HOW FAR IS AN ULTRAFLAT SEQUENCE OF UNIMODULAR POLYNOMIALS FROM BEING CONJUGATE-RECIPROCAL?

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Abstract. In this paper we study ultraflat sequences (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ in general, not necessarily those produced by Kahane in his paper [Ka]. We examine how far is a sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ from being conjugate reciprocal. Our main results include the following.

Theorem. *Given a sequence (ε_n) of positive numbers tending to 0, assume that (P_n) is a (ε_n)-ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. The coefficients of P_n are denoted by $a_{k,n}$, that is,*

$$P_n(z) = \sum_{k=0}^n a_{k,n} z^k \,, \qquad , k = 0, 1, \dots, n, \quad n = 1, 2, \dots \,.$$

Then

$$\sum_{k=0}^n k^2 |a_{k,n} - \overline{a}_{n-k,n}|^2 \geq \left(\frac{1}{3} + \delta_n \right) n^3 \,,$$

where (δ_n) is a sequence of real numbers converging to 0.

1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k z^k, \;\; a_k \in \mathbb{C} \,, \;\; |a_k| = 1 \right\} .$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^n a_k z^k, \;\; a_k \in \{-1, 1\} \right\} .$$

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The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| < \sqrt{n+1} < \max_{z \in \partial D} |P_n(z)| \,. \tag{1.1}$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1} \,, \qquad z \in \partial D? \tag{1.2}$$

Obviously (1.2) is impossible if $n \geq 1$. So one must look for less than (1.2), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1} \,, \qquad z \in \partial D \,, \tag{1.3}$$

or equivalently

$$\max_{z \in \partial D} \big| |P_n(z)| - \sqrt{n+1} \big| \leq \varepsilon \sqrt{n+1} \,.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if*

$$(1-\varepsilon_{n_k})\sqrt{n_k+1} \leq |P_{n_k}(z)| \leq (1+\varepsilon_{n_k})\sqrt{n_k+1} \,, \qquad z \in \partial D \,, \tag{1.4}$$

or equivalently

$$\max_{z \in \partial D} \big| |P_{n_k}(z)| - \sqrt{n_k+1} \big| \leq \varepsilon_{n_k} \sqrt{n_k+1} \,.$$

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z \in \partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1} \,, \tag{1.4}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where

$$\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right) . \tag{1.5}$$

Thus the Erdős conjecture (1.4) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$.

An extension of Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS].

2. New Results

In this paper we study ultraflat sequences (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ in general, not necessarily those produced by Kahane in his paper [Ka]. With trivial modifications our results remain valid even if we study ultraflat sequences (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$. It is left to the reader to formulate these analogue results. We examine how far an ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is from being conjugate reciprocal. Our main results are formulated by the following theorems. In each of Theorems 2.1 – 2.3 we assume that (ε_n) is a sequence of positive numbers tending to 0, and the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is (ε_n)-ultraflat.

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k , \qquad a_k \in \mathbb{C} ,$$

then its conjugate polynomial is defined by

$$Q_n^*(z) := z^n \overline{Q}_n(1/z) := \sum_{k=0}^{n} \overline{a}_{n-k} z^k .$$

Theorem 2.1. *We have*

$$\int_{\partial D} (|P_n'(z)| - |P_n^{*\prime}(z)|)^2 \, |dz| = 2\pi \left(\frac{1}{3} + \gamma_n\right) n^3 ,$$

where (γ_n) *is a sequence of real numbers converging to 0.*

Theorem 2.2. *If the coefficients of* P_n *are denoted by* $a_{k,n}$*, that is*

$$P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k , \qquad k = 0, 1, \ldots , n, \quad n = 1, 2, \ldots ,$$

then

$$\sum_{k=0}^{n} k^2 |a_{k,n} - \overline{a}_{n-k,n}|^2 \geq \left(\frac{1}{3} + \delta_n\right) n^3 ,$$

where (δ_n) *is a sequence of real numbers converging to 0.*

Theorem 2.3. *We have*

$$\int_{\partial D} |P_n(z) - P_n^*(z)|^2 \, |dz| \geq 2\pi \left(\frac{1}{3} + \gamma_n \right) n \,,$$

where (γ_n) *is a sequence of real numbers converging to 0. Using the notation of Theorem 2.2, in terms of the coefficients of* P_n*, we have*

$$\sum_{k=0}^{n} |a_{k,n} - \overline{a}_{n-k,n}|^2 \geq \left(\frac{1}{3} + \delta_n \right) n \,,$$

where (δ_n) *is a sequence of real numbers converging to 0.*

Remark 2.4 Theorem 2.4 tells us much more than the non-existence of an ultraflat sequence of conjugate reciprocal unimodular polynomials. It measures how far such an ultraflat sequence is from being a sequence of conjugate reciprocal polynomials.

3. Lemmas

To prove the theorems in Section 2, we need two lemmas. The first one can be checked by a simple calculation.

Lemma 3.1. *Let* P_n *be an arbitrary polynomial of degree* n *with complex coefficients having no zeros on the unit circle. Let*

$$f_n(z) := \frac{zP_n'(z)}{P_n(z)} \qquad \text{and} \qquad f_n^*(z) := \frac{zP_n^{*\prime}(z)}{P_n^*(z)} \,.$$

Then

$$\overline{f_n(z)} + f_n^*(z) = n \,, \qquad z \in \partial D \,.$$

Our next lemma may be found in [MMR] (page 676) and is due to Malik.

Lemma 3.2. *Let* P_n *be an arbitrary polynomial of degree* n *with complex coefficients. We have*

$$\max_{z \in \partial D} \left(|P_n'(z)| + |P_n^{*\prime}(z)| \right) \leq n \max_{z \in \partial D} |P_n(z)| \,.$$

Lemma 3.3 (Bernstein's Inequality in $L_2(\partial D)$). *If* Q_n *is a polynomial of degree at most* n *with complex coefficients, then*

$$\int_{\partial D} |Q_n'(z)|^2 \, |dz| \leq n^2 \int_{\partial D} |Q_n(z)|^2 \, |dz| \,.$$

4. Proof of the Theorems

Proof of Theorem 2.1. Lemma 3.2 combined with the ultraflatness of (P_n) implies that

$$|P_n'(z)| + |P_n^{*\prime}(z)| \leq n \max_{z \in \partial D} |P_n(z)| \leq (1+\varepsilon_n)(n+1)^{3/2}$$

for every $z \in \partial D$. Lemma 3.1 combined with the ultraflatness of P_n imply

$$|P_n'(z)| \frac{1}{(1-\varepsilon_n)\sqrt{n+1}} + |P_n^{*\prime}(z)| \frac{1}{(1-\varepsilon_n)\sqrt{n+1}} \geq \frac{|P_n'(z)|}{|P_n(z)|} + \frac{|P_n^{*\prime}(z)|}{|P_n^*(z)|} \geq n\,,$$

that is

$$|P_n'(z)| + |P_n^{*\prime}(z)| \geq (1-\varepsilon_n) n^{3/2}$$

for every $z \in \partial D$. We conclude that

$$(1-\varepsilon_n)^2 n^3 \leq (|P_n'(z)| + |P_n^{*\prime}(z)|)^2 \leq (1+\varepsilon_n)^2 (n+1)^3\,, \qquad z \in \partial D\,.$$

Multiplying the expression in the middle out and integrating on ∂D with respect to $|dz|$, we obtain

$$\begin{aligned} 2\pi(1-\varepsilon_n)^2 n^3 &\leq \int_{\partial D} |P_n'(z)|^2\,|dz| + \int_{\partial D} |P_n^{*\prime}(z)|^2\,|dz| + 2\int_{\partial D} |P_n'(z)P_n^{*\prime}(z)|\,|dz| \\ &\leq 2\pi(1+\varepsilon_n)^2 n^3\,. \end{aligned}$$

Note that

$$\begin{aligned} \text{(2.1)} \qquad \int_{\partial D} |P_n'(z)|^2\,|dz| &= \int_{\partial D} |P_n^{*\prime}(z)|^2\,|dz| = 2\pi \sum_{k=1}^{n} k^2 \\ &= 2\pi \frac{n(n+1)(2n+1)}{6} \sim \frac{2\pi}{3} n^3\,. \end{aligned}$$

Hence

$$\int_{\partial D} |P_n'(z)||P_n^{*\prime}(z)|\,|dz| = 2\pi \left(\frac{1}{6} + \delta_n\right) n^3$$

with constants δ_n converging to 0. Integrating the equation

$$(|P_n'(z)| - |P_n^{*\prime}(z)|)^2 = |P_n'(z)|^2 + |P_n^{*\prime}(z)|^2 - 2|P_n'(z)P_n^{*\prime}(z)|\,,$$

and using observation (2.1) we obtain the theorem. □

Proof of Theorem 2.2. Parseval Formula and the triangle inequality give

$$\begin{aligned} \sum_{k=0}^{n} k^2 |a_{k,n} - \overline{a}_{n-k,n}|^2 &= \int_{\partial D} |P_n'(z) - P_n^{*\prime}(z)|^2\,|dz| \\ &\geq \int_{\partial D} (|P_n'(z)| - |P_n^{*\prime}(z)|)^2\,|dz|\,, \end{aligned}$$

and the theorem follows from Theorem 2.2. □

Proof of Theorem 2.3. Applying Theorem 2.1, the triangle inequality, and the Bernstein inequality in L_2 for $P_n - P_n^*$ (see Lemma 3.3), we obtain

$$2\pi\left(\frac{1}{3}+\gamma_n\right)n^3 = \int_{\partial D}(|P_n'(z)| - |P_n^{*\prime}(z)|)^2\,|dz| \leq \int_{\partial D}|P_n'(z) - P_n^{*\prime}(z)|^2\,|dz|$$
$$\leq n^2\int_{\partial D}|P_n(z) - P_n^*(z)|^2\,|dz|\,,$$

where (γ_n) is a sequence of real numbers converging to 0. Now the first part of the theorem follows after dividing by n^2. To see the second part we proceed as in the proof of Theorem 2.2 by using Parseval's formula.

□

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