Nonrational conformal field theory

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1. Introduction

In these notes we will discuss the problem to develop a mathematical theory of a certain class of conformal field theories (CFT) which contain the unitary CFT. In similar attempts of this kind the focus mostly was on the so-called *rational* CFT. The author believes that this restriction is unnatural and may obscure where the real issues are. From a physical point of view it seems that *rational* CFT are exceptional rather than generic, owing their existence to some remarkable arithmetic accidents. Although the rational CFT are certainly a mathematically rich and interesting subject in its own right, it seems to the author that the simplifications resulting from rationality obscure what CFTs really are.

The present approach will be based on the so-called gluing construction of the conformal blocks in which one constructs large classes of conformal blocks from the conformal blocks associated to the three punctured sphere. Some aspects of the resulting “Lego-Teichmüller game” are well-understood in the case of rational CFT including relations to modular tensor categories, modular functors etc., see [BK1] and references therein. However, it seems to the author that the gluing construction had not yet been developed for the case of arbitrary Riemann surfaces before. This may be due to the fact that key mathematical results concerning Riemann surface theory like [RoS] have become available only recently. The author was also unable to find a satisfactory treatment of the consequences of projectiveness of the canonical connection on spaces of conformal blocks within this framework. We will outline an approach to CFT based on the gluing construction that properly deals with these issues.

Of particular importance for us will be to find a proper generalization of the concept of a modular functor which does not assume finite-dimensionality of the spaces of conformal blocks. This immediately raises the issue to control convergence of expansions w.r.t. to a basis for (sub-) spaces of the space of conformal blocks by means of suitable topologies. Also for other reasons it will be seen to be of foundational importance to have a nondegenerate hermitian form, or, in good cases, a scalar product on the spaces of conformal blocks. This is not only required for the construction of correlation functions out of the conformal blocks, it also serves the task to select a subspace of "tempered" conformal blocks among the space of all solutions to the conformal Ward identities. This is one of the main issues which makes the nonrational case much more subtle and interesting than the rational case: As we will illustrate by an example one will generically find that the space of tempered conformal blocks is much smaller than the space of all solutions to the conformal Ward identities. However, the latter contains a subspace of "factorizable" conformal blocks – those that have a reasonable behavior at all boundaries of the moduli space $\mathfrak{M}(\Sigma)$ of complex structures on a given two-dimensional surface Σ. In the example discussed below it turns out that the space of all factorizable conformal blocks can be fully understood[1] provided one understands the much smaller space of all tempered conformal blocks.

The variant of the concept of a modular functor that will be proposed below is based on the consideration of stable surfaces[2] only. This is not usually done in the context of modular functors related to rational CFT, where cutting the surface into pieces containing discs etc. is also allowed. One of the issues that arise is to properly formulate the distinguished role played by insertions of the vacuum representation. This turns out to be somewhat more subtle in nonrational cases.

An important issue is the existence of a canonical nondegenerate hermitian form on spaces of conformal blocks. We will propose a generalization of known relations between the canonical hermitian form and other data characterizing modular functors like the so-called fusion transformation in Section 6. Existence of a scalar product on spaces of conformal blocks seems to be an open question even for many rational CFT. In Section 6 we will present arguments indicating that the hermitian form gives a scalar product whenever one restricts attention to the conformal blocks associated to *unitary* representations.

It may also be worth mentioning the analogies between CFT and the theory of automorphic forms [Wi, Fr1, Fr2] in which, very roughly speaking, the role of the automorphic forms is taken by the conformal blocks. These analogies play an important role in certain approaches to the geometric Langlands-correspondence, see [Fr2] for a review. An ingredient of the classical theory of automorphic forms that does not seem to have a good counterpart within CFT at the moment is a good analog of the scalar product on spaces of automorphic forms. This structure is the foundation for doing harmonic analysis on spaces of automorphic forms. The author believes that the scalar products on spaces of conformal blocks discussed in this paper provide a natural analog of such a structure.

[1] Via meromorphic continuation.

[2] Surfaces X with $2g-2+n>0$, with g being the genus of X and n the number of marked points.

In any case, one of my aims in this paper will be to advertise the harmonic analysis on spaces of conformal blocks as an attractive future field of mathematical research, naturally generalizing the theory of automorphic forms and the harmonic analysis on real reductive groups.

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2. Constraints from conformal symmetry

2.1 Motivation: Chiral factorization of physical correlation functions

A point of view shared by many physicists is that a conformal field theory is characterized by the set of its n-point correlation functions

$$\big\langle\, V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\,\big\rangle_X\,, \tag{2.1}$$

which can be associated to any Riemann surface X with n marked points $z_1,\dots,z_n$ and a collection of vertex operators $V_k(z_k,\bar{z}_k)$ $k=1,\dots,n$. The vertex operators $V_k(z_k,\bar{z}_k)$ are in one-to-one correspondence with states V_k in representations $\mathfrak{R}_k$ of the conformal symmetry $\mathrm{Vir}\times\mathrm{Vir}$ by the state-operator correspondence.

A lot of work on CFT was stimulated by the observation that conformal symmetry combined with physical consistency requirements constrain the correlation functions of a CFT strongly.

We will assume that the representations $\mathfrak{R}_k$ factorize as $\mathfrak{R}_k=R_k\otimes R_k'$. It is then sufficient to know the correlation functions in the case that the vectors $V_k\in R_k$ factorize as $V_k=v_k\otimes v_k'\in R_k\otimes R_k'$. The notation $\hat{\Sigma}$ will be used as a short-hand for the topological surface Σ with marked points z_k "decorated" by the representations R_k, R_k'. There are general arguments which indicate that the correlation functions should have a holomorphically factorized structure

$$\big\langle\, V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\,\big\rangle_X\,=\int\limits_{\mathbb{F}_{\hat{\Sigma}}\times\mathbb{F}_{\hat{\Sigma}}} d\mu_{\hat{\Sigma}}(S,S')\;\mathcal{F}_S(v;X)\,\overline{\mathcal{F}}_{S'}(v';X)\,. \tag{2.2}$$

This decomposition disentangles the relevant dependencies by encoding them into the following objects:

- The conformal blocks $\mathcal{F}_S(v;X)$ and $\bar{\mathcal{F}}_{S'}(v';X)$ depend holomorphically and antiholomorphically on the complex structure of the Riemann surface X, respectively. The set $\mathbb{F}_{\hat{\Sigma}}$ of labels S that the integration is extended over will be specified more explicitly below. They furthermore depend on the vectors $v = \bigotimes_{k=1}^{n} v_k \in \bigotimes_{k=1}^{n} R_k$ and $v' = \bigotimes_{k=1}^{n} v'_k \in \bigotimes_{k=1}^{n} R'_k$, respectively.

- The measure $d\mu_{\hat{\Sigma}}(S, S')$ does not depend on the complex structure of the Riemann surface X but only on its topological type Σ. together with the assignment of representations R_k, R'_k to the punctures z_k.

Given that the correlation functions of a CFT factorize as in (2.2), it has turned out to be fruitful to approach the construction of correlation function in three steps:

- First construct the conformal blocks $\mathcal{F}_S(v; X)$ by exploiting the constraints coming from the conformal symmetry of the theory.
- Describe the restrictions on the measure $d\mu_{\hat{\Sigma}}(S, S')$ that follow from basic physical consistency requirements (locality, crossing symmetry, modular invartiance).
- Identify the solution to these requirements which fulfils further *model-specific* conditions.

We will in the following mainly focus on the first two of these items. Concerning the third let us only remark that the specification of the chiral symmetries will in general not be sufficient to determine the CFT. One may think e.g. of the CFTs with N=2 superconformal symmetry where one expects to find multi-parametric families of such CFTs in general.

2.2 Vertex algebras

Vertex algebras V represent the chiral symmetries of a CFT. We will require that these symmetries form an extension of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}\,. \tag{2.3}$$

A convenient formalism for describing extensions of the conformal symmetry generated by the Virasoro algebra is provided by the formalism of vertex algebras, see [B, FLM, K, FBZ]. The symmetries are generated from the modes of the "currents" denoted $Y(A, z)$, with formal Laurent-expansion of the form

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}\,, \tag{2.4}$$

There is a canonical Lie algebra $U'(V)$ which can be attached to a vertex algebra V, see [FBZ, Section 4.1]. $U'(V)$ is generated from the expansion coefficients A_n introduced in (2.4).

2.3 Representations of vertex algebras

As indicated above, one wants to assign representations of the the vertex algebra V to the marked points of Σ. Representations M of the vertex algebra V must in particular be representations of the Lie-algebra $U'(V)$ generated from the coefficients A_n, see [FBZ, Section 5] for more details.

Note that V can be considered as a representation of itself, the so-called vacuum representation which is generated from a distinguished vector $v_\circ$ such that $Y(v_\circ, z) = \mathrm{id}$. This realizes the idea of state-operator correspondence: The currents $Y(A,z)$ are in one-to-one correspondence with the states $A \equiv A_{-1}v_\circ$ that they generate from the "vacuum" $v_\circ$ via $\lim_{z\to 0} Y(A,z)v_\circ$. The energy-momentum tensor $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is identified with $Y(A,z)$ for $A = L_{-2}v_\circ$.

We will mainly be interested in unitary representations M of the vertex algebra V. To define the notion of a unitary representation of V we need to say how hermitian conjugation should act on the generators of $U'(V)$. Formally this means that we need to assume that V is equipped with a $*$-structure, a conjugate linear anti-automorphism $* : A_n \to A_n^*$ of $U'(V)$ such that $*^2 = \mathrm{id}$. M is a unitary representation of V if it has the structure of a Hilbert space with scalar product $\langle .,. \rangle_M$ such that $A_n^\dagger = A_n^*$. We will assume in particular that $L_n^* = L_{-n}$, as it is usually done to define unitary representations of the Virasoro algebra.

Let $\mathbb{U}$ be the set of all (equivalence classes) of *irreducible* unitary representations with positive energy of V. Following the terminology from Lie group theory we will call $\mathbb{U}$ the unitary dual of V. General unitary representations M can then be parametrized by measures μ_M on $\mathbb{U}$. The corresponding Hilbert space $\mathfrak{H}_M$ consists of all families of vectors $v = (\, v_u \,;\, u \in \mathbb{U}\,)$ such that $v_u \in M_u$ for μ_M-almost all $u \in \mathbb{U}$ which are square-integrable w.r.t.

$$\| \, v \, \|^2 \; = \; \int_{\mathbb{U}} d\mu_M(u) \; \| \, v_u \, \|_{M_u}^2 \, . \tag{2.5}$$

We will in the following sometimes restrict attention to the example of the Virasoro algebra itself. This may be motivated by the observation that in physics the presence of a large symmetry is a lucky accident rather than generic, but most of our discussion can be generalized to other vertex algebras as well.

2.4 Conformal blocks

We will start by recalling a definition of the conformal blocks that has become standard in the mathematical literature.

2.4.1 Definition

Let X be a Riemann surface of genus g with n marked points $P_1, \dots, P_n$ and choices of local coordinates t_k near P_k, $k = 1, \dots, n$ such that the value $t_k = 0$ parametrizes the point $P_k \in X$.

Definition 1. *A Virasoro conformal block is a linear functional $F : M_X \equiv \prod_{r=1}^n M_r \to \mathbb{C}$ that satisfies the following invariance condition:*

$$F\big(\, T[\eta] \cdot v\,\big) \;=\; 0\,, \tag{2.6}$$

for all $v \in M_X$ and all meromorphic vectors fields η on X that have poles only at $z_1, \dots, z_n$. The operator $T[\eta]$ is defined as

$$T[\eta] \;=\; \sum_{k=1}^{n} \sum_{n\in\mathbb{Z}} \eta_n^{(k)} L_n^{(k)}\,, \qquad L_n^{(k)} = \mathrm{id} \otimes \dots \otimes \underset{(\mathrm{k-th})}{L_n} \otimes \dots \otimes \mathrm{id}\,, \tag{2.7}$$

where the $\eta_n^{(k)}$ are the Laurent expansion coefficients of η near P_k, $\eta(t_k) = \sum_{n\in\mathbb{Z}} \eta_n^{(k)}\, t_k^{n+1}$.

The definition of conformal blocks for a general conformal vertex algebra V is given in [FBZ]. In addition to the conformal invariance condition formulated in Definition 1 one imposes conditions which express invariance w.r.t. the symmetries generated by the other currents that generate the vertex algebra V.

We then denote by $\mathcal{C}_V(X, R)$ the space of all conformal blocks associated to a vertex algebra V, a Riemann surface X and the assignment R of a representation M_k to each of the marked points z_k on X.

2.4.2 Insertions of the vacuum representation

Let us consider the case that one of the marked points $z_0, \dots, z_n$ is decorated by the vacuum representation V. If e.g. $R_0 = V$ we may compare the space $\mathcal{C}_V(X, R)$ to the space $\mathcal{C}_V(X', R')$ where X' is the Riemann surface obtained from X by "filling" the marked point z_0, and with representations $R_k \in \mathrm{Rep}(V)$, $k = 1, \dots, n$ assigned to the marked points $z_1, \dots, z_n$, respectively. It can then be shown that the spaces $\mathcal{F}(X, R)$ and $\mathcal{F}(X', R')$ are canonically isomorphic [FBZ, Theorem 10.3.1]. The isomorphism is defined by demanding that

$$F'(v) \;=\; F(v_\circ \otimes v)\,, \tag{2.8}$$

holds for all $v \in \bigotimes_{k=1}^n R_k$. In other words: Insertions of the vacuum do not change the space of conformal blocks. This innocent looking fact will be referred to as the "propagation of vacua". It has important consequences.

2.4.3 Deformations of the complex structure of X

A key point that needs to be understood about spaces of conformal blocks is the dependence on the complex structure of X. There is a canonical way to represent infinitesimal variations of the complex structure on the spaces of conformal blocks. By combining the definition of conformal blocks with the so-called "Virasoro uniformization" of the moduli space $\mathfrak{M}_{g,n}$ one may construct a representation of infinitesimal motions on $\mathfrak{M}_{g,n}$ on the space of conformal blocks.

The "Virasoro uniformization" of the moduli space $\mathfrak{M}_{g,n}$ may be formulated as the statement that the tangent space $T\mathfrak{M}_{g,n}$ to $\mathfrak{M}_{g,n}$ at X can be identified with the double quotient

$$T\mathfrak{M}_{g,n} \;=\; \Gamma\big(X\setminus\{x_1,\dots,x_n\},\Theta_X\big)\;\Big\backslash\;\bigoplus_{k=1}^{n}\mathbb{C}(\!(t_k)\!)\partial_k\Big/\bigoplus_{k=1}^{n}\mathbb{C}[\![t_k]\!]\partial_k\,, \tag{2.9}$$

where $\Gamma(X\setminus\{x_1,\dots,x_n\},\Theta_X)$ is the set of vector fields that are holomorphic on $X\setminus\{x_1,\dots,x_n\}$, while $\mathbb{C}(\!(t_k)\!)$ and $\mathbb{C}[\![t_k]\!]$ are formal Laurent and Taylor series respectively.

Let us then consider $F(T[\eta]\cdot v)$ with $T[\eta]$ being defined in (2.7) in the case that $\eta\in\bigoplus_{k=1}^{n}\mathbb{C}(\!(t_k)\!)\partial_k$ and $L_rv_k=0$ for all $r>0$ and $k=1,\dots,n$. The defining invariance property (2.6) together with $L_rv_k=0$ allow us to define

$$\delta_\vartheta F(v) = F(T[\eta_\vartheta]\cdot v)\,, \tag{2.10}$$

where δ_ϑ is the derivative corresponding a tangent vector $\vartheta\in T\mathfrak{M}_{g,n}$ and η_ϑ is any element of $\bigoplus_{k=1}^{n}\mathbb{C}(\!(t_k)\!)\partial_k$ which represents ϑ via (2.9). Generalizing these observations one is led to the conclusion that derivatives w.r.t. to the moduli parameters of $\mathfrak{M}_{g,n}$ are (projectively) represented on the space of conformal blocks, the central extension coming from the central extension of the Virasoro algebra (2.3).

It is natural to ask if the infinitesimal motions on $\mathfrak{M}_{g,n}$ defined above can be integrated. The space of conformal blocks would then have the structure of a holomorphic vector bundle with a projectively flat connection[3]. This would in particular imply that locally on $\mathfrak{M}_{g,n}$ one may define families of conformal blocks $X\to F_X$ such that the functions $X\to F_X(v)$ depend holomorphically on the complex structure μ on X.

Examples where this property has been established in full generality are somewhat rare, they include the WZNW-models, the minimal models and certain classes of rational conformal field theories in genus zero. However, from a physicists point of view, a vertex algebra whose conformal blocks do not have this property is pathological. We are not going to assume integrability of the canonical connection in the following.

[3] Projective flatness means flatness up to a central element.

2.5 Correlation functions vs. hermitian forms

Let us return to our original problem, the problem to construct correlation functions $\langle V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\rangle_X$. Assuming a holomorphically factorized structure as in (2.2), it seems natural to identify $\mathbb{F}_{\hat{\Sigma}}$ with an index set for a "basis"[4] $\{\mathcal{F}_S(v;X); S\in\mathbb{F}_{\hat{\Sigma}}\}$ for a subspace $\mathfrak{F}(\hat{\Sigma})$ of the space of solutions to the conformal Ward identities (2.6) that is defined as follows.

Let us focus attention on the dependence of $\mathcal{F}_S(v;X)$ w.r.t. the label S by using the notation $f_{v,X}(S)\equiv\mathcal{F}_S(v;X)$. The measure $d\mu_{\hat{\Sigma}}$ on $\mathbb{F}_{\hat{\Sigma}}\times\mathbb{F}_{\hat{\Sigma}}$ introduced in (2.2) allows one to consider the space $\mathfrak{F}(\hat{\Sigma})$ of functions on $\mathbb{F}_{\hat{\Sigma}}$ such that

$$\int\limits_{\mathbb{F}_{\hat{\Sigma}}\times\mathbb{F}_{\hat{\Sigma}}} d\mu_{\hat{\Sigma}}(S,S')\;(f(S))^*f(S')\;<\;\infty\,. \tag{2.11}$$

By definition, the space $\mathfrak{F}(\hat{\Sigma})$ comes equipped with a hermitian form $H_{\hat{\Sigma}}$ which allows one to represent the correlation functions in the form

$$\big\langle V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\big\rangle_X\;=\;H_{\hat{\Sigma}}(f_{v',X},f_{v,X})\,. \tag{2.12}$$

The elements of the space $\mathfrak{F}(\hat{\Sigma})$ are identified with elements of a subspace of the space of conformal blocks by associating to each $f\in\mathfrak{F}(\hat{\Sigma})$ a solution $\mathcal{F}_f$ to the conformal Ward identities via

$$\mathcal{F}_f(v;X)\;\equiv\int\limits_{\mathbb{F}_{\hat{\Sigma}}\times\mathbb{F}_{\hat{\Sigma}}} d\mu_{\hat{\Sigma}}(S,S')\;(f(S))^*\,\mathcal{F}_{S'}(v;X)\,. \tag{2.13}$$

We are therefore confronted with the task to construct suitable hermitian forms on subspaces of the space of conformal blocks which allow us to represent the correlation functions in the form (2.12).

3. Behavior near the boundary of moduli space

It is of particular importance for most applications of CFT within physics to understand the behavior of correlation functions near the boundaries of the moduli space $\mathfrak{M}(\Sigma)$ of complex structures on a given two-dimensional surface Σ. Such boundaries may be represented by surfaces on which a closed geodesic c was shrunk to zero length, thereby pinching a node. Two cases may arise:

(A) Cutting X along c produces two disconnected surfaces X_1 and X_2 with boundary.

[4] Possibly in the sense of generalized functions.

(B) Cutting X along c produces a connected surface X' with boundary whose genus is smaller than the genus of X (“pinching a handle”).

In the following we will propose certain assumptions which ensure existence of an interpretation of the CFT in question as a quantum field theory with Hilbert space

$$\mathfrak{H}_{\rm CFT} \,=\, \int_{\mathbb{U}^2}^{\oplus} d\mu(r,r')\; R_r \otimes R_{r'}\,. \tag{3.1}$$

These assumptions may be loosely formulated as follows.

In case (A) it is required that there exists a representations of the correlation functions by “inserting complete sets of intermediate states”, schematically

$$\big\langle\, V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\,\big\rangle_X \,=\, \sum_{v\in\mathcal{B}} \,\big\langle\, 0\,|\,\mathsf{O}_{X_1}\, q^{L_0}\bar{q}^{\bar{L}_0}\,|\,v\,\big\rangle_{X_1}\big\langle\, v\,|\,\mathsf{O}_{X_2}\,|\,0\,\big\rangle_{X_2}\,, \tag{3.2}$$

where $\mathsf{O}_{X_r}:\mathfrak{H}_{\rm CFT}\to\mathfrak{H}_{\rm CFT}$, $r=1,2$ are certain operators associated to the surfaces X_1 and X_2, respectively, and the summation is extended over the vectors v which form a basis $\mathcal{B}$ for the space of states $\mathfrak{H}_{\rm CFT}$ of the conformal field theory in question.

In case (B) it is required that there exists a representations of the correlation functions as a trace

$$\big\langle\, V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\,\big\rangle_X \,=\, {\rm tr}_{\mathfrak{H}_{\rm CFT}}\big(q^{L_0}\bar{q}^{\bar{L}_0}\mathsf{O}_{X'}\big)\,, \tag{3.3}$$

where $\mathsf{O}_{X'}:\mathfrak{H}_{\rm CFT}\to\mathfrak{H}_{\rm CFT}$ is a certain operator associated to the surface X'.

One may formulate these two conditions more precisely by demanding that the conformal blocks which appear in (2.2) can be obtained by the gluing construction that we will now describe in more detail.

3.1 Gluing of Riemann surfaces

For the following it will be more convenient to consider Riemann surfaces whose boundary components are represented by holes with parametrized boundaries rather than marked points with choices of local coordinates around them. Conformal invariance allows to relate the two ways of representing boundary components, see [RaS] for a mathematical discussion of some of the issues involved.

3.1.1

Let X' be a possibly disconnected Riemann surface with $2m+n$ parametrized boundaries which are labelled as $C_1^+,C_1^-,\dots,C_m^+,C_m^-,B_1,\dots,B_n$. We will assume that the parametrizations of the boundaries $C_r^\pm$, $r = 1, \dots, m$ extend holomorphically to give coordinates $t_r^\pm$ for annular neighborhoods $A_r^\pm$ of $C_r^\pm$ such that the boundaries $C_r^\pm$ are represented by the circles $|t_r^\pm| = 1$, while the coordinates of points in the interior of $A_r^\pm$ satisfy $|t_r^\pm| < 1$. We will furthermore assume that the annuli $A_r^\pm$ are mutually non-intersecting.

We may then define a new Riemann surface X by identifying all the points P_1, P_2 which satisfy

$$t_r^+(P_1)\, t_r^-(P_2) \,=\, q_r \,, \tag{3.4}$$

for given complex numbers q_r such that $|q| < 1$ and all $r = 1, \dots, m$. The annuli $A_r^\pm$ are thereby mapped to annuli A_r embedded into the new surface X.

One may apply this construction to a family X'_t of surfaces of the kind above with a set of parameters collectively denoted $t = (t_1, \dots, t_k)$. This yields a family $X_{q,t}$ of Riemann surfaces that is labelled by the $m+k$ parameters $q = (q_1, \dots, q_m)$ and t. The family of surfaces obtained in this way contains the nodal surfaces X_d which are obtained when at least one of the q_r equals zero. If X'_t is stable, i.e. if its disconnected components all have a number n of punctures larger than $2 - 2g$, one gets the nodal surfaces X_d that represent the points of the Deligne-Mumford compactification $\overline{\mathfrak{M}}(\Sigma)$ of the moduli space $\mathfrak{M}(\Sigma)$ of complex structures on surfaces Σ homotopic to X.

3.1.2

It will be important for us to notice that there exists a universal family of this kind: A family X_p of surfaces such that for any other family Y_q of surfaces which contains a nodal surface Y_{q_0} isomorphic to X_d there exists a holomorphic map $p = \varphi(q)$, defined in some neighborhood of the point q_0, such that Y_q and $X_{\varphi(q)}$ are isomorphic (related by a holomorphic map).

More precisely let us consider families $\pi_{\mathcal{U}} : \mathcal{X} \to \mathcal{U}$ of surfaces degenerating into a given nodal surface X_d. This means that π is holomorphic and that $X_p \equiv \pi^{-1}(p)$ is a possibly degenerate Riemann surface for each point p in a neighborhood $\mathcal{U}$ of the boundary component $\partial\overline{\mathfrak{M}}_{g,n}$ containing the nodal surface X_d. Following [RoS] we will call families $\pi_{\mathcal{U}} : \mathcal{X} \to \mathcal{U}$ as above an unfolding of the degenerate surface X_d. The surface X_d is called the central fiber of the unfolding $\pi_{\mathcal{U}}$.

Let us call a family $\pi_{\mathcal{U}} : \mathcal{X} \to \mathcal{U}$ universal if for any other family $\pi_{\mathcal{V}} : \mathcal{Y} \to \mathcal{V}$ which has a central fiber Y_d isomorphic to X_d, there exists a unique extension of the isomorphism $f : Y_d \to X_d$ to a pair of isomorphisms (holomorphic maps) (φ, ϕ), where $\varphi : \mathcal{V} \to \mathcal{U}$ and $\phi : \mathcal{Y} \to \mathcal{X}$ such that $\pi_{\mathcal{U}} \circ \phi = \varphi \circ \pi_{\mathcal{V}}$.

Theorem 1. — [RS] —
A nodal punctured Riemann surface X_d admits a universal unfolding if and only if it is stable,

i.e. iff $n > 2 - 2g$.

It is no loss of generality to assume that the family surfaces X_p are obtained by the gluing construction, so $X_p \equiv X_{q,t}$.

3.1.3

It is possible to apply the gluing construction in the cases where $X' = \coprod_{p=1}^{2g-2+n} S_p$ is the disjoint union of three-holed spheres. One thereby gets families of surfaces X_q parametrized only by the gluing parameters $q = (q_1, \ldots, q_m)$ with m being given as $m = 3g - 3 + n$. The different possibilities to get surfaces X by gluing three punctured spheres can be parametrized by the choice of a cut system, i.e. a collection $\mathrm{c} = \{c_1, \ldots, c_{3g-3+n}\}$ of nonintersecting simple closed curves on X. X' is reobtained by cutting X along the curves c_r, $r = 1, \ldots, m$. The coordinates $q = (q_1, \ldots, q_m)$ parametrize a neighborhood $\mathcal{U}_c$ of the point in $\overline{\mathfrak{M}}_{g,n}$ represented by the maximally degenerate surface X_c corresponding to $q = 0$. It is known [M, HV] that one may cover all of $\overline{\mathfrak{M}}_{g,n}$ with the coordinate patches $\mathcal{U}_c$ if one considers all possible cut systems c on X.

One may similarly use the coordinates $\tau = (\tau_1, \ldots, \tau_m)$ such that $q_r = e^{-\tau_r}$ as system of coordinates for subsets of the Teichmüller space $\mathfrak{T}_{g,n}$. One should note, however, that the coordinates τ are not unambiguously determined by the cut system c. Indeed, the coordinates τ' obtained by $\tau'_r = \tau_r + 2\pi i k_r$, $k_r \in \mathbb{Z}$, would equally well do the job.

In order to resolve this ambiguity, one may refine the cut system c by introducing a marking σ of X, a three-valent graph on X such that each curve of the cut system intersects a unique edge of σ exactly once.

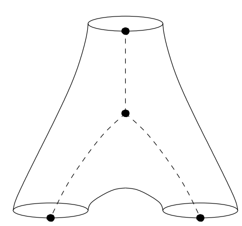

Figure 1: Standard marking of a three holed sphere.

With the help of the graph σ one may then define a "fundamental domain" $\mathcal{V}_\sigma$ for the variables τ as follows: Let us introduce a standard graph like the one depicted in Figure 1 on each of the three-holed spheres S_p. We may assume that our coordinates are such that the standard graphs in the three holed spheres intersect the boundaries of the annuli A_r in the points $P_r^{\pm}$ given by $t_r^{\pm}(P_r^{\pm}) = |q|$, respectively. On each annulus consider the curve $\gamma_r : [0, 1] \to A_r$,

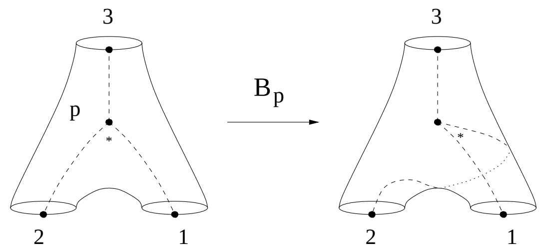

Figure 2: The B-move

where $t_r^+(\gamma(\theta)) = |q|e^{\mp\theta}$. The curve γ_r connects the points $P_r^{\pm}$, winding around the annulus A_r a number of times specified by the integer part of $\mathrm{Im}(\tau)/2\pi$. We thereby get a three valent connected graph $\sigma'(\tau)$ on X. The fundamental domain $\mathcal{V}_\sigma$ for the coordinates τ may then be defined by the requirement that the graph $\sigma'(\tau)$ is homotopic to the given graph σ.

3.1.4

It is clear from the definitions that each marking σ determines a cut system $c = c(\sigma)$. The set $\mathcal{M}$ of markings may be regarded as a cover $c : \mathcal{M} \to \mathcal{C}$ of the set $\mathcal{C}$ of all cut systems. The subgroup $\mathrm{MCG}(\Sigma)_c$ of the mapping class group which preserves a cut system c acts transitively on the set of all markings which correspond to the same cut system. The group $\mathrm{MCG}(\Sigma)_c$ is generated by the Dehn twists along the curves c_r, $r = 1, \dots, 3g-3+n$ representing the cut system, together with the braiding transformations of the three holed holed spheres obtained by cutting X along the curves c_r, $r = 1, \dots, 3g-3+n$. An example for the braiding transformations is graphically represented in Figure 2.

3.2 Gluing of conformal blocks

3.2.1

Let us keep the set-up from paragraph 3.1.1. Let R be an assignment of representations of a vertex algebra V to the boundary components of X_t' which is such that representation R_r^- assigned to boundary component C_r^- is the dual of the representation R_r^+ assigned to boundary component C_r^+ for all $r = 1, \dots, m$. Let $F_t \in \mathcal{C}_V(X_t'; R)$ be a family of conformal blocks assigned to the family X_t of surfaces with an assignment R of representations to the boundary components of X_t as above. Let finally

$$e_\tau \equiv \bigotimes_{r=1}^{m} e_r(\tau_r)\,, \qquad e_r(\tau_r) \equiv \sum_{\imath \in \mathcal{I}_r} e_{r,\imath}^- \otimes e^{-\tau_r L_{0,r}} e_{r,\imath}^+\,,$$

where $\{e^+_{r,\imath};\imath\in\mathcal{I}_r\}$ and $\{e^-_{r,\imath};\imath\in\mathcal{I}_r\}$ are bases for R^+_r and R^-_r, respectively, which are dual to each other in the sense that $\langle e^-_{r,\imath},e^+_{r,\jmath}\rangle_{R^+_r}=\delta_{\imath,\jmath}$, with $\langle.,.\rangle_{R^+_r}$ being the dual pairing.

We may then consider the expression

$$G_{t,\tau}(v)=F_t(v\otimes e_\tau)\,, \tag{3.5}$$

where $v_1\otimes\ldots\otimes v_n$. As it stands, the expression is defined as a formal power series in powers of $e^{-\tau_r}$.

3.2.2

To proceed, we will need to introduce a nontrivial assumption:

> The series in (3.5) have a finite radius of convergence. The resulting domains of definition $\mathcal{D}_\sigma$ of the conformal blocks $G_{t,\tau}$ cover the neighborhoods $\mathcal{U}_{c(\sigma)}\subset\overline{\mathfrak{M}}_{g,n}$ which form an atlas of $\overline{\mathfrak{M}}_{g,n}$ according to [M, HV].

This assumption about the analyticity of the conformal blocks $G_{t,\tau}$ obtained from the gluing construction is fundamental for most of the rest. On the other hand it is not yet clear how large the class of vertex algebras is for which they are satisfied.

For rational CFT it is to be expected that the conformal blocks satisfy a closed system of differential equations which allow one to show convergence and existence of an analytic continuation. The situation is more difficult in the case of nonrational CFT. Validity of the assumption above is known [TL] in the case of the Virasoro algebra for surfaces with $g=0$. This leads one to suspect that the assumption above should be valid in genereral, as it should always be possible to decompose the conformal blocks of a vertex algebra V into Virasoro conformal blocks. This issue is further discussed in Remark 1 below.

3.2.3 Conformal Ward identities

We want to check that the expression in (3.5) satisfies the conformal Ward identities. To this aim it suffices to notice that η can be split as the sum of $\eta_{\rm out}$ and $\eta_{\rm in}$, where $\eta_{\rm out}$ is holomorphic in X', and $\eta_{\rm in}$ is holomorphic in $\bigcup_{r=1}^{3g-3+n}A_r$. This means that $G_{t,\tau}(T[\eta]v)$ can be represented as

$$G_{t,\tau}\big(T[\eta]v\big)\;=\;F_t\big(T[\eta_{\rm out}](v\otimes e_\tau)\big)+F_t\big(v\otimes T[\eta_{\rm in}]e_\tau\big)\,. \tag{3.6}$$

We have $F_t(T[\eta_{\rm out}]w)\,=\,0$ for all $\eta_{\rm out}$ that are holomorphic in X'_t by the conformal Ward identities satisfied by F_t. It is furthermore possible to show that the vectors $e_r(\tau_r)$ are invariant

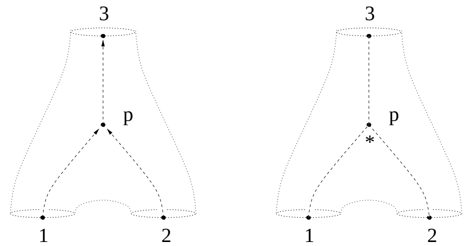

Figure 3: Two representations for the decoration on a marking graph

under the action of the holomorphic maps $\eta_{\rm in}^r \equiv \eta_{\rm in}|_{A_r}$,

$$T[\eta_{\rm in}^r]e_r(\tau_r) \,=\, 0\,, \qquad r=1,\dots,m\,.$$

The conformal Ward identities follow from the combination of these two observations.

Keeping in mind Theorem 1 from [RoS] it follows from the conformal Ward identities that the conformal blocks defined in (3.5) do not depend on the choices involved in the gluing construction. The resulting families $G_{t,\tau}$ of conformal blocks are therefore well-defined in a neighborhood $\mathcal{V}_\sigma$ of the component of $\partial\mathfrak{T}_{g,n}$ that is obtained by $\tau_r\to\infty$ for $r=1,\dots,m$.

It seems to the author that the foundational importance for CFT of the work [RoS] was hitherto not as widely appreciated as it should be. It explains in particular why it is absolutely preferable to formulate CFT in terms of stable surfaces.

3.2.4 Decorated marking graphs

In the gluing construction above one assigns elements of a basis for the space of conformal blocks of the three punctured sphere to each vertex of a marking graph. The definition of a basis for this space will generically depend on the choice of a distinguished boundary component of the three punctured sphere in question. In order to parametrize different bases in spaces of conformal blocks one needs to choose for each vertex $p\in\sigma_0$ a distinguished edge emanating from it. The distinguished edge will sometimes be called “outgoing”. As a convenient graphical representation we will use the one introduced in Figure 3. The term marking will henceforth be used for graphs σ as defined above together with the choice of a distinguished edge emanating from each vertex.

3.2.5

The following data label the conformal blocks on X that can be constructed by means of the gluing construction. We will denote by σ_0 and σ_1 the sets of vertices and edges of σ, respectively.

- The marking σ.
- An assignment ρ of representation labels $r_e \in \mathbb{U}$ to the edges $e \in \sigma_1$.
- An assignment w of conformal blocks $w_p \in \mathcal{C}_V(S_p, \rho_p)$ to each vertex $p \in \sigma_0$ of σ with assignment ρ_p of representations to the edges that emanate from p determined by ρ.

We will use the notation $G_{\sigma\tau}(\delta)$ for the family of conformal blocks which is essentially uniquely defined by the data σ and the "decoration" $\delta = (\rho, w)$.

3.3 Correlation functions

We are now in the position to formulate our requirements concerning the behavior of the correlation functions $\langle V_n(z_n, \bar{z}_n) \dots V_1(z_1, \bar{z}_1)\rangle_X$ near the boundary of moduli space more precisely. Let us consider a marking σ. The marking σ determines a maximally degenerate surface $X_{c(\sigma)}$ and a neighborhood $\mathcal{U}_{c(\sigma)} \subset \overline{\mathfrak{M}}_{g,n}$ of $X_{c(\sigma)}$. Let us recall the set-up from subsection 2.5, in particular the representation (2.12). We will adopt the following two requirements:

3.3.1 First requirement: Factorization of conformal blocks

The functions $\mathcal{F}_S(v; X)$ can be identified with the values $G_{\sigma\tau}(\delta|v)$ of the conformal blocks obtained from the gluing construction provided that a suitable identification between the labels S and the decorations δ is adopted. This means in particular that the space $\mathbb{F}_{\hat{\Sigma}}$ of all labels S should be identified with the *sub*-space $\mathcal{I}_{\sigma,R}$ of the space $\mathcal{I}_\sigma$ of all decorations δ which contains all decorations δ_R with fixed assignment R of representations to the external egdes[5] of σ.

The space $\mathcal{I}_\sigma$ of all decorations can be described more explicitly as

$$\mathcal{I}_\sigma \equiv \prod_{\rho \in \mathbb{U}^{\sigma_1}} \bigotimes_{p \in \sigma_0} \mathcal{I}_p(\rho_p) \,, \tag{3.7}$$

where $\mathbb{U}^{\sigma_1}$ is the vector space of all assignments ρ of elements $r_e \in \mathbb{U}$ to the edges $e \in \sigma_1$ and $\mathcal{I}_p(\rho_p)$ is the index set for a basis in $\mathcal{C}(S_p, \rho_p)$, $p \in \sigma_0$.

The role of the functions $f_{v,X}$ from subsection 2.5 is then taken by the functions $E^R_{\sigma\tau}(v)$ which map a decoration $\delta_R \in \mathcal{I}_{\sigma,R}$ to $G_{\sigma\tau}(\delta_R|v)$. We will later find it more convenient to consider the functions $E_{\sigma\tau}(v) : \mathcal{I}_\sigma \to \mathbb{C}$ which map an *arbitrary* decoration $\delta \in \mathcal{I}_\sigma$ to $G_{\sigma\tau}(\delta|v)$. This is natural in view of the fact that the assignment R of representations to boundary components is part of the data contained in the decoration δ. The space of all complex-valued functions on $\mathcal{I}_\sigma$ will be denoted $\mathfrak{F}_\sigma$.

[5] The edges ending in the boundary components of Σ

3.3.2 Second requirement: Factorization of the hermitian form $H_{\hat{\Sigma}}$

In order to have a representation for the correlation functions in the form of (2.12) there must exist a hermitian form $H_\sigma^{R'R}$ on suitable subspaces of $\mathfrak{F}_\sigma \times \mathfrak{F}_\sigma$ such that

$$\big\langle V_n(z_n,\bar{z}_n)\dots V_1(z_1,\bar{z}_1)\big\rangle_X \,=\, H_\sigma^{R'R}\big(\,E_{\sigma\tau}(v')\,,\,E_{\sigma\tau}(v)\,\big)\,. \tag{3.8}$$

Our second main requirement is that the hermitian form $H_\sigma^{R'R}$ can be factorized as

$$H_\sigma^{R'R} \,=\, \int\limits_{\mathbb{U}^{\sigma_1}\times\mathbb{U}^{\sigma_1}} d\mu_{R'R}(\rho',\rho)\,\bigotimes_{p\in\sigma_0} H_p^{\rho'_p\rho_p}\,, \tag{3.9}$$

where

- the measure $d\mu_{R'R}$ has support only when the assignment of representations to the external edges given by ρ' and ρ coincides with R' and R, respectively, and

- the hermitian forms $H_p^{\rho'_p\rho_p}$ are defined on certain subspaces of the spaces $\mathfrak{F}(\rho'_p)\times\mathfrak{F}(\rho_p)$ of complex-valued functions on $\mathcal{I}(\rho'_p)\times\mathcal{I}(\rho_p)$.

Assuming that the hermitian forms $H_p^{\rho'\rho}$ can be represented in the form

$$H_p^{\rho'\rho}(f,g) \,=\, \sum_{\imath',\imath\in\mathbb{I}} (f(\imath'))^*\, H_{\imath'\imath}(\rho',\rho)\, g(\imath)\,, \tag{3.10}$$

one may in particular represent the three point functions $\langle V_3(y_3,\bar{y}_3)V_2(y_2,\bar{y}_2)V_1(y_1,\bar{y}_1)\rangle_{S_{0,3}}$ as

$$\begin{aligned}\big\langle V_3(\infty)V_2(1)V_1(0)\big\rangle_{S_{0,3}} \,&= \\ =\sum_{\imath',\imath\in\mathbb{I}} H_{\imath'\imath}(\rho',\rho)\; &G_{\imath'}(\,\rho'\,|\,v'_3\otimes v'_2\otimes v'_1)\; G_\imath(\,\rho\,|\,v_3\otimes v_2\otimes v_1)\,.\end{aligned} \tag{3.11}$$

In (3.11) we have used the standard model for $S_{0,3}$ as $\mathbb{P}^1\setminus\{0,1,\infty\}$ and assumed that $y_3=\infty$, $y_2=1$, $y_1=0$. $\{G_{\imath'}(\rho');\imath'\in\mathcal{I}(\rho')\}$ and $\{G_\imath(\rho);\imath\in\mathcal{I}(\rho)\}$ are bases for $\mathcal{C}(S_{0,3}.\rho')$ and $\mathcal{C}(S_{0,3}.\rho)$, respectively.

3.4 Conformal blocks as matrix elements

For future use, let us note that in the case that X has genus zero there is a convenient reformulation of the gluing construction of the conformal blocks in terms of chiral vertex operators.

3.4.1 Chiral vertex operators

The chiral vertex operators are families of operators $\mathsf{Y}^{r_2}_{r_3r_1}(\mathfrak{v}_2|z)$, $\mathfrak{v}_2 \in R_2$, $z \in \mathbb{C} \setminus \{0\}$ mapping the representation R_{r_1} to the dual $\bar{R}_{r_3}$ of R_{r_3}. They are defined such that the conformal blocks $G^{(3)}$ associated to the three punctured sphere $S^{(3)}$ are related to the matrix elements of $\mathsf{Y}^{r_2}_{r_3r_1}(\mathfrak{v}_2|z)$ as

$$F^{(3)}(\,\rho\,|\,\mathfrak{v}_3 \otimes \mathfrak{v}_2 \otimes \mathfrak{v}_1) \,=\, \big\langle\, \mathfrak{v}_3\,,\, \mathsf{Y}^{r_2}_{r_3r_1}(\mathfrak{v}_2|\,1\,)\,\mathfrak{v}_1\,\big\rangle_{R_3}\,. \tag{3.12}$$

It is assumed that the assignment $\rho : k \mapsto r_k$, $k \in \{1, 2, 3\}$ is in correspondence to the numbering of boundary components introduced in Figure 3. A simplified diagrammatical representation is introduced in Figure 4.

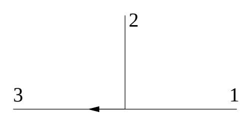

Figure 4: Diagrammatical representation for chiral vertex operators.

The chiral vertex operators $\mathsf{Y}^{r_2}_{r_3r_1}(\mathfrak{v}_2|z)$ are well-defined by (3.12) in the sense of formal power series in z.

3.4.2

There are two different ways to glue two decorated three-punctured spheres such that the outgoing boundary component of the first is glued to one of the incoming boundary components of the other. To each of the two gluing patterns we may associate two natural ways to compose chiral vertex operators, namely

$$\mathsf{Y}^{r_3}_{r_4r_{21}}(\,\mathfrak{v}_3\,|\,1\,)\mathsf{Y}^{r_2}_{r_{21}r_1}(\,\mathfrak{v}_2\,|\,z\,)\,\mathfrak{v}_1 \quad \text{and} \quad \mathsf{Y}^{r_{32}}_{r_4r_1}\big(\,\mathsf{Y}^{r_3}_{r_{32}r_2}(\,\mathfrak{v}_3\,|\,1-z\,)\,\mathfrak{v}_2\,|\,1\,\big)\,\mathfrak{v}_1\,, \tag{3.13}$$

respectively. A diagrammatic representation for these two ways to compose chiral vertex operators is given in Figure 5, respectively.

Figure 5: Diagrammatic representation for the compositions in (3.13).

Let us call a marking σ on a surface X of genus 0 admissible if the outgoing boundary components of one pair of pants are always glued to an incoming boundary component of another. The resulting markings σ distinguish a unique outgoing boundary component of the surface X. We will assign the representation R_n to this boundary component. Using the rule for compositions of chiral vertex operators recursively we can construct an operator $\mathsf{Y}_{X,\sigma} : R_{n-1}\otimes\ldots\otimes R_1 \to \bar{R}_n$ such that the conformal block associated to σ can be represented as

$$G_\sigma(\mathfrak{v}_n \otimes \ldots \otimes \mathfrak{v}_1) = \big\langle \mathfrak{v}_n \, , \, \mathsf{Y}_{X,\sigma}(\mathfrak{v}_{n-1} \otimes \ldots \otimes \mathfrak{v}_1) \big\rangle_{R_n} \, ,$$

for all $\mathfrak{v}_k \in R_k$, $k = 1, \ldots, n$. The fact that the matrix elements above represent the conformal blocks follows from the observation that the composition of chiral vertex operators is equivalent to the gluing operatation.

Remark 1. The issues of convergence of the power series defined by the gluing construction and convergence of the power series representing the chiral vertex operators are closely related. In rational CFT one can settle this issue with the help of the differential equations satisfied by the conformal blocks, see e.g. [TK]. In the case of the Virasoro algebra one may use analytic arguments for deriving such results [TL]. The typical situation seems to be that the power series representing, for example, $\langle \mathfrak{v}_{n+1}, \mathsf{Y}^{r_n}_{r_{n+1}s_{n-1}}(\, \mathfrak{v}_n \,|\, z_n \,) \ldots \mathsf{Y}^{r_1}_{s_1 r_0}(\, \mathfrak{v}_1 \,|\, z_1 \,)\, \mathfrak{v}_0 \rangle$ converge provided that the variables z_k are *radially* ordered $|z_n| > \cdots > |z_1|$.

4. From one boundary to another

For a given pair of markings σ_2, σ_1 it may happen that the domains $\mathcal{D}_{\sigma_r}$, $r = 1, 2$ in which the corresponding conformal blocks can be defined by means of the gluing construction have a nontrivial overlap, $\mathcal{D}_{\sigma_2} \cap \mathcal{D}_{\sigma_1} \neq \emptyset$. Assume that τ_1 and τ_2 parametrize the same point in $\mathcal{D}_{\sigma_2} \cap \mathcal{D}_{\sigma_1} \subset \mathfrak{T}_{g,n}$. We then have two possible ways to represent the correlation function $\langle \, V_n(z_n, \bar{z}_n) \ldots V_1(z_1, \bar{z}_1) \, \rangle_X$ in the form (3.8), namely either as $H_{\sigma_1}(G_{\sigma_1\tau_1}, G_{\sigma_1\tau_1})$ or as $H_{\sigma_2}(G_{\sigma_2\tau_2}, G_{\sigma_2\tau_2})$, respectively. It is a natural requirement to demand that these two representations agree,

$$H_{\sigma_1}\big(\, G_{\sigma_1\tau_1} \, , \, G_{\sigma_1\tau_1} \, \big) \, = \, H_{\sigma_2}\big(\, G_{\sigma_2\tau_2} \, , \, G_{\sigma_2\tau_2} \, \big) \, . \tag{4.1}$$

These constraints generalize what is often called crossing symmetry, locality and modular invariance. Keeping in mind our assumption that the domains $\mathcal{D}_{\sigma_r}$ cover the neighborhoods $\mathcal{U}_{c(\sigma)} \subset \overline{\mathfrak{M}}_{g,n}$ which form an atlas of $\overline{\mathfrak{M}}_{g,n}$ we arrive at an unambiguous definition of the correlation functions on all of $\mathfrak{M}_{g,n}$.

In order to analyze the conditions further, we need to introduce another assumption:

The families $G_{\sigma_1\tau_1}$ and $G_{\sigma_2\tau_2}$ are linearly related.

This assumption will be formulated more precisely below. It is, on the one hand, absolutely necessary for the validity of (4.1) at least in the case of rational CFT [MS1][6]. The assumption above is, on the other hand, a rather deep statement about a given vertex algebra V from the mathematical point of view, especially if V is not rational.

What simplifies the analysis somewhat is the fact that the relations between pairs of markings can be reduced to a few simple cases associated to Riemann surfaces of low genus $g = 0, 1$ and low number of marked points $n \leq 4$. In order to explain how this reduction works we will begin by briefly reviewing the necessary results from Riemann surface theory.

4.1 The modular groupoid

The physical requirements above boil down to understanding the relations between conformal blocks associated to pairs $[\sigma_2, \sigma_1]$ of markings. In order to break down understanding such relations to a sort of Lego game it will be very useful to observe that all transitions between two markings can be factorized into a simple set of elementary moves. One may formalize the resulting structure by introducing a two-dimensional CW complex $\mathcal{M}(\Sigma)$ with set of vertices $\mathcal{M}_0(\Sigma)$ given by the markings σ, set of edges $\mathcal{M}_1(\Sigma)$ associated to the elementary moves. It will then be possible to identify the set $\mathcal{M}_2(\Sigma)$ of "faces" (relations between the elementary moves) that ensure simply-connectedness of $\mathcal{M}(\Sigma)$, as we are now going to describe in more detail.

4.1.1 Generators

The set of edges $\mathcal{M}_1(\Sigma)$ of $\mathcal{M}(\Sigma)$ will be given by elementary moves denoted as (pq), Z_p, B_p, F_{pq} and S_p. The indices $p, q \in \sigma_0$ specify the relevant three holed spheres within the pants decomposition of Σ that is determined by σ. The move (pq) will simply be the operation in which the labels p and q get exchanged. Graphical representations for the elementary moves Z_p, B_p, F_{pq} and S_p are given in Figures 6, 2, 7 and 8, respectively.

Proposition 1. [BK2] *Any two markings σ, σ' can be connected to each other by a path composed out of the moves (pq), Z_p, B_p, F_{pq} and S_p.*

4.1.2 Relations

It is useful to notice that *any* round trip can be broken up into elementary ones. A complete list of these elementary round trips was first presented in [MS1], see [BK2] for a mathematical

[6] It seems likely that the argument in [MS1] can be generalized to the nonrational case if there exists an analytic continuation of the correlation functions from the euclidean section $\bar{\tau} = \tau^*$ to functions analytic in independent variables $\tau, \bar{\tau}$.

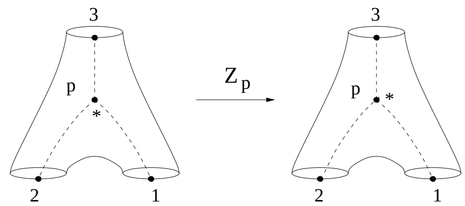

Figure 6: The Z-move

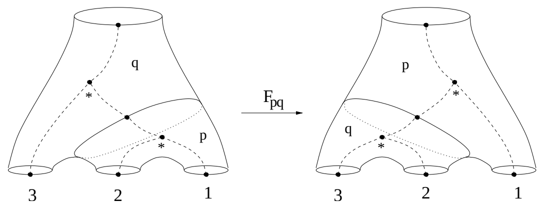

Figure 7: The F-move

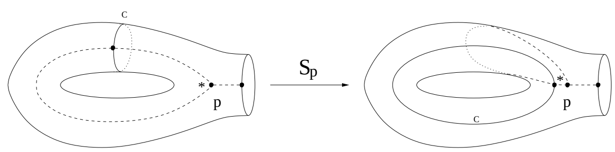

Figure 8: The S-move

treatment.

To simplify notation we will write $\pi_2 \sim \pi_1$ if the round trip π in question can be represented as $\pi = \pi_2 \circ \pi_1^{-1}$.

Relations supported on surfaces of genus zero.

$$g = 0, \quad s = 3: \quad Z_p \circ Z_p \circ Z_p \sim \mathrm{id}\,. \tag{4.2}$$

$$g = 0, \quad s = 4: \quad \begin{aligned} &\mathrm{a)} \quad F_{qp} \circ B_p \circ F_{pq} \;\sim\; (pq) \circ B_q \circ F_{pq} \circ B_p\,, \\ &\mathrm{b)} \quad F_{qp} \circ B_p^{-1} \circ F_{pq} \;\sim\; (pq) \circ B_q^{-1} \circ F_{pq} \circ B_p^{-1}, \\ &\mathrm{c)} \quad A_{pq} \circ A_{qp} \;\sim\; (pq)\,. \end{aligned} \tag{4.3}$$

$$g = 0, \quad s = 5: \quad F_{qr} \circ F_{pr} \circ F_{pq} \;\sim\; F_{pq} \circ F_{qr}. \tag{4.4}$$

We have used the abbreviation

$$A_{pq} \;\equiv\; Z_q^{-1} \circ F_{pq} \circ Z_q^{-1} \circ Z_p\,. \tag{4.5}$$

Relations supported on surfaces of genus one. In order to write the relations transparently let us introduce the following composites of the elementary moves.

$$g=0,\quad s=3: \qquad \begin{aligned} &\text{a)} \quad B'_p \;\equiv\; Z_p^{-1}\circ B_p\circ Z_p^{-1}, \\ &\text{b)} \quad T_p \;\equiv\; Z_p^{-1}\circ B_p\circ Z_p\circ B_p\,, \end{aligned} \tag{4.6}$$

$$g=0,\quad s=4: \qquad B_{qp}\;\equiv\; Z_q^{-1}\circ F_{qp}^{-1}\circ B'_q\circ F_{pq}^{-1}\circ Z_q^{-1}\circ (pq)\,, \tag{4.7}$$

$$g=1,\quad s=2: \qquad S_{qp}\;\equiv\; (F_{qp}\circ Z_q)^{-1}\circ S_p\circ (F_{qp}\circ Z_q)\,. \tag{4.8}$$

It is useful to observe that the move T_p, represents the Dehn twist around the boundary component of the trinion $\mathfrak{t}_p$ numbered by $i=1$ in Figure 3.

With the help of these definitions we may write the relations supported on surfaces of genus one as follows:

$$g=1,\quad s=1: \qquad \begin{aligned} &\text{a)} \quad S_p^2 \;\sim\; B'_p, \\ &\text{b)} \quad S_p\circ T_p\circ S_p \;\sim\; T_p^{-1}\circ S_p\circ T_p^{-1}. \end{aligned} \tag{4.9}$$

$$g=1,\quad s=2: \quad B_{qp}\;\sim\; S_{qp}^{-1}\circ T_q^{-1}T_p\circ S_{pq}\,. \tag{4.10}$$

Theorem 2. – [BK2] – *The complex $\mathcal{M}(\Sigma)$ is connected and simply connected for any e-surface Σ.*

4.2 Representation of the generators on spaces of conformal blocks

Definition 2. *We will say that a vertex algebra V has the factorization property if*

(i) there exists an analytic continuation of the family $G_{\sigma\tau}(\rho,w)$ into a domain in $\mathfrak{T}_{g,n}$ that contains the fundamental domain $\mathcal{V}_\sigma$ in $\mathfrak{T}_{g,n}$.

(ii) For each pair (σ_2,σ_1) of markings related by one of the elementary transformations Z_p, B_P, F_{pq} and S_p there exists a nontrivial intersection $\mathcal{D}_{\sigma_2\sigma_1}\subset\mathfrak{T}_{g,n}$ of the domains $\mathcal{D}_{\sigma_2}$ and $\mathcal{D}_{\sigma_1}$ within which the conformal blocks $G_{\sigma_1\tau_1}(\rho_1,w_1)$ and $G_{\sigma_2\tau_2}(\rho_2,w_2)$ can be uniquely defined by the gluing construction and analytic continuation. There exists a relation of the form

$$G_{\sigma_1\tau_1}(\rho_1,w_1)\;=\;\int d\mu_\Sigma(\rho_2)\sum_{w_2}F_{\sigma_1\sigma_2}(\rho_1|\rho_2)_{w_1}^{w_2}\,G_{\sigma_2\tau_2}(\rho_2,w_2)\,, \tag{4.11}$$

which holds whenever τ_2, τ_1 parameterize the same point in $\mathcal{D}_{\sigma_2\sigma_1}$.

Let us note that the conjecture can be verified by elementary means in the cases of the moves Z_p and B_p. These are the moves which do not change the cut system. Existence of a relation like

(4.11) for the cases that σ_2 and σ_1 are related by a F_{pq}- or S_p-move is a deep statement. It is, on the one hand, a requirement without which a CFT can hardly be of physical relevance. It may, on the other hand, be hard to prove mathematically that a given vertex operator algebra has this property. Statements of this type are presently only known in the case of certain rational vertex algebras (from the differential equations satisfied by the conformal blocks) or in the nonrational case of the Virasoro algebra reviewed in Section 6.

4.2.1

The relation (4.11) suggests to consider an operator $\mathsf{M}_{\sigma_1\sigma_2}$ between certain subspaces of $\mathfrak{F}_{\sigma_2}$ and $\mathfrak{F}_{\sigma_1}$, respectively, which is defined such that

$$(\mathsf{M}_{\sigma_2\sigma_1}f)_{w_1}(\rho_1) \,=\, \int d\mu_\Sigma(\rho_2)\sum_{w_2} F_{\sigma_1\sigma_2}(\rho_1|\rho_2)_{w_1}^{w_2}\, f_{w_2}(\rho_2)\,. \tag{4.12}$$

The conditions (4.1) can be seen to be equivalent to the following condition of invariance of the family of hermitian forms H_σ,

$$H_{\sigma_1}\big(\,\mathsf{M}_{\sigma_1\sigma_2}f_{\sigma_2}\,,\,\mathsf{M}_{\sigma_1\sigma_2}g_{\sigma_2}\,\big) \,=\, H_{\sigma_2}\big(\,f_{\sigma_2}\,,\,g_{\sigma_2}\,\big)\,. \tag{4.13}$$

Combined with the factorization (3.9) one gets nontrivial restrictions on the family of hermitian forms $H^{\rho'\rho}$ on $\mathfrak{F}(\rho')\times\mathfrak{F}(\rho)$.

4.3 Representation of the relations on spaces of conformal blocks

4.3.1

The operators Z_p, B_p, F_{pq} and S_p defined in the previous subsection are not independent. The faces ϖ of the complex $\mathcal{M}(\Sigma)$ correspond to round trips in $\mathfrak{T}_{g,n}$ starting and ending in the fundemantal domain $\mathcal{V}_\sigma$ associated to some marking σ. The realization of such round trip on the conformal blocks can be realized by means of parallel transport w.r.t. the canonical connection on spaces of conformal blocks, in general. The operator $\mathsf{U}(\pi)$ which is associated to a round trip therefore has to be proportional to the identity. It does not have to be equal to the identity since the canonical connection is not flat but only projectively flat. One thereby gets a relation $\mathsf{U}(\varpi)=\zeta_\varpi$ between the "generators" Z_p, B_p, F_{pq} and S_p for every round trip ϖ that one can compose out of the elementary moves (pq) Z_p, B_p, F_{pq} or S_p.

4.3.2 Insertions of the vacuum

Taking into account the propagation of vacua implies that the operators Z_p, B_p, F_{pq} must simplify considerably if one of the representations inserted is the vacuum. We must have, in particular, the relations

$$\mathsf{F}\cdot\big[\mathfrak{H}^{0}_{\bar{r}_3 r_3}\otimes\mathfrak{H}^{r_3}_{r_2 r_1}\big]=\mathsf{Z}\cdot\mathfrak{H}^{r_1}_{r_3 r_2}\otimes\mathfrak{H}^{0}_{\bar{r}_1 r_1} \tag{4.14}$$

$$\mathsf{F}\cdot\big[\mathfrak{H}^{r_3}_{0\bar{r}_3}\otimes\mathfrak{H}^{r_3}_{r_2 r_1}\big]=\mathfrak{H}^{\bar{r}_2}_{0 r_2}\otimes\mathfrak{H}^{r_3}_{r_2 r_1} \tag{4.15}$$

$$\mathsf{B}\cdot\mathfrak{H}^{\bar{r}}_{0r}=\mathfrak{H}^{\bar{r}}_{r0} \tag{4.16}$$

$$\mathsf{B}\cdot\mathfrak{H}^{0}_{r\bar{r}}=\mathsf{T}\cdot\mathfrak{H}^{0}_{\bar{r}r}\,. \tag{4.17}$$

We are using the notation $\mathfrak{H}^{r_3}_{r_2 r_1}$ for $\mathcal{C}(S_{0,3},\rho)$ if $\rho=(r_3,r_2,r_1)$, with labelling of boundary components in correspondence to the decoration introduced in Figure 3.

Some of the relations in which these operators appear trivialize accordingly. By rescaling the operators Z_p, B_p, F_{pq} and S_p, if neccessary, one can achieve that $\zeta_\varpi=1$ for some other relations. With the help of these observations it is easy to see that one may assume $\xi_\varpi=1$ for each of the relations (4.2)-(4.4) associated to Riemann surfaces of genus zero.

In the case of genus one let us observe that the relation (4.10) trivializes if one of the two external representations is the vacuum representation. One may furthermore always redefine S_p such that $\xi_\varpi=1$ in the case of the relation which corresponds to (4.9), b). One is left with the relation (4.9), a). The arguments well-known from rational CFT [MS1, BK1] lead one to the conclusion that the corresponding relation is

$$\mathsf{S}_p^2\;=\;e^{-\pi i\frac{c}{2}}\,\mathsf{B}'_p\,, \tag{4.18}$$

where c is the central charge of the Virasoro algebra introduced in (2.3).

5. Notion of a stable modular functor

We shall now formulate an abstract framework that we believe to be suitable for large classes of not necessarily rational CFT. This framework can be seen as a variant of the concept of a modular functor from [S]. Combined with the gluing construction of the conformal blocks it will be shown to yield a concrete realization of the point of view of Friedan and Shenker [FS] who proposed to view the partition function of a CFT as a hermitian metric on a projective line bundle over the moduli space of Riemann surfaces, with expectation value of the stress-energy tensor being the canonical connection.

An important feature is that we will assume existence of a scalar product on the spaces of conformal blocks. The topology defined by the scalar product gives us control on the possible infinite-dimensionality of these spaces. Existence of a scalar product may seem to be an overly strong assumption, but we will discuss in the following sections why we believe that the class of vertex algebras that is covered by our formalism is rather large.

5.1 Towers of representations of the modular groupoid

Definition 3. *A tower of representations of the modular groupoid assigns to a topological surface Σ the following objects:*

- *the family of Hilbert spaces $\left[\mathfrak{H}_\sigma\right]_{\sigma\in\mathcal{M}_0(\Sigma)}$ of the form*

$$\mathfrak{H}_\sigma \;\equiv\; \int_{\mathbb{U}^{\sigma_1}}^{\oplus} d\mu_\sigma(\rho) \bigotimes_{p\in\sigma_0} \mathfrak{F}(\rho_p)\,, \tag{5.1}$$

  where $d\mu_\sigma(\rho)$ is the product measure $d\mu_\sigma(\rho) = \prod_{e\in\sigma_1} d\mu_{\rm Pl}(r_e)$ if $\rho : \sigma_1 \ni e \to r_e \in \mathbb{U}$.

- *For each pair $[\sigma_2, \sigma_1]$ of markings a unitary operator $\mathsf{M}_{\sigma_2\sigma_1} : \mathfrak{H}_{\sigma_1} \to \mathfrak{H}_{\sigma_2}$ such that*

$$\mathsf{M}_{\sigma_1\sigma_3} \cdot \mathsf{M}_{\sigma_3\sigma_2} \cdot \mathsf{M}_{\sigma_2\sigma_1} \;=\; \xi_{\sigma_3\sigma_2\sigma_1} \cdot \mathrm{id}\,, \qquad \mathsf{M}_{\sigma_1\sigma_2} \cdot \mathsf{M}_{\sigma_2\sigma_1} \;=\; 1\,, \tag{5.2}$$

  where $\xi_{\sigma_3\sigma_2\sigma_1} \in \mathbb{C}$, $|\xi_{\sigma_3\sigma_2\sigma_1}| = 1$.

This assignment is such that the following requirements hold:

Disjoint union: *Let $X = X' \sqcup X''$ be the disjoint union of X' and X'', and let $\sigma_i = \sigma_i' \sqcup \sigma_i''$, $i = 1, 2$ be two markings on X. Then*

$$\mathcal{H}_\sigma \;=\; \mathcal{H}_{\sigma'} \otimes \mathcal{H}_{\sigma''}\,, \tag{5.3}$$

$$\mathsf{M}_{\sigma_2\sigma_1} \;=\; \mathsf{M}_{\sigma_2'\sigma_1'} \otimes \mathsf{M}_{\sigma_2''\sigma_1''}\,. \tag{5.4}$$

Gluing: *Assume that X' is obtained from X by gluing the boundary components B_2 and B_1. Let σ be a marking on X with edges e_2 and e_1 ending in B_2 and B_1, respectively, and let σ' be the marking on X obtained from σ by gluing e_2 and e_1.*

There is a dense subset $\mathcal{T}_\sigma$ of $\mathcal{H}_\sigma$ such that for each $\Psi \in \mathcal{T}_\sigma$ the following gluing projection is well defined:

$$\mathsf{G}_{e_2e_1}\Psi \;=\; \Psi|_{s_{e_2}=s_{e_1}}\,,$$

where $\Psi|_{s_{e_2}=s_{e_1}}$ is obtained from Ψ by restricting it to the subset of $\mathcal{I}_\sigma$ where $r_{e_1} = r_{e_2}$.

The projection $\mathsf{G}_{e_2e_1}$ is then required to be compatible with the representations of the modular groupoids of X and X' in the following sense: It is required that $\mathsf{M}_{\sigma_2\sigma_1}$ maps from $\mathcal{T}_{\sigma_1}$ to $\mathcal{T}_{\sigma_2}$ and that

$$\mathsf{G}_{e_2e_1}\mathsf{M}_{\sigma_2\sigma_1}\Psi \;=\; \mathsf{M}_{\sigma_2'\sigma_1'}\mathsf{G}_{e_2e_1}\Psi\,, \tag{5.5}$$

holds for all pairs (σ_2', σ_1') of markings on X obtained from the corresponding markings (σ_2, σ_1) on X by gluing, and all $\Psi \in \mathcal{T}_{\sigma_1}$.

Propagation of vacua: *Let X' be obtained from X by gluing a disc into the boundary component $B_\circ$. Let σ be a marking of X, $e_\circ$ be the edge of σ that ends in $B_\circ$ and $p_\circ$ be the vertex from which $e_\circ$ emanates. One gets a marking σ' on X' by deleting $e_\circ$ and $p_\circ$ and gluing the other two edges that emanate from $p_\circ$ into a single edge of σ'.*

There then exist dense subsets $\mathcal{T}_\sigma$ and $\mathcal{T}_{\sigma'}$ of $\mathcal{H}_\sigma$ and $\mathcal{H}_{\sigma'}$, respectively, together with projection mappings $P_{\sigma,B_\circ} : \mathcal{T}_\sigma \mapsto \mathcal{T}_{\sigma'}$ such that

$$\mathsf{P}_{\sigma_2,B_\circ}\mathsf{M}_{\sigma_2\sigma_1}\Psi \;=\; \mathsf{M}_{\sigma_2'\sigma_1'}\mathsf{P}_{\sigma_1,B_\circ}\Psi\,, \tag{5.6}$$

holds for all pairs (σ_2, σ_1) of markings on X and the corresponding markings (σ_2', σ_1') on X' defined above.

The requirements concerning disjoint union and gluing imply that the representations of the modular groupoids are constructed out of the representatives of the elementary moves B_p, Z_p, F_{pq} and S_p. A tower of representations of the modular groupoid is therefore characterized by the following data:

- The measure set $\mathbb{U}$ (labels of unitary representations of V), equipped with a measure $d\mu_{\rm Pl}$.
- The Hilbert spaces $\mathfrak{H}_\sigma^{(3)}(\rho)$ associated to the markings σ on the three punctured sphere $\Sigma_{0,3}$ with assignment $\rho : k \mapsto r_k$, $k \in \{1, 2, 3\}$.
- The operators Z, B, F and S mapping $\mathfrak{H}_{\sigma_1}(\rho)$ to $\mathfrak{H}_{\sigma_2}(\rho)$ with respective markings σ_1 and σ_2 being chosen as depicted in figures 6-8.

Let us furthermore remark that the constraints imposed on these data by the propagation of vacua requirement are related to (4.14)-(4.17). The precise relationship can be subtle if the vacuum representation is not contained in support of $d\mu_{\rm PL}$ as it may happen for nonrational CFT (see Section 6 for an example). The definition of the projection mappings $P_{\sigma,B_\circ}$ then involves analytic continuation w.r.t. the conformal dimensions of the representations, and the compatibility condition (5.6) requires that the dependence of Z, B, F and S on the labels of external representations has a sufficiently large domain to analyticity.

5.2 Unitary modular functors

Given a tower of representations of the modular groupoids there is a canonical way to construct a corresponding modular functor, as we shall now explain. The main issue is to eliminate the apparent dependence on the choice of the marking σ.

Each of the spaces $\mathfrak{H}_\sigma$ becomes a representation of the mapping class group $\mathrm{MCG}(\Sigma)$ by choosing for each $m \in \mathrm{MCG}(\Sigma)$ a sequence π_m of elementary moves that connects σ to $m(\sigma)$. Taking advantage of the fact that the isomorphism $\mathfrak{H}_\sigma \simeq \mathfrak{H}_{m(\sigma)}$ is canonical one gets an operator $\mathsf{M}(m)$ on $\mathfrak{H}_\sigma$.

It is easily seen that for each pair $[\sigma_2, \sigma_1]$ there exist numbers $\zeta_{\sigma_2\sigma_1}$ which satisfy

$$\zeta_{\sigma_1\sigma_3} \cdot \zeta_{\sigma_3\sigma_2} \cdot \zeta_{\sigma_2\sigma_1} \;=\; \xi_{\sigma_3\sigma_2\sigma_1} \cdot \mathrm{id}\,. \tag{5.7}$$

Indeed, given a fixed reference marking σ_0 one may take e.g. $\zeta_{\sigma_2\sigma_1} \equiv \xi^{-1}_{\sigma_2\sigma_0\sigma_1}$. This means that one can use the numbers $\zeta_{\sigma_2\sigma_1}$ to define a *projective* holomorphic line bundle $\mathcal{L}_V$ over $\mathfrak{T}_{g,n}$. To this aim, use the $\mathcal{V}_\sigma$ as local coordinate patches, with transition functions $\zeta_{\sigma_2\sigma_1}$. Projectiveness follows from the nontriviality of the phase $\xi_{\sigma_3\sigma_2\sigma_1}$ associated to triples of markings.

Definition 4. *Let $\mathfrak{H}(\Sigma)$ be the Hilbert space whose elements Φ are collections of vectors $\Psi_\sigma \in \mathfrak{H}_\sigma$ such that*

$$\Psi_{\sigma_2} \;=\; \zeta^{-1}_{\sigma_2\sigma_1}\, \mathsf{M}_{\sigma_2\sigma_1} \Psi_{\sigma_1}\,, \tag{5.8}$$

holds for all pairs of markings σ_1, σ_2 and a given collection of complex numbers $\zeta_{\sigma_2\sigma_1}$ of modulus one which satisfy (5.7).

For a given collection of numbers $\eta_\sigma \in \mathbb{C}$, $\sigma \in \mathcal{M}_0(\Sigma)$, $|\eta_\sigma| = 1$, let us call the operation $\Psi_\sigma \to \eta_\sigma \Psi_\sigma$ for all $\Psi_\sigma \in \mathfrak{H}_\sigma$ a gauge transformation. We will identify the Hilbert spaces $\mathfrak{H}(\Sigma)$ related by gauge transformations.

Let $\rho(\Sigma)$ be the family of mapping class group representations $(\rho_\sigma(R))_{\sigma \in \mathcal{M}_0(\Sigma)}$ on the spaces $\mathfrak{H}_\sigma(R)$ modulo the equivalence relation $\sim$ that is induced by the identifications (5.8).

The assignment $\Sigma \to (\mathfrak{H}(\Sigma), \mathsf{M}(\Sigma))$ will be called a stable unitary projective functor.

5.3 Similarity of modular functors

For rational CFT there exist deep results on the equivalence of modular functors from conformal field theory to similar objects coming from quantum group theory [Fi]. In order to formulate analogous statements about nonrational CFT we will propose the following natural notion of similarity of modular functors.

Definition 5. *We will call two modular functors $\mathcal{F}$ and $\mathcal{F}'$ with data*

$$\big[\, \mathbb{U}\;,\; d\mu_{\rm Pl}\,,\; \mathfrak{H}^{(3)}_\sigma(\rho)\,,\; \mathsf{Z}\;,\; \mathsf{B}\;,\; \mathsf{F}\;,\; \mathsf{S}\;\big]$$
$$\big[\, \mathbb{U}'\,,\; d\nu_{\rm Pl}\,,\; \mathfrak{K}^{(3)}_\sigma(\rho')\,,\; \mathsf{Z}'\,,\; \mathsf{B}'\,,\; \mathsf{F}'\,,\; \mathsf{S}'\,\big]$$

similar iff the following conditions are satisfied:

- *There exists a bijection between $\mathbb{U}$ and $\mathbb{U}'$. The measures $d\mu_{\rm Pl}$ and $d\nu_{\rm Pl}$ are equivalent, i.e. there exists a positive function $m(r)$ on $\mathbb{U}$ such that*

$$d\mu_{\rm Pl}(r) \;=\; m(r) d\nu_{\rm Pl}\,.$$

- *There exist families of invertible operators* $\mathsf{E}^{0,3}(\rho) : \mathfrak{H}^{(3)}(\rho) \to \mathfrak{K}^{(3)}(\rho')$, *the dependence on each representation label* $r_k \in \mathbb{U}$, $k = 1, 2, 3$ *measurable w.r.t.* $d\mu_{\rm Pl}(s)$ *such that the operators* $\mathsf{E}_\sigma : \mathfrak{H}_\sigma \to \mathfrak{K}_\sigma$ *defined as*

$$\mathsf{E}_\sigma \equiv \int_{\mathbb{U}^{\sigma_1}}^{\oplus} d\nu_\sigma(\rho) \prod_{p\in\sigma_0} \mathsf{E}^{0,3}(\rho_p) \tag{5.9}$$

  are invertible.

- *The resulting operators* $\mathsf{E}_\sigma : \mathfrak{H}_\sigma \to \mathfrak{K}_\sigma$ *satisfy*

$$\mathsf{M}'_{\sigma_2\sigma_1} = \mathsf{E}_{\sigma_2} \cdot \mathsf{M}_{\sigma_2\sigma_1} \cdot \mathsf{E}_{\sigma_1}^{-1} .$$

5.4 Friedan-Shenker modular geometry

Let us temporarily restrict attention to surfaces X with one marked point at position $z \in X$, decorated with the vacuum representation V. We will assume that the values of the conformal blocks $G_{\sigma\tau}(\delta|v)$ at a given vector $v \in V$ may be considered as a family $(G_{\sigma\tau}(v))_{\tau\in\mathcal{V}_\sigma}$ of elements of the Hilbert space $\mathfrak{H}_\sigma$.

Out of $(G_{\sigma\tau}(v))_{\tau\in\mathcal{V}_\sigma}$ one may then define a collection of vectors $\{ \Psi_{\sigma;\sigma'}(v|\tau) \,;\, \sigma' \in \mathcal{M}_0(\Sigma) \}$, where $\Psi_{\sigma;\sigma'}(v|\tau) \in \mathfrak{H}_{\sigma'}$ for all $\sigma' \in \mathcal{M}_0(\Sigma)$ such that the conditions

$$\Psi_{\sigma;\sigma_2}(v|\tau) = \zeta^{-1}_{\sigma_2\sigma_1} \mathsf{M}_{\sigma_2\sigma_1} \Psi_{\sigma;\sigma_1}(v|\tau) \quad \text{and} \quad \Psi_{\sigma;\sigma}(v|\tau) = G_{\sigma\tau}(v) \in \mathfrak{H}_\sigma \tag{5.10}$$

are satisfied. Indeed, consistency of the definition of $\Psi_{\sigma;\sigma'}(v|\tau)$ implied by (5.10) follows from (5.2) and (5.7). Let $(\Phi_{\sigma\tau}(v))_{\tau\in\mathcal{V}_\sigma}$ be the holomorphic family of vectors in $\mathfrak{H}(\Sigma)$ which is associated by Definition 4 to the collection $\{ \Psi_{\sigma;\sigma'}(v|\tau) \,;\, \sigma' \in \mathcal{M}_0(\Sigma) \}$.

Given two markings σ_2, σ_1 such that $\mathcal{V}_{\sigma_2} \cap \mathcal{V}_{\sigma_1} \neq \emptyset$ it is easy to see that the families of vectors $(\Phi_{\sigma_2\tau_2}(v))_{\tau_2\in\mathcal{V}_{\sigma_2}}$ and $(\Phi_{\sigma_1\tau_1}(v))_{\tau_1\in\mathcal{V}_{\sigma_1}}$ are related as

$$\Psi_{\sigma_2\tau}(v) = \zeta_{\sigma_2\sigma_1} \Psi_{\sigma_1\tau}(v) \tag{5.11}$$

if τ_2 and τ_1 parametrize the same point in $\mathcal{V}_{\sigma_2} \cap \mathcal{V}_{\sigma_1}$. Indeed, we had defined $\mathsf{M}_{\sigma_2\sigma_1}$ in (4.11),(4.12) such that $G_{\sigma_1\tau_1}(v) = \mathsf{M}_{\sigma_1\sigma_2} G_{\sigma_2\tau_2}(v)$. This implies

$$\Psi_{\sigma_1;\sigma_1}(v|\tau_1) = G_{\sigma_1\tau_1}(v) = \mathsf{M}_{\sigma_1\sigma_2} G_{\sigma_2\tau_2}(v) = \mathsf{M}_{\sigma_1\sigma_2} \Psi_{\sigma_2;\sigma_2}(v|\tau_2) \overset{(5.8)}{=} \zeta_{\sigma_1\sigma_2} \Psi_{\sigma_2;\sigma_1}(v|\tau_2) .$$

This means that for each σ one may regard the family $(\Psi_{\sigma\tau}(v))_{\tau\in\mathcal{V}_\sigma}$ as a local holomorphic section of the projective line bundle $\mathcal{L}_V$ over $\mathfrak{T}_{g,n}$.

The invariance conditions (4.13) imply that the family of hermitian forms H_σ defines a hermitian form H on $\mathfrak{H}(\Sigma)$. Objects of particular interest for the case at hand are the partition function $Z_g(X)$,

$$Z_g(X) \equiv H\big(\Psi_{\sigma\tau}(v_\circ)\,,\,\Psi_{\sigma\tau}(v_\circ)\big) \tag{5.12}$$

and the expectation values $\langle\!\langle\, Y(A,z)\,\rangle\!\rangle$ of local fields $Y(A,z)$ from the vertex algebra V,

$$\langle\!\langle\, Y(A,z)\,\rangle\!\rangle \equiv \frac{H\big(\Psi_{\sigma\tau}(A)\,,\,\Psi_{\sigma\tau}(A)\big)}{H\big(\Psi_{\sigma\tau}(v_\circ)\,,\,\Psi_{\sigma\tau}(v_\circ)\big)}\,. \tag{5.13}$$

Following [FS] we will regard the partition function $Z_g(X)$ as a hermitian metric $\mathcal{H}$ on the projective line bundle $\mathcal{L}_V$. It follows easily from (2.10) that

$$\delta_\vartheta \log Z_g(X) = \big\langle\!\big\langle\, Y(T[\eta_\vartheta]v_\circ, z)\,\big\rangle\!\big\rangle\,, \tag{5.14}$$

where $T[\eta] = \sum_{n\in\mathbb{Z}} \eta_n L_n$, δ_ϑ is the derivative corresponding a tangent vector $\vartheta \in T\mathfrak{M}_{g,0}$ and η_ϑ is any element of $\mathbb{C}(\!(t)\!)\partial_t$ which represents ϑ via (2.9). Equation (5.14) can be seen as a more precise formulation of the claim from [FS] that the expectation value of the stress-energy tensor coincides with the connection on the projective line bundle $\mathcal{L}_V$ which is canonically associated with the metric $\mathcal{H}$. We have thereby reconstructed the main ingredients of the formulation proposed by Friedan and Shenker [FS] within the framework provided by the gluing construction.

6. Example of a nonrational modular functor

There is considerable evidence for the claim that the most basic example of a vertex algebra, the Virasoro algebra, yields a realization of the framework above. The results of [TL] are essentially equivalent to the construction of the corresponding modular functor in genus 0. In the following section we shall review the main characteristics of this modular functor.

6.1 Unitary positive energy representations of the Virasoro algebra

The unitary highest weight representations R_Δ of the Virasoro algebra are labelled by the eigenvalue Δ of the Virasoro generator L_0 on the highest weight vector. It will be convenient to parametrize Δ as follows

$$\Delta_\alpha = \alpha(2\delta - \alpha), \qquad \text{where} \qquad c = 1 + 24\delta^2\,. \tag{6.1}$$

The representations $R_\alpha \equiv R_{\Delta_\alpha}$ are unitary iff $\Delta \in [0,\infty)$. In terms of the parametrization (6.1) one may cover this range by assuming that

$$\alpha \in \mathbb{U} \equiv [0,\delta] \cup \big(\delta + i\mathbb{R}^+\big)\,. \tag{6.2}$$

The representation R_α for $\alpha = 0$ corresponds to the vacuum representation V. The set parametrizes the unitary dual of the Virasoro algebra. In order to indicate an important analogy with the representation theory of noncompact Lie groups we shall call the family of representations R_α with $\alpha \in \delta + i\mathbb{R}^+$ the principal series of representations, which constitute the tempered dual $\mathbb{T}$ of the Virasoro algebra. Pursusing these analogies it seems natural to call the family of representations R_α with $\alpha \in [0, \delta]$ the complementary series.

6.1.1 *Free field representation*

The Fock space $\mathcal{F}$ is defined to be the representation of the commutation relations

$$[\mathsf{a}_n, \mathsf{a}_m] = \frac{n}{2}\delta_{n+m} \,, \tag{6.3}$$

which is generated from the vector $\Omega \in \mathcal{F}$ characterised by $a_n\Omega = 0$ for $n > 0$. There is a unique scalar product $(\,.\,,\,.\,)_{\mathcal{F}}$ on $\mathcal{F}$ such that $\mathsf{a}_n^\dagger = \mathsf{a}_{-n}$ and $(\,\Omega\,,\,\Omega\,)_{\mathcal{F}} = 1$.

Within $\mathcal{F}$ we may define a one-parameter family of representations $\mathcal{F}_p$ of the Virasoro algebra by means of the formulae

$$\begin{aligned} L_n(p) \;&=\; 2(p + in\delta)a_n + \sum_{k\neq 0,n} a_k a_{n-k} \,, \qquad n \neq 0 \,, \\ L_0(p) \;&=\; p^2 + \delta^2 + 2\sum_{k>0} a_{-k}a_k \,. \end{aligned} \tag{6.4}$$

The representation $\mathcal{F}_p$ is unitary w.r.t. the scalar product $(\,.\,,\,.\,)_{\mathcal{F}}$ if $p \in \mathbb{R}$. It is furthermore known [Fr] to be irreducible and therefore isomorphic to R_α if $p = -i(\alpha - \delta)$ for all $\alpha \in \mathbb{U}$.

6.2 Construction of Virasoro conformal blocks in genus zero

In the case of the Virasoro algebra there exists a unique conformal block G associated to the three punctured sphere which satisfies

$$G(\,\rho\,|\,v_{\alpha_3} \otimes v_{\alpha_2} \otimes v_{\alpha_1}\,) \;=\; 1 \,, \tag{6.5}$$

v_{α_k}, $k \in \{1, 2, 3\}$ being the highest weight vectors of the representations R_{α_k}, respectively. The corresponding family of operators $\mathsf{Y}^{\alpha_2}_{\alpha_3\alpha_1}(\mathfrak{v}_2|z) : R_{\alpha_1} \to R_{\alpha_3}$ is uniquely characterized by its member corresponding to $\mathfrak{v}_2 = v_{\alpha_2}$, which will be denoted $\mathsf{Y}^{\alpha_2}_{\alpha_3\alpha_1}(z)$.

6.2.1 Free field construction of chiral vertex operators

Let us introduce the (left-moving) chiral free field $\varphi(x) = \mathsf{q} + \mathsf{p}x + \varphi_<(x) + \varphi_>(x)$, by means of the expansions

$$\varphi_<(x) = i\sum_{n<0}\frac{1}{n}\,\mathsf{a}_n\,e^{-inx}, \qquad \varphi_>(x) = i\sum_{n>0}\frac{1}{n}\,\mathsf{a}_n\,e^{-inx}, \tag{6.6}$$

The operators q and p are postulated to have the following commutation and hermiticity relations

$$[\mathsf{q},\mathsf{p}] = \frac{i}{2}, \qquad \mathsf{q}^\dagger = \mathsf{q}, \qquad \mathsf{p}^\dagger = \mathsf{p}, \tag{6.7}$$

which are naturally realized in the Hilbert-space

$$\mathfrak{H}^{\rm F} \equiv L^2(\mathbb{R}) \otimes \mathcal{F}\,. \tag{6.8}$$

Diagonalizing the operator p corresponds to decomposing $\mathfrak{H}^{\rm F}$ as direct integral of irreducible unitary representations of the Virasoro algebra,

$$\mathcal{M} \simeq \int_{\mathbb{T}}^{\oplus} d\alpha\; R_\alpha\,. \tag{6.9}$$

The basic building blocks of all constructions will be the following objects:

NORMAL ORDERED EXPONENTIALS :

$$\mathsf{E}^\alpha(x) \equiv \mathsf{E}^\alpha_<(x)\mathsf{E}^\alpha_>(x), \qquad \begin{aligned} \mathsf{E}^\alpha_<(x) &= e^{\alpha\mathsf{q}}\,e^{2\alpha\varphi^+_<(x)}\,e^{\alpha x\mathsf{p}} \\ \mathsf{E}^\alpha_>(x) &= e^{\alpha x\mathsf{p}}\,e^{2\alpha\varphi^+_>(x)}\,e^{\alpha\mathsf{q}} \end{aligned} \tag{6.10}$$

SCREENING CHARGES:

$$\mathsf{Q}(x) \equiv e^{-\pi b\mathsf{p}}\int_0^{2\pi} dx'\;\mathsf{E}^b(x+x')\;e^{-\pi b\mathsf{p}}\,. \tag{6.11}$$

The following property is of considerable importance:

POSITIVITY: The screening charges are densely defined positive operators, i.e.

$$\big(\,\psi\,,\,Q(\sigma)\psi\,\big)_{\mathcal{M}} \geq 0$$

holds for ψ taken from a dense subset of $\mathfrak{H}$.

Out of the building blocks introduced in the previous subsection we may now construct an important class of chiral fields,

$$\mathsf{h}^\alpha_s(\sigma) = \mathsf{E}^\alpha(\sigma)\,\big(\mathsf{Q}(\sigma)\big)^s\,, \tag{6.12}$$

Positivity of Q allows us to consider these objects for *complex* values of s and α.

One of the most basic properties of the $\mathsf{h}_s^\alpha(w)$ are the simple commutation relations with functions of the operator p,

$$\mathsf{h}_s^\alpha(w) f(\mathsf{p}) \;=\; f\big(\mathsf{p} - i(\alpha + bs)\big)\mathsf{h}_s^\alpha(w)\,. \tag{6.13}$$

By projecting onto eigenspaces of p one may therefore define a family of operators $\mathsf{h}_{\alpha_3\alpha_1}^{\alpha_2}(w)$: $R_{\alpha_1} \to R_{\alpha_3}$. Specifically, for each $\mathfrak{w} \in \mathcal{F}$ and each $\alpha \in \delta + i\mathbb{R}$ let us define a distribution $\mathfrak{w}_\alpha$ on dense subspaces of $\mathcal{M}$ by the relation $(\,\mathfrak{w}_\alpha\,,\,\mathfrak{v}\,)_{\mathcal{M}} = (\,\mathfrak{w}\,,\,\mathfrak{v}_\alpha\,)_{\mathcal{F}}$ if $\mathfrak{v}$ is represented via (6.9) by the family of vectors $\mathfrak{v}_\alpha$, $v_\alpha \in R_\alpha$. This implies that the matrix elements of the operators $\mathsf{h}_{\alpha_3\alpha_1}^{\alpha_2}(w)$ are determined by the relation

$$\big(\,\mathfrak{w}\,,\,\mathsf{h}_{\alpha_3\alpha_1}^{\alpha_2}(w)\,\mathfrak{u}\,\big)_{\mathcal{F}} \;=\; \big(\,\mathfrak{w}_{\alpha_3}\,,\,\mathsf{h}_s^{\alpha_2}(w)\,\hat{\mathfrak{u}}\,\big)_{\mathcal{M}}\,, \tag{6.14}$$

where $bs = \alpha_3 - \alpha_1 - \alpha_2$ and $\hat{\mathfrak{u}}$ is any vector in $\mathcal{M}$ represented by the family of vectors $\hat{\mathfrak{u}}_\alpha$ such that $\hat{\mathfrak{u}}_{\alpha_1} = \mathfrak{u}$.

The uniqueness of the conformal block $G^{(3)}$ implies that the operator $\mathsf{Y}_{\alpha_3\alpha_1}^{\alpha_2}(\mathfrak{v}_2|\,z\,)$ must be proportional to $\mathsf{h}_{\alpha_3\alpha_1}^{\alpha_2}(\mathfrak{v}_2\,|\,z)$ via

$$\mathsf{Y}_{\alpha_3\alpha_1}^{\alpha_2}(\mathfrak{v}_2|\,z\,) \;=\; N_{\alpha_3\alpha_1}^{\alpha_2}\,\mathsf{h}_{\alpha_3\alpha_1}^{\alpha_2}(\mathfrak{v}_2\,|\,z)\,. \tag{6.15}$$

The explicit formula for the normalizing factor $N_{\alpha_3\alpha_1}^{\alpha_2}$ was found in [TL].

6.3 Factorization property

The results of [TL] show that the conformal blocks in genus zero satisfy the factorization property with linear relations (4.11) composed from the elementary transformations F_{pq}, B_p and Z_p whose representatives can be calculated explicitly.

F: Let X be a four-punctured sphere and let σ_s, σ_u be the two markings depicted in Figure 7. We will denote the respective assignments of representation labels to the edges of σ_s and σ_u by $\rho_s(\alpha_s)$ and $\rho_u(\alpha_u)$, respectively, leaving implicit the assignment of labels $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ to the external edges with numbering being indicated in Figure 7. The operator $\mathbf{F}_{pq}$ may then be represented as the integral operator

$$G_{\sigma_s\tau_2}(\rho_s(\alpha_s)) \;=\; \int d\mu_{\rm Pl}(\alpha_u)\;\mathrm{F}_{\alpha_s\alpha_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]\;G_{\sigma_u\tau_1}(\rho_u(\alpha_u))\,. \tag{6.16}$$

The explicit expression for the kernel $\mathrm{F}_{\alpha_s\alpha_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ can be found in [TR, TL]. More illuminating is probably the observation that the kernel $\mathrm{F}_{\alpha_s\alpha_u}$ is closely related to the $6j$ symbols of the modular double [Fa, BT] of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$,

$$\mathrm{F}_{\alpha_s\alpha_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] \;=\; \frac{\nu_{\alpha_3\alpha_s}^{\alpha_4}\,\nu_{\alpha_2\alpha_1}^{\alpha_s}}{\nu_{\alpha_u\alpha_1}^{\alpha_4}\,\nu_{\alpha_3\alpha_2}^{\alpha_u}}\,\big\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\big|\begin{smallmatrix}\alpha_s\\ \alpha_u\end{smallmatrix}\big\}\,. \tag{6.17}$$

The explicit formula for the normalizing factors $\nu_{\alpha_2\alpha_1}^{\alpha_s}$ can be found in [TR].

B: Let X be a three-punctured sphere and let σ_2, σ_1 be the two markings depicted on the left and right halves of Figure 2, respectively. Let ρ be the assignment $\rho : k \to \alpha_k$, $k = 1, 2, 3$ of representation labels to edges as numbered in Figure 2. We then have

$$G_{\sigma_2}(\rho) \,=\, \mathrm{B}_{\alpha_3\alpha_2\alpha_1} G_{\sigma_1}(\rho)\,, \qquad \mathrm{B}_{\alpha_3\alpha_2\alpha_1} \,\equiv\, e^{\pi i(\alpha_3(Q-\alpha_3)-\alpha_1(Q-\alpha_1)-\alpha_2(Q-\alpha_2))}\,. \tag{6.18}$$

Z: Z is represented by the identity operator.

An important part of the statements above may be reformulated as the claim that the modular functor $\mathcal{F}_{\text{Teich}}$ is similar in the sense of Definition 5 to a modular functor $\mathcal{F}_{\text{Qgrp}}$ that is constructed in close analogy to the construction of Reshetikhin-Turaev from the representations of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ introduced in [PT1, Fa] and studied in more detail in [PT1, BT].

6.4 The Hilbert space structure

As explained above, we need a pair $\left[\,\mathfrak{H}_\sigma^{(3)}(\rho)\,,\, d\mu_{\text{Pl}}\right]$ of objects in order to characterize the Hilbert space structures on the spaces of conformal blocks.

Hilbert space $\mathfrak{H}_\sigma^{(3)}(\rho)$: It is well-known that the space of conformal blocks on the three punctured sphere is at most one-dimensional. More precisely we have:

$$\mathfrak{H}_\sigma^{(3)}(\rho) \,\simeq\, \mathbb{C}, \qquad \rho : k \mapsto \alpha_k, \quad k \in \{1, 2, 3\}\,, \tag{6.19}$$

if $\alpha_i \neq 0$ for $i = 1, 2, 3$. If $\alpha_i = 0$ for some $i \in \{1, 2, 3\}$ and if $k, l \in \{1, 2, 3\}$ are not equal to i we have

$$\mathfrak{H}_\sigma^{(3)}(\rho) \,\simeq\, \begin{cases} \mathbb{C} \ \text{ if } \alpha_k = \alpha_l \ \text{ or } \ \alpha_k = 2\delta - \alpha_l\,, \\ \emptyset \ \text{ otherwise.} \end{cases} \tag{6.20}$$

As a standard basis for $\mathfrak{H}_\sigma^{(3)}(\rho)$ we shall use the conformal block $G^{(3)}(\rho)$ that is uniquely defined by the normalization (6.5). The Hilbert space structure on the one-dimensional space $\mathfrak{H}_s i^{(3)}(\rho)$ is then described by the numbers

$$D(\alpha_3, \alpha_2, \alpha_1) \,\equiv\, \|\, G^{(3)}(\rho)\,\|^2\,, \tag{6.21}$$

that are given explicitly by the formula

$$D(\alpha_3, \alpha_2, \alpha_1) \,=\, \left| \frac{\Gamma_b(\alpha_{123} - Q)}{\Gamma_b(Q)} \prod_{k=1}^{3} \frac{\Gamma_b(\alpha_{123} - 2\alpha_k)}{\Gamma_b(2\alpha_k)} \right|^2\,, \tag{6.22}$$

where $\Gamma_b(x) \equiv \Gamma_2(x|b, b^{-1})$ with $\Gamma_2(x|\omega_1, \omega_2)$ being the Barnes Double Gamma function, and we have used the abbreviation $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$.

The measure $d\mu_{\rm Pl}$ on $\mathbb{U}$ will then be equal to

$$d\mu_{\rm Pl}(\alpha) \,=\, d\alpha \sin(2b(\alpha-\delta))\,\sin(2b^{-1}(\delta-\alpha)) \quad \text{on} \;\; \delta+i\mathbb{R}^+\,, \tag{6.23}$$

with $d\alpha$ being the standard Lebesque measure on $\delta + i\mathbb{R}^+$.

6.5 Extension to higher genus

Claim 1. *There exists a unique extension of the $g = 0$ modular functor $\mathcal{F}_{\rm Vir}$ to $g > 0$ that is compatible with the propagation of vacua.*

The proof of this claim has not appeared in the literature yet. Let us therefore briefly sketch the path along which the author has arrived at the claim above.

The main observation to be made is that there exists a unitary modular functor $\mathcal{F}_{\rm Teich}$ whose restriction to $g = 0$ is similar to $\mathcal{F}_{\rm Vir}$ in the sense of Definition 5. $\mathcal{F}_{\rm Teich}$ was constructed in [TT].[7]

Uniqueness is in fact quite easily seen by noting that arguments well-known from rational conformal field theory [MS2] carry over to the case at hand and allow us to derive an explicit formula for the coefficients $S_{\alpha\beta}(\gamma)$ in terms of $\mathrm{F}_{\alpha_s\alpha_u}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ and $\mathrm{B}_{\alpha_3\alpha_2\alpha_1}$, namely

$$\begin{aligned} &F_{0\beta}\big[\begin{smallmatrix}\alpha_1 & \alpha_1\\ \alpha_1 & \alpha_1\end{smallmatrix}\big] S_{\alpha_1\alpha_2}(\beta) \,=\\ &\quad=\, S_{0\alpha_2}(0)\int d\mu_{\rm PL}(\gamma)\; e^{\pi i(2\Delta_{\alpha_2}+2\Delta_{\alpha_1}-2\Delta_\gamma-\Delta_\beta)}\; \mathrm{F}_{0\gamma}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\big]\mathrm{F}_{\gamma\beta}\big[\begin{smallmatrix}\alpha_1 & \alpha_1\\ \alpha_2 & \alpha_2\end{smallmatrix}\big]\,. \end{aligned} \tag{6.24}$$

6.6 Remarks

It is often natural to first focus attention on the subspace of "tempered" conformal blocks which are obtained from the gluing construction by using three point conformal blocks associated to representations from the tempered dual $\mathbb{T}$ only. The formulation of the theory as a modular functor applies straightforwardly to this case.

However, in the case of the Virasoro algebra we may observe rather nice analytic properties of the conformal blocks when considered as functions of the representation labels α_k [TR]. The dependence w.r.t. the external representations is *entire analytic*, while the dependence w.r.t. the internal representations is *meromorphic*. The poles are given by the zeros of the Kac determinant.

[7] The key step in the verification of our claim above is to notice that the restriction of $\mathcal{F}_{\rm Teich}$ to surfaces with $g = 0$ is similar to the modular functor $\mathcal{F}_{\rm Qgrp}^{g=0}$ coming from the harmonic analysis on the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that was mentioned above. This establishes the existence of an extension of $\mathcal{F}_{\rm Vir}$ to $g > 0$.

The factorization property of the analytically continued conformal blocks follows from the corresponding property of the tempered conformal blocks. Analytic continuation w.r.t. the representation labels therefore allows us to generate a large class of conformal blocks with reasonable behavior at the boundaries of the Teichmüller spaces from the tempered conformal blocks. We will call this class of conformal blocks the factorizable conformal blocks. It is not clear to the author how this class compares to the set of *all* solutions to the conformal Ward identities.

7. Existence of a canonical scalar product?

We propose that for each vertex algebra V whose conformal blocks have the factorization property there always exists a distinguished choice for H_σ, canonically associated to V, which is "diagonal", i.e. such that H_σ is of the form

$$H_\sigma \;=\; \int\limits_{\mathbb{U}^{\sigma_1}} \prod_{e\in\sigma_1} d\mu_{\rm Pl}(r_e) \bigotimes_{p\in\sigma_0} H_p^{\bar{\rho}_p\rho_p}\,, \tag{7.1}$$

where $\bar{\rho}_p$ is the decoration of S_p obtained from ρ_p by replacing each representation by its dual, σ_1 is the set of edges and σ_0 is the set of vertices of the graph σ. In rational CFT this case is often referred to as the CFT corresponding to the "diagonal modular invariant". We propose the terminology "V-minimal model" for the corresponding CFT.

Whenever the hermitian form H is positive definite we get a *scalar product* on the space of conformal blocks. We will subsequently argue that this is always the case if the representations in question are unitary.

7.1 Existence of a canonical hermitian form from the factorization property

Note that $\dim\mathfrak{H}_{r_2r_1}^{r_3} = 1$ whenever one of the representations R_{r_i}, $i = 1, 2, 3$ coincides with the vacuum representation, and the two other representations are R and $\bar{R}$, with $\bar{R}$ being the dual of R. This implies that there is a unique (up to a constant) conformal block associated to the diagram on the left of Figure 9 if the representation associated to the edges with label 0 is the vacuum representation and if the representations associated to the edges with labels $1, \bar{1}, 2, \bar{2}$ are chosen as $R_1, \bar{R}_1, R_2, \bar{R}_2$, respectively. This conformal block will be denoted as $G^{\sigma_1}_{0\tau}\big[\begin{smallmatrix} r_2 & \bar{r}_1 \\ \bar{r}_2 & r_1\end{smallmatrix}\big]$.

Let us, on the other hand, use the notation $G^{\sigma_2}_{r_3\tau}\big[\begin{smallmatrix} \bar{r}_2 & r_2 \\ \bar{r}_1 & r_1\end{smallmatrix}\big]_{\imath\bar{\imath}}$ for the conformal blocks associated to the diagram on the right of Figure 9 in the case that the representations associated to the edges with labels $1, 2, \bar{1}, \bar{2}$ are chosen as above. The indices $\imath, \bar{\imath}$ are associated to the vertices enclosed in little circles in a manner that should be obvious.

Bear in mind that we are considering vertex algebras whose conformal blocks have the factorization property. It follows in particular that the conformal blocks $G^{\sigma_1}_{0\tau}\big[\begin{smallmatrix} r_2 & \bar{r}_1 \\ \bar{r}_2 & r_1\end{smallmatrix}\big]$ and $G^{\sigma_2}_{r_3\tau}\big[\begin{smallmatrix} \bar{r}_2 & r_2 \\ \bar{r}_1 & r_1\end{smallmatrix}\big]_{\bar{\imath}\imath}$

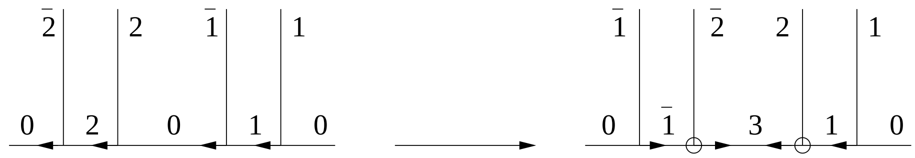

Figure 9: Simplified representation for the markings involved in the relation (7.2).

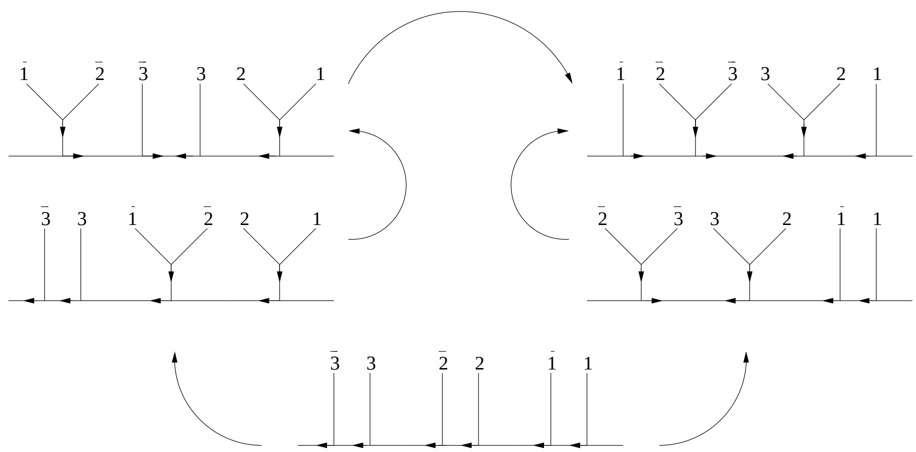

Figure 10: Proof of invariance under F.

are related by an expansion of the form

$$G^{\sigma_1}_{0\tau}\left[\begin{smallmatrix} r_2 & \bar r_1\\ \bar r_2 & r_1\end{smallmatrix}\right] \;=\; \int d\mu_{12}(r_3) \sum_{\imath,\bar\imath\in\mathbb{I}_p} D_{\bar\imath\imath}(r_3|r_2,r_1)\, G^{\sigma_2}_{r_3\tau}\left[\begin{smallmatrix}\bar r_2 & r_2\\ \bar r_1 & r_1\end{smallmatrix}\right]_{\bar\imath\imath}. \tag{7.2}$$

Conjecture 1.
There exists a subset $\mathbb{T}$ *of* $\mathbb{U}$ *parametrizing "tempered" representations such that for* $r_1, r_2 \in \mathbb{T}$ *the measure* $d\mu_{12}$ *is supported in* $\mathbb{T}$*. In this case there exists a factorization of the form*

$$d\mu_{12}(r_3)\; D_{\imath\bar\imath}(r_3|r_2,r_1) \;=\; d\mu_{\rm Pl}(r_3)\; D_{\bar\imath\imath}(r_3,r_2,r_1)\,, \tag{7.3}$$

with $d\mu_{\rm Pl}$ *being independent of* r_2*,* r_1 *such that the hermitian forms on spaces of conformal blocks constructed via (7.1) from* $d\mu_{\rm Pl}(s)$ *and* $D_{\bar\imath\imath}(r_3,r_2,r_1)$ [8] *satisfy the invariance property (4.13).*

In other words, the data appearing in the relationship (7.2) characterize a hermitian form on spaces of conformal blocks canonically associated with any vertex algebra V that has the factorization property.

[8] $H^{\bar\rho\rho}_p(f,g) = \sum_{\bar\imath,\imath\in\mathbb{I}}(f(\imath'))^* D_{\bar\imath\imath}(r_3,r_2,r_1)g(\imath)$

7.1.1

Let us note that validity of the conjecture above is known in the case of rational CFT's. Indeed, let us keep in mind that according to §3.3.2 above one may express the three point function in the V-minimal model in terms of the numbers $D_{\imath\bar{\imath}}(r_3, r_2, r_1)$ introduced in the conjecture above. In the case that the operators Z_p, B_p and F_{pq} are represented by matrices it is easy to figure out an expression of $D_{\imath\bar{\imath}}(r_3, r_2, r_1)$ in terms of the matrix elements of Z_p and F_{pq}. This expression coincides with the formula for the three point function that was obtained as a special case of the general formalism developed in [FRS4] for the description of correlation function in rational CFT. Invariance of the corresponding hermitian form follows from the relations satisfied by the operators Z_p, B_p F_{pq}, S_p that were discussed in the previous section.

Our main point is of course to propose that a similar relationship also holds in nonrational cases. And indeed, given that there exists a factorization of the form (7.3) it is not hard to show invariance under F by considering the sequence of transformations indicated in Figure 10. Invariance under B is verified similarly. Invariance under Z follows from the invariance under F thanks to (4.14). The conjecture is furthermore supported by the results from [TL] reviewed in Section 6 above.

7.2 Unitary fusion

There is a generalization of the tensor product for unitary representations of certain vertex algebras that has the virtue to make unitarity of the resulting representation manifest. The underlying theory is closely related to the theory of superselection sectors from algebraic quantum field theory [FRS, Ha]. We will in the following briefly discuss a reformulation called "Connes-fusion" [Wa].

In order to simplify the exposition, we will restrict attention to the case of the Virasoro algebra with $c > 25$. What follows is a sketch of the picture that would result from using the results of [TL] within a theory of "Connes-fusion" of representations of $\mathrm{Diff}(S_1)$ along the lines of [Wa]. The author believes that similar things can be done for many other vertex algebras, which would allow one to show that the canonical hermitian form proposed in the previous subsection is positive definite in the case of unitary representations.

7.2.1

It is temporarily useful to replace the states v_m within the representations R_m, $m = 1, 2$ by the vertex operators $\mathsf{V}_m \equiv \mathsf{V}_m(v_m)$ which generate the states v_m from the vacuum, $v_m = \mathsf{V}_m\Omega$. We want to think of representations R_m, $m = 1, 2$ as representations of $\mathrm{Diff}(\mathrm{I}_m)$ associated to the intervals I_m, respectively. The vertex operators V_m should therefore commute with the action of $\mathrm{Diff}(\mathrm{I}_m^c)$ as

$$\pi_{R_m}(g)\mathsf{V}_m = \mathsf{V}_m\pi_0(g) \quad \text{for all} \quad g\in\mathrm{Diff}(\mathrm{I}_m^c)\,, \tag{7.4}$$

with $\pi_0(g)$ being the action of $\mathrm{Diff}(\mathrm{I}_m^c)$ on the vacuum representation V. Operators with such a property can be constructed from the chiral vertex operators $\mathsf{Y}^{\alpha_2}_{\alpha_3\alpha_1}(\mathfrak{v}_2|z)$ as

$$\mathsf{V}_m \;\equiv\; \int_{\mathrm{I}_m} dx\; f_m(x)\,\mathsf{Y}^{r_m}_{r_m 0}(\mathfrak{v}_m|e^{ix})\,, \qquad m=1,2\,, \tag{7.5}$$

f being a smooth function with support in I_m. Operators like the one defined in (7.5) will be unbounded in general, but bounded operators can be obtained by taking the phase of its polar decomposition.

Let us then consider the spaces $\mathcal{V}_m$ of *bounded* intertwiners $\mathsf{V}_m : V \to R_m$ which satisfy (7.4). On $\mathcal{V}_1\otimes\mathcal{V}_2$ define an inner product by

$$\big\langle\, w_1\otimes w_2\,,\, v_1\otimes v_2\,\big\rangle \;=\; \big\langle\,\Omega\,,\,\mathsf{W}_2^*\mathsf{V}_2\,.\mathsf{W}_1^*\mathsf{V}_1\,\Omega\big\rangle \tag{7.6}$$

The Hilbert space completion of $\mathcal{V}_1\otimes\mathcal{V}_2$ is denoted $R_1\boxtimes R_2$. We observe that the scalar product of the "fused" representation is defined in terms of the conformal block $G^{\sigma_1}_{0\tau}\big[\begin{smallmatrix} r_2 & \bar{r}_1\\ \bar{r}_2 & r_1\end{smallmatrix}\big]$ that had previously appeared in the relations (7.2).

7.2.2

In the case of the Virasoro algebra one may deduce the validity of relations (7.2) from (6.16) by analytically continuing α_s to the value $\alpha_s=0$. This allows one to write $\|\,v_1\otimes v_2\,\|^2$ in the form

$$\|\,v_1\otimes v_2\,\|^2 \;=\; \int_{\mathbb{U}} d\mu_{21}(\alpha_s)\;\|\,\mathsf{V}^{(2)}_{\alpha_s}(v_2,v_1)\,\|^2_{R_{\alpha_s}}\,, \tag{7.7}$$

where $\mathsf{V}^{(2)}_{\alpha_s}(v_2,v_1)$ is a certain vector in the *irreducible* representation R_{α_s} that may be written as

$$\mathsf{V}^{(2)}_{\alpha_s}(v_2,v_1)\;\equiv\;\mathsf{V}^2_{s1}v_1\;=\;\int_{\mathrm{I}_2} dx\; f_2(x)\,\mathsf{Y}^{\alpha_2}_{\alpha_s\alpha_1}(\mathfrak{v}_2|e^{ix})\,\mathsf{V}_1\,\Omega\,,$$

provided that V_2 can be represented in the form (7.5). Note that the space $R_1\boxtimes R_2$ is naturally a representation of $\mathrm{Diff}(\mathrm{I}_1)\times\mathrm{Diff}(\mathrm{I}_2)$. Equation (7.7) is interpreted as an expression for the unitary equivalence

$$R_1\,\boxtimes\,R_2\;\simeq\;\int_{\mathbb{U}}^{\oplus} d\mu_{21}(s)\;R_s \tag{7.8}$$

which implies in particular the fact that the representation of $\mathrm{Diff}(\mathrm{I}_1)\times\mathrm{Diff}(\mathrm{I}_2)$ on $R_1\boxtimes R_2$ can be extended to a representation of $\mathrm{Diff}(S^1)$. The factorization $d\mu_{21}(s)=d\mu_{\rm Pl}(s)D(r_3,r_2,r_1)$ then allows us to rewrite (7.8) as

$$R_1\,\boxtimes\,R_2\;\simeq\;\int_{\mathbb{U}}^{\oplus} d\mu_{\rm Pl}(s)\;R_s\otimes\mathrm{Hom}(R_1\boxtimes R_2;R_s)\,, \tag{7.9}$$

where $\mathrm{Hom}(R_1 \boxtimes R_2; R_s)$ is the one-dimensional Hilbert space of intertwiners with metric given by $D(r_3, r_2, r_1)$.

7.3 Associativity of unitary fusion

It should be possible to show on general grounds that the fusion operation is associative,

$$(R_1 \boxtimes R_2) \boxtimes R_3 \;\equiv\; R_1 \boxtimes R_2 \boxtimes R_3 \;\equiv\; R_1 \boxtimes (R_2 \boxtimes R_3)\,. \tag{7.10}$$

Indeed, let us consider

$$\|\,(v_1 \otimes v_2) \otimes v_3\,\|^2 \quad \text{and} \quad \|\,v_1 \otimes (v_2 \otimes v_3)\,\|^2 \tag{7.11}$$

The left hand side and the right hand side of (7.11) can be represented respectively as

$$\int_{\mathbb{U}} d\mu_{12}(\alpha_s)\,\big\langle\,\Omega, \mathsf{V}_3^*\mathsf{V}_3\,\mathsf{V}_{21}^*(\alpha_s)\mathsf{V}_{21}(\alpha_s)\,\Omega\,\big\rangle\,, \qquad \mathsf{V}_{21}(\alpha_s) \;\equiv\; \mathsf{V}_{s1}^2\mathsf{V}_1\,, \tag{7.12}$$

$$\int_{\mathbb{U}} d\mu_{23}(\alpha_u)\,\big\langle\,\Omega, \mathsf{V}_{32}^*(\alpha_u)\mathsf{V}_{32}(\alpha_u)\,\mathsf{V}_1^*\mathsf{V}_1\,\Omega\,\big\rangle\,, \qquad \mathsf{V}_{32}(\alpha_u) \;\equiv\; \mathsf{V}_{u2}^3\mathsf{V}_2\,. \tag{7.13}$$

It is useful to note that the compositions of chiral vertex operators which appear in (7.12) and (7.13) correspond to the diagrams on the left and right in the middle line of Figure 10, respectively. From this diagrammatic representation it is easily seen that (7.12) and (7.13) are both equal to

$$\langle\,\Omega, \mathsf{V}_3^*\mathsf{V}_3\,\mathsf{V}_2^*\mathsf{V}_2\,\mathsf{V}_1^*\mathsf{V}_1\,\Omega\,\rangle \;\equiv\; \|\,v_1 \otimes v_2 \otimes v_3\,\|^2\,, \tag{7.14}$$

corresponding to the diagram on the bottom of Figure 10, which makes the associativity of the fusion operation manifest. By using (7.7) one may rewrite (7.12) and (7.13) respectively in the form

$$\int_{\mathbb{U}} d\mu_{123}(\alpha_4)\,\Big\|\,\mathsf{V}^{(3)}_{\alpha_4,s}(v_3,v_2,v_1)\,\Big\|^2\,, \qquad \mathsf{V}^{(3)}_{\alpha_4,s}(v_3,v_2,v_1) \;\equiv\; \int_{\mathbb{U}} d\mu_{12}(\alpha_s)\;\mathsf{V}^{(2)}_{\alpha_4}\big(\,v_3\,,\,\mathsf{V}^{(2)}_{\alpha_s}(v_2,v_1)\,\big) \tag{7.15}$$

$$\int_{\mathbb{U}} d\mu_{123}(\alpha_4)\,\Big\|\,\mathsf{V}^{(3)}_{\alpha_4,u}(v_3,v_2,v_1)\,\Big\|^2\,, \qquad \mathsf{V}^{(3)}_{\alpha_4,u}(v_3,v_2,v_1) \;\equiv\; \int_{\mathbb{U}} d\mu_{23}(\alpha_u)\;\mathsf{V}^{(2)}_{\alpha_4}\big(\,\mathsf{V}^{(2)}_{\alpha_u}(v_3,v_2)\,,\,v_1\,\big)\,. \tag{7.16}$$

These relations may both be seen as expressions for the unitary equivalences

$$(\,R_1 \,\boxtimes\, R_2\,) \,\boxtimes\, R_3 \;\simeq\; \int_{\mathbb{U}}^{\oplus} d\mu_{\rm Pl}(\alpha)\;R_\alpha \otimes L^2(\mathbb{U}, d\mu^{\alpha}_{(12)3})\,, \tag{7.17}$$

$$R_1 \,\boxtimes\, (\,R_2 \,\boxtimes\, R_3\,) \simeq \int_{\mathbb{U}}^{\oplus} d\mu_{\rm Pl}(\alpha)\;R_\alpha \otimes L^2(\mathbb{U}, d\mu^{\alpha}_{1(23)})\,, \tag{7.18}$$

where

$$
\begin{aligned}
d\mu^{\alpha}_{(12)3}(\alpha_s) &= d\mu_{\rm Pl}(\alpha_s)\; D(\alpha_s,\alpha_2,\alpha_1)D(\alpha,\alpha_3,\alpha_s)\\
d\mu^{\alpha}_{1(23)}(\alpha_u) &= d\mu_{\rm Pl}(\alpha_u)\; D(\alpha_u,\alpha_3,\alpha_2)D(\alpha,\alpha_u,\alpha_1)
\end{aligned}
$$

It should be noted that the Hilbert spaces $L^2(\mathbb{U}, d\mu^{\alpha}_{(12)3})$ and $L^2(\mathbb{U}, d\mu^{\alpha}_{1(23)})$ which appear in (7.17) and (7.18), respectively, are nothing but different models for Hilbert-subspaces of $\mathrm{Hom}(R_1 \boxtimes R_2 \boxtimes R_3; R_\alpha)$. These Hilbert spaces are canonically isomorphic to the spaces of conformal blocks $\mathcal{H}_{\sigma_1}$ and $\mathcal{H}_{\sigma_2}$ associated to the markings on the left and right of Figure 7, respectively. It therefore follows from the associativity of the fusion product that there exists a one-parameter family of unitary operators $\mathsf{F} : \mathcal{H}_{\sigma_1} \to \mathcal{H}_{\sigma_2}$ that represents the unitary equivalence between the representations (7.17) and (7.18), respectively.

7.4 Discussion

The author believes that the link between the hermitian form on spaces of conformal blocks and unitary fusion has not received the attention it deserves. More specifically, there are two reasons why the authors believes that the connection between the unitary fusion and the hermitian form on spaces of conformal blocks is worth noting and being better understood:

On the one hand, it offers an explanation for the positivity of the coefficients $D_{\bar{u}}(r_3, r_2, r_1)$ defining the hermitian form H_V in the unitary case, thereby elevating it to a scalar product.

If, on the other hand, one was able to show on a priori grounds that the representation $\mathrm{Diff}(\mathrm{I}_1) \times \mathrm{Diff}(\mathrm{I}_2)$ on $R_1 \boxtimes R_2$ can be extended to a representation of $\mathrm{Diff}(S^1)$ then one might use this as a basis for a conceptual proof of the factorization property (4.11) in genus 0 along the lines sketched above.

It does not seem to be possible, however, to give a simple explanation of the factorization (7.3) in Conjecture 1 from this point of view. This deep property seems to require new ideas for its explanation. We see it as a hint towards an even deeper level of understanding CFT in its relation to the harmonic analysis of $\mathrm{Diff}(S_1)$, or some extension thereof.

8. Outlook

First we would like to stress that the class of nonrational CFT covered by the formalism described in this paper is expected to contain many CFTs of interest. To illustrate this claim, let us formulate the following conjecture.

8.1 Modular functors from W-algebras and Langlands duality

We would finally like to formulate a conjecture. Let $W_k(\mathfrak{g})$ be the W-algebra constructed in [FF1, FF2]

Conjecture 2. *There exists a family of stable unitary modular functors*

$$\big(\,\Sigma\,,\,\mathfrak{g}\,,\,k\,\big)\;\longmapsto\;\big(\mathfrak{H}_{\mathfrak{g},k}(\Sigma)\,,\,\mathsf{M}_{\mathfrak{g},k}(\Sigma)\,\big)$$

that is canonically isomorphic to either

> *the space of conformal blocks for certain classes of unitary representations of the W-algebra $\mathcal{W}_k(\mathfrak{g})$ with its natural unitary mapping class group action,*

or

> *the space of states obtained via the quantization of the higher Teichmüller spaces [FG1, FG2] together with its canonical mapping class group action*

such that Langlands duality holds: There is a canonical isomorphism

$$\big(\mathfrak{H}_{\mathfrak{g},k}(\Sigma)\,,\,\mathsf{M}_{\mathfrak{g},k}(\Sigma)\,\big)\;\simeq\;\big(\mathfrak{H}_{{}^{\rm L}\mathfrak{g},\check{k}}(\Sigma)\,,\,\mathsf{M}_{{}^{\rm L}\mathfrak{g},\check{k}}(\Sigma)\,\big)\,,$$

with ${}^{\rm L}\mathfrak{g}$ being the Langlands dual to the Lie algebra $\mathfrak{g}$ and $\check{k}$ being related to k via $(k+h^\vee)r^\vee = (\check{k}+h^\vee)^{-1}$, $h^\vee$ being the dual Coxeter number.

8.2 Boundary CFT

It seems interesting to note a link to boundary CFT. In the case of the V-minimal model one expects following Cardy's analysis [Ca] to find a one-to-one correspondence between conformal boundary conditions and irreducible representations. There should in particular exist a distinguished boundary condition $B_\circ$ which corresponds to the vacuum representation.

This boundary condition is fully characterized by the measure appearing in the expansion of the corresponding boundary state into the Ishibashi-states $|r\rangle\rangle$ which preserve the full chiral algebra V,

$$|B_\circ\rangle\;=\;\int_{\mathbb{U}}d\mu_{\rm B_\circ}(r)\;|r\rangle\rangle\,. \tag{8.1}$$

It is not hard to see that the two-point function $\langle V_2(z_2,\bar{z}_2)V_1(z_1,\bar{z}_1)\rangle_{B_\circ}$ in the presence of a boundary with condition $B_\circ$ is proportional to $G^{\sigma_1}_{\circ\tau}\big[\begin{smallmatrix} r_2 & \bar{r}_1\\ \bar{r}_2 & r_1\end{smallmatrix}\big]$. The expansion (7.2) describes the

OPE of the two fields V_1, V_2. It easily follows from these observations that the one-point function (in a suitable normalization) coincides with the Plancherel-measure $d\mu_{\rm Pl}(s)$,

$$d\mu_{{\rm B}_\circ}(r) \;=\; d\mu_{\rm Pl}(r)\,. \tag{8.2}$$

We take this observation as an intriguing hint concerning the generalization of our considerations to boundary CFT.

8.3 Nonrational Verlinde formula?

In the case of rational CFT one can deduce a lot of useful relations [MS1, MS2] between the defining data of a modular functor from the relations (4.2)-(4.4), (4.9)-(4.10) and (4.14)-(4.17). These relations give the key to some derivations of the famous Verlinde formula. Much of this remains intact in the nonrational case, as the example of formula (6.24) illustrates.

A fundamental difference comes from the fact that the vacuum representation is *not* in the support of $\mu_{\rm Pl}$. This implies that objects like ${\rm F}_{\alpha\beta}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big]$ or $S_{\alpha\beta}(\gamma)$ are not necessarily well-defined at $\beta=0$. This means that many of the relations valid in rational CFT do not have obvious counterparts in the nonrational case.

As a particularly interesting example let us note that in the case of the minimal models one has the relation [Ru]

$${\rm F}_{0r}\big[\begin{smallmatrix}r_2 & r_1\\ r_2 & r_1\end{smallmatrix}\big]\,{\rm F}_{r0}\big[\begin{smallmatrix}r_2 & r_2\\ r_1 & r_1\end{smallmatrix}\big] \;=\; \frac{S_{0r}S_{00}}{S_{0r_2}S_{0r_1}}\,,\qquad S_{r_1r_2}\equiv S_{r_1r_2}(0)\,. \tag{8.3}$$

As explained above, the left hand side does not have an obvious counterpart in the nonrational case in general. However, in the case of the Virasoro algebra with $c>25$ it turns out that

$${\rm F}''_{\alpha 0}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] \;=\; \lim_{\beta\to 0}\,\beta^2\,{\rm F}_{\alpha\beta}\big[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\big] \tag{8.4}$$

exists and satisfies the relation

$${\rm F}_{0\alpha}\big[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\big]\,{\rm F}''_{\alpha 0}\big[\begin{smallmatrix}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{smallmatrix}\big] \;=\; \frac{B_0\,B(\alpha)}{B(\alpha_2)B(\alpha_1)}\,, \tag{8.5}$$

where $B(\alpha)=\sin 2b(\alpha-\delta)\sin 2b^{-1}(\delta-\alpha)$. Equation (8.5) can be verified with the help of the exlicit expressions for the objects involved.

The relation (8.5) is particularly suggestive in view of the fact that $S_0{}^r$ gets identified with the so-called quantum dimension in the correspondence between modular functors and modular tensor categories [BK1]. What appears on the right hand side of (8.5) is related to the measure $d\mu_{\rm Pl}$ via $d\mu_{\rm Pl}(\alpha)=d\alpha\,B(\alpha)$, with $d\alpha$ being the standard Lebesque measure on $\mathbb{T}$.

This measure can be seen as the most natural counterpart of the quantum dimension in the nonrational case. This is seen most clearly when considering the quantum group structure[9] associated to a rational modular functor [Pf]. The quantum dimension represents the weight of a representation in the Plancherel (or Peter-Weyl) decomposition of the space of functions on the quantum group. As mentioned above, there is a quantum group "dual" to the modular functor defined by the representation theory of the Virasoro algebra with $c > 25$ [PT1, TR]. The natural measure appearing in the decomposition of the space of functions on the corresponding quantum group is precisely $d\mu_{\rm Pl}$ [PT1].

It is clearly an important open task for the future to analyze more systematically the set of relations that can be obtained in such a way.

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