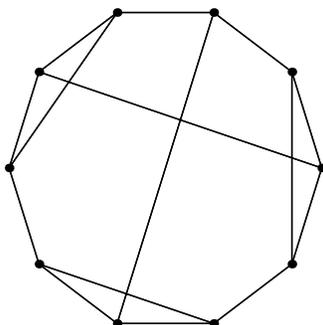


Figure 10: A planar (3,3)-graph on 10 vertices, which can be obtained from  $K_4$  by expanding all but one of the vertices to triangles.

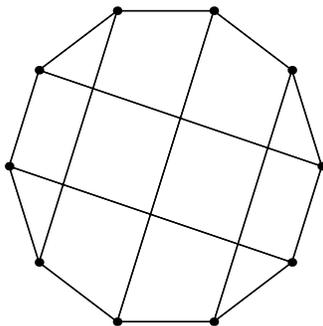


Figure 11: A planar (3,3)-graph on 10 vertices.

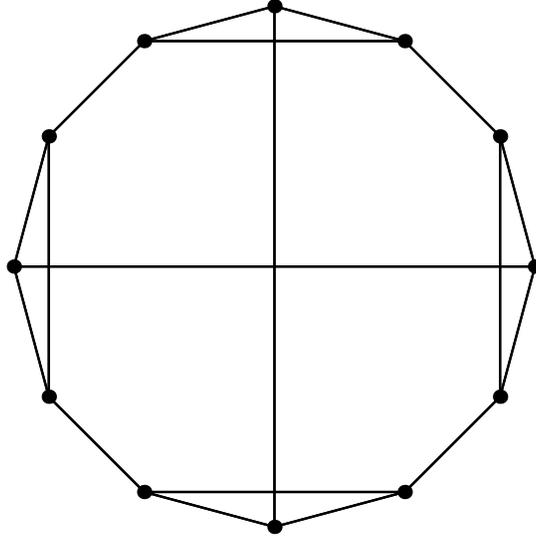


Figure 12: The truncated tetrahedron graph, which is a planar  $(3,3)$ -graph on 12 vertices.

$n = 12$

There are two planar  $(3,3)$ -graphs on 12 vertices (see Figures 12-13). The graph in Figure 12 is isomorphic to the truncated tetrahedron graph, that is, the graph obtained from  $K_4$  by expanding each vertex to a triangle.

$n = 14$

There are 509 connected cubic graphs on 14 vertices. Of these, 34 have diameter 3. All 34 are nonplanar, including the Heawood graph (see Figure 14), which is the unique  $(3,6)$ -cage (that is, the minimal-order 3-regular graph whose shortest cycle has length 6).

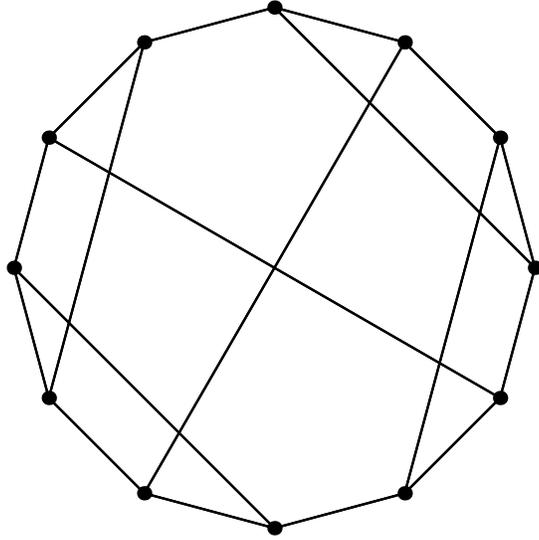


Figure 13: A planar (3,3)-graph on 12 vertices.

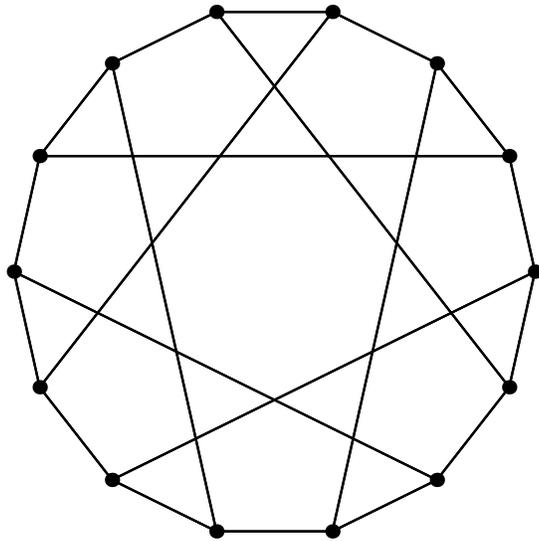


Figure 14: The Heawood graph, which is a nonplanar (3,3)-graph on 14 vertices.

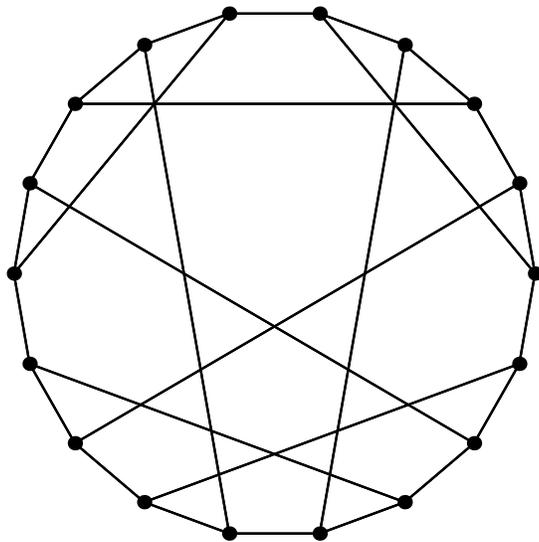


Figure 15: The unique (3,3)-graph on 18 vertices. It is nonplanar.

$n = 16$

There are 4060 connected cubic graphs on 16 vertices. Of these, 14 have diameter 3. All 14 are nonplanar.

$n = 18$

There are 41301 connected cubic graphs on 18 vertices. Only one has diameter 3, and it is nonplanar (see Figure 15).

$n = 20$

There are 510489 connected cubic graphs on 20 vertices. Only one has diameter 3 (see Figure 1). It is the graph  $C_5 * F_4$ , which is nonplanar, being contractible to  $K_5$ . See [7] for a definition of the graph product  $*$ .

Vertices	Connected Cubic Graphs	Hamiltonian	Diameter 3	Planar (3,3)-Graphs
4	1	1	0	0
6	2	2	0	0
8	5	5	3	3
10	19	17	15	6
12	85	80	34	2
14	509	474	34	0
16	4060	3841	14	0
18	41301	39635	1	0
20	510489	495991	1	0

Table 2: Summary of Results.

## Summary

Table 2 summarizes our results for unlabeled graphs. The numbers of Hamiltonian connected cubic graphs and diameter-3 connected cubic graphs are given in [27].

## Further Research

Proof by exhaustive search (with or without computer aid), even if practical, is often not as satisfying as other methods. In this case (unlike Appel and Haken’s well-known computer-aided proof of the Four Color Theorem [1]), the existing catalog of connected cubic graphs [27] is known to be complete and correct, having confirmed both theoretical results by Robinson and Wormald [30, 31] and constructive results by others [2, 12, 13, 17, 18, 29]. Still, the nature of the results suggests that a clever proof by contradiction may be found for orders greater than 12.

The results could also be extended to other  $(\Delta, D)$ -graphs.

In fact, the existing catalog [27] can be used to find all the planar (3,2)-graphs. There are only five (3,2)-graphs: two on 6 vertices (which we already considered), two on 8 vertices (both nonplanar), and one on 10 vertices (the

nonplanar Petersen graph). So  $C_3 \times K_2$  is the only planar (3,2)-graph.

For most higher values of  $\Delta$  and  $D$ , better bounds than the Moore bound are known when one considers only planar graphs [19]. But even these bounds are large, making exhaustive computer search of existing catalogs impractical or even impossible. Catalogs of connected regular graphs of degrees greater than three are available (see [5] and [6]), but they are limited to graphs on 13 vertices or fewer. Gunnar Brinkmann [11] has improved upon McKay and Royle's results by using a faster algorithm to extend the catalog of connected cubic graphs to include all graphs on at most 24 vertices. He has also extended to even more vertices, but only for graphs with restricted girth.

A famous conjecture of Berge, proved by Zhang [38], would be useful in applying Hilbig's Theorem when  $\Delta = 4$ .

**Theorem 6 (Zhang, 1985)** *Every 4-regular graph contains a 3-regular subgraph.*

Hence to find the planar  $(4,D)$ -graphs on a fixed even number  $N$  of vertices, one could add 1-factors to the planar  $(3,D')$ -graphs on  $N$  vertices for all  $D'$  with  $D' \geq D$  (since adding edges does not increase the diameter) and  $D' \leq \frac{N-2}{k} + 1$  (where  $k$  is the connectivity; see [36]). If  $k$  is not known *a priori*, we can take  $k = 1$  (again, finite diameter  $\iff$  connected). From the resulting graphs, select those that are planar and have diameter  $D$ .

The following theorem shows that when  $\Delta$  is even, a connected  $\Delta$ -regular graph is 2-edge-connected, a property that appears as a hypothesis for many theorems about degree and diameter.

**Theorem 7** *Every regular graph of even degree is bridgeless.*

The proof is an easy exercise and is thus omitted (see [21], p. 60).

Petersen [28] obtained a characterization of regular graphs of even degree.

**Theorem 8 (Petersen, 1891)** *A graph is 2-factorable  $\iff$  it is regular of even degree.*

With these theorems and the ideas described in this paper, perhaps using Hilbig's Theorem in conjunction with a canonical labeling algorithm (to significantly reduce the number of cases) would aid in cataloging other planar  $(\Delta, D)$ -graphs.

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