Optimal investment for investors with state dependent income, and for insurers

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Abstract

An optimal control problem is considered where a risky asset is used for investment, and this investment is financed by initial wealth as well as by a state dependent income. The objective function is accumulated discounted expected utility of wealth, where the utility function is nondecreasing and bounded. This problem is investigated for constant as well as for stochastic discount rate, where the stochastic model is a time homogeneous finite state Markov process. We prove that the Bellman equation to this optimization problem has a classical solution and give a verification argument. Based on this we deal with the problem of optimal investment for an insurer with an insurance business modelled by a compound Poisson or a compound Cox process, under the presence of constant as well as (finite state space Markov) stochastic interest rate.

Key words: Optimal investment, Lundberg risk process, Markov modulated risk process, Bellman equation

1 Introduction and Summary

Assume that a tradeable asset is given with continuous price process modelled as a geometric Brownian motion:

$$dZ(t) = aZ(t)dt + bZ(t)dW(t),\ Z(0) = z,$$

with $a, b > 0$, and $W(t), t \geq 0$, a standard Wiener process. Let $X(t)$ be the wealth of an investor at time t satisfying

$$dX(t) = c(X(t))dt + \theta(t)dZ(t),\ X(0) = s. \tag{1}$$

Here, $s \geq 0$ is the *initial wealth*, $c(X(t))$ a positive income per time unit depending on current wealth $X(t)$, and $\theta(t)$ denotes the number of shares of asset

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$Z(t)$ held at time t. The strategy $\theta = (\theta(t), t \geq 0)$ constitutes a stochastic process which is adapted, i.e. $\theta(t)$ is a function of $X(u), u < t$, and such that the stochastic integrals

$$\int_0^t \theta(u) dZ(u)$$

are well defined for all $t > 0$ (e.g. θ predictable). We shall call these strategies *admissible*. We do not impose a budget constraint, i.e. we assume that at any time we can borrow an arbitrary amount of money without paying interest. This assumption can be justified when all payments are discounted.

For a positive bounded non decreasing function $g(x)$ with $g(x) < g(\infty), x > 0$, and $\lambda > 0$ let

$$G(\theta) = E\left[\int_0^\tau g(X(t)) e^{-\lambda t} dt\right],$$

where $\tau = \inf\{t > 0 : X(t) < 0\}$ is the time of ruin. One could consider λ as a subjective discount factor and the function $g(x)$ as a utility function. We are interested in the optimal trading strategy $\widehat{\theta}$ which maximizes $G(\theta)$:

$$\widehat{\theta} = \arg\max G(\theta) \tag{2}$$

This optimization problem looks rather artificial, the model and the objective function might seem quite odd. We use it here as a tool for solving optimal investment problems for insurers. A function $g(x)$ with $g(x) < g(\infty)$ for all $x > 0$ will be called *strictly increasing at infinity.*

If we have a positive constant interest rate ρ which is the same for borrowing and lending, then we obtain a similar problem in which the dynamics of the wealth process is slightly changed:

$$dX(t) = (c(X(t)) + \rho X(t))\, dt + \theta(t) \left[dZ(t) - \rho Z(t) dt\right], X(0) = s.$$

This yields the same type of problem: change $c(x)$ into $c(x) + \rho x$ and a to $a - \rho$.

If for given trading strategy θ we denote $m(t) = \theta(t) Z(t)$ the amount of money invested at time t, then the dynamics of the wealth process is

$$dX(t) = (c(X(t)) + m(t)a)\, dt + m(t) b dW(t), X(0) = s,$$

and therefore $G(\theta)$ does not depend on $Z(0) = z$. In particular the value function of the above problem $V(x)$ is a function of *current wealth* x alone. It should satisfy the following intuitive conditions:

$$\begin{aligned} V'(x) &> 0, V''(x) < 0, x > 0, \\ g(0) - \lambda V(0) + c(0) V'(0) &= 0, && (3) \\ \lambda V(\infty) &= g(\infty). && (4) \end{aligned}$$

Condition (3) is motivated by the fact that when $X(t) = 0$ the amount invested $m(t)$ should be zero; for otherwise we would have immediate bankruptcy. The function $V(x)$ satisfies the following Bellman equation:

$$\sup_A \left\{ g(x) - \lambda V(x) + (c(x) + Aa) V'(x) + \frac{1}{2} A^2 b^2 V''(x) \right\} = 0, x > 0. \tag{5}$$

The first two terms correspond to the running cost and the discounting, respectively, and the last two are of the form $LV(x)$, where L is the infinitesimal generator for $X(t)$ when $m(t)$ is constant. After solving for the sup we obtain

$$g(x) - \lambda V(x) + c(x)V'(x) = \frac{1}{2}\frac{a^2}{b^2}\frac{V'(x)^2}{V''(x)}, x > 0. \tag{6}$$

We shall prove that equation (6) has a smooth solution $V(x)$ satisfying (3). We shall also prove a verification theorem stating that the solution $V(x)$ of (6) is the value function of our problem, and the maximizer

$$A(x) = -\frac{a}{b^2}\frac{V'(x)}{V''(x)}$$

defines the optimal investment strategy in feedback form via

$$\theta(t) = A(X(t))/Z(t).$$

$m(t) = A(X(t))$ is the optimal amount of money invested into the tradeable asset, and $X(t)$ is the current wealth at time t which is the result of the trading strategy $\theta(u), u < t$. Notice that the requirement $A(0) = 0$ implies $V''(0) = -\infty$, and this singularity causes problems in the proof for the existence of a solution as well as in numerical calculations.

We shall also solve the optimization problem for the case of stochastic discount rate $\lambda(t)$ for which we adopt the Markov chain model of Norberg [3]. Assume that $\lambda_1, ..., \lambda_I$ are positive discount rates, and $i(t)$ is a time homogeneous Markov process on the set $1, ..., I$ with intensities $b_{ij}, i, j = 1, ..., I$, and let $\lambda(t) = \lambda_{i(t)}$. For later use we shall assume that the utility function $g(x)$ as well as the premium income depend on the current state of the world, i.e. if $i(t) = i$ then we shall use a positive non decreasing utility function $g_i(x)$ and $c_i(x)$ as income rates per time unit. Again we assume that the functions $g_i(x)$ are strictly increasing at infinity. To simplify things we assume that $g_i(\infty) = \gamma/\lambda_i, i = 1, ..., I$. Optimization is done with respect to the following objective function:

$$H(\theta) = E\left[\int_0^\tau g_{i(t)}(X(t)) \exp\left(-\int_0^t \lambda(u)du\right) dt\right] = \max!$$

The state dependent utility functions $g_i(x)$, income rates $c_i(x)$ and subjective interest rates λ_i might be used for modeling different subjective views and incomes in states representing social status (such as *working, unemployed, disabled, dead with widow*). The specific values for $g_i(\infty)$ do not have an economic interpretation. Again we are interested in finding the optimal investment strategy

$$\widehat{\theta} = \arg\max H(\theta).$$

The value function $V_i(x)$ is the maximal possible value for $H(\theta)$ when $i(0) = i$

and $X(0) = x$. The natural conditions are

$$\begin{aligned} V_i'(x) &> 0, V_i''(x) < 0, x > 0, \\ 0 &= g_i(0) - \lambda_i V_i(0) + c_i(0) V_i'(0) + \sum_{j=1}^{I} b_{ij} V_j(0), \text{ and} \qquad (7) \\ \lambda_i V_i(\infty) &= g_i(\infty), i = 1, ..., I. \qquad (8) \end{aligned}$$

For arbitrary values of $g_i(\infty)$ we would get

$$\lambda_i V_i(\infty) = g_i(\infty) - \sum_j b_{ij}/\lambda_j g_j(\infty), i = 1, ..., I.$$

The Bellman equation for the value function of this problem is a system of interacting differential equations:

$$\begin{aligned} & g_i(x) - \lambda_i V_i(x) + c_i(x) V_i'(x) + \sum_{j=1}^{I} b_{ij} V_j(x) \qquad (9) \\ = & \frac{1}{2} \frac{a^2}{b^2} \frac{V_i'(x)^2}{V_i''(x)}, x > 0, i = 1, ..., I. \end{aligned}$$

Also for this set of equations, we prove that there is a smooth solution

$$(V_1(x), ..., V_I(x))$$

to (9) satisfying the natural conditions, under the assumption that the positive non decreasing bounded functions $g_i(x)$ have a continuous first derivative. The optimal trading strategy is given by

$$A_i(x) = -\frac{a}{b^2} \frac{V_i'(x)}{V_i''(x)} \text{ and } \theta(t) = A_i(X(t))/Z(t) \text{ if } i(t) = i, i = 1, ..., I.$$

The proof of the existence results is based on monotonicity arguments which seem to be suitable for this type of stochastic control problems. The corresponding proof in [2] was based on contraction arguments. Also this somewhat artificial optimization problem is merely a useful tool for solving optimal investment problems in insurance in which we will consider insurance business modelled by 1) a compound Poisson or 2) a compound Cox process.

In the first case, we consider a compound Poisson process $S(t),\ t \geq 0$, with constant Poisson intensity λ and claim size distribution Q with density $p(x)$, and model the risk process of an insurance company (or an insurance portfolio) by

$$dR(t) = (c + \rho R(t))dt - dS(t),\ R(0) = s.$$

Here, c is the (positive) premium intensity for the portfolio and $\rho \geq 0$ a deterministic constant interest rate. Again, we consider a market index $Z(t)$ which is used for investment:

$$dZ(t) = Z(t)(adt + bdW(t)),\ \ Z(0) = z,$$

with fixed known parameters $b > 0, a > \rho$ and a standard Wiener process $W(t)$. At time t the insurer will hold $\theta(t)$ shares of the index, resulting in a technical result

$$dX(t) = dR(t) + \theta(t)\left[dZ(t) - \rho Z(t)dt\right], \; X(0) = s.$$

Our aim is to maximize the survival probability

$$\delta(s) = P\{X(t) \geq 0 \text{ for all } t \geq 0\}$$

over all possible admissible strategies θ. Only predictable strategies are admissible; this means in particular that the value of an admissible strategy at time t may depend on the history of the processes $Z(u)$ and $R(u)$ up to time t, but it may not depend on the size of a claim occurring at time t. Again we shall not consider a budget constraint. This optimization problem has been solved in [2] for the case $\rho = 0$. Here, we could allow for more general positive state dependent premium intensity $c(x)$, not only for $c(x) = c + \rho x$. The value function $V(x, z)$ which is the largest possible survival probability when $X(0) = x, Z(0) = z$, will depend on x, not on z. Write the dynamics of $X(t)$ as follows:

$$\begin{aligned} dX(t) &= (c + \rho X(t))dt - dS(t) + \theta(t)[dZ(t) - \rho Z(t)dt] \\ &= (c + \rho X(t))dt - dS(t) + m(t)[(a - \rho)dt + bdW(t)]. \end{aligned}$$

Since S and W are Levy processes, the optimal value V will now depend only on $X(0) = x : V = V(x)$. It should satisfy the natural conditions

$$\begin{aligned} V'(x) &> 0, V''(x) < 0, x > 0, \\ V(x) &= 0 \text{ for } x < 0, \\ \lambda V(0) &= cV'(0), \text{ and} \\ V(\infty) &= 1. \end{aligned}$$

The Bellman equation for $V(x)$ is

$$\lambda E[V(x - Y) - V(x)] + (c + \rho x)V'(x) = \frac{1}{2}\frac{(a - \rho)^2}{b^2}\frac{V'(x)^2}{V''(x)}, \; x > 0. \tag{10}$$

Here, Y is a generic claim size with distribution Q. This equation has a smooth solution which satisfies the natural conditions and solves the optimization problem. The necessary verification theorem for this case is given in [2]; we repeat it here with a complete proof.

In the second case, we assume that the intensity of claims and the interest rates are stochastic and modelled as a finite state space Markov process as in [3]. Assume that $\lambda_1, ..., \lambda_I$ and $\rho_1, ..., \rho_I < a$ are possible levels of intensity and interest rate, respectively, and that $i(t)$ is a time homogeneous Markov process on the set $\{1, ..., I\}$ with intensities $b_{ij}, i, j = 1, ..., I$. The stochastic intensity $\lambda(t)$ and the stochastic interest rate $\rho(t)$ are given by

$$\begin{aligned} \lambda(t) &= \lambda_{i(t)} \text{ and} \\ \rho(t) &= \rho_{i(t)}. \end{aligned}$$

We could also allow more general non decreasing state dependent premium income functions $c_i(x)$ here, not only $c_i(x) = c + \rho_i x$. The claims process is modelled as a compound Cox process, i.e. given $\lambda(t), t \geq 0$, the claims occur according to a non homogeneous Poisson process with intensity $\lambda(t), t \geq 0$ (see Grandell [1], Chapter 4). Admissible investment strategies are now all predictable processes $\theta(t)$ which are based on the observation of the processes $R(u), Z(u), i(u), u < t$, or $R(u), Z(u), \lambda(u), \rho(u), u < t$. Again, $V_i(x)$ is the maximal survival probability when $i(0) = i$ and $X(0) = x$. The natural conditions are

$$\begin{aligned} V_i'(x) &> 0, V_i''(x) < 0, x > 0, \\ V_i(x) &= 0, x < 0, \\ \lambda_i V_i(0) &= cV_i'(0) + \sum_{j=1}^{I} b_{ij} V_j(0), \text{ and} \\ V_i(\infty) &= 1, i = 1, ..., I. \end{aligned}$$

The Bellman equations for the value functions $V_i(x)$ are

$$\begin{aligned} & \lambda_i E[V_i(x - Y) - V_i(x)] + (c + \rho_i x)V_i'(x) + \sum_{j=1}^{I} b_{ij} V_j(x) \qquad (11) \\ = & \frac{1}{2} \frac{(a - \rho_i)^2}{b^2} \frac{V_i'(x)^2}{V_i''(x)}, x > 0, i = 1, ..., I. \end{aligned}$$

Again, Y is a generic claim size with distribution Q. We show that the system of equations (11) has a smooth solution $(V_1(x), ..., V_I(x))$ satisfying the natural conditions, provided Q has a continuous density $p(x)$, and an optimal investment strategy is determined by

$$\theta(t) = A_i(X(t-))/Z(t)$$

when $i(t) = i$, where

$$A_i(x) = -\frac{a}{b^2} \frac{V_i'(x)}{V_i''(x)}, i = 1, ..., I.$$

2 Results and Proofs

2.1 Optimal investment with income and fixed discount rate

We shall standardize the Bellman equation: Multiply both sides of (6) by b^2/a^2 and denote the changed constant λ and the changed function $c(x)$ again by λ and $c(x)$. Furthermore, replace $g(x)$ by $\lambda g(x)$,(3) by

$$\lambda(g(0) - V(0)) + c(0)V'(0) = 0 \qquad (12)$$

and (8) by $V(\infty) = g(\infty)$. Then equation (6) is given in *normal form*:

$$\lambda(g(x) - V(x)) + c(x)V'(x) = \frac{1}{2}\frac{V'(x)^2}{V''(x)}, x > 0. \tag{13}$$

Theorem 1 *Assume that both, the bounded non decreasing function $g(x)$ and the positive function $c(x)$, are continuously differentiable, and that $g(x)$ is strictly increasing at infinity. Then there exists an increasing bounded and concave function $V(x)$ with $V(x) > g(x)$, $x > 0$, which is twice continuously differentiable on $(0, \infty)$ and is a solution of (13).*

Proof. We first show that (13) is equivalent to the following system of interacting differential equations which hold for $x > 0$:

$$\lambda(V(x) - g(x)) - c(x)V'(x) = \frac{1}{2}\sqrt{U(x)}V'(x) \tag{14}$$

$$\sqrt{U(x)}\left((\lambda + \frac{1}{2} - c'(x))V'(x) - \lambda g'(x)\right) + c(x)V'(x) = \frac{1}{4}U'(x)V'(x) \tag{15}$$

Assume first that $V(x)$ is a bounded twice continuously differentiable increasing concave solution of (13). Then

$$U(x) = \left(\frac{V'(x)}{V''(x)}\right)^2$$

is well defined for $x > 0$, and (14) holds. Differentiating that equation and using

$$\sqrt{U(x)}V''(x) = -V'(x)$$

we obtain that $U(x)$ satisfies (15). Assume now that $U(x), V(x)$ is a solution of the system (14), (15) with non decreasing and twice differentiable function $V(x)$. Differentiating (14) yields for $x > 0$

$$\begin{aligned} & \sqrt{U(x)}\left((\lambda - c'(x))V'(x) - \lambda g'(x) - c(x)V''(x) - \frac{1}{2}\sqrt{U(x)}V''(x)\right) \\ = & \frac{1}{4}U'(x)V'(x). \end{aligned}$$

Comparing this with (15) we obtain

$$\sqrt{U(x)}\left(-\frac{1}{2}\sqrt{U(x)}V''(x) - c(x)V''(x)\right) = \sqrt{U(x)}\frac{1}{2}V'(x) + c(x)V'(x)$$

or

$$\left(\sqrt{U(x)} + 2c(x)\right)\left(\sqrt{U(x)}V''(x) + V'(x)\right) = 0,\ x > 0.$$

Since the first factor is always positive, the second must be zero, hence

$$\sqrt{U(x)} = -\frac{V'(x)}{V''(x)}$$

and consequently (13) must hold. Furthermore, $V''(x) < 0$ for $x > 0$. Now define $U_0(x) \equiv 0$ and $V_n(x), U_{n+1}(x), n = 0, 1, 2, \ldots$ recursively by

$$U_{n+1}(0) = 0, V_n(\infty) = g(\infty),$$

and for $x > 0$

$$\lambda(g(x) - V_n(x)) + c(x)V_n'(x) = -\frac{1}{2}\sqrt{U_n(x)}V_n'(x), \tag{16}$$

$$\begin{aligned} &\sqrt{U_{n+1}(x)}\left((\lambda + \frac{1}{2} - c'(x))V_n'(x) - \lambda g'(x)\frac{c(x) + \frac{1}{2}\sqrt{U_{n+1}(x)}}{c(x) + \frac{1}{2}\sqrt{U_n(x)}}\right) \\ &+ c(x)V_n'(x) = \frac{1}{4}U_{n+1}'(x)V_n'(x). \end{aligned} \tag{17}$$

The expression in (17) contains the term

$$\frac{c(x) + \frac{1}{2}\sqrt{U_{n+1}(x)}}{c(x) + \frac{1}{2}\sqrt{U_n(x)}}$$

which is not present in (15). This term is needed to obtain a non decreasing sequence $U_n(x)$. Later we shall see that $U_n(x)$ converges, and hence the above term will converge to one.

For a given function $U_n(x)$ the equation (16) has to be solved for $V_n(x)$:

$$V_n'(x) = \frac{\lambda(V_n(x) - g(x))}{c(x) + \frac{1}{2}\sqrt{U_n(x)}}, x > 0. \tag{18}$$

For a given function $V_n(x)$ the equation (17) has to be solved for $U_{n+1}(x)$:

$$\begin{aligned} \frac{1}{4}U_{n+1}'(x) &= c(x) + \\ &\sqrt{U_{n+1}(x)}\left(\lambda + \frac{1}{2} - c'(x) - \frac{g'(x)\left(c(x) + \frac{1}{2}\sqrt{U_{n+1}(x)}\right)}{V_n(x) - g(x)}\right). \end{aligned} \tag{19}$$

Lemma 10 in Section 3 below, applied to

$$c_1(x) = \left(c(x) + \frac{1}{2}\sqrt{U_{n+1}(x)}\right)^{-1} \le \left(c(x) + \frac{1}{2}\sqrt{U_n(x)}\right)^{-1} = c_2(x)$$

yields that $V_{n+1}(x) \ge V_n(x)$ for all $x > 0$ whenever $U_{n+1}(x) \ge U_n(x)$, $x > 0$, and that $V_n(x) > g(x), x > 0$. Lemma 11 in Section 3 below, applied to

$$k_1(x) = -\frac{g'(x)}{V_{n-1}(x) - g(x)} \le -\frac{g'(x)}{V_n(x) - g(x)} = k_2(x)$$

yields that $U_{n+1}(x) \geq U_n(x)$ for all $x > 0$ provided $V_n(x) \geq V_{n-1}(x)$, $x > 0$, and that $U_n(x)$ is bounded from above by the solution $u(x)$ of the differential equation

$$\frac{1}{4}u'(x) = c(x) + \sqrt{u(x)}\left(\lambda + \frac{1}{2} - c'(x)\right),\ x > 0,\ u(0) = 0.$$

Since $U_1(x) \geq U_0(x), x > 0$, the sequences $U_n(x)$ and $V_n(x)$ are non decreasing and bounded, they converge to limits $U(x)$ and $V(x)$, respectively. Therefore the sequences $U_n'(x)$ and $V_n'(x)$ are converging too, and hence the pair $U(x), V(x)$ is a solution for the system (14) and (15) which has the asserted properties: Clearly $V(x) > g(x)$ and thus $V'(x) > 0$. Furthermore, $U(x) > 0$ for $x > 0$, and hence $V''(x) > 0, x > 0$. ■

The following verification theorem states that a concave solution of the Bellman equation (6) solves our first control problem.

Theorem 2 *Assume that the functions $g(x)$ and $c(x)$ satisfy the conditions of Theorem 1, and that $V(x)$ is an increasing and concave solution of the Bellman equation (6) with $V(\infty) = g(\infty)$. Let*

$$\theta_0(t) = -\frac{a}{b^2}\frac{V'(X(t))}{V''(X(t))Z(t)}$$

and $\theta_1(t)$ be an arbitrary predictable (w.r. to the process $X(t)$) trading strategy for which

$$\int_0^t \theta_1(u)dZ(u) \tag{20}$$

is well defined for all t. If $x \geq 0$ is an initial surplus and if $V_0(x), V_1(x)$ are the quantities

$$E\left[\int_0^\tau g(X(t))e^{-\lambda t}dt\right]$$

with investment strategies $\theta_0(t)$ and $\theta_1(t)$, respectively, then $V_0(x) \geq V_1(x)$.

Proof. Since $V(x)$ is smooth, $\theta_0(t)$ is predictable, and the stochastic integrals (20) are defined for $\theta_0(t)$ too. Let $X_i(t)$ be the process with trading strategy $\theta_i(t)$, $i = 0, 1$, i.e.

$$dX_i(t) = c(X_i(t))dt + \theta_i(t)aZ(t)dt + \theta_i(t)bZ(t)dW(t),\ X_i(0) = x, i = 0, 1.$$

Let

$$\begin{aligned} Z_i(t) &= \lambda\int_0^t e^{-\lambda u}g(X_i(u))du + e^{-\lambda t}V(X_i(t\wedge\tau_i)), Z_i(0) = V(x),\ i = 0, 1, \\ W_i(t) &= Z_i(t\wedge\tau_i), i = 0, 1, \end{aligned}$$

where τ_i is the ruin time of the process $X_i(t)$. Then with $X_i(t) = x_i$ and $A_i = \theta_i(t)Z(t), i = 0, 1$, we have

$$\begin{aligned} e^{\lambda t} dW_i(t) &= (\lambda g(x_i) - \lambda V(x_i))\, dt + V'(x_i) dX_i(t) + \frac{1}{2} V''(x_i) A_i b^2 dt, \\ &= \left(\lambda(g(x_i) - V(x_i)) + V'(x_i)(c + A_i a) dt + \frac{1}{2} V''(x_i) A_i^2 b^2 \right) dt \\ &+ V'(x_i) A_i b dZ(t), i = 0, 1. \end{aligned}$$

According to (5) the process $X_0(t)$ is a local martingale, while $X_1(t)$ is a local supermartingale. Notice that relation (5) also holds for $x = 0$ because of (3). Since both processes are bounded and continuous, they are a proper martingale and a proper supermartingale, respectively. So for $t \geq 0$

$$E[W_0(t)] = V(x) = E[W_1(0)] \geq E[W_1(t)].$$

Since $V(x)$ is bounded, we have by dominated convergence

$$\lim_{t\to\infty} E[Z_i(t)] = \lambda E\left[\int_0^{\tau_i} e^{-\lambda t} g(X_i(t)) dt\right] + E\left[e^{-\lambda \tau_i} V(0) 1_{\{\tau_i < \infty\}}\right], i = 0, 1.$$

Since the process $X_0(t)$ never gets negative because of $A(u)$ continuous and $A(0) = 0$ we have $P\{\tau_0 < \infty\} = 0$ and so

$$\begin{aligned} V(x) &= E\left[\int_0^{\tau_0} e^{-\lambda t} g(X_0(t)) dt\right] \\ &\geq E\left[\int_0^{\tau_1} e^{-\lambda t} g(X_1(t)) dt\right] + E\left[e^{-\lambda \tau_i} V(0) 1_{\{\tau_i < \infty\}}\right] \\ &\geq E\left[\int_0^{\tau_1} e^{-\lambda t} g(X_1(t)) dt\right] \end{aligned}$$

which proves optimality of $\theta_0(t)$. ■

2.2 Optimal investment with income and stochastic discount rate

Again we standardize the Bellman equation: Multiply both sides of (9) by b^2/a^2 and denote the changed constants λ_i and the changed functions $c_i(x)$ again by λ_i and $c_i(x)$. Furthermore, replace $g_i(x)$ by $\lambda g_i(x)$,(7) by

$$\lambda_i(g_i(0) - V_i(0)) + c_i(0) V_i'(0) + \sum_{j=1}^{I} b_{ij} V_j(0) = 0, i = 1, ..., I \tag{21}$$

and (4) by $V_i(\infty) = g_i(\infty), i = 1, ..., I$. Then equation (9) is given in normal form:

$$\lambda_i(g_i(x) - V_i(x)) + c_i(x) V_i'(x) + \sum_{j=1}^{I} b_{ij} V_j(x) = \frac{1}{2} \frac{V_i'(x)^2}{V_i''(x)}, x > 0, i = 1, ..., I. \tag{22}$$

Theorem 3 *Assume that the functions $g_i(x), i = 1, ..., I$, $c_i(x)$ are continuously differentiable and positive, bounded, strictly increasing at infinity and non decreasing with $g_1(\infty) = ... = g_I(\infty)$, and that the functions $c_i(x)$ are positive and continuously differentiable. Then there exist continuous bounded functions $V_i(x)$ with $V_i'(x) > 0, V_i''(x) < 0, x > 0$,*

$$V_i(x) > g_i(x) \text{ for } x > 0, i = 1, ..., I,$$

which are twice continuously differentiable on $(0, \infty)$ such that $(V_1(x), ..., V_I(x))$ is a solution of the system of equations (22).

Proof. Define $g_i^*(x)$ by the system of linear equations

$$\lambda_i g_i(x) + \sum_{j \neq i} b_{ij} g_j^*(x) = (\lambda_i - b_{ii}) g_i^*(x), i = 1, ..., I.$$

The matrix C with entries $b_{ij} - \lambda_i \delta_{ij}$ is the nontrivial part of an intensity matrix of a Markov process on the states $\{0, ..., I\}$ which is eventually absorbed in state 0, so C^{-1} exists and has all its entries negative or zero. So the functions $g_i^*(x)$ are linear combinations of the functions $\lambda_j g_j(x)$ with non negative coefficients, and therefore these $g_i^*(x)$ are non decreasing. Furthermore, $g_i^*(\infty) = g_i(\infty), i = 1, ..., I$. Let $V_{0,i}(x)$ be the solution of the set of equations

$$\begin{aligned} \lambda_i(g_i(x) - V_i(x)) + c_i(x)V_i'(x) + b_{ii}V_i(x) + \sum_{j \neq i} b_{ij} g_j^*(x) &= \frac{1}{2}\frac{V_i'(x)^2}{V_i''(x)}, \qquad (23) \\ V_i(\infty) &= g_i(\infty), \; i = 1, ..., I, \end{aligned}$$

and for $n = 1, 2, ...$ define $V_{n,i}(x)$ as the solution to

$$\begin{aligned} &\lambda_i(g_i(x) - V_i(x)) + c_i(x)V_i'(x) + b_{ii}V_i(x) + \sum_{j \neq i} b_{ij} V_{n-1,j}(x) \qquad (24) \\ = \;& \frac{1}{2}\frac{V_i'(x)^2}{V_i''(x)}, V_i(\infty) = g_i(\infty), \; i = 1, ..., I. \end{aligned}$$

Notice that the equations (23) and (24) are no longer interacting, and the existence of a solution follows from Theorem 1 applied for

$$\begin{aligned} \lambda &= \rho_i := \lambda_i - b_{ii} > 0, \\ g(x) \; &: = \frac{1}{\rho_i}\left(\lambda_i g_i(x) + \sum_{j \neq i} b_{ij} g_j^*(x)\right) = g_i^*(x) \end{aligned}$$

for equation (23) and

$$\begin{aligned} \lambda &= \rho_i, \\ g(x) &= g_{n,i}(x) := \frac{1}{\rho_i}\left(\lambda_i g_i(x) + \sum_{j \neq i} b_{ij} V_{n-1,j}(x)\right), \end{aligned}$$

for equation (24), respectively. The function $g(x)$ is non decreasing in each case since $g_i^*(x)$ and $V_{n,j}(x)$ are non decreasing for all j, and $b_{ij} \geq 0$ for all $j \neq i$, and $g(x)$ has a continuous derivative. Recall that $V_{n,i}(x) > g_{n,i}(x)$, in particular $V_{0,i}(x) > g_i^*(x)$ and hence

$$\begin{aligned} g_{1,i}(x) &= \frac{1}{\rho_i}\left(\lambda_i g_i(x) + \sum_{j\neq i} b_{ij} V_{0,j}(x)\right) \\ &\geq \frac{1}{\rho_i}\left(\lambda_i g_i(x) + \sum_{j\neq i} b_{ij} g_j^*(x)\right) = g_i^*(x). \end{aligned}$$

So we obtain from section (2.1) that

$$V_{1,i}(x) \geq V_{0,i}(x), i = 1, ..., I,$$

and then by induction

$$V_{n+1,i}(x) \geq V_{n,i}(x), i = 1, ..., I, n = 1, 2, ...$$

since the utility functions $g_{n,i}(x)$ in the corresponding optimal control problems of type (2) are increased when n is increased. Hence, the bounded monotone sequences $V_{n,i}(x)$ converge to some functions $V_i(x), i = 1, ..., I$, which are non decreasing, concave, satisfy

$$V_i(x) \geq V_{0,i}(x) > g_i(x),$$

and

$$V_i(\infty) = g_i(\infty),\ i = 1, ..., I.$$

Notice that

$$g_{n,i}(\infty) = \frac{1}{\rho_i}\left(\lambda_i g_i(\infty) + \sum_{j\neq i} b_{ij} V_{n-1,j}(\infty)\right) = g_i(\infty).$$

We now show that

$$\text{the functions } V''_{n,i}(x) \text{ are bounded on compact subsets of } (0, \infty), \tag{25}$$

uniformly in $n = 1, 2, ...$This will imply that the functions $V'_{n,i}(x)$ will converge (along a subsequence) to continuous functions $v_i(x)$ such that

$$V_i(x) = g_i(\infty) - \int_x^\infty v_i(u)du, i = 1, ..., I. \tag{26}$$

Hence the functions $V_i(x)$ are continuously differentiable. This implies that the limits

$$\lim_{n\to\infty} g_{n,i}(x) = g_i^*(x) := \frac{1}{\lambda_i - b_{ii}}\left(\lambda_i g_i(x) + \sum_{j\neq i} b_{ij} V_j(x)\right)$$

are bounded, non decreasing, and continuously differentiable. Let $V_i^*(x)$ be the value function for the optimal control problem (2) with utility function $g_i^*(x)$. For fixed i the function $V_i^*(x)$ is twice continuously differentiable on $(0,\infty)$, and it satisfies the equation

$$\rho_i(g_i^*(x) - V(x)) + c_i(x)V'(x) = \frac{1}{2}\frac{V'(x)^2}{V''(x)}$$

or

$$\lambda_i(g_i(x) - V(x)) + c_i(x)V'(x) + b_{ii}V(x) + \sum_{j\neq i} b_{ij}V_j(x) = \frac{1}{2}\frac{V'(x)^2}{V''(x)},$$

where the functions V_j are the limits in (26). Furthermore, $g_{n,i}(x) \leq g_i^*(x)$ implies $V_i(x) \leq V_i^*(x)$, and for arbitrary admissible trading strategy $\theta(t)$ with corresponding wealth process $X(t)$ we have by dominated convergence

$$\lim_{n\to\infty} E\left[\int_0^\tau g_{n,i}(X(t))e^{-\lambda_i t}dt\right] = E\left[\int_0^\tau g_i^*(X(t))e^{-\lambda_i t}dt\right]$$

which implies $V_i^*(x) \leq V_i(x)$. Hence the vector $(V_1^*, ..., V_I^*)$ is a solution of (9).

To prove (25) let

$$U_{n,i}(x) = \left(-\frac{V'_{n,i}(x)}{V''_{n,i}(x)}\right)^2, n = 1,2,..., i = 1,...,I$$

(compare the proof of Theorem 1). The functions $V_{n,i}(x)$ are concave, so $V'_{n,i}(x) \leq V'_{n,i}(0)$, and

$$V'_{n,i}(0) = \rho_i\frac{V_{n,i}(0) - g_{n,i}(0)}{c_i(0)} \leq \rho_i\frac{V_{n,i}(\infty)}{c_i(0)} = \rho_i\frac{g_i(\infty)}{c_i(0)}.$$

Therefore, the nonnegative functions $V'_{n,i}(x)$ are uniformly bounded. So we have to show that

$$\begin{array}{l}\text{the functions } U_{n,i}(x) \text{ are uniformly bounded}\\ \text{away from zero on compact intervals } \subset (0,\infty).\end{array} \qquad (27)$$

The function $U_{n,i}(x)$ solves the equation

$$\frac{1}{4}U'(x) = c_i(x) + \sqrt{U(x)}\left(\rho_i + \frac{1}{2} - c_i'(x) - \frac{g'_{n,i}(x)\left(c_i(x) + \frac{1}{2}\sqrt{U(x)}\right)}{V_{n,i}(x) - g_{n,i}(x)}\right).$$

According to Lemma 11 we have to find a continuous lower bound $G(x)$ satisfying

$$-\frac{g'_{n,i}(x)}{V_{n,i}(x) - g_{n,i}(x)} \geq G(x) \text{ for all } x \geq 0 \text{ and } n = 1,2,... . \qquad (28)$$

Then $U_{n,i}(x) \geq U_i(x)$, where $U_i(x)$ is the solution to

$$\begin{aligned}
\frac{1}{4}U'(x) &= c_i(x) + \sqrt{U(x)}\left(\rho_i + \frac{1}{2} - c_i'(x) + G(x)\left(c_i(x) + \frac{1}{2}\sqrt{U(x)}\right)\right), \\
U(0) &= 0,
\end{aligned}$$

which is continuous and positive on $(0, \infty)$. This would prove (27). For (28) we have to find a continuous upper bound $H(x)$ satisfying

$$\frac{g_{n,i}'(x)}{V_{n,i}(x) - g_{n,i}(x)} \leq H(x) \text{ for all } x \geq 0 \text{ and } n = 1, 2, \ldots .$$

Since

$$\begin{aligned}
g_{n,i}'(x) &= \frac{1}{\rho_i}\left(\lambda_i g_i'(x) + \sum_{j \neq i} b_{ij} V_{n-1,j}'(x)\right) \\
&\leq \frac{1}{\rho_i}\left(\lambda_i g_i'(x) + \sum_{j \neq i} b_{ij} \rho_j \frac{g_j(\infty)}{c_j(0)}\right)
\end{aligned}$$

for all $x \geq 0$, we have to show that

$$V_{n,i}(x) - g_{n,i}(x), n = 1, 2, \ldots$$

has a continuous positive lower bound for all $x \geq 0$. Let $v_{n,i}(x)$ be the solution to the equation

$$\rho_i(g_{n,i}(x) - V(x)) + c_i(x)V'(x) = 0, \; V(\infty) = g_{n,i}(\infty).$$

Then, as in the proof of Theorem 1, we apply Lemma 6 to

$$\begin{aligned}
c^{(1)}(x) &= \frac{\rho_i}{c_i(x) + \frac{1}{2}\sqrt{U_{n,i}(x)}}, \\
c^{(2)}(x) &= \frac{\rho_i}{c_i(x)} \geq c^{(1)}(x), \\
V_{n,i}'(x) &= c^{(1)}(x)(V_{n,i}(x) - g_{n,i}(x)), \text{ and} \\
v_{n,i}'(x) &= c^{(2)}(x)(v_{n,i}(x) - g_{n,i}(x)),
\end{aligned}$$

to obtain

$$V_{n,i}(x) - g_{n,i}(x) \geq v_{n,i}(x) - g_{n,i}(x).$$

With

$$F_i(x) = \rho_i \int_0^x \frac{dy}{c_i(y)}$$

we have according to Lemma 10

$$v_{n,i}(x) = \exp(F_i(x))\left[\int_{F_i(x)}^{z_i} g_{n,i}(F_i^{-1}(y))e^{-y}dy + g_{n,i}(\infty)e^{-z_i}\right],$$

and hence

$$\begin{aligned} & v_{n,i}(x) - g_{n,i}(x) \\ \geq & \frac{\lambda_i}{\rho_i} e^{F_i(x)} \left[\int_{F_i(x)}^{z_i} \left(g_i(F_i^{-1}(y)) - g_i(x) \right) e^{-y} dy + \left(g_i(\infty) - g_i(x) \right) e^{-z_i} \right], \\ x \geq & 0, n = 1, 2, ..., i = 1, ..., I, \end{aligned}$$

which is the positive continuous lower bound we were looking for. ■

Remark 4 *The verification argument, i.e. a solution of the optimization problem with a solution of the Bellman equation (9), is completely analogous to the one in section (2.1). If* $(V_1(x), ..., V_I(x))$ *is a solution in the sense of theorem (3), then the investment strategy*

$$\theta(t) = -\frac{a}{b^2} \frac{V_i'(X(t))}{V_i''(X(t))Z(t)} \text{ if } i(t) = i, i = 1, ..., I, \tag{29}$$

is optimal.

2.3 Optimal investment for Poisson type insurers

Theorem 5 *Assume that the claim size* Y *has a continuous density* $p(y)$*. Then the equation (10) has a solution* $V(x)$ *which is nonnegative, non decreasing, concave with* $V(\infty) = 1$*, continuous on* $[0, \infty)$ *and twice continuously differentiable on* $(0, \infty)$.

Proof. We shall define a sequence of optimization problems with value functions V_n, $n = 1, 2, ...$ which correspond to the optimization of investment up to the $n-$th claim. Let $V_0(s)$ be the survival probability without investment which satisfies the differential equation

$$\lambda E[V_0(x - Y) - V_0(x)] + (c + rx)V_0'(x) = 0.$$

For $n = 1, 2, ...$ define $g_n(x) = E[V_{n-1}(x - Y)]$, and let $V_n(x)$ be the value function of the problem

$$E\left[\int_0^\tau g_n(X(t)) \exp(-\lambda t) dt\right] = \max \tag{30}$$

The function $g_1(x)$ is non decreasing, continuously differentiable, and it satisfies $g_1(x) < g_1(\infty) = 1$. For $n = 1, 2, ...$ the functions $V_n(x)$ are increasing, concave, continuous on $[0,\infty)$ and twice continuously differentiable on $(0, \infty)$ satisfying $V_n(\infty) = g_n(\infty)$. One can show by induction that the functions $g_n(x)$ are non decreasing and continuously differentiable with $g_n(x) < g_n(\infty) = 1$, and that

$$V_{n+1}(x) \geq V_n(x), x \geq 0. \tag{31}$$

For $n = 0$ (31) follows from Lemma (10) for $c_1(x) = c + rx$ and

$$c_2(x) = c_1(x) - \frac{1}{2}\frac{V_1'(x)}{V_1''(x)}.$$

If the assertion (31) is true for n, then $g_{n+1}(x) \le g_{n+2}(x)$ for $x \ge 0$, which implies $V_{n+2}(x) \ge V_{n+1}(x)$ for $x \ge 0$ since for all n the function $V_n(x)$ is the value function of the above control problem (30). Since all functions $V_n(x)$ are bounded by 1, they converge to some function $V(x)$. Let $g(x) = E[V(x - Y)]$. Since the functions $V_n(x)$ are uniformly Lipschitz,

$$V_n'(x) \le V_n'(0) \le \frac{1}{c}$$

(see (18) and recall that $U_n(0) = 0$), the function $V(x)$ is uniformly Lipschitz, too. Hence by Lemma 12 the function $g(x)$ is continuously differentiable. Let $V^*(x)$ be the solution of our stochastic control problem (30) with utility function $g(x)$. From $V_n(x) \le V(x)$ we obtain $g_n(x) \le g(x)$ and hence $V^*(x) \ge V(x)$. On the other hand, for all admissible strategies $\theta(t)$ with resulting wealth process $X(t)$ we have by dominated convergence

$$\lim_{n\to\infty} E\left[\int_0^\tau g_n(X(t))\exp(-\lambda t)dt\right] = E\left[\int_0^\tau g(X(t))\exp(-\lambda t)dt\right]$$

which implies $V^*(x) \le V(x)$. Hence $V(x) = V^*(x)$ is twice continuously differentiable on $(0, \infty)$, and it satisfies the equation (10). ■

The verification argument states that from a smooth solution to the Bellman equation one can derive a solution to the optimization problem.

Theorem 6 *Assume that the Bellman equation (10) has a solution $V(x)$ which is nonnegative, non decreasing, concave with $V(\infty) = 1$, continuous on $[0, \infty)$ and twice continuously differentiable on $(0, \infty)$. Then for $s \ge 0$ and arbitrary admissible trading strategy $\theta(t)$ with corresponding risk process*

$$dX(t) = (c + \rho X(t))dt - dS(t) + \theta(t)[dZ(t) - \rho Z(t)dt], X(0) = s,$$

we have

$$V(s) \ge P\{X(t) \ge 0 \text{ for all } t \ge 0\},$$

and this upper bound is attained by the strategy defined by

$$\begin{aligned} \theta^*(t) &= A(X^*(t))/Z(t), \\ dX^*(t) &= (c + \rho X^*(t))dt - dS(t) + \theta^*(t)\left[dZ(t) - \rho Z(t)dt\right], X^*(0) = s, \\ A(x) &= -\frac{a}{b^2}\frac{V'(x)}{V''(x)}, x > 0,\ A(0) = 0. \end{aligned}$$

Proof. We first consider the case $\rho = 0$. Consider the following three admissible strategies: $\theta(t)$ arbitrary and $\theta^*(t)$ defined above, and for $\varepsilon > 0$ to be specified later

$$\theta_1(t) = \theta(t) + \varepsilon^2/Z(t).$$

With these strategies define the following three reserve processes

$$\begin{aligned}dX_0(t) &= cdt - dS(t) + \theta^*(t)dZ(t), X_0(0) = s,\\ dX_1(t) &= cdt - dS(t) + \theta_1(t)dZ(t), X_1(0) = s + \varepsilon,\\ dX_2(t) &= cdt - dS(t) + \theta(t)dZ(t), X_2(0) = s,\end{aligned}$$

and τ_0, τ_1, τ_2 their ruin times. Define the processes

$$W_j(t) = V(X_j(t \wedge \tau_j)), j = 0, 1, 2.$$

The process $W_0(t)$ is a martingale, while $W_1(t), W_2(t)$ are supermartingales. This implies that for $t \geq 0$

$$E[W_0(t)] = V(s) \geq W[X_2(t)]. \tag{32}$$

For $t \to \infty$, the processes $W_j(t)$ converge to 0 on $\{\tau_j < \infty, X_j(\tau_j) < 0\}$, and to $V(0)$ on $\{\tau_j < \infty, X_j(\tau_j) = 0\}$. Since between claims the process $X_0(t)$ evolves according to

$$dX_0(t) = cdt + A(X_0(t))(adt + bdW(t))$$

with continuous $A(x)$ satisfying $A(0) = 0$, we have

$$P\{\tau_0 < \infty, X_0(\tau_0) = 0\} = 0.$$

Now let $t \to \infty$ in (32). Then

$$V(s) = \lim_{t\to\infty} E[X_0(t)1_{\{\tau_0=\infty\}}] \leq P\{\tau_0 = \infty\},$$

and

$$\begin{aligned}V(s+\varepsilon) &\geq \lim_{t\to\infty} E[X_1(t)1_{\{\tau_1=\infty\}}] + V(0)P\{\tau_1 < \infty, Y_1(\tau_1) = 0\}\\ &\geq \lim_{t\to\infty} E[X_1(t)1_{\{\tau_1=\infty,\tau_2=\infty\}}].\end{aligned}$$

Since

$$X_1(t) - X_2(t) = \varepsilon + \varepsilon^2(at + bW(t)) \to \infty$$

we have $X_1(t) \to \infty$ on $\{\tau_2 = \infty\}$ and hence

$$V(s+\varepsilon) \geq P\{\tau_1 = \infty \text{ and } \tau_2 = \infty\}.$$

Using

$$\begin{aligned}P\{\tau_1 < \tau_2\} &\leq P\{\varepsilon + a\varepsilon^2 t + b\varepsilon^2 W(t) < 0 \text{ for some } t \geq 0\}\\ &= \exp(-2a^2/(b^2\varepsilon))\end{aligned}$$

we obtain with $\varepsilon \to 0$ our assertion

$$P\{\tau_0 = \infty\} \geq V(s) \geq P\{\tau_2 = \infty\}.$$

In the case $\rho > 0$ consider the three processes

$$\begin{aligned} dX_0(t) &= (c + \rho X_0(t))dt - dS(t) + \theta^*(t)\,[dZ(t) - \rho Z(t)dt]\,, X_0(0) = s, \\ dX_1(t) &= (c + \rho X_1(t))dt - dS(t) + \theta(t)\,[dZ(t) - \rho Z(t)dt]\,, X_1(0) = s + \varepsilon, \\ dX_2(t) &= (c + \rho X_2(t))dt - dS(t) + \theta(t)\,[dZ(t) - \rho Z(t)dt]\,, X_2(0) = s, \end{aligned}$$

and the same processes $W_j(t), j = 0, 1, 2$. As above,

$$V(s) \leq P\{\tau_0 = \infty\}.$$

The relation $X_1(t) - X_2(t) = \varepsilon \exp(\rho t)$ yields that $X_1(t) \to \infty$ on $\{\tau_2 = \infty\}$, and hence

$$V(s + \varepsilon) \geq \lim_{t \to \infty} E[X_1(t) 1_{\{\tau_2 = \infty\}}] = P\{\tau_2 = \infty\}.$$

With $\varepsilon \to 0$ we obtain our assertion

$$P\{\tau_0 = \infty\} \geq V(s) \geq P\{\tau_2 = \infty\}.$$

■

Remark 7 *For a numerical solution of equation (10) we can use the two interacting differential equations which are equivalent to (10) and similar to (14) and (15):*

$$g(x) = E[V(x - Y)]$$

$$\lambda[V(x) - g(x)] - c(x)V'(x) = \frac{1}{2}\sqrt{U(x)}V'(x) \tag{33}$$

$$\sqrt{U(x)}\left[(\lambda + \frac{1}{2} - c'(x))V'(x) - \lambda g'(x)\right] + c(x)V'(x) = \frac{1}{4}U'(x)V'(x). \tag{34}$$

Starting with $U(0) = g(0) = 0$ and $V(0) = 1$, say. A solution satisfying the boundary condition $V(\infty) = 1$ can be obtained via $\overline{V}(x) = V(x)/V(\infty)$.

Remark 8 *There are cases in which the above interacting differential equations separate. The first case is when $P(Y < s) = 0$. The second is when the distribution Q of Y is an exponential distribution: $Y \sim Exp(\theta)$. In this case,*

$$g'(x) = \theta(V(x) - g(x)),$$

and with (33)

$$g'(x) = \frac{\theta}{\lambda}\left[\frac{1}{2}\sqrt{U(x)}V'(x) + c(x)V'(x)\right],$$

which, inserted in (34), yields

$$\sqrt{U(x)}\left\{(\lambda + \frac{1}{2} - c'(x)) - \theta\left[\frac{1}{2}\sqrt{U(x)} + c(x)\right]\right\} + c(x) = \frac{1}{4}U'(x),$$

or, in terms of $A(x)=\sqrt{U(x)}$:

$$A'(x)A(x)=\left(2c(x)+2A(x)\left(\lambda+\frac{1}{2}-c'(x)-\theta c(x)\right)-A^2(x)\theta\right)$$

In particular, if $c(x)\equiv c>0$ *then the last equation reads*

$$A'(x)=2cA(x)\left(\frac{1}{A(x)}-R\right)\left(\frac{1}{A(x)}+\gamma\right),$$

where $R>0$ *and* $-\gamma<0$ *are the two distinct solutions of the Lundberg equation*

$$\lambda+rc+\frac{1}{2}=\frac{\lambda\theta}{\theta-r}.$$

The last differential equation has a solution with $A(0)=0$ *satisfying the following transcendental equation*

$$\left(\frac{1}{A(x)}-R\right)^{\gamma}\left(\frac{1}{A(x)}+\gamma\right)^{R}=\exp(-(R+\gamma)x).$$

A function $u(x)$ *with* $u(0)=1$ *satisfying* $-u(x)/u'(x)=A(x)$ *is*

$$u(x)=\frac{\exp(-Rx)}{(1+\gamma A(x))^R}.$$

The function $u(x)$ *is related to* $V'(x)$ *and* $V(x)$ *via*

$$V'(x)=\frac{\lambda}{c}V(0)u(x),$$

and

$$1-V(x)=\frac{\lambda\int_x^\infty u(y)dy}{c+\lambda\int_0^\infty u(y)dy}.$$

2.4 Optimal investment for Cox type insurers

Theorem 9 *Assume that the functions* $g_i(x)$ *and* $c_i(x)$ *satisfy the conditions in 3, that the claim size* Y *has a continuous density* $p(y)$*, and that the interest rates* $\rho_1,...,\rho_I$ *are nonnegative. Then the equation (9) has a solution* $(V_1(x),...,V_I(x))$ *with functions* $V_i(x)$ *which are nonnegative, non decreasing, concave, continuous on* $[0,\infty)$ *and twice continuously differentiable on* $(0,\infty)$*, with* $V_i(\infty)=1$ *and* $V_i(x)=0$ *for* $x<0$.

Proof. Let $V_{0,i}(x), i=1,...,I$, be the survival probabilities of the risk process without investment which satisfy the system of interacting differential equations

$$\begin{aligned}\lambda_i E[V_{0,i}(x-Y)-V_{0,i}(x)]+(c+r_ix)V_{0,i}'(x)+\sum_{j=1}^{I}b_{ij}V_{0,j}(x) &= 0, \quad (35)\\ V_{0,i}(\infty) &= 1,\end{aligned}$$

and $V_{0,i}(x) = 0, x < 0$ (see [1], p. 84, (14)). The functions $V_{0,i}(x)$ are uniformly Lipschitz, and so the functions $g_{1,i}(x) = E[V_{0,i}(x-Y)], i = 1,...,I$, are continuously differentiable according to Lemma 12. For $n = 1,2,3...$ let $V_{n,i}(x), i = 1,...,I$, be a solution of the system

$$\lambda_i[g_{n,i}(x) - V_{n,i}(x)] + (c + r_i x)V'_{n,i}(x) + \sum_{j=1}^{I} b_{ij}V_{n,j}(x) \qquad (36)$$
$$= \frac{1}{2}\frac{V'_{n,i}(x)^2}{V''_{n,i}(x)},\ V_{n,i}(\infty) = 1, i = 1,...,I,$$

satisfying $V_{n,i}(x) = 0$ for $x < 0$, where $g_{n,i}(x) = E[V_{n-1,i}(x-Y)]$. By induction the functions $g_{n,i}(x), i = 1,...,I, n = 1,2,...$ are non decreasing, have continuous derivatives and, by Theorem 3, (36) has a smooth solution . If

$$V_{1,i}(x) \geq V_{0,i}(x), i = 1,...,I, \qquad (37)$$

then, as in the proof of Theorem 5, the functions $V_{n,i}(x), n = 1,2,...$ converge to functions $V_i(x)$ for which $(V_1(x),...,V_I(x))$ is a solutions to equation (9). To prove (37), apply Lemma 10 b) for

$$\begin{aligned} f(x) &= V_{1,i}(x) - V_{0,i}(x), \\ c(x) &= \frac{\lambda_i}{c_i(x)}, \text{ and} \\ g(x) &= \frac{c_i(x)}{\lambda_i}\left(\sum_{j\neq i} b_{ij}V_{1,j}(x) - \frac{1}{2}\frac{V'_{1,i}(x)^2}{V''_{1,i}(x)}\right) \geq 0. \end{aligned}$$

Notice that $f(\infty) = g(\infty) = 0$, the functions $c(x)$ and $g(x)$ are continuously differentiable on $(0,\infty)$, and the function $g(x)$ is bounded. Then $f(x) \geq 0$ which is (37). ■

The verification argument is similar to the one given in the Poisson case.

3 Auxiliary Lemmas

Lemma 10 *Let $c(x)$, $g(x)$ be functions which are continuously differentiable on $(0,\infty)$, $c(x)$ positive, $g(x)$ bounded with*

$$\lim_{x\to\infty} g(x) =: g(\infty),$$

and let

$$F(x) = \int_0^x c(y)dy \text{ with } F(\infty) = z \leq \infty.$$

a) The differential equation

$$f'(x) = f(x) - g(x),\ f(\infty) = g(\infty)$$

has the unique solution

$$f_0(x) = e^x \int_x^\infty g(y) e^{-y} dy$$

b) The differential equation

$$f'(x) = c(x)\left(f(x) - g(x)\right),\; f(\infty) = g(\infty)$$

has the unique solution

$$f(x) = e^{F(x)} \int_{F(x)}^{z} g(F^{-1}(y)) e^{-y} dy + g(\infty) e^{-z}.$$

c) If $c_1(x) \leq c_2(x)$ for $x > 0$, both positive, if $g(x)$ is non decreasing with $g(x) < g(\infty), x > 0$, and if for $i = 1, 2$ $f_i(x)$ is the solution of

$$f_i'(x) = c_i(x)(f_i(x) - g(x)),\; f_i(\infty) = g(\infty)$$

then $f_1(x) \geq f_2(x)$ for $x > 0$. Furthermore, $f_i(x) > g(x)$ for $x > 0$.

Proof. a) and b) are easy. For c) we first note that the inequality $f_i(x) > g(x)$ for $x > 0$ follows from $g(y) \geq g(x)$ for $y > x$ and

$$\int_{F_i(x)}^{\infty} \exp(F_i(x)) e^{-y} dy = 1,$$

and the fact that the set $\{y > F(x) : g(F^{-1}(y) > g(x)\}$ is not a Lebesgue null set. To prove the inequality consider the functions

$$\phi_i(x) = f_i(-\log(x)), 0 \leq x \leq 1, i = 1, 2.$$

which are continuous on $[0, 1]$, continuously differentiable on $(0, 1]$, with $\phi_1(0) = \phi_2(0)$. With

$$\Phi(x, y) = -\frac{1}{x} c_1(-\log(x))[y - g(-\log(x))]$$

we obtain

$$\begin{aligned} \phi_1'(x) &= \Phi(x, \phi_1(x)), 0 < x \leq 1, \\ \phi_2'(x) &\leq \Phi(x, \phi_2(x)), 0 < x \leq 1. \end{aligned}$$

Theorem XI in ([4], p. 69; the uniqueness condition on p. 81 holds with $\omega = 0$) yields $\phi_1(x) \leq \phi_2(x)$ or $f_1(x) \leq f_2(x)$. ■

Lemma 11 *Let*

$$f(x), g(x), h(x), k(x), k_1(x), k_2(x)$$

be continuous functions on $[0, \infty)$, $f(x)$ and $h(x)$ positive. Then *a) The initial value problem*

$$u'(x) = f(x) + \sqrt{u(x)}\left(g(x) + k(x)(h(x) + \sqrt{u(x)})\right), u(0) = 0,$$

has a positive solution on $(0, \infty)$ *which is moreover (even locally) unique.*
b) If $k_1(x) \leq k_2(x)$ *for all* $x \geq 0$, *then the solutions* $u_i(x)$ *of*

$$u_i'(x) = f(x) + \sqrt{u_i(x)}\left(g(x) + k_i(x)(h(x) + \sqrt{u_i(x)})\right), u_i(0) = 0, i = 1, 2$$

satisfy $u_1(x) \leq u_2(x)$ *for all* $x \geq 0$.

Proof. Let $\Phi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$,

$$\Phi(x, y) := f(x) + \sqrt{|y|}\left(g(x) + k(x)(h(x) + \sqrt{|y|})\right).$$

According to Peano's Theorem, the initial value problem $u' = \Phi(x, u), u(0) = 0$, has a solution which we may assume to be extended to some maximal interval $[0, T)$. The assumption $T < \infty$ provides the existence of some $C > 0$ such that

$$|\Phi(x, y)| \leq C(1 + |y|) \quad for \quad 0 \leq x \leq T, y \in \mathbb{R},$$

which implies $|u(x)| \leq e^{Cx} - 1 \quad for \quad 0 \leq x < T$, so that u does not approach the "boundary" of $[0, \infty) \times \mathbb{R}$ near T. This contradicts the maximality of $[0, T)$. Thus, $T = \infty$.
Now let $k_1 \leq k_2$, and let u_1, u_2 denote solutions of the two initial value problems stated in part b), on some interval $[0, T], T < \infty$. We will prove that

$$0 < u_1(x) \leq u_2(x) \quad for \quad 0 < x \leq T. \tag{38}$$

Then, choosing formally $k_1 := k_2 := k$, we see that (38) completes the proof of part a), since it implies positivity and uniqueness of the solution u. Moreover, (38) clearly proves part b).
For proving (38), let Φ_1 be defined as Φ above, with k_1 in place of k. The differential equations imply, for $0 < x \leq T$,

$$\begin{aligned} u_2'(x) - \Phi_1(x, u_2(x)) &= (k_2(x) - k_1(x))\sqrt{|u_2(x)|}\left(h(x) + \sqrt{|u_2(x)|}\right) \\ &\geq 0 = u_1'(x) - \Phi_1(x, u_1(x)). \end{aligned} \tag{39}$$

Together with the initial conditions, (39) implies $u_2 \geq u_1$ on $[0, T]$ according to ([4],Thm.8.XI,p.69), if a uniqueness condition in the sense of ([4],10.(3),p.81) is satisfied. To obtain it, observe first that, due to the differential equations and to the positivity of f, some $\varepsilon, \delta > 0$ exist such that $u_i' \geq \delta$ on $[0, \varepsilon]$, so

$$(x, u_i(x)) \in D(\Phi_1) := \{(t, y) : 0 < t \leq \varepsilon, \delta x \leq y < \infty\}$$

for $0 < x \leq \varepsilon$ and $i = 1, 2$. Since we can find some $C > 0$ such that

$$\Phi_1(x, y) - \Phi_1(x, \bar{y}) \leq \frac{C}{\sqrt{x}}(y - \bar{y}) \quad \text{for} \quad (x, y), (x, \bar{y}) \in D(\Phi_1), y \geq \bar{y},$$

we obtain the desired uniqueness condition on $(0,\varepsilon]$, so that (38) holds on $(0,\varepsilon]$. Moreover, u_1 and u_2 are positive on $[\varepsilon,T]$, because otherwise some minimal $\xi \in (\varepsilon,T]$ would exist such that $u_i(\xi)=0$ and therefore $u_i'(\xi)\le 0$ (for $i=1$ resp. 2), which contradicts $u_i'(\xi)=f(\xi)>0$. Consequently, $u_i(x)\ge\eta>0$ for $x\in[\varepsilon,T], i=1,2$. Since Φ_1 satisfies a (global) Lipschitz condition with respect to y on $[\varepsilon,T]\times[\eta,\infty)$, a uniqueness condition is fulfilled also on $[\varepsilon,T]$. This completes the proof of (38), and thus, of the lemma. ■

Lemma 12 *Let $V(x)$ be a non decreasing function which is uniformly Lipschitz on $[0,\infty)$ satisfying $V(\infty)=1$. If $p(x)$ is a function, integrable on $[0,\infty)$ and continuous on $(0,\infty)$, then the function*

$$g(x)=\int_0^x V(x-y)p(y)dy$$

is continuously differentiable on $(0,\infty)$.

Proof. Since $V(x)$ is uniformly Lipschitz, there exists a measurable bounded function $f(x)$ such that for all $x\ge 0$

$$V(x)=1-\int_x^\infty f(y)dy.$$

Then with

$$F(x)=\int_0^x p(y)dy$$

we have

$$\begin{aligned} g(x) &= \int_0^x V(x-y)p(y)dy \\ &= F(x)-\int_0^x\int_{x-y}^\infty f(u)du\,p(y)dy \\ &= F(x)-\int_0^x F(x-y)f(y)dy \end{aligned}$$

which shows that $g(x)$ is differentiable with continuous derivative

$$g'(x)=p(x)-\int_0^x p(x-y)f(y)dy.$$

■

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