

# Embedding a graph into the torus in linear time

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## Abstract

A linear time algorithm is presented that, for a given graph  $G$ , finds an embedding of  $G$  in the torus whenever such an embedding exists, or exhibits a subgraph  $\Omega$  of  $G$  of small branch size that cannot be embedded in the torus.

## 1 Introduction

Let  $K$  be a subgraph of  $G$ , and suppose that we are given an embedding of  $K$  in some surface. The *embedding extension problem* asks whether it is possible to extend the embedding of  $K$  to an embedding of  $G$  in the same surface, and any such embedding is an *embedding extension* of  $K$  to  $G$ . An *obstruction* for embedding extensions is a subgraph  $\Omega$  of  $G - E(K)$  such that the embedding of  $K$  cannot be extended to  $K \cup \Omega$ . The obstruction is *small* if  $K \cup \Omega$  is homeomorphic to a graph with a small number of edges. If  $\Omega$  is small, then it is easy to verify (for example, by checking all the possibilities

embedding extension  
problem

embedding extension

obstruction

small

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\*Supported in part by the Ministry of Science and Technology of Slovenia, Research Project P1-0210-101-94.

for the rotation systems of  $K \cup \Omega$ ) that no embedding extension to  $K \cup \Omega$  exists, and hence  $\Omega$  is a good verifier that there are no embedding extensions of  $K$  to  $G$  as well.

It is known [21] that the general problem of determining the genus, or the non-orientable genus of graphs is NP-hard. However, for every fixed surface there is a polynomial time algorithm which checks if a given graph can be embedded in the surface. Such algorithms were found first by Filotti *et al.* [5]. Unfortunately, even for the torus their algorithm has time complexity estimated only by  $\mathcal{O}(n^{188})$ . A special polynomial time algorithm for embedding cubic graphs in the torus has been published by Filotti [4]. Robertson and Seymour developed an  $\mathcal{O}(n^3)$  algorithm using graph minors (with recent improvement by B. Reed to  $\mathcal{O}(n^2 \log n)$ ) [18, 19, 20].

The main result of this paper is a linear time algorithm that, for a given graph  $G$ , finds an embedding of  $G$  in the torus whenever such an embedding exists, or exhibits a small obstruction  $\Omega$  in  $G$  that cannot be embedded in the torus (Theorem 8.1). Moreover, obtained obstruction  $\Omega$  is minimal in the sense that every proper subgraph of  $\Omega$  admits an embedding in the torus. Obstructions for some related embedding extension problems in the torus are characterized in the same way. As a consequence, our algorithm proves finiteness of the number of forbidden subgraphs for embeddability in the torus, a special case of Robertson and Seymour's generalized Kuratowski theorem [16].

These and other auxiliary results of this paper are used as the corner stones in the design of linear time algorithms for the embeddability of graphs in general surfaces [12].

2-cell embeddings in orientable surfaces can be described combinatorially [7] by specifying a *rotation system*: for each vertex  $v$  of the graph  $G$  we have the cyclic permutation  $\pi_v$  of its neighbors, representing their circular order around  $v$  on the surface. In order to make a clear presentation of our algorithm, we have decided to use this description only implicitly. Whenever we say that we have an embedding, we mean such a combinatorial description. Whenever used, it is easy to see how one can combine the embeddings of some parts of the graph described this way into an embedding of larger subgraphs.

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by Cook and Reckhow [3]. More precisely, our model is the *unit-cost* RAM where operations on integers, whose value is  $O(n)$ , need only

rotation system

unit-cost

constant time ( $n$  is the order of the given graph).

## 2 Basic definitions

Let  $K$  be a connected subgraph of  $G$ . A vertex of  $K$  of degree in  $K$  different from 2 is a *main vertex* of  $K$ . A *branch* of  $K$  is any simple walk in  $K$  (possibly closed) whose endpoints are main vertices but no internal vertex on this walk is a main vertex. A *bridge* of  $K$  in  $G$  (also called a  *$K$ -bridge* in  $G$ ) is a subgraph of  $G$  which is either an edge  $e \in E(G) \setminus E(K)$  (together with its endpoints) which has both endpoints in  $K$ , or it is a connected component of  $G - V(K)$  together with all edges (and their endpoints) between this component and  $K$ . Each edge of a  $K$ -bridge  $B$  having an endpoint in  $K$  is a *foot* of  $B$ . The vertices of  $B \cap K$  are the *vertices of attachment* of  $B$ . If a  $K$ -bridge is attached to a single branch of  $K$ , it is said to be *local*. The number of branches of  $K$  is called the *branch size* of  $K$ .

A *block* of a graph  $G$  is either a set of  $p \geq 1$  parallel edges whose removal disconnects the graph or a maximal 2-connected subgraph of  $G$ . In a 2-connected graph  $G$ , one can also define the concept of 3-connected components. If  $G = H \cup K$ , where  $H$  and  $K$  are edge-disjoint graphs, has exactly 2 vertices in common, and each of  $H$  and  $K$  contains at least 2 edges, then the pair  $\{H, K\}$  is a *2-separation* of  $G$ . Denote by  $x, y$  the vertices of  $V(H) \cap V(K)$ . The 2-separation is *elementary* if either  $H - \{x, y\}$  or  $K - \{x, y\}$  is non-empty and connected, and either  $H$  or  $K$  is a block. Assume that the 2-separation  $\{H, K\}$  of  $G$  is elementary. Denote by  $H'$  and  $K'$  the graphs obtained from  $H$  and  $K$ , respectively, by adding to each of them a new edge between  $x$  and  $y$ . Graphs  $H'$  and  $K'$  are called *split components* of the 2-separation. The added edges are *virtual edges*. It is easy to verify that  $H'$  and  $K'$  are both blocks, and we may repeat the process on their elementary 2-separations until no further elementary 2-separations are possible. It turns out [22] that the obtained graphs are either 3-connected, cycles, or parallel edges. Each of the graphs obtained in this way is called a *3-connected component* of  $G$ . The set of 3-connected components of the graph is uniquely determined although different choices of 2-separations may have been used during the process of constructing them (cf. [22] for details). Every 3-connected component consists of several edges of  $G$  and several virtual edges. It is obvious by construction that each edge of  $G$  belongs to exactly

main vertex  
branch  
bridge  
 $K$ -bridge  
foot  
vertices of attachment  
local  
branch size  
block  
2-separation  
elementary  
split components  
virtual edges  
3-connected component

one 3-connected component, and that each virtual edge has a corresponding virtual edge in some other 3-connected component. The 3-connected components of  $G$  can be viewed as subgraphs of  $G$ , where each virtual edge corresponds to a path in  $G$ .

A linear time algorithm for obtaining the 3-connected components of a 2-connected graph was devised by Hopcroft and Tarjan [8].

There are well known linear time algorithms that for a given graph determine whether the graph is planar or not. The first such algorithm was obtained by Hopcroft and Tarjan [9]. There are several other linear time planarity algorithms (e.g. Booth and Lueker [1], Fraysseix and Rosenstiehl [6], Williamson [23, 24]). Extensions of original algorithms produce also an embedding (rotation system) whenever the given graph is found to be planar [2], or find a small obstruction — a subgraph homeomorphic to  $K_5$  or  $K_{3,3}$  — if the graph is non-planar [23, 24]. The subgraph homeomorphic to  $K_5$  or  $K_{3,3}$  is called a *Kuratowski subgraph* of  $G$ .

Kuratowski subgraph

**Lemma 2.1** *There is a linear time algorithm that, given a graph  $G$ , either exhibits an embedding of  $G$  in the plane, or finds a Kuratowski subgraph of  $G$ .*

Let  $K$  be a subgraph of  $G$  with a given 2-cell embedding in some surface. Main vertices and open branches of  $K$  are called *basic pieces* of  $K$ . Let  $F$  be a face of  $K$ . A basic piece of  $K$  is *singular* in  $F$  if it appears more than once on the facial walk  $\partial F$  of  $F$ . The face  $F$  is *singular* if it contains a singular basic piece. If  $\partial F$  contains exactly  $k$  singular branches (and no other singular parts apart from their endpoints), then  $F$  is *k-singular*. We say that  $F$  is *at most 1-singular* if it is either non-singular, 1-singular, or the only singularity is a vertex that appears on  $\partial F$  exactly twice. Further, a face  $F$  of  $K$  is *weakly 2-singular*, if  $\partial F$  contains a singular branch, say  $e$ , and a singular main vertex  $x$  which is not an endpoint of  $e$  and appears on  $\partial F$  exactly twice. Moreover, apart from  $x$ ,  $e$ , and the ends of  $e$ , there are no other singular vertices or branches on  $\partial F$ .

basic pieces

singular

singular

k-singular

at most 1-singular

weakly 2-singular

Suppose that  $K \subseteq G$  is embedded in some surface. Distinct  $K$ -bridges  $B_1$  and  $B_2$  *overlap* in a face  $F$  of  $K$  if  $B_1 \cup B_2$  cannot be embedded in  $F$ . If each of  $B_1$  and  $B_2$  admits an embedding in  $F$ , then their overlapping in  $F$  depends only on appearances of vertices of attachment on  $\partial F$  to which  $B_1$  and  $B_2$ , respectively, are attached. Therefore we will, in the sequel, distinguish only between embeddings of a bridge that do not use exactly the same appearances of vertices of attachment on  $\partial F$ .

overlap

### 3 Some preliminary results

In this section we will prove some auxiliary results which will enable us to simplify embedding extension problems. First we will see that one can restrict attention to bridges that are attached only to two basic pieces.

**Lemma 3.1** *Let  $\Sigma$  be a closed surface. Suppose that  $K$  is a subgraph of  $G$  and let  $\sigma_1, \sigma_2, \sigma_3$  be distinct basic pieces of  $K$ . If the number of  $K$ -bridges in  $G$  that are attached to  $\sigma_1, \sigma_2$  and  $\sigma_3$  is greater than  $6 - 2\chi(\Sigma)$ , then  $G$  cannot be embedded in  $\Sigma$ .*

**Proof.** Suppose that we have an embedding with  $p$  such bridges. By contracting  $\sigma_1, \sigma_2, \sigma_3$  to points, we get an embedding in  $\Sigma$  of a graph containing  $K_{3,p}$ . It is well known (cf. [7]) that the genus of  $K_{3,p}$  is  $\lceil (p-2)/4 \rceil$  and the non-orientable genus is  $\lceil (p-2)/2 \rceil$ . It follows that  $p \leq 6 - 2\chi(\Sigma)$ .  $\square$

**Lemma 3.2** *Let  $K$  be a 2-connected non-planar graph. Suppose that  $K_0$  is a Kuratowski subgraph of  $K$  such that there are no local  $K_0$ -bridges in  $K$ . If  $K$  is embedded in the torus then it has at most one singular face. Moreover, no basic piece appears on a boundary walk of a face more than twice, and a possible singular face is at most 2-singular.*

**Proof.** Suppose that there are two singular faces,  $F_1$  and  $F_2$ . Let  $x_i$  be a singular basic piece on  $\partial F_i$  and let  $\gamma_i$  be a closed curve in  $F_i \cup \{x_i\}$  joining two appearances of  $x_i$  on  $\partial F_i$  ( $i = 1, 2$ ). Since  $K$  is 2-connected,  $\gamma_1$  and  $\gamma_2$  are non-contractible. If they are non-homotopic, then  $x_1 = x_2$  and it is easy to see that  $K$  is planar. This contradicts our assumption. Otherwise, let  $K = K_1 \cup K_2$  where  $K_1$  and  $K_2$  are the two subgraphs in the cylinders between  $\gamma_1$  and  $\gamma_2$ . Note that  $K_1 \cap K_2 = \{x_1, x_2\}$  and that  $x_1, x_2 \in V(K_0)$ . Since  $K$  is 2-connected,  $x_1 \neq x_2$ .  $K_0$  is a subdivision of a 3-connected graph, this implies that  $K_2 - x_1 - x_2$  (or  $K_1 - x_1 - x_2$ ) contains no main vertices of  $K_0$ . Since  $K_0$  is non-planar,  $K_2$  contains a segment of a branch of  $K_0$  joining  $x_1$  and  $x_2$ . Since  $F_1 \neq F_2$ ,  $K_2$  also contains edges of  $K \setminus E(K_0)$ . Such edges then belong to a local  $K_0$ -bridge in  $K$ . A contradiction.

Suppose that  $x$  is a basic piece that appears three times on  $\partial F$ . Then  $K$  can be extended by adding a vertex joined to  $x$  and all basic pieces between

two appearances of  $x$ . This graph satisfies the conditions of the lemma, while its embedding has two singular faces, a contradiction.

The last assertion is proved in the same way. □

Let  $G$  be a 2-connected graph and  $K \subseteq G$ . Then  $G$  is 3-connected modulo  $K$  if for every vertex set  $X \subseteq V(G)$ ,  $|X| = 2$ , every connected component of  $G - X$  contains a main vertex of  $K$ . This is obviously equivalent to the following condition: If  $G^+(K)$  is the graph obtained from  $G$  by adding a new vertex whose neighbors are the main vertices of  $K$ , then  $G^+(K)$  is 3-connected. On the other hand, if  $K$  is homeomorphic to a 3-connected graph, then  $G$  is 3-connected modulo  $K$  if and only if it is 3-connected.

3-connected modulo  $K$

**Proposition 3.3** ([10]) *Let  $K \subseteq G$  and let  $e$  be a branch of  $K$  joining main vertices  $x, y$  of  $K$ . Suppose that  $G$  is 3-connected modulo  $K$ . There is a linear time algorithm that either replaces  $e$  by a branch  $e'$  joining  $x$  and  $y$  such that  $e'$  is internally disjoint from  $K - e$  and such that there are no local bridges of  $K - e + e'$  attached to  $e'$ , or finds a Kuratowski subgraph  $L$  of  $G$  such that  $L \cap K \subseteq e$ .*

## 4 Obstructions and 2-restricted embeddings

Let  $K$  be a subgraph of  $G$  that is 2-cell embedded into the torus. We want to extend the embedding of  $K$  to an embedding of  $G$ . If this is not possible, we would like to exhibit a small obstruction. Unfortunately, obstructions can be arbitrarily large, even if we consider only minimal obstructions whose every proper subgraph is no longer an obstruction. In this case, it can be shown that the obstruction has a special structure, so that, although large, it behaves like a small subgraph.

There are two ways how to circumvent the difficulties with large obstructions. One can replace the obstruction by a small subgraph using an operation called *squashing* (see [11] for details). This operation changes the graph  $G$  into a graph  $G'$  and leaves  $K$  unchanged. It ensures that any embedding of  $G'$  extending any 2-cell embedding of  $K$  gives rise to an embedding of the original graph  $G$  extending the same embedding of  $K$ , and conversely.

squashing

Another possibility is to change the subgraph  $K$  so that the obstruction becomes small. While the structure of large obstructions always permits such

a change, a special care has to be taken. Namely, during our embedding extension algorithm, a subgraph  $K$  of  $G$  (which is an obstruction for planar embeddability) is embedded in all possible ways in the torus. Every such embedding is then tested if it permits an embedding extension to  $G$ . If not, an obstruction  $\Omega$  is obtained. If  $\Omega$  is small, then it is combined with the obstructions for previous embeddings of  $K$ . If  $\Omega$  is not small, then  $K$  is changed and the whole process would have to resume from the beginning. To avoid that, only a special kind of change of  $K$  is permitted, and it turns out that this is sufficient. This change replaces the interiors of two branches  $e, f$  of  $K$  by two disjoint paths in  $\Omega \cup e \cup f$  so that the obtained subgraph  $K'$  is homeomorphic to  $K$ . The 2-cell embeddings of  $K$  and 2-cell embeddings of  $K'$  are in a natural correspondence (determined by the same local rotations at the main vertices). The obstruction  $\Omega$  can be replaced by a small obstruction  $\Omega'$  that obstructs extensions of the considered embedding of  $K'$ . Furthermore, if an obstruction  $\Phi$  for previously tested embeddings of  $K$  intersects the part of  $G$  where changes have been done, we replace part of  $\Phi$  by  $\Omega'$ . This way we make sure that the combined obstruction obtained from the previously tested embeddings of  $K$  remains an obstruction for the corresponding embeddings of  $K'$ . We will refer to such a change of  $K$  and  $\Omega$  as a *compression* of a large obstruction. compression

The latter approach is more appropriate from the algorithmic point of view. In our algorithm, large obstructions can only be obtained from the 2-restricted embedding extension problem (discussed below) and from the corner embedding extension problem (Section 7). In both cases, applied algorithms (Proposition 4.1 and Theorem 7.1) take care of necessary compressions of possible large obstructions. Therefore, obtained obstructions are always small.

Suppose that we have a small obstruction  $\Omega$ . To indicate that the subgraph  $K$  might have been changed (during compression), we also say that  $\Omega$  is *nice*. nice

Suppose that  $\Omega_1$  and  $\Omega_2$  are subgraphs of the same  $K$ -bridge  $B$  in  $G$ . It is shown in [14, Corollary 3.6] that one can construct, in time proportional to  $|E(B)|$ , a subgraph  $\Omega'$  of  $B$  such that either  $\Omega'$  is an obstruction for extending any 2-cell embedding of  $K$  (in which case  $\Omega'$  is small; it has at most 11 branches), or  $\Omega'$  has the following properties:

1.  $\Omega'$  has the same feet as  $\Omega_1 \cup \Omega_2$ .

2.  $\Omega' - V(K)$  is connected.
3.  $\Omega'$  obstructs as strong as  $\Omega_1 \cup \Omega_2$ .
4. If  $K \cup B$  is 3-connected modulo  $K$ , then there are no local  $(K \cup \Omega')$ -bridges in  $K \cup B$ .
5.  $K \cup \Omega'$  has at most  $5\theta - 9$  branches that are contained in  $\Omega'$ , where  $\theta$  is the number of feet of  $\Omega_1 \cup \Omega_2$ .

The operation of replacing  $\Omega_1$  and  $\Omega_2$  by  $\Omega'$  is called *combining* of  $\Omega_1$  and  $\Omega_2$ . combining

If  $\Omega_1$  and  $\Omega_2$  are obstructions, let  $\Omega'$  be an obstruction obtained from  $\Omega_1$  and  $\Omega_2$  by combining  $\Omega_1 \cap B$  and  $\Omega_2 \cap B$  for each  $K$ -bridge  $B$  in  $G$ . This operation will be referred to as *combining* of obstruction. By the above, combining combining can be done in linear time, and  $\Omega'$  obstructs at least as strong as each of  $\Omega_1$  and  $\Omega_2$ .

Most of our procedures for embedding extensions reduce to several applications of a relatively simple embedding extension problem that can be defined as follows. Denote by  $\mathcal{B}$  a set of  $K$ -bridges in  $G$ . Suppose that  $\mathcal{B}$  contains no local  $K$ -bridges and that for every  $B \in \mathcal{B}$ , at most two different embeddings in faces of  $K$  are allowed. Moreover, none of the allowed embeddings uses more than one appearance of each basic piece. The following result is proved in [13].

**Proposition 4.1** *Let  $G, K, \mathcal{B}$  be as above, and let  $Q = \cup\{B \mid B \in \mathcal{B}\}$ . There is a linear time algorithm which either finds an embedding extension of  $K$  to  $K \cup Q$ , or returns a nice obstruction  $\Omega \subseteq Q$  for those embedding extensions that use only allowed embeddings of bridges in  $\mathcal{B}$ . Moreover, at most two  $K$ -bridges in  $K \cup \Omega$  have only one allowed embedding.*

The same result holds also in case when a small number of bridges is allowed to have more than two (though a small number of) different embeddings.

## 5 Embedding distribution of types

Let  $K$  be a subgraph of  $G$  that is 2-cell embedded in some surface and let  $\mathcal{P}$  be the set of all basic pieces of  $K$ . If  $B$  is a  $K$ -bridge, let  $T \subseteq \mathcal{P}$  be the set of basic pieces of  $K$  that  $B$  is attached to. We say that  $B$  is of *type*  $T$ . type

In general, a bridge of type  $T$  can be embedded in two or more faces of  $K$ , and in some faces in several different ways.

Let  $F$  be a face of  $K$ . For  $T \subseteq \mathcal{P}$ , let  $\pi_1, \dots, \pi_k$  be the appearances of basic pieces from  $T$  on  $\partial F$ . An *embedding scheme* for the type  $T$  in  $F$  is a subset of  $\pi_1, \dots, \pi_k$  in which at least one appearance of every basic piece from  $T$  occurs. Note that non-singular faces admit at most one embedding scheme for  $T$ . embedding scheme

There is a natural partial ordering among the embedding schemes for the type  $T \subseteq \mathcal{P}$  in  $F$ , induced by the set inclusion. More precisely, if  $\delta$  and  $\delta'$  are embedding schemes for  $T$  in the same face  $F$ , then  $\delta \preceq \delta'$  if every appearance of a basic piece in  $\delta$  also participates in  $\delta'$ .

For example, suppose that  $\partial F$  contains singular branches  $e$  and  $f$ . Let  $a$  be an endvertex of  $e$ . Denote by  $e_1, e_2, f_1, f_2, a_1, a_2$  appearances on  $\partial F$  of  $e, f$ , and  $a$ , respectively. Let  $B$  be a bridge of type  $T = \{a, e, f\}$ . Then  $T$  admits the following embedding schemes in  $F$ :  $\delta_{ijk} = \{a_i, e_j, f_k\}$  ( $i, j, k \in \{1, 2\}$ ),  $\delta_{.jk} = \{a_1, a_2, e_j, f_k\}$  ( $j, k \in \{1, 2\}$ ),  $\delta_{i.k}, \delta_{ij.}, \delta_{..k}, \delta_{.j.}, \delta_{i..}, \delta_{...}$  (defined similarly, with the dot representing the selection of both appearances). Then, for example,  $\delta_{ijk} \preceq \delta_{.jk} \preceq \delta_{..k} \preceq \delta_{...}$  for every  $i, j$ , and  $k$ .

An embedding of a bridge  $B$  in the face  $F$  is *compatible* with a given embedding scheme  $\delta$  for the type  $T$  in  $F$  if  $B$  is of type  $T$  and is attached only to appearances of basic pieces on  $\partial F$  listed in  $\delta$ . If all appearances are used, then the embedding is *strictly compatible* with  $\delta$ . Note that compatibility is monotone with respect to  $\preceq$ , i.e., if  $\delta \preceq \delta'$  and an embedding of  $B$  is compatible with  $\delta$ , then it is also compatible with  $\delta'$ . compatible

An *embedding distribution* for a type  $T \subseteq \mathcal{P}$  is a selection of embedding schemes for the type  $T$ , possibly in different faces. embedding distribution

Partial order  $\preceq$  can be extended to embedding distributions (where it becomes a quasi-order). If  $\Delta(T)$  and  $\Delta'(T)$  are embedding distributions for  $T \subseteq \mathcal{P}$ , then  $\Delta(T) \preceq \Delta'(T)$  if for every embedding scheme  $\delta \in \Delta(T)$  in a face  $F$ , there is an embedding scheme  $\delta' \in \Delta'(T)$  in the same face  $F$  such that  $\delta \preceq \delta'$ . An embedding of a bridge  $B$  of type  $T$  in the face  $F$  is *compatible* with embedding distribution  $\Delta(T)$  if the embedding of  $B$  is compatible with some  $\delta \in \Delta(T)$ . compatible

Suppose that  $\mathcal{T} = \{T_1, T_2, \dots, T_s\}$  are all types of  $K$ -bridges. An *embedding distribution of types* is a family of embedding distributions  $\Delta = \{\Delta(T_1), \Delta(T_2), \dots, \Delta(T_s)\}$ . The order  $\preceq$  also relates embedding distributions of types in the obvious way. Let  $\mathcal{B}$  be a set of  $K$ -bridges embedded in embedding distribution of types

the surface extending the given embedding of  $K$ . We say that this embedding is *compatible* with the embedding distribution  $\Delta$  if every bridge  $B$  in  $\mathcal{B}$  is embedded compatibly with the embedding distribution  $\Delta(T) \in \Delta$ , where  $T$  is the type of  $B$ . compatible

Suppose that we have an embedding distribution  $\Delta$ . We are interested in embedding extensions of  $K$  to  $G$  compatible with  $\Delta$ . Let  $\mathcal{T}' \subseteq \mathcal{T}$  be the types of  $K$ -bridges that do not contain singular (open) branches of  $K$  and such that there is at least one  $K$ -bridge in  $G$  of this type. In the sequel we will prove that for each type  $T \in \mathcal{T}'$  we can find bridges  $B_1(T), B_2(T), \dots, B_{s(T)}(T)$  ( $s(T)$  will be small and usually we will have  $s(T) \leq 7$ ) of type  $T$  having the following property. Let  $\mathcal{R}$  be the set of these bridges taken over all types  $T \in \mathcal{T}'$ . Then every embedding of  $\mathcal{R}$  compatible with  $\Delta$  satisfies:

- ( $\star$ ) If the embedding of  $\mathcal{R}$  can be extended to an embedding of  $G$  compatible with  $\Delta$  and cannot be extended to an embedding of  $G$  that is compatible with some  $\Delta' \prec \Delta$ , then for every  $T \in \mathcal{T}'$  and  $\delta \in \Delta(T)$ , one of the bridges  $B_1(T), \dots, B_{s(T)}(T)$  is embedded strictly compatible with  $\delta$ .

The bridges  $B_1(T), \dots, B_{s(T)}(T)$  will be called *representatives* for the type  $T$ . representatives

Embedding distribution of types will be used in our algorithms in the following way. For every possible embedding distribution  $\Delta$  we will try to extend the given embedding of  $K$  to an embedding of  $G$ , compatible with  $\Delta$ . Embedding distributions will be selected in increasing order, i.e., if an embedding distribution  $\Delta$  is being considered and  $\Delta' \prec \Delta$ , then  $\Delta'$  has already been considered. Given  $\Delta$ , we determine representatives of types (as explained below). For every embedding of the representatives compatible with  $\Delta$ , we try to extend the embedding to an embedding of  $G$ . Because of ( $\star$ ) and our choice of embedding distributions in monotone order, we can consider only embeddings of the representatives that are not compatible with any  $\Delta' \prec \Delta$ . Obtaining an embedding, we stop and return the embedding (and our mission is completed). Otherwise, a nice obstruction is obtained in the way described at particular places where this procedure is applied. Finally, the obstructions are combined into a nice obstruction for embedding extensions strictly compatible with  $\Delta$ . We will refer to this process as the *procedure of embedding distribution of types*. procedure of embedding distribution of types

From now on we assume that the surface is the torus. In the rest of this section we will show how to get representatives for a type  $T \in \mathcal{T}'$ , given

an embedding distribution  $\Delta(T)$ . We will assume that  $K$  satisfies some additional assumptions:

**(K1)**  $K$  is 3-connected up to vertices of degree 2.

**(K2)**  $K$  is non-planar.

**(K3)** There are no local  $K$ -bridges in  $G$ .

Moreover, we will assume that every  $K$ -bridge  $B$  of type  $T \in \mathcal{T}'$  has at least one embedding compatible with  $\Delta(T)$ . This condition can be verified in linear time, and in case of negative answer, a corresponding bridge can be used as a witness that there is no embedding extension compatible with  $\Delta$ .

If  $T$  contains three or more basic pieces, let  $Q_1, \dots, Q_k$  be the bridges of type  $T$ . If  $k > 6$ , then by Lemma 3.1, the bridges  $Q_1, \dots, Q_7$  cannot extend the embedding of  $K$ . Therefore the bridges  $Q_1, \dots, Q_7$  taken as the representatives  $B_1(T), \dots, B_7(T)$  will satisfy  $(\star)$  no matter which representatives will be taken for other types. On the other hand, if  $k \leq 6$ , let  $s(T) = k$  and let  $Q_1, \dots, Q_k$  be the representatives for the type  $T$ .

Suppose now that  $T = \{\zeta_1, \zeta_2\}$ , where  $\zeta_1$  and  $\zeta_2$  are main vertices. In this case all bridges  $B$  of type  $T$  with the property that  $B + \zeta_1\zeta_2$  is planar, can be embedded parallel to each other in just one of the faces and all attached to the same appearances of  $\zeta_1$  and  $\zeta_2$ . We take one of them as a representative. Note that by the minimality of  $\Delta$ , we can assume that only one embedding scheme for such bridges is provided. If some bridge of type  $T$  does not have the above property, it must always be attached to four or more appearances of basic pieces. As above, we take all (or at most 7) of such bridges as additional representatives for  $T$ .

If  $T = \{\zeta_1, \zeta_2\}$ , where  $\zeta_1$  and  $\zeta_2$  are open branches of  $K$ , we know that  $\zeta_1$  and  $\zeta_2$  are non-singular since  $T \in \mathcal{T}'$ . If  $\Delta(T)$  contains just one embedding scheme, let  $B_1(T)$  be an arbitrary bridge of type  $T$  and  $s(T) = 1$ . Otherwise, since  $\zeta_1$  and  $\zeta_2$  are non-singular,  $\Delta(T)$  contains exactly two embedding schemes  $\delta_1, \delta_2$ , and they correspond to distinct faces  $F_1, F_2$  of  $K$ , respectively. First, we try to embed all bridges of type  $T$  in  $F_1$ . If the test fails, then we find two overlapping bridges,  $B_1$  and  $B_2$ . Then, let  $s(T) = 2$ ,  $B_1(T) = B_1$ ,  $B_2(T) = B_2$ . (Cf. [15] for details how to perform these steps in linear time.) In these two cases, it is clear that the chosen representatives satisfy  $(\star)$ . Otherwise, let  $s(T) = 2$  and let  $B_1(T)$  and  $B_2(T)$  be the left-most and the

right-most bridge, respectively, according to the obtained embedding in  $F_1$ . In this case, having an embedding of  $B_1 \cup B_2$  that is not strictly compatible with  $\Delta(T)$ , all bridges of type  $T$  can be embedded between  $B_1$  and  $B_2$  without obstructing any embedding of other bridges. Since embedding distributions  $\delta$  are considered in monotone order, this shows that  $(\star)$  is satisfied for this type.

The last case is when  $T = \{\zeta_1, \zeta_2\}$ , where  $\zeta_1$  is a main vertex of  $K$  and  $\zeta_2$  an open branch. We distinguish two cases. By the minimality property of  $\Delta$  and since  $\zeta_2$  is non-singular, we may assume that every  $\delta \in \Delta(T)$  contains exactly one appearance of  $\zeta_1$ . (If some bridge of type  $T$  would be, under some embedding extension, attached to two appearances of  $\zeta_1$ , it could be re-embedded so that only one of the appearances would be used.) If  $\Delta(T)$  contains one or two embedding schemes (for one or for two faces), we proceed as in the previous case. Since  $K$  satisfies (K1)–(K3), the only remaining case is when  $\Delta(T)$  contains three embedding schemes (Lemma 3.2; in order to apply this lemma, we use (K2) to get a Kuratowski subgraph  $K_0$  in  $K$  and then we use (K1) and Proposition 3.3 to eliminate local  $K_0$ -bridges in  $K$ ). Two of them are in a singular face  $F_1$  and the third one is in another face. We say that such a type  $T$  is *3-problematic*. Since this case is slightly less obvious, we will assume that for all non-3-problematic types in  $\mathcal{T}'$  representatives have already been chosen. 3-problematic

Suppose first that  $T$  is the only 3-problematic type in  $\Delta$  with  $\zeta_2 \in T$ . We will construct a sequence  $B_1(T), B_2(T), \dots, B_k(T)$  of representatives for the type  $T$  as follows. Initially, the sequence is empty. If some  $\delta \in \Delta(T_1)$ ,  $T_1 \in \mathcal{T}$ ,  $T_1 \neq T$ , obstructs realizations of one of the embedding schemes  $\delta_1, \delta_2, \delta_3 \in \Delta(T)$ , then we are satisfied with the empty sequence ( $k = 0$ ). The minimality of  $\Delta$  assures that there will be no embedding extension compatible with  $\Delta$  and hence  $(\star)$  holds. Note that we allow the case when  $T_1$  is another 3-problematic type or when  $T_1 \notin \mathcal{T}'$  even though in such cases we do not have representatives for  $T_1$ .

Suppose that  $\partial F_1$  is composed of  $\zeta_2, P_1, \zeta_1, P_2, \zeta_1, P_3$  in this order. There are three cases to consider.

**Case 1.** For some type  $T_1 \in \mathcal{T}' \setminus \{T\}$  such that  $\zeta_2 \in T_1$ , there is an embedding scheme  $\delta \in \Delta(T_1)$  that requires that some bridge of type  $T_1$  is embedded in  $F_1$  and has attachments on  $\zeta_2$  and  $P_2$ . Note that we already have the representatives for  $T_1$ . For every representative  $B$  of type  $T_1$  embedded in  $F$  compatibly with  $\delta$  we will extend the already constructed sequence

$B_1(T), B_2(T), \dots$  by a small number of additional bridges as follows. The bridge  $B$  in  $F_1$  splits the bridges of type  $T$  in four classes. The first two classes contain bridges of type  $T$  that do not overlap with  $B$  in  $F_1$  and can be embedded together with  $B$  so that they are attached to one, or the other occurrence of  $\zeta_1$  on  $\partial F_1$  (but not both), respectively. The third class contains bridges that overlap with  $B$  in  $F_1$ . The fourth class, if not empty, contains bridges that can be embedded in  $F_1$  together with  $B$  in two different ways. They are attached to  $\zeta_1$  and  $\zeta_2 \cap B$  (which must be just a vertex). In the first two classes, try to find a pair of overlapping bridges. If such bridges do not exist, take the “extreme” bridges (with respect to their simultaneous embedding in  $F_1$ ). From the last two classes, if they are non-empty, take an arbitrary bridge as a representative. We extend the sequence  $B_1(T), B_2(T), \dots$  by all these representatives. As before, for such a choice  $(\star)$  holds.

**Case 2.** For a singular branch  $\zeta \subseteq P_2$ , the embedding distribution  $\Delta$  requires that a bridge of type  $T_1 \supseteq \{\zeta, \zeta_2\}$  will be embedded in  $F_1$  in such a way that it is attached to  $\zeta_2$  and to  $P_2$ . Note that the type  $T_1 \notin \mathcal{T}'$ . We will assume that the second occurrence of  $\zeta$  on  $\partial F_1$  is contained in  $P_3$ . If the other occurrence of  $\zeta$  is contained in  $P_1$ , then the situation is symmetrical, and if both occurrences of  $\zeta$  are contained in  $P_2$ , then the situation is exactly the same as in any subcase of Case 1 (by taking as  $B$  an arbitrary bridge of type  $T_1$  and selecting an arbitrary embedding of  $B$ ). Let  $B_1$  be a bridge of type  $T_1$  that is attached to a vertex  $x$  on  $\zeta_2$  as close as possible to  $P_1$ . It is clear that in every embedding extension compatible with  $\Delta$ , at least one bridge of type  $T_1$  that is attached to  $x$  is embedded so that it is attached to the appearance of  $\zeta$  on  $P_2$  (by minimality of  $\Delta$ ). Then we determine representatives for  $T$  in the same way as in any subcase of Case 1 with a path in  $B_1$  from  $\zeta_1$  to  $x$  playing the role of the bridge  $B$  from Case 1. This will cover the possibility when  $B_1$  is embedded in  $F_1$  and is attached to the segment  $P_2$  of  $\partial F_1$ . On the other hand, if some other bridge  $B_2$  of type  $T_1$  that is attached to  $x$  is embedded that way, we get the same representatives from Case 1. This establishes that our choice of representatives for  $T$  is independent of the choice of  $B_1$ .

**Case 3.** Suppose that according to  $\Delta$  no type  $T_1 \in \mathcal{T} \setminus \{T\}$  such that  $\zeta_2 \in T_1$  and such that a piece of  $P_2$  is contained in  $T_1$ , is supposed to be embedded in  $F_1$  with bridges attached at  $\zeta_2$  and  $P_2$ . In this case, every embedding of bridges of type  $T$  can be changed (without interfering with other types embedded compatibly with  $\Delta$ ) in such a way that one of the appearances

of  $\zeta_1$  is not used. By the minimality of  $\Delta$ , such embeddings do not exist, and we have  $(\star)$  for the empty sequence of representatives for  $T$ .

This completes the analysis of the case when there is only one 3-problematic type that contains  $\zeta_2$ .

Suppose now that  $T' = \{\zeta'_1, \zeta_2\}$  with  $\zeta'_1 \neq \zeta_1$  is another 3-problematic type such that  $T'$  shares the branch  $\zeta_2$  with  $T = \{\zeta_1, \zeta_2\}$ . Recall that embeddings of  $T$  and  $T'$  are in a singular face  $F_1$  and in a non-singular face  $F_2$  (Lemma 3.2) where  $\partial F_1 \cap \partial F_2 \supseteq \zeta_1 \cup \zeta'_1 \cup \zeta_2$ . By (K3), neither  $\zeta_1$  nor  $\zeta'_1$  is an endpoint of  $\zeta_2$ . If the two occurrences of  $\zeta_1, \zeta'_1$  on  $\partial F_1$  interlace, then one can show that  $K$  is planar, contradicting (K2). On the other hand, if  $\zeta_1$  and  $\zeta'_1$  do not interlace, then already (K1) and (K3) lead to a contradiction.

It follows from the construction that for every embedding of the set  $\mathcal{R}$  of the representatives compatible with  $\Delta$ ,  $(\star)$  is satisfied. Moreover, suppose that we have an embedding of  $\mathcal{R}$  that is compatible with  $\Delta$  but not compatible with any  $\Delta' \prec \Delta$ . If  $B$  is a bridge of type  $T \in \mathcal{T}'$  that is not in  $\mathcal{R}$ , then  $B$  can be regarded as having at most two embeddings compatible with  $\Delta$  that extend the embedding of  $\mathcal{R}$ . If  $B$  has three admissible embeddings, then the representatives for  $T$  have been determined in Case 2. In this case the vertex  $x$  determines which embeddings are admissible. This property will later enable us to use the 2-restricted embedding algorithm (Proposition 4.1) for extending an embedding of  $\mathcal{R}$  by bridges of types from  $\mathcal{T}'$ .

The above discussion verifies that our procedure determines representatives for all types that do not contain singular branches of  $K$ .

## 6 At most 1-singular case

In this section we solve the embedding extension problem for the torus in the case when the embedding of  $K$  leaves at most 1-singular faces. The solution illustrates the use of procedure of embedding distribution of types developed in the previous section. Note that by Lemma 3.2 there is at most one 1-singular face if we assume that  $K$  satisfies conditions (K1)–(K3).

**Proposition 6.1** *Let  $G$  be a graph and let  $K$  be its subgraph satisfying (K1)–(K3) embedded in the torus. There is a linear time algorithm that either finds an embedding extension from  $K$  to  $G$ , or discovers a nice obstruction for such embeddings.*

**Proof.** First we will consider the case when there is a 1-singular face  $F$ . Denote its facial walk  $\partial F$  as  $aeb e^-$  (where  $e^-$  is the traversal of the singular branch  $e$  in the opposite direction). Let  $x = a \cap \bar{e}$  and  $y = b \cap \bar{e}$  be the endpoints of  $e$ . Now we apply the procedure of embedding distribution of types (Section 5) to transform our problem to a small number of simpler subproblems. Recall that, at each iteration of this procedure, we are given an embedding of  $K$  extended with the representatives for every bridge type that does not contain the open singular branch  $e$ . We have three cases to consider, and in each case we will assume that previous cases do not apply.

1. The selected embedding of the representatives dissects the face  $F$  into non-singular faces. Then all faces are non-singular and the 2-restricted embedding algorithm (Proposition 4.1) can be used to finish the task. (Since the embedding of the representatives is fixed, we do not need to take care of local bridges.)
2. There is a representative  $B$  of some type attached to the interior of  $a$  and to the vertex  $y$  (or, similarly, a representative attached to the interior of  $b$  and to the vertex  $x$ ). Suppose that there is a bridge  $B'$  attached to  $e$  and to the interior of  $b$ . Then the appropriate paths in bridges  $B$  (connecting  $a$  and  $y$ ) and in  $B'$  (connecting  $e$  and  $b$ ) remove the singularity of  $F$ .  $B'$  has at most three different embeddings, and for each of them we can conclude as in the previous case.

On the other hand, if there are no  $K$ -bridges attached to  $e$  and to the interior of  $b$ , then, under every embedding extension compatible with the current embedding distribution of types, there will be a face where  $y$  appears twice on its border. Then all the bridges can be re-embedded so that they will all use only one appearance of  $y$ . Thus the face  $F$  can be considered as a non-singular face and, again, the 2-restricted embedding algorithm can be applied.

3. Otherwise, the bridges attached only to  $x$  and to the interior of  $b$  or attached only to  $y$  and to the interior of  $a$  have embeddings out of  $F$ . Moreover, since  $K$  is a subdivision of a 3-connected graph and there are no local  $K$ -bridges in  $G$ , bridges attached entirely to  $a$  or entirely to  $b$  must be embedded in  $F$ . The same is true for bridges attached to  $e$ . For all these bridges we can use 1-cylinder algorithm from [15] to

get their simultaneous embedding in  $F$  or to discover a nice obstruction. (The 1-cylinder algorithm of [15] is similar to an application of the procedure of embedding distribution of types. It always reduces the problem to a small number of 2-restricted embedding extension problems.) The remaining bridges have at most two different (compatible) embeddings, and we can use the 2-restricted embedding algorithm to obtain an embedding extension or to discover a nice obstruction. Since none of these other bridges can be embedded in  $F$ , the combination of both embeddings gives an embedding of  $G$  extending the embedding of  $K$ . On the other hand, any of the two possible obstructions is an obstruction for the particular extension subproblem.

As the procedure of embedding distribution of types suggests, if we obtain an embedding extension in any of the subproblems, then we are done. Otherwise, the obstructions obtained in each subproblem are combined into a single nice obstruction.

The case when the face  $F$  in the initial embedding of  $K$  is only weakly 1-singular can be solved using a similar procedure. In Case 1, the algorithm is the same. Otherwise, the same method as in the second part of Case 2 applies.

Finally, if all faces of  $K$  are non-singular, the 2-restricted embedding algorithm can be used at the very beginning. □

## 7 2-singular face embedding

Let  $K$  be a subgraph of  $G$  such that no  $K$ -bridge in  $G$  is local. Suppose that  $K$  is 2-cell embedded in the torus and that  $F$  is a face of  $K$  with singular branches  $e$  and  $f$  which appear on  $\partial F$  in the interlaced order:  $e, f, e^-, f^-$ . Suppose that branches  $e$  and  $f$  of  $K$  and their endpoints are the only singular pieces of  $F$ . Let  $\mathcal{B}$  be a set of  $K$ -bridges in  $G$  and let  $Q := \bigcup_{B \in \mathcal{B}} B$ . Is it possible to embed all members of  $\mathcal{B}$  in  $F$  simultaneously? In this section we provide an answer to this question. In its solution we will use a more specific embedding problem: is it possible to embed all the members of  $\mathcal{B}$  in  $F$  in such a way that every bridge is attached to at most one of the occurrences of each basic piece. We shall refer to such an embedding as a *simple embedding*. An obstruction  $\Omega \subseteq Q$  is a *corner obstruction* if it has no simple embedding

simple embedding  
corner obstruction

in  $F$ . The following theorem is proved in [11].

**Theorem 7.1** *Let  $G, K, F, \mathcal{B}$ , and  $Q$  be as above. There is a linear time algorithm that either finds a simple embedding extension of  $K$  to  $K \cup Q$  in  $F$ , or returns a nice corner obstruction  $\Omega \subseteq Q$  for such embedding extensions.*

Next we apply Theorem 7.1 to solve the problem of embedding  $Q$  in  $F$ .

**Proposition 7.2** *Let  $G, K, F, \mathcal{B}$ , and  $Q$  be as above. Suppose that  $K$  satisfies (K1)–(K3) and that  $K \cup Q$  is 3-connected up to vertices of degree 2. There is a linear time algorithm that either finds an embedding of  $Q$  in  $F$ , or returns a nice obstruction  $\Omega \subseteq Q$  for such embeddings.*

**Proof.** By using Theorem 7.1 we either find a (simple) embedding or get a corner obstruction  $\Omega_0$ . If  $\Omega_0$  itself has no embedding in  $F$ , then  $\Omega = \Omega_0$  does the job. (This is checked easily since  $\Omega_0$  is nice.) Otherwise,  $\Omega_0$  has an embedding in  $F$ . Then we may assume that  $\Omega_0 \cup \partial F$  is connected. Since  $K \cup Q$  is 3-connected, we may assume that  $K \cup \Omega_0$  is also 3-connected (up to vertices of degree 2).

Choosing one of the embeddings, we can eliminate local  $(K \cup \Omega_0)$ -bridges in  $K \cup Q$  (Proposition 3.3). (If the removal of local bridges is not successful, we get an obstruction, and its combination with  $\Omega_0$  yields a required obstruction  $\Omega$ .) For every embedding of  $\Omega_0$  we repeat the procedure outlined below. As a result, we either get an embedding of  $Q$ , or we get obstructions for all possibilities. Their combination gives rise to a requested nice obstruction  $\Omega$ .

Let us now fix an embedding of  $\Omega_0$  in  $F$ . Since  $\Omega_0$  is a corner obstruction, one of the  $K$ -bridges in  $K \cup \Omega_0$  is not simple embedded. If such a bridge is attached to all four appearances of  $e$  and  $f$  on  $\partial F$ , then  $F$  is dissected by  $\Omega_0$  into closed 2-cells. Thus we may apply Proposition 4.1. The other possibility is that one of the bridges is attached to both appearances of  $e$  (or  $f$ ). In this case,  $K \cup \Omega_0$  satisfies (K1)–(K3) and hence Section 6 solves the problem.  $\square$

## 8 Embedding a graph in the torus

We finally reached the point where we can put together the preliminary material of previous sections to get a linear time algorithm for embedding graphs in the torus.

**Theorem 8.1** *There is a linear time algorithm that, for a given graph  $G$ , finds an embedding of  $G$  in the torus whenever such an embedding exists, or exhibits a subgraph  $\Omega$  of  $G$  of small branch size that cannot be embedded in the torus. Moreover,  $\Omega$  is minimal in the sense that every proper subgraph of  $\Omega$  can be embedded in the torus.*

**Proof.** The proof will be given by means of an algorithm consisting of five steps.

*Step 1: Planarity and 2-connectivity.* By Lemma 2.1 we either find an embedding of  $G$  in the plane (in which case we are done since this also determines an embedding of  $G$  in the torus), or we find a Kuratowski subgraph  $K_0$  of  $G$ . If two distinct blocks of  $G$  are non-planar, then their Kuratowski subgraphs determine a required obstruction for embeddability of  $G$  in the torus. Otherwise, we may assume that  $G$  is 2-connected (usually, planarity algorithms take care of this).

*Step 2: 3-connectivity.* Suppose that  $G$  is not 3-connected and that  $\{K, H\}$  is a 2-separation. Split  $G$  and denote by  $K'$  and  $H'$  the obtained split components and by  $e$  the added virtual edge. If  $H'$  is planar, then every embedding of  $K'$  in the torus can be extended to an embedding of  $G$ . In such a case it suffices to consider the graph  $K'$  instead of  $G$ .

Suppose that  $K$  and  $H$  are both planar. Their plane embeddings determine embedding of  $K$  and  $H$  into the cylinder with ends of  $e$  embedded on distinct boundary components of the cylinder. The two cylinders can be pasted together along their boundaries to get the torus such that corresponding endpoints of  $e$  are pairwise identified. This determines an embedding of  $G$  in the torus.

Now we may assume that  $K$  and  $H'$  are non-planar and that  $K_0 \subseteq K$ . Let  $K_1$  be a Kuratowski subgraph in  $H'$ . If a branch of  $K_1$  contains both endpoints of  $e$ , then we may assume that  $e \in E(K_1)$ . In this case, the virtual edge  $e$  can be replaced by a path  $P$  in  $K$  disjoint from  $K_0$ , or a pair of internally disjoint paths  $P = P_1 \cup P_2$  joining both endpoints of  $e$  with a  $K_0$ . The new subgraph is again denoted by  $K_1$ . If the endpoints of  $e$  are not contained in a branch of  $K_1$  we leave  $K_1$  unchanged. In both cases, let  $K_2 = K_0 \cup K_1$ . If  $K_2$  cannot be embedded in the torus, it is a small obstruction since  $|V(K_0) \cap V(K_1)| \leq 2$ . Suppose that  $K_2$  is toroidal. Since  $K_5$  and  $K_{3,3}$  are 3-connected,  $K_1$  is embedded in a single face  $F$  of  $K_0$  in the torus. Clearly, this face has both vertices of  $V(K_0) \cap V(K_1)$  singular and appearing

on the boundary of  $F$  in the interlaced order, and  $K_1$  is attached to each of their appearances. This implies, in particular, that we have the case when  $e$  was replaced by two paths  $P_1, P_2$  and that both of these paths are trivial.) This makes  $F$  dissected into non-singular faces. Other faces of  $K_0$  are non-singular by Lemma 3.2. Consequently, every embedding of  $K_2$  in the torus has only non-singular faces. Add to  $G$  a new vertex  $w$  and join it to all main vertices of  $K_2$ . In the obtained graph  $G'$ ,  $K_2$  lies in a single 3-connected component  $L$ . If some other 3-connected component of  $G$  is non-planar, then a Kuratowski subgraph in this component and  $K_2$  determine a small obstruction. Otherwise, embeddability of  $G$  in the torus can be reduced to embeddability of  $L - w$  which is 3-connected modulo  $K_2$ . By Proposition 3.3, we can efficiently eliminate local  $K_2$ -bridges (or we discover another Kuratowski subgraph whose union with  $K_2$  is a small obstruction). After that, Proposition 4.1 can be used to get an embedding of  $L - w$  in the torus (that can further be extended to an embedding of  $G$ ), or to discover for each of the embeddings of  $K_2$  a nice obstruction for extending this embedding. In the latter case, combining these obstructions completes our task.

Let us recall that the 3-connected components of  $G$  are structured in a tree-like way. Therefore, using Hopcroft and Tarjan's linear time algorithm [8] for obtaining 3-connected components of  $G$ , we can either finish our work (in linear time, as explained above), or reduce the problem to 3-connected case. From now on we will thus assume that  $G$  is 3-connected.

*Step 3: Removal of local bridges.* In this step we will try to change the graph  $K_0 \subseteq G$  into a graph homeomorphic to  $K_0$ , having the same main vertices and such that there are no local bridges with respect to the new subgraph. This can be achieved in linear time by using Proposition 3.3. More precisely, we either get a Kuratowski subgraph  $L$  of  $G$  such that  $L \cap K_0 \subseteq e$  for some branch  $e$  of  $K_0$ , or we achieve our goal. In the former case, let  $\Omega = (K_0 - e) \cup L \cup P_1 \cup P_2$ , where  $P_1$  and  $P_2$  are segments on  $e$  from each of its endpoints to the first vertex of  $L \cap e$  (if any; otherwise  $P_1 = P_2 = e$ ). If  $P_1$  or  $P_2$  is non-trivial (not just a vertex), then  $\Omega$  is a small obstruction for the torus embeddability of  $G$ . (To see this, embed  $L$  in the torus and observe that  $(K_0 - e) \cup P_1 \cup P_2$  cannot be embedded in any of the faces of  $L$ .) Otherwise, we can apply the 2-restricted embedding algorithm of Section 4 for every embedding of  $\Omega$  in the same way as described in Step 2 (the end of the case where  $K$  and  $H'$  were non-planar). Therefore, in the following steps we will assume that  $K_0$  has no local bridges.

*Step 4: Reduction to embedding extension problems.* We will reduce the torus embeddability problem of  $G$  (having the subgraph  $K_0$  constructed in Steps 1–3) to a small number of embedding extension problems. We will try to extend every possible embedding of  $K_0$  into the torus to an embedding of  $G$ . Having found an embedding in any of the cases, we get an embedding of the original graph  $G$ . On the other hand, if we are unsuccessful in each case, we will combine the obtained nice obstructions  $\Omega_1, \dots, \Omega_N$  (one for each embedding extension problem) into a small obstruction for the torus embeddability of  $G$ .

Let us now fix an embedding of  $K_0$  in the torus. Note that the embedding is 2-cell since  $K_0$  is a Kuratowski subgraph. We will the procedure of embedding distribution of types as described in Section 5. Therefore we reduce the problem of extending the embedding of  $K_0$  to  $G$  to a small number of subproblems. An obstruction  $\Omega_i$  ( $1 \leq i \leq N$ ) for extending the chosen embedding of  $K_0$  is obtained by combining obstructions obtained in each of the subproblems.

*Step 5: Embedding extensions.* This step has to be repeated for each of the subproblems that are generated in Step 4. We assume that  $K_0$  is embedded in the torus, and that for each type of bridges of  $K_0$ , its embedding distribution is chosen. Moreover, for every type  $T$  of bridges such that  $T$  does not contain a singular branch, representatives of types are obtained as explained in Section 5. Note that conditions (K1)–(K3) that are required in this process are guaranteed by our preliminary reductions in Steps 1–3. Additionally, the representatives are already embedded according to the selected embedding distribution.

Denote by  $K$  the union of  $K_0$  together with all representatives. We can assume that  $K$  satisfies conditions (K1)–(K3). By the above, an embedding of  $K$  is fixed. It remains to solve the embedding extension problem with respect to  $K \subseteq G$ . If the embedding of  $K$  is at most 1-singular, then we can apply results of Section 6. Suppose now that this is not the case. Then we have a (weakly) 2-singular face  $F$  with singular branches (or vertices)  $e$  and  $f$ . Its facial walk can be written as  $aebfce^-df^-$ .

Let  $G_1$  be the union of  $K$  and all  $K$ -bridges in  $G$  that have an attachment in the interior of  $e$  or in the interior of  $f$ . Check the embedding extendibility of  $K$  to  $G_1$  by using the algorithm of Proposition 7.2. Note that  $K$ -bridges in  $G_1$  are necessarily embedded in  $F$ . Obtaining an obstruction, we have an obstruction for embedding extensions of  $K$  to  $G$  as well. Suppose now

that an embedding of  $G_1$  extending the embedding of  $K$  has been found. As mentioned at the end of Section 5, all the remaining  $K$ -bridges have at most two different embeddings extending the embedding of  $K$  (compatible with the chosen embedding distribution of types). Moreover, some of their embeddings in  $F$  may be blocked by the presence of  $K$ -bridges in  $G_1$ . Note that these blockages are independent of the way in which  $G_1$  is embedded since the bridges that are not in  $G_1$  are attached and embedded in  $F$  locally to one of the segments  $a$ ,  $b$ ,  $c$ , or  $d$  (since the embedding distribution of types left a (weakly) 2-singular face). Using the 2-restricted embedding algorithm of Section 4 (respecting the constraints imposed by embedded bridges of  $G_1$ ), we either find an embedding extension of  $K$  to  $G$ , or get an obstruction  $\Omega_1$ . Note that  $\Omega_1$  is an obstruction for extending the embedding of  $G_1$  to  $G$ . Therefore  $\Omega_1$  is not necessarily an obstruction for extending the embedding of  $K$  (with respect to the chosen embedding distribution). Suppose that  $B'$  is a  $K$ -bridge in  $K \cup \Omega_1$  that has an embedding in  $F$  and outside  $F$ , but after adding  $G_1$ , its embedding in  $F$  is blocked. Then the bridge  $B$  of  $G$  containing  $B'$  was uniquely embeddable in the 2-restricted embedding extension problem solved above. By Proposition 4.1, at most two such bridges participate in  $\Omega_1$ . Therefore,  $\Omega_1$  can simply be extended by adding at most two paths in  $G_1$  that obstruct their embeddings in  $F$ . The obtained subgraph is an obstruction for extending the embedding of  $K$  to an embedding of  $G$  with respect to the chosen embedding distribution of types.

The above proof shows that in the case when an obstruction  $\Omega$  is obtained, the branch size of  $\Omega$  is bounded by certain constant. It is clear that once we have a small obstruction, we can get a minimal one in constant time by successively removing superfluous branches (using the above algorithm).  $\square$

## 9 Concluding remarks

Let  $G$  be a (simple) graph with  $n$  vertices and  $m$  edges. Euler's formula implies that  $G$  cannot be embedded in the torus if  $m > 3n$ . Therefore we may assume that the size of the input for our torus embedding algorithm is  $\mathcal{O}(n)$  and hence the linearity of our algorithm means time complexity  $\mathcal{O}(n)$ . The constant in the  $\mathcal{O}(n)$  bound that follows from our algorithm is not very small. An upper bound on this constant can be obtained from the proof. Although

this bound is enormous, there are reasons to believe that our algorithm is practically applicable.

There are several possibilities to improve particular steps of the algorithm. We have decided not to include such enhancements since they would increase the length of the presentation.

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