3 MEASUREMENT OF MEMBERSHIP FUNCTIONS: THEORETICAL AND EMPIRICAL WORK

Taner Bilgiç
I. Burhan Türkşen

Abstract: This chapter presents a review of various interpretations of the fuzzy membership function together with ways of obtaining a membership function. We emphasize that different interpretations of the membership function call for different elicitation methods. We try to make this distinction clear using techniques from measurement theory.

3.1 INTRODUCTION AND PREVIEW

Since Zadeh (1965) introduced the notion of fuzzy sets one of the main difficulties has been with the *meaning and measurement* of membership functions. Particularly, lack of a consensus on the meaning of membership functions has created some confusion. This confusion is neither bizarre nor unsound. After all fuzzy sets are totally characterized by their membership functions and in order to diffuse this cloud of confusion and for a sound theory of fuzzy sets a rigorous semantics together with practical elicitation methods for membership functions are necessary.

There are various interpretations as to where fuzziness might stem from. Depending on the interpretation of fuzziness one subscribes to, the meaning attached to the membership function changes. It is the objective of this chapter to review the various interpretations, theoretical models for various interpretations, and elicitation methods of membership functions.

We first start with the formal (i.e., mathematical) definition of a membership function[1]. A fuzzy (sub)set, say F, has a membership function μ_F[2] defined as a function from a well defined universe (the referential set), X, into the unit interval as: $\mu_F : X \to [0, 1]$ (see Figure 3.1).

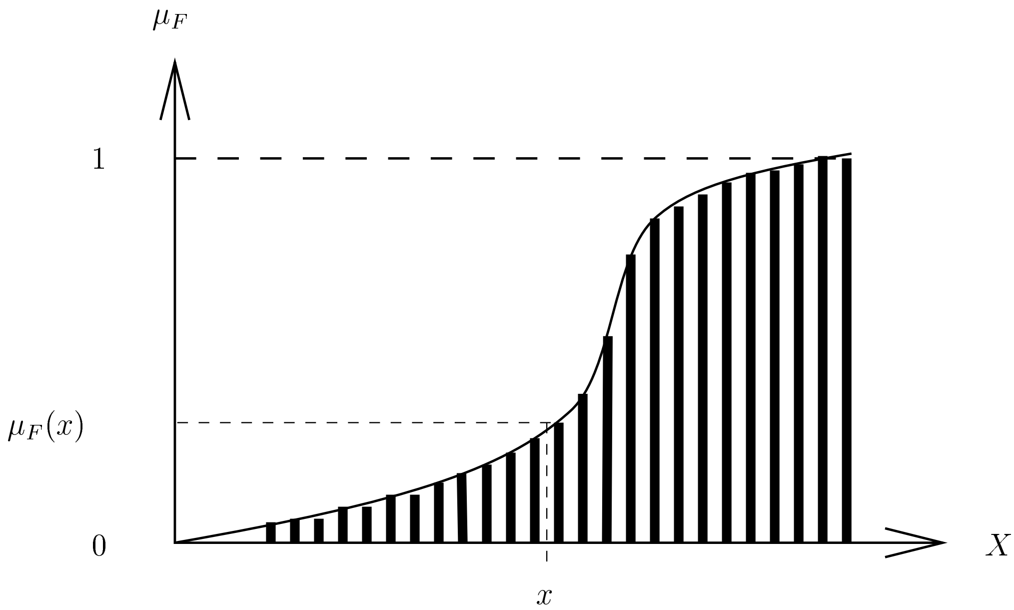

Figure 3.1 A membership function

Hence, the vague predicate “John (x) is tall (T)” is represented by a number in the unit interval, $\mu_T(x)$. There are several possible answers to the question “What does it mean to say $\mu_T(x) = 0.7$?”

likelihood view 70% of a given population declared that John is tall (see Section 3.2.1).

random set view 70% of a given population described “tall” as an interval containing John’s height (see Section 3.2.2).

similarity view John’s height is away from the prototypical object which is truly “tall” to the degree 0.3 (a normalized distance, see Section 3.2.3).

utility view 0.7 is the *utility* of *asserting* that John is tall (see Section 3.2.4).

measurement view When *compared* to others, John is taller than some and this fact can be encoded as 0.7 on some scale (see Section 3.2.5).

In Section 3.2, each of the interpretations mentioned above for the concept of grade of membership are taken up one-by-one, emphasizing their differences as well as similarities. Consequences of subscribing to a certain interpretation are also highlighted. It is important to realize that these interpretations are hypotheses about where *fuzziness* stems from and each interpretation suggests a calculus for manipulating membership functions. The calculus of fuzzy sets as described by Zadeh (1965) and his followers are sometimes appropriate as the calculus of fuzziness but sometimes inappropriate depending on the interpretation. After presenting a critical discussion of each interpretation, we give a measurement-theoretic interpretation, on the basis of which lies the comparative view. Measurement-theoretic view is covered in some depth in Section 3.2.5 as it sheds light to the foundations of every other interpretation.

In Section 3.3, we summarize experimental research conducted with the aim of constructing the membership functions and/or verifying the axioms of the fuzzy set theory. This section mentions various methods of obtaining membership functions as they are used in experimental research of fuzzy set theory. The interplay of the elicitation method and the assumed interpretation of fuzzy sets is emphasized.

We close the chapter with a quick summary of the review in Section 3.4.

In general, the aim of this chapter is to make the reader familiar with different (sometimes competing) interpretations of fuzzy sets and methods of elicitation for practical applications. The controversies and arguments, as well as further references are usually given as end notes for the interested reader to follow up.

3.2 INTERPRETATIONS OF GRADE OF MEMBERSHIP

When someone is introduced to the fuzzy set theory, the concept of graded membership sounds fairly intuitive since this is just an extension of a well known concept: membership in a set to "graded membership" in a set. However, the second step is quite mind boggling:

"How can graded membership be measured?"

This question has been considered, albeit in the context of many-valued logics, by many people from different disciplines[3]. Although more than two thousand years ago, Aristotle commented on an "indeterminate truth value", the interest in formal aspects of many-valued logics has started in early 1900's (McCall & Ajdukiewicz 1967, Rosser & Turquette 1977). But the *meaning* of multiple truth values has not been explained to satisfaction. For some, this is sufficient to discard many-valued logics all together (Kneale 1962, French 1984). On the other hand, the intellectual curiosity never let go of the subject (Scott 1976).

Anyone who is to use fuzzy sets must answer the following questions:

1. What does graded membership mean?
2. How is it measured?
3. What operations are meaningful to perform on it?

To answer the first question one has to subscribe to a certain view of fuzziness. Mainly there has been two trends in the interpretations of fuzziness: those who think that fuzziness is subjective as opposed to objective and those who think that fuzziness stems from the individual, as opposed to from a group of people (or sensors, etc.) as depicted in Figure 3.2.

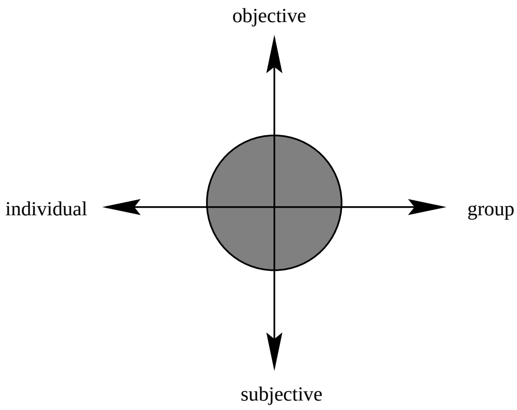

Figure 3.2 Main interpretations of fuzziness

We first summarize some of these views and then present a measurement-theoretic development which sheds light into all three questions.

Both the likelihood and the random set views of the membership function implicitly assume that there are *more than one* evaluator or experiments are *repeated*. Therefore, if one thinks of membership functions as "meaning representation", they come close to the claim that "meaning is essentially objective" and fuzziness arises from inconsistency or error. On the other hand, during the initial phases of the development of fuzzy sets, it has been widely accepted that membership functions are *subjective* and context dependent (Zadeh 1965, 1975). The similarity and utility views of the membership function differ from the others in their espousing a subjective interpretation. The measurement view is in the connection of subjective and objective views in the sense that the problem can be defined in both ways depending on the observer(s) who is (are) making the comparison. The comparisons can be results of subjective evaluations or results of "precise" (or idealized) measurements.

3.2.1 The likelihood view

Particularly Hisdal (1985, 1988) advocates a *likelihood* view of the concept of a grade of membership[4]. Hisdal's work is a typical example of creating a *descriptive* (interpretive is the word she uses) theory of fuzzy sets.

Hisdal's TEE (Threshold, Error, assumption of Equivalence) model for membership functions considers several *sources* of fuzziness:

1. Errors in measurement (which are essentially statistical in nature).

2. Incomplete information.

3. Interpersonal contradictions.

Furthermore, Hisdal argues that, *for a singe subject* equipped with perfect information (about, say, John's height) there cannot be any fuzziness! In this case John would be called 'tall' if his height is greater than a well-defined threshold or not.

Hisdal considers three ways of eliciting the membership function:

1. Label experiment: John is tall where tall $\in \{\text{very tall}, \text{tall}, \text{short}\}$.

2. Yes-No experiment: Is John tall?

3. Grade of membership experiment: What is the degree of belonging of John to the set of tall people?

Essentially, Hisdal's argument goes as follows: when asked to what degree a person is tall, the subject who has an imprecise measurement of the person's height (say she knows that it is within 180 ± 5 cm.) constructs an *error curve* to quantify the possible error around 180 cm. Then, she would argue as: "in 75% of the cases the subject's height lie in my quantization interval for 'tall' ". Therefore

$$\mu_{\text{tall}}(180cm) = P(\text{tall}|x = 180cm) = 0.75$$

where $P(\text{tall}|x = 180cm)$ denotes the likelihood that the label "tall" is assigned to the subject in question given that the subject's height is 180 cm.

Hisdal postulates that a subject's grade of membership is the modification of her (crisp) answers to the Label or Yes-No experiments *modified* by her *estimate* of the error curve.

These arguments essentially apply to the first type of fuzziness: errors in measurement[5]. Hisdal also goes on to explore the connectives which are valid under this interpretation. She argues that min and max are not always appropriate as conjunction and disjunction, respectively. This should come as no surprise since in this probabilistic setting, we know that joint probabilities are not always simply recovered from the constituent probabilities.

Like Hisdal, Mabuchi (1992) also defines membership function as likelihood but assumes that the likelihood is determined by the number of *evidence grounds* of all necessary grounds which support an attribute's membership. The construction of membership functions follows closely that of a checklist paradigm (Bandler & Kohout 1986, Kohout & Bandler 1993) and max-min calculus is not supported as disjunction and conjunction.

Thomas (1979, 1995) elaborates on a likelihood semantics for fuzzy sets. Thomas subscribes to the philosophical point of view that "meaning is essentially objective and is a convention among the users of a language." Furthermore, Thomas also opposes to the determinism and idealism that measurement theory imposes. He claims that measurement is essentially a vague process. These views lead Thomas to equate the likelihood function to the membership function and hence provide a likelihood semantics for fuzzy set theory. Like Hisdal, Thomas also finds out that the min-max calculus as proposed by Zadeh is not sufficient in this framework.

3.2.2 Random set view

The definition given for a membership function in Section 3.1 is called the *vertical* representation (Dubois & Prade 1989). One can also view the membership function *horizontally* where a fuzzy set F on a referential set X is represented in terms of its "level-cuts" (see Figure 3.3):

$$\{F_\alpha : \alpha \in (0,1]\}$$

where $F_\alpha = \{x : \mu_F(x) \geq \alpha\}$. Zadeh (1971) defines the membership function as:

$$\mu_F(x) = \sup\{\alpha \in (0,1] : x \in F_\alpha\}.$$

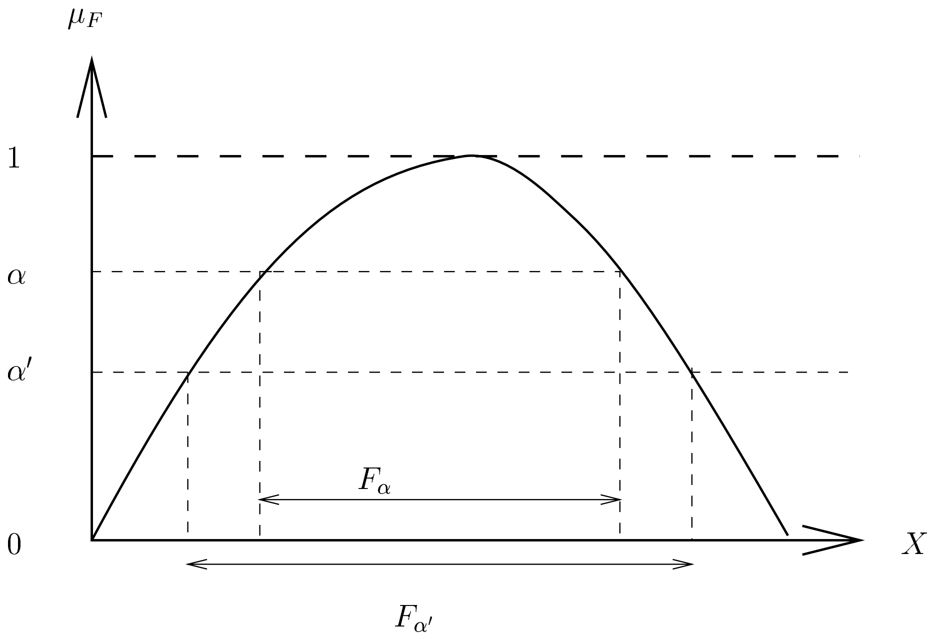

Figure 3.3 The horizontal representation of a membership function

In this representation we view the membership function as a nested family of level-cuts[6]. Note that each level-set, F_α, is a set in the classical sense.

The membership function can also be represented as an integral:

$$\mu_F(x) = \int_0^1 \mu_{F_\alpha}(x) d\alpha$$

where $F_\alpha(x) = 1$ if $x \in F_\alpha$ and 0 otherwise (Dubois & Prade 1989). Of course, one must make continuity and measurability assumptions so that the integral is well-defined. In that sense, it is a specialized view of the membership function (see

Section 3.2.5 for more details on continuity and Archimedean assumptions). However, in this view, the membership function can be viewed as a uniformly distributed random set consisting of the Lebesgue measure on $[0,1]$ and the set-valued mapping $F_\alpha : (0,1] \to X$.

Therefore, for $\mu_F(x) = 0.7$, the random set view is: 70% of the population defined an *interval* on the referential set, X, as an interval containing x (e.g. x = John's height) on the basis of an evaluation F (e.g. tallness). The remaining 30% defined intervals which *excluded* x to be F.

3.2.3 Similarity View

The similarity view of the membership functions naturally arise in prototype theory where membership is a notion of being *similar* to a *representative* of the category (Rosch & Mervis 1975, Lakoff 1987). In that sense, membership function measures the *degree of similarity* of an element to the set in question. It is assumed that there exists a *perfect* example of the set (or the category) that belongs to the set to the full degree. Others belong to the set to a degree measured by their relative *distance* to the perfect example.

Osherson & Smith (1981, 1982) claim that in rating the prototypicality of a stimulus in some category, subjective probability is sufficient. They critically discuss the (in)adequacy of fuzzy operators like conjunction and implication. There is agreement on the need to account gradients in category membership, however, a major task is to validate to what extent do people use gradient category membership in day-to-day cognitive tasks. There are two theories on the subject:

- People assign borderline stimuli to the category in which it has the greatest membership. Thereby economizing on cognitive effort (Lakoff 1973).
- People are mainly "thresholders" assigning an item to a category if the stimuli are greater than a certain threshold (Kochen 1975).

Pipino, van Gich & Tom (1981) report evidence on both types of behaviour for the same cognitive tasks.

Kempton (1981) reviews and defends the similarity view by extending traditional cognitive anthropological folk classification methods using methods based on fuzzy set theory.

Zysno (1981) discusses scaling the membership function. His approach stems from a similarity view of the membership function. He takes a single human being as the *measurement device* and assumes that fuzziness arises from the insufficient cognitive abilities of this person who is faced with the task of "comparing the object with a certain prototype or imaginable ideal." This approach naturally leads to a notion of *distance* which Zysno relates to the membership function as:

$$\mu(x) = \frac{1}{1 + f(d_x)}$$

where d_x is the distance of object x from the ideal and $f(d_x)$ is a certain function of the distance. Then he proposes alternatives for the function $f(d_x)$ and empirically evaluates them.

A theoretical discussion of this method is given by Zimmermann & Zysno (1985) in which a mathematical model of a membership function is defined on the real line where membership is a function of the distance between an object and an ideal object. The proposed model is also extensively tested in an experimental setting.

However, when one talks about distance, there is a more substantial assumption that comes with it: a metric space. Many empirical researchers take the notion of a metric space on which a distance measure can be defined for granted. What needs to be discussed is the properties of this metric space and its correspondence to psychophysical laws.

Zwick, Carlstein & Budescu (1987) study different measures of similarity among fuzzy concepts. They consider similarity measures defined both on metric spaces and in a set theoretic manner. They consider 19 such measures and try to experimentally verify the performance of these measures. They report that all measures successfully yield similar and dissimilar concepts in a crude manner. However, when the aim is to distinguish between degrees of similarities metric measures perform better. From a measurement-theoretic point of view this is expected. In a crude categorization, one is interested only about similar objects not with *how similar* they are. This can be done on an ordinal scale. However, when the aim is to find out how similar the concepts are then one needs to move to interval and ratio scales on which metric measures are meaningful and carry more information.

Ruspini (1991) also considers a similarity semantics for fuzzy set theory. He builds the notion of similarity on a certain modal logic in which the accessibility relation of the modal logic is a similarity relation in a metric space. This method provides a formalism of fuzzy set theory based on the possible world semantics of modal logics (Chellas 1980)[7].

The notion of similarity seems to require some sort of topology (or a metric space) on which it can be defined (Tversky 1977). Most often than not, someone who subscribes to the similarity view implicitly assumes a continuous, metric space. However this assumption needs a critical examination before committing to the similarity view. Possibly the best way to put similarity view under a critical framework is via Measurement Theory (c.f. Section 3.2.5). We will have more to say on this point in that section.

3.2.4 View from utility theory

Giles (1988) offer a decision-theoretic interpretation for the membership function. He argues that a set is equivalent to a *property* and claims that a sound meaning to the membership function can be given by considering it *together* with the problem of *fuzzy reasoning*. Hence, he is following the path of logic to come up with a semantic theory for the concept of a graded truth value.

He defines a fuzzy sentence as “a sentence to which we attribute a degree of belief” which is a function of *possible worlds* (or possible states of mind, nature etc.). Giles considers sentences that are *asserted* rather than merely uttered. For example, when one asserts that “John is tall” Giles assumes that there exists a pay-off function related to this assertion. This pay-off function offers more if the statement is closer to truth (e.g., you are more credible among your peers if you speak the truth!). This assumption drives the *utility theory* approach to the semantics of the membership function. (Giles 1988, p. 304) claims that “Utilities of assertions carry the meanings of assertions”. Of course, when defined this way, the utility approach yields a membership function on an interval scale (unique up to positive linear transformations) (Fishburn 1970).

He dismisses the effects of context on the meaning of membership function (as in a child who is 160 cm in height is a tall child but not a tall person) by claiming that one should try to represent the *average* meaning of the assertion in a *normal* society.

When viewed this way, the connectives for the theory are no more truth functional! Therefore, none of the triangular norms and conorms (cf. Chapter 2 of this handbook) are candidates for disjunction and conjunction. A semantic approach to fuzzy set theory, as Giles attempted, defies the *algebraic* view in which classical connectives are simply extended to many-valued settings in a truth functional way.

3.2.5 View from measurement theory

The three problems posed at the beginning of Section 3.2 can be taken up by using the techniques of measurement theory (Krantz, Luce, Suppes & Tversky 1971, Roberts 1979, Narens 1985, Suppes, Krantz, Luce & Tversky 1989, Luce, Krantz, Suppes & Tversky 1990). Measurement theory is concerned with *representation* and *meaningfulness* of the particular representation.

Most often than not, mathematical structures are investigated in their own right without any application in mind. This is essential to the growth of human mathematical enterprise. On the other hand, fuzzy set theory is an *empirical science*; it claims that it represents a natural phenomena that we observe in our everyday lives: fuzziness. Therefore a theory of fuzzy sets must have clear correspondences in our daily lives as well. Measurement theory bridges that gap by taking a phenomenon which can be modelled as an algebraic structure (e.g. fuzziness) and by mapping it into a *numerical* structure (hence capturing the essential core of the *measurement* process). In such a theory, one can discuss the representation of a qualitative structure and the meaningfulness of such a representation[8].

We utilize measurement theory to *formally*[9] answer the general questions that we posed at the beginning of this chapter. The aim is to *bring to the surface* the axioms that are implicit in subscribing to a particular interpretation of membership functions and to discuss them as to their testability, validity and/or intuitiveness.

We start with Saaty’s (1986) formulation who espouses a ratio scale. After discussing the assumptions of this formulation, we argue that ratio scales are neither *natural* nor acceptable for membership functions as Saaty claims them to be.

We then differentiate between two related but *different* measurement problems: *(i)* membership measurement (Section 3.2.5) and *(ii)* property ranking (Section 3.2.5). We argue that, although the first problem received much attention in the literature the second one is closely related to the question of “which connectives to use?”. Then, we combine the two problems to arrive at an axiomatization for fuzzy set theory (Section 3.2.5)[10]. In the Appendix to this chapter, we give definitions, and representation and uniqueness results for ordered algebraic structures which can be taken to model fuzzy set theory. These results provide a technical background to facilitate the discussion of the measurement-theoretic frameworks for fuzzy set theory in Sections 3.2.5–3.2.5. The reader who is interested in technical details can use the Appendix as a reference.

Saaty's measurement on a ratio scale. Saaty (1986) thinks that it is hopeless to seek to determine the membership function through direct scaling (fundamental measurement) and espouses a derived scale. He proceeds with assuming a finite set $\mathcal{A}$ with n elements called the alternatives and a set of criteria, $\mathcal{C}$. He defines a binary relation on $\mathcal{A}$ denoted by $\succ_c$ and read as "more preferred than with respect to criteria c". Then he *assumes* that $\succ_c$ is complete and there *exists* a representation, say P_c for $\succ_c$ and continues to *add* axioms on this *fundamental* scale! The first axiom he adds is what he calls the *reciprocal axiom* which states:

$$P_c(A_i, A_j) = 1/P_c(A_j, A_i)$$

for all A_i and A_j in $\mathcal{A}$. This implies that the *initial fundamental scale*, P_c, is measured at least on a *ratio scale*. Saaty does not show how this can be done and, therefore, implicitly takes all the axioms that lead to a ratio scale for granted.

However, the basic tenet of measurement theory (as in any axiomatization of a theory) is to lay out assumptions as clear as possible so that their validity can be tested and/or they can be accepted on normative grounds. By assuming that such preference comparisons can be measured on a ratio scale without justifying this position does not provide a sound formalism for fuzzy set theory[11].

Membership measurement. There are two important (but different) measurement problems in fuzzy set theory. The first kind deals with measuring the degree of membership of *several subjects or objects* in a single fuzzy set. This problem has been studied in (Yager 1979, Norwich & Türkşen 1982b, Norwich & Türkşen 1984, Türkşen 1991, Bollmann-Sdorra, Wong & Yao 1993, Bilgiç 1995) among others.

In this problem, there is a single fuzzy set (or a fuzzy term), F, and a finite number of agents, objects, elements, etc. in A. The question is: "to what degree an agent from A belongs to fuzzy set F?". The first measurement problem takes the relation "an agent is more F than another agent", where F is a fuzzy term (typically an adjective). The important thing to notice is the fact that this representation compares several agents over a *single* fuzzy term, F. The resulting representation measures the degree to which each agent belongs to the fuzzy set F.

To capture this graded membership concept, consider a binary relation, $\succsim_F$ on A with the following interpretation:

$$a \succsim_F b \Longleftrightarrow a \text{ belongs to } F \text{ at least as much as } b \text{ belongs to } F$$

or equivalently,

$$a \succsim_F b \Longleftrightarrow a \text{ is at least as } F \text{ as } b \text{ is } F.$$

Examples of such sentences are:

- Mary is at least as intelligent as John
- The new generation is less political than the old one

- This task is more important than the other

Norwich & Türkşen(1982b, 1984) consider the relation, $\succsim_F$, and discuss its possible representations.

The structure is assumed to be bounded (i.e., there exists elements a^- and a^+ in A such that for all other a in A, $a^+ \succsim_F a \succsim_F a^-$). This amounts to assuming that there are agents for which the fuzzy term F is completely true or completely false. If F is "tall", for example, there may be some elements of A for which "tall" and "not tall" are completely unambiguous. However, if F is "cute" or "bright" etc. it is very hard to find agents who are absolutely "cute" or "not cute".

If the boundary condition is accepted, then it is clear that an ordinal scale representation exists for $\langle A, \succsim_F\rangle$ once the transitivity and connectedness of $\succsim_F$ are accepted. The ordinal scale for measuring fuzziness allows only the comparison of membership values and no other numerical manipulation.

Norwich & Türkşen (1982b) also consider the difference measurement problem. Discussing the axioms of an algebraic difference structure $\langle A \times A, \succsim_F\rangle$ for membership in a fuzzy set, they argue that membership can be measured on an interval scale. The extra information required in this case is to be able to compare *pairs* of agents (e.g., John is taller than James than Jason is taller than Jack.).

The interval scale uniqueness for this representation guarantees that averaging is a meaningful operation as well as comparison but the notion of an *origin* is still arbitrary (i.e., the bounds do *not* invariably map to the same numbers in the numerical domain.

On the basis of the above arguments it can be shown that a function, $\mu_F : A \to [0, 1]$ exists such that:

$$a \succsim_F b \Longleftrightarrow \mu_F(a) \geq \mu_F(b).$$

The uniqueness is either ordinal or interval depending on the axioms that are accepted and $\mu_F(a)$ measures the degree with which a belongs to F. In the case of an interval scale, the following representation also holds:

$$(a, b) \succsim_F (c, d) \Longleftrightarrow \mu_F(a) - \mu_F(b) \geq \mu_F(c) - \mu_F(d).$$

In (Norwich & Türkşen 1984, Section 2.2) it is argued that ratio scales are not likely to arise in fuzzy set theory. They suggest that there is a difficulty with the *concatenation* of psychological attributes and hence no operational meaning can be assigned to the concatenation operator (addition) of *extensive* measurement which results in a ratio scale[12].

Property ranking. A complementary problem is to consider *many* fuzzy terms for a *single* agent. This way one can introduce the connectives into the measurement-theoretic setting.

The justification of using different operators as connectives for fuzzy sets are considered by many authors. Bellman & Giertz (1973) give the first axiomatic justification for using min and max operators followed by Fung & Fu (1975). Yager (1979) also considers the same problem and proposes axioms from which min and max follow as the only solutions. The introduction of triangular norms, conorms and parametrized families of functions as possible candidates of connectives triggered more research

(Yager (1980), Dubois & Prade (1980, 1982), Weber (1983), Alsina (1985) among others).

Bollmann-Sdorra *et al.* (1993) take up the same problem and clearly distinguish the membership measurement and property ranking problems. Bollmann-Sdorra *et al.* justify the use of min and max operators as intersection and union, respectively.

Bilgiç (1995) and Bilgiç & Türkşen (1995a) consider the same problem. They start with a very weak algebraic structure for which only an ordinal scale representation exists. Then, they continue to add axioms on this structure on an "as-needed" basis and show how ratio and absolute scale representations are obtained[13]. At each addition of a new axiom they discuss the testability and validity of the new axiom for fuzzy set theory.

This approach is in accord with some linguistic theories where it is claimed that although adjectives (like tall, clever etc.) linguistically precede their comparative forms (taller, cleverer etc.), the comparative forms precede the simple forms *logically* (Sapir 1944, Palmer 1981). In particular, Kamp (1975, p. 127) argues that:

> $\cdots$ when we learn a language like English we learn the meanings of individual adjectives and, moreover, the semantic function which this comparative-forming operation performs, *in general*, so that we have no difficulty in understanding, on first hearing, the meaning of the comparative of an adjective which we had thus far only encountered in the positive. If this is so then the meaning of an adjective must be such that the comparative can be understood as a semantic transformation of that meaning into the right binary relation.

This amounts to saying that the meaning of adjectives (fuzzy terms) requires its comparative form for formal analysis; an idea perfectly reflected in measurement theory.

It should be stressed that the problem as defined in this manner *does not assume* that the subject is evaluated on a scale for each individual fuzzy term. Hence, the disjunction in fuzzy set theory is *not* the combination of two fuzzy scales from the membership measurement problem of Section 3.2.5. Such a formulation is briefly reviewed in Section 3.4 which uses conjoint measurement techniques.

The qualitative relation of concern is[14]:

$$F_i \succsim_a F_j \iff \text{an agent } a \text{ is more } F_i \text{ than s/he is } F_j.$$

Examples of such expressions are:

- John is taller than he is clever.
- Inventory is higher than it is low.
- Coffee is at least as unhealthy as it is tasty.
- Her last novel is more political than it is confessional.

It should be noticed that there is a *single subject* in such sentences. Therefore, the treatment of this section excludes those sentences like "I am taller than you are strong". General two-subject sentences are discussed in Section 3.2.5.

The problem is formalized by imposing an algebraic, qualitative structure $\langle \mathcal{F}_a, \succsim_a, \oplus_a \rangle$ where $\mathcal{F}_a$ is a (countable) set of fuzzy terms related to a *single subject* a:

$$\mathcal{F}_a = \{F_1, F_2, F_3, \ldots\},$$

$\succsim_a$ is an ordering of those terms and $\oplus_a$ is intended to model the disjunction of two fuzzy terms[15].

While membership measurement can at best be performed on an interval scale, *formally* as strong as absolute scale representations exist for the property ranking problem (cf. Theorem 1 in the Appendix).

The formal existence of scales that are stronger than interval scales poses the problem of coming up with a *natural origin*. However, since the ranking of properties of an individual is highly a subjective act of the observer, there cannot be universally accepted bounds on the measurement scale (Norwich & Türkşen 1984). This makes all the scales resulting from the measurement, *relative* to the observer and amounts to concluding that the formally attainable absolute or ratio scales are not likely to arise in the framework of fuzzy set theory (Bilgiç & Türkşen 1995a).

Of all the axioms imposed on the fuzzy set structure, the boundaries and the Archimedean axiom are the most problematic (cf. Appendix to this chapter for formal definitions of these axioms).

The boundary axiom asserts the existence of (unique) *maximal and minimal* fuzzy terms and requires that for a given subject, there exists a fuzzy term which is more suitable to the subject than any other fuzzy term. Although this sounds reasonable for some cases, it is unlikely that it will hold for *all* fuzzy terms.

The Archimedean axiom (cf, Definition 3 in the Appendix) means that if Mary is cleverer than she is angry, there should be a *finite* amount of anger which, when attributed to Mary, will make her at least as angry as she is clever. Admittedly this might hold for *some* cases. However consider the statement “alcohol intake is more hazardous to liver than it is enjoyable.” Archimedean axiom states that a finite amount of joy would make alcohol intake more enjoyable than it is hazardous. Is this quantity really finite?

It is important to notice that all the representations that are discussed in this section are “two-way” representations in the sense that the conditions to be imposed on the qualitative structure should be both *necessary and sufficient* for a representation on the numerical domain. If “one-way” representations are considered, then many of the problematic structural axioms like Archimedean and continuity that are necessary for the representation in the real line can be omitted. However, this approach does not give rise to the fruitful concept of *meaningfulness*.

If one only considers one-way representations then one can use t-conorms for all representations since they satisfy all the conditions of the theorems and *more*. However, such usage must be made with care. The numbers resulting from using a t-conorm as the disjunction can only be compared to each other in the ordinal sense and no other arithmetic operation (however tempting) can be performed on them unless one is ready to accept (or empirically verify) the Archimedean and continuity axioms.

If everyone cannot agree on the bounds of the system, the ratio and absolute scale representations that arise may be considered to be *relative* to the observer.

On the other hand, without the Archimedean axiom, there does not seem to be a representation that has a uniqueness stronger than ordinal scale.

All the representation theorems yield a numerical function, say ν_a, that takes on values in the unit interval:

$$F_i \succsim_a F_j \Longleftrightarrow \nu_a(F_i) \geq \nu_a(F_j).$$

This function represents the ordering of attributes (ranking of properties) that can be associated to a single subject a and *does not necessarily* represent the *belonging* of the subject to the fuzzy set F.

Simultaneous measurement of membership and property ranking. It is almost impossible to treat separately the measurement of membership functions and the selection of connectives to be used in the combination of membership functions.

The scales resulting from membership measurement and property ranking problems do not necessarily measure the same entity. The degree with which a subject belongs to a certain fuzzy set may not be equal to the degree that fuzzy set is associated with him/her[16].

One way to pursue is to *combine* the membership measurement and property ranking problems by introducing a new structure where the resulting measurement scale necessarily measures the membership degree in a fuzzy set. This path is followed by (Bollmann-Sdorra *et al.* 1993, Bilgiç 1995, Bilgiç & Türkşen 1995b)[17].

Bollmann-Sdorra *et al.* (1993) consider an ordered algebraic structure which justify the use of *min* as conjunction and *max* as disjunction. Their representation is ordinal.

Bilgiç & Türkşen (1995b) introduce two ways this combination can be made. In the first one, the two different problems are simply cast into a bounded semigroup structure (cf. Appendix). The consequences of this model are analyzed. It is argued that since accepting the Archimedean axiom (cf. Definition 3 in the Appendix) can be very hard for some fuzzy terms, the ratio scale representations are not likely to arise.

Apparently, in the literature of the ordered algebraic structures, there are very strong (absolute scale) representations for similar structures that one can use to represent a fuzzy set structure (cf. Appendix). Figure 3.4 summarizes the representation theorem presented in the Appendix (Theorem 1)[18].

However, representations stronger than ordinal scales require unacceptable structural axioms like the strong monotonicity (SM) and the Archimedean (Ar) conditions (cf. Definition 3 in the Appendix).

If one gives up the Archimedean axiom and endows the structure with other axioms (particularly the Idempotency axiom, cf. Figure 3.4), some weak representations can be obtained. These are ordinal scale representations and particularly the function *max* as originally suggested by Zadeh can be recovered as the unique disjunction satisfying some reasonable axioms.

However, ordinal scale representations exist at the cost of accepting that the equivalence of two fuzzy propositions is *transitive*. This fact is known to yield a class of paradoxes usually called the heaps paradox or sorites. The equivalence of fuzzy terms is not necessarily transitive. In order to introduce the *intransitivity* of equivalence in fuzzy set theory Bilgiç & Türkşen (1995b) propose another model in which the weak ordering axiom is replaced by the "interval ordering" axiom which results in a threshold representation. Unfortunately, this representation has some peculiar uniqueness characteristics[19].

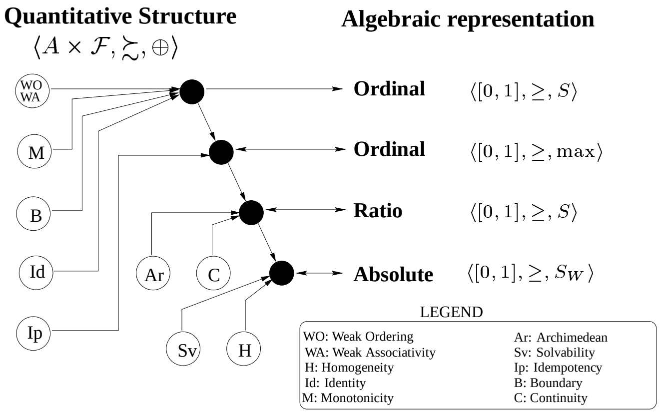

Figure 3.4 Summary of representations

The measurement models show that measurement of membership functions in fuzzy set theory is *formally* possible. However, the acceptability of each formal model needs to be critically analyzed.

On the other hand, once the Archimedean axiom is omitted it is not possible to come up with representations stronger than ordinal scale. This suggests that if one is not ready to accept the Archimedean axiom the only meaningful operation that can be performed on the measurement scale is the comparison (i.e., the results are on an ordinal scale). Any other arithmetic operation is simply meaningless. This suggests that triangular norms and conorms can be used to model connectives in fuzzy set theory but their results should not be attached any cardinal significance.

Archimedean axiom is necessary when a representation into real numbers is sought. If a representation into field extensions of the real number system is considered, then Archimedean condition is not necessary anymore (Narens 1985). Such a path has been followed by Nola & Gerla (1986). However, this approach has consequences of a philosophical nature as to why degrees of truth are not real numbers and why do they require non-standard analysis (Robinson 1966). This is still an open question.

If one accepts the Archimedean axiom and continuity, the results about the additive generators of Archimedean triangular norms and conorms simply state that using these in the unit interval and using addition on the extended real numbers amount to the same thing. Therefore, using an Archimedean triangular norm or a conorm as conjunction or disjunction of fuzzy sets is simply a matter of preference or convenience. For disjunction, one can equivalently use addition on extended reals. Therefore, the justification of continuous, Archimedean triangular norms and conorms require some unnatural structural axioms and in the end they amount to ordinary, additive extensive measurement. In order to move away from additivity one either has to give up the Archimedean axiom which as we have seen does not lead to strong representations (but nevertheless recovers max) or give up some other structural axiom. Luce *et al.* (1990) show that if one accepts the Archimedean axiom then associativity (in presence of positivity) amounts to additivity. Hence, the first candidate to give up seems to be the associativity (Bilgiç 1996). The attempts of Fodor (1993) and Fodor & Keresztfalvi (1995) to generalize triangular norms by dropping associativity require a measurement-theoretic discussion.

As for giving up the continuity, the main thrust behind the concept of a fuzzy set is continuous gradation. Hence moving away from continuity would be against the very nature of fuzziness.

Conjoint Measurement. One particular difficulty with the notion of fuzzy terms is that they can be *composed of* two or more simple terms (e.g. the term 'comfort' may constitute two parts 'humidity' and 'temperature'). Türkşen (1991) applies *conjoint measurement* techniques to this problem with the aim of showing that the scale for the higher order term (comfort) is constructed from the individual scales (humidity and temperature) using triangular norms and conorms. Under the assumption of *decomposability* Türkşen shows that all the triangular norms and conorms preserve the monotonicity of the building scales.

Türkşen (1991) considered the problem of *justifying* triangular norms, conorms, and parametrized families of them as valid under conjoint measurement. In so doing, he considers only "one-way" representations. To find both necessary and sufficient conditions that the main operations such as triangular norms, conorms and the parametrized families must follow and to discuss their acceptability in fuzzy set theory requires further investigation.

A critique of measurement theory. The measurement-theoretic discussion presented in this section relies on the *representational* view of measurement. There are three main objections to the representational theory of measurement (RTM) from psychologists and philosophers alike (Michell 1990, Henry E. Kyburg 1992):

1. RTM cannot be applied
   - because it uses an axiomatic approach, and
   - because it cannot give a satisfactory account of actual scientific measurement practice.
2. Errors of measurement cannot be incorporated into the RTM framework
3. Metaphysical objections:
   - RTM is wrongly *liberal* in what it accepts as measurement.
   - Whether numbers are primitive or obtained as results of a measurement process (realism on numbers or not).

RTM is motivated by the belief that numbers are not *real* and tries to show how numbers find their way into measurement (Michell 1990). The role of numbers in measurement is to *represent*. However, Michell (1990) espouses another view. He agrees with the methods of RTM in mapping an empirical system to a numerical system but, in his view, numbers are physical quantities rather than a mere abstraction. He considers that a numerical system which satisfies the properties of real numbers as the *only* possible candidate for measurement. And hence other numerical systems (ordinal scales for example) cannot be considered to yield a measurement scale.

Luce (1996) counters these criticisms and defends the RTM from several vantage points. He points out the main objective of the RTM as:

> "Contemporary" measurement theory builds on top of the classical view and introduces the, now well established, notion of *scale types*. Its purpose is primarily to inform the sciences—especially the behavioral and social ones—of the full range of measurement possibilities and to provide concrete contributions to those sciences (Luce 1996, p. 79).

Although there are valid points in the critique of the RTM, particularly about representing the errors of measurement, RTM is, by far, the most widely accepted account of measurement (see also Bilgiç & Türkşen (1997), in this connection).

3.3 ELICITATION METHODS

In the experimental research concerning membership functions, we detect two main trends: one branch tries to experimentally *validate* the postulates of fuzzy set theory. The second branch adopts a certain interpretation of the concept of the 'grade of membership' and then seeks ways of its elicitation. Since in both cases one needs ways of eliciting membership functions, we will summarize elicitation methods together with experimental validation results.

Mainly, there appears to be six methods used in experiments with the aim of constructing membership functions (Norwich & Türkşen 1982a, Chameau & Santamarina 1987a, Türkşen 1991):

- polling: do you agree that John is tall? (Yes/No).
- direct rating (point estimation): classify color A according to its darkness, classify John according to his tallness. In general, the question is: "How F is a?".
- reverse rating: identify the person who is tall to the degree 0.6? In general, identify a who is F to the degree $\mu_F(a)$.
- interval estimation (set valued statistics): give an interval in which you think color A lies, give an interval in which you think the height of John lies.
- membership function exemplification: what is the degree of belonging of color A to the (fuzzy) set of dark colors? What is the degree of belonging of John to the set of tall people? In general, "To what degree a is F?".
- pairwise comparison: which color, A or B, is darker (and by how much?)

3.3.1 Polling

In polling one subscribes to the point of view that fuzziness arise from *interpersonal* disagreements. The question "Do you agree that a is F?" is asked to *different* individuals. The answers are polled and an average is taken to construct the membership function.

Hersh & Carmazza (1976) use this approach[20] in their experiment 1. They presented their subjects a phrase (like small, very small, large, very large etc.) and then showed 12 squares in random order. The subjects responded by 'yes' or 'no' depending on whether they think that the phrase applies to the shown square or not. The subjects were also shown the 12 squares at the beginning of the experiment in ascending order so that every subject was operating within approximately *the same context*. Although the experimental results justify $(1 - \mu(x))$ for the connective NOT, the 'squaring the membership function' is not justified for VERY. Instead of squaring, VERY seems to *shift* the membership function (a result also reported in (Zysno 1981, Norwich & Türkşen 1984, Chameau & Santamarina 1987a)). They also report a reasonably good match between the membership function of "either small or large" and the *maximum* of the membership functions of small and large.

Polling is also one of the natural ways of eliciting membership functions for the likelihood interpretation.

3.3.2 Direct Rating

Direct rating seems to be the most straightforward way to come up with a membership function. This approach subscribes to the point of view that fuzziness arise from *individual subjective* vagueness[21]. The subject is required to classify a with respect to F *over and over again* in time. The experiment has to be carefully designed so that it will be harder for the subject to remember past answers. Hersh & Carmazza (1976) use this approach in their experiment 2. Essentially, this is a repetition of their experiment 1 for a single subject repeated over days. They report similar results for negation, disjunction and emphasizing (VERY) for each individual as in the group membership experiment. Stemming from responses of one particular subject they differentiate between *linguistic* and *logical* interpretations of membership functions.

One can also use direct rating to *compare* the answers of a subject against a predefined membership function Türkşen (1988, 1991). In that case the subject is asked "How F is a?" and the experimenter has a perfect knowledge of the evaluation for F (e.g. subject's height is known to the experimenter but not to the subject). The same question is asked to the same subject over and over again, and the membership is constructed using the assumption of probabilistic errors and by estimating a few key parameters as is usual for this type of construction.

Chameau & Santamarina (1987a) also discuss this method (which they call membership exemplification, however we reserve that term for another method). They use *several* subjects and aggregate their answers as opposed to asking a single subject same questions over and over again as is done in Norwich & Türkşen (1984) experiments. Norwich & Türkşen (1982c) and Chameau & Santamarina (1987a) report that this method results in membership functions with a wider spread (more fuzzy) when compared to polling and pairwise comparison.

Thole, Zimmermann & Zysno (1979) consider the measurement of membership functions and the justification of connectives within an empirical setting based on measurement theory. They argue that since the numerical membership scale is *bounded* the scale *has to be* an absolute scale. They mention the biases that one can have in direct elicitation methods, particularly the “end effect” which is a common problem in all bounded scales (including probability). They opt for an indirect elicitation method in the spirit of measurement theory and admit that only interval scales can be constructed with indirect methods. Then they suggest a combination of a direct rating technique and a Thurstonian scaling method. This way they create two scales and then try to identify the relationship of the two scales. They also report that neither the minimum nor the product operators are adequate representations of conjunction[22].

Nowakowska (1977) considers scaling of membership functions. He uses direct rating and poses the problem in terms of psychophysical scaling. He clarifies the assumptions to be made in order to perform direct rating.

Direct rating methods require evaluations to be measured on at least interval scales (Türkşen 1988, 1991).

3.3.3 Reverse Rating

In this method, the subject is given a membership degree and then asked to identify the object for which that degree corresponds to the fuzzy term in question (Türkşen 1988, 1991). This method can be used for individuals by repeating the same question for the same membership function as well as for a group of individuals.

Once the subject’s (or subjects’) responses are recorded the conditional distributions can be taken to be normally distributed and the unknown parameters (mean and variance) can be estimated as usual. This method also requires evaluations to be made on at least interval scales.

Chameau & Santamarina (1987a) consider reverse rating as a valuable tool to *verify* the membership function obtained by using another approach rather than an acquisition method.

3.3.4 Interval Estimation

Interval estimation subscribes to the *random set*-view of the membership function (cf. Section 3.2.2). The subject is asked to give an *interval* that describes the Fness of a. Let I_i be the set-valued observation (the interval) and m_i the frequency with which I_i is observed. Then $R = (I_i, m_i)$ defines a random set in the sense of Section 3.2.2 (Dubois & Prade 1989, 1991). Notice that this method is more appropriate to situations where there is a clear linear ordering in the measurement of the fuzzy concept like in tallness, heat, time, etc.

Chameau & Santamarina (1987a) find this approach of elicitation particularly advantageous over polling and direct rating in which the answer mode is necessarily crisp (Yes/No). Interval estimation is a relatively simple way of acquiring the membership function and it results in membership functions that are “less fuzzy” (the spread is narrower) when compared to direct rating and polling.

Interval estimation subscribes to the *uncertainty* view of membership functions as opposed to the *vagueness* view and in that sense it brings the issues of uncertainty modelling using fuzzy set theory, random sets, possibility measures and their relations to probability theory (Dubois & Prade 1993). Dubois & Prade (1986) and Civanlar &

Trussel (1986) describe statistical methods to come up with the membership function while subscribing to the random-set interpretation.

Zwick (1987) also consider the random set interpretation and uses the law of comparative judgments to assess the membership function.

Recently, set-valued statistics has been proposed as a more "natural" way of handling data than point-valued statistics. The set-valued statistics methods also assume a random-set or likelihood interpretation of the membership function. Such methods are analyzed in Kruse & Meyer (1987) in some detail.

Li & Yen (1995, Chapter 4) discuss various methods of elicitation methods. Their main assumption states fuzziness as a form of uncertainty. As a result of this view the first three methods they propose are suitable for the random set view of the fuzzy sets. They ask their subjects for intervals which include the fuzzy concept at hand. They do not differentiate between single subject versus multiple subject fuzzy concepts. And they do not discuss the issue of commensurability between two measurements they obtain for different agents and different fuzzy concepts.

3.3.5 Membership Exemplification

In terms of membership function exemplification, Hersh & Carmazza (1976) performed a test for the direct elicitation of the membership function. Hersh & Carmazza ordered twelve squares in ascending order and indicated each square with an ordinal number. They asked the subjects "write the number(s) which is appropriate for "large", "very large", "small" etc. The results are at variance with direct rating and polling most likely because there is no *repetition* in this elicitation method to normalize the effects of error or "noise".

Kochen & Badre (1974) also report experimental results on exemplification (they call it *anchoring*) which makes the resulting membership functions more precise (than without exemplification).

Zysno (1981) uses exemplification in an empirical setting. He asks 64 subjects from 21 to 25 years of age to rate 52 different statements of age with respect to one of the four sets: very young man, young man, old man, and very old man. He utilizes a scale from 0 to 100 to collect the answers. Since, at the outset, he has some hypothesis on the nature of the membership function (see Section 3.2.3), he mainly tries to test those hypotheses.

Kulka & Novak (1984) test for whether people use min-max or algebraic product and sum when combining fuzzy concepts. They use the method of exemplification (using ellipses and rectangles) in their experiments. Their conclusions are weak and they call for more experiments to come up with stronger conclusions.

However, the use of computer graphics to give an example membership function to be modified by the subject greatly enhanced this procedure as is usually witnessed in commercial applications of "fuzzy expert system shells".

3.3.6 Pairwise Comparison

Kochen & Badre (1974) report experimental results for the "precision" of membership functions using pairwise comparison method. They report experimental evidence for the precision of *greater, very much greater* and *much greater* in decreasing precision, in that order. How the addition of *very* makes the adjective more precise is highlighted.

Oden (1979) discusses the use of fuzzy set theory in psycholinguistic theories. He considers *comparisons* of the form: "which is a better example of a bird: an eagle or a pelican?", and after the answer to this question (say, an eagle is chosen): "How *much more* of a bird is an eagle than a pelican?". But, by asking for the strength of the preference directly, Oden falls for a pitfall for which many researchers have been cautioning us for a long time (see e.g. (Luce & Raiffa 1957, p. 32, Fallacy 3)).

Chameau & Santamarina (1987a) also use the same pairwise comparison technique and report it to be as robust as polling and direct rating. Following Saaty (1974), they require the subjects to provide pairwise comparisons *and the strength* of preference. This yields a non-symmetric full matrix of relative weights. The membership function is found by taking the components of the eigenvector corresponding to the maximum eigenvalue. The values are also *normalized*. Chameau & Santamarina (1987a) also find the requirement that evaluations be on a ratio scale to be unnatural (cf. Section 3.2.5).

However, they espouse a "comparison-based point estimation" which determines the position of a set of stimuli on the reference axis by pairwise comparison and the membership is calculated by *aggregating* the values provided by several subjects. Although the subjects of Chameau & Santamarina (1987a) experiments ranked this method almost as good as the interval estimation method (which was ranked as the best method), this method also needs the unfortunate assumption of a ratio scale. Furthermore, pairwise comparison requires many comparison experiments in a relatively simple domain.

3.3.7 Fuzzy Clustering Methods

Most of the elicitation techniques described so far subscribe to the subjective interpretation of the membership function. However, a particularly important concern in practice is approaches to construct membership functions from a *given set of data*. When adequate amount of data is already collected and a model based on fuzzy sets is required to analyze the problem, the membership functions must be constructed from the available data.

Fuzzy clustering techniques (Ruspini 1969, Bezdek & Harris 1979, Sugeno & Yasukawa 1993) are appropriate tools for such an analysis. Usually the analysis proceeds as follows (Nakanishi, Turksen & Sugeno 1993):

- Apply clustering on the output data and then project it into the input data, generate clusters and select the variables associated with input-output relations. The clustering method determines clusters on the data space based on Euclidean norm.

- Form the membership functions for the variables selected (i.e., determine the shape of the membership functions). There is a procedure (Nakanishi *et al.* 1993, Section 2.3.1) that lets one to select four parameters which completely characterizes trapezoidal membership functions[23].

- Select the input variables by dividing the data into three subgroups. Use two groups in the model building for selection of effective (important) variables and cross validation for data set independence. The third group is used as the test data to validate the goodness of the model.

Since most fuzzy clustering techniques are based on Euclidean norm, formation of fuzzy clusters, and hence membership functions with non-Euclidean norms requires further investigation.

3.3.8 Neural-fuzzy Techniques

With the advance in neural network models and their applications to machine learning a considerable interest has been on unifying neural networks and fuzzy set theory (Rocha 1982, Takagi 1990, Kosko 1991). In this section we briefly mention several studies in which neural networks have been utilized to come up with membership functions from a given set of data.

Takagi & Hayashi (1991) discuss a Neural Network that generates nonlinear, *multi-dimensional* membership functions which is a membership function generating module of a larger system that utilized fuzzy logic. They claim that the advantage of using nonlinear multi-dimensional membership functions is in its effects in reducing the number of fuzzy rules in the rule base.

Yamakawa & Furukawa (1992) present an algorithm for learning membership functions using a model of the *fuzzy neuron*. Their method uses example-based learning and optimization of cross-detecting lines. They assign *trapezoidal* membership functions and automatically come up with its parameters. The context is handwriting recognition. They also report some computational results for their algorithm.

On the experimental side, Erickson, Lorenzo & Woodbury (1994) claim that fuzzy membership functions and fuzzy set theory *better explain* the classification of taste responses in brain stem. They analyze previously published data and allow each neuron to belong to *several* classifications *to a degree*. This degree is measured by the neuron's response to the stimuli. They show that their model based on fuzzy set theory explains the data better than other statistical models.

Furukawa & Yamakawa (1995) describe two algorithms that yield membership functions for a fuzzy neuron and their application to recognition of hand writing. The crossing points of two (trapezoidal) membership functions are *optimized* for the task at hand.

3.3.9 General Remarks

In general, Chameau & Santamarina (1987a, 1987b) report good agreement between direct rating, interval estimation and membership exemplification with the comment that, in most of the cases, fuzzy sets obtained by exemplification method are wider (fuzzier) than the ones obtained by other methods. The main difficulty with the point-estimation method is the contradiction between fuzziness of the perception and the crispness of the response mode. This difficulty is overcome by the interval estimation method which in turn needs a minimum number of assessors or assessments. However, Chameau & Santamarina (1987a) report that as low as five assessments are sufficient. Exemplification yields membership functions without further processing which is an advantage. The way they carry out the pairwise comparison method assumes a ratio scale for the measurements which is hardly justified (cf. Section 3.2.5).

The assessors that took part in the experiments of Chameau & Santamarina (subjectively) rated the interval estimation method as the best in terms of expected consistency and expected quality. The age of the assessors affected their response, particularly in the "old-not old" task.

In all of the elicitation methods, Chameau & Santamarina obtain the membership functions based on *averaging or aggregation* of the responses from several assessors. In that sense they do not subscribe to the individualistic interpretation of fuzziness. Chameau & Santamarina justify this approach by assuming that fuzziness *is a property of the phenomenon* rather than a property attributed by the observer, which may be disputed.

One important issue in constructing membership functions is the *context*. It has always been emphasized that fuzzy set theory and particularly the membership functions are context dependent. Hersh, Carmazza & Brownell (1979) discuss the effects of *changing context* in determining the membership functions, a problem which eluded linguists for a long time (Kamp 1975). They report that the frequency of occurrence of the elements does not effect the location and form of the membership function (i.e., if one asks "Is John tall?" over and over again the resulting membership is consistently of the same form). On the other hand, the number of unique elements significantly affects the form of the membership function (i.e., the answer to the question "Is John tall?" varies when there is only one more person to consider versus when there is more than one more person to consider. The context changes!). This result tends to suggest that the membership function is not only a function of the object from the universe of discourse but the discourse as well. That is, if X is the universe of discourse and $x \in X$, the membership function for the fuzzy subset F of X is actually of the form:

$$\mu_F(x) = f_F(x, X).$$

Dombi (1990) takes up the problem of determining the membership function. He focuses on providing a theoretical basis for membership construction which is relevant to the problem at hand, described with only a few parameters that are *meaningful*. He avoids discussing problems of context dependency, measurement and uncertainty.

He postulates five axioms of which the most unnatural is the fourth one: μ is a rational function of polynomials of the following form

$$\mu(x) = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{A_o x^m + A_1 x^{m-1} + \cdots + a_m} \qquad (m \neq 0).$$

Furthermore, Dombi requires that a membership function be such that $(n + m)$ is minimal.

Based on these assumptions he derives a *mathematical form* for the membership function and verifies his result using the empirical data obtained by Zimmermann & Zysno (1985).

He also discusses the connections of the proposed form to logical connectives. Particularly, he considers aggregation operators (which generalize triangular norms) and derives expressions for the conjunction and disjunction.

This work is interesting in the sense that it starts the analysis from the membership function, postulates a *form* for membership function and then *derives* conjunction and disjunction. On the other hand, the axioms (or constraints, rather) are not further discussed for their rationality.

Chen & Otto (1995) consider constructing *continuous* membership functions from a given set of discrete points (evaluations). The rationale is that: one can only answer *finite* amount of questions from which a continuous membership function has to be constructed. They propose that the membership function can be constructed on an interval scale. When one requires membership functions to be continuous and convex[24], curve fitting methods might yield membership functions that are outside the unit interval and non-convex. In order to obtain continuous membership functions that are invariably bounded, convex and continuous Chen & Otto propose a *constrained interpolation* method. This method, which minimally considers the semantic issues, is a powerful way of obtaining a continuous membership function from discrete data, a problem frequently encountered in practice.

Kruse, Gebhardt & Klawonn (1994, Section 2.8.3) briefly discuss elicitation of membership functions. They acknowledge the elicitation as a major problem in fuzzy set theory. They maintain that the fuzzy concepts can only be measured on an ordinal scale and they emphasize that "the initial choice of precise numbers for membership functions is not important". They claim that the fuzzy methods are *insensitive* to precise measurement of membership functions and what is more important is the *qualitative properties* of the phenomena modelled using fuzzy sets.

3.4 SUMMARY

Membership functions are the building blocks of fuzzy set theory. We outlined five possible interpretations of the membership function and critically discussed each interpretation. We believe that a satisfactory treatment of the meaning of membership function can be given within measurement theory. The view from measurement theory provides insights into the foundations of the fuzzy set theory. In that framework one can lay down the axioms and thoroughly discuss them as to their applicability to the theory.

The results of the measurement-theoretic investigation indicates some difficulties in the foundations related to boundaries and to some of the *cardinal scales* (ratio and absolute scales). It is unlikely that the bounds on the membership scale are *universal*. In asserting this we do not only think that no consensus on bounds is available but also there are no normative grounds on which the bounds *should* exist. This observation brings about the *subjectivity* of membership evaluations. Furthermore, the requirements of ratio and absolute scales seem to be too strict for membership functions. Hence, the only *meaningful* operation on membership functions turn out to be the comparison (ordinal scale).

On the experimental side, there has been numerous studies for eliciting membership functions. These are mainly characterized by whether they espouse an *individual* or a *group* interpretation of fuzziness and whether they espouse an objective or a subjective view of fuzziness. Experimental results for both interpretations are reported using various methods of elicitation. The experimental results are also available for various interpretations discussed in Section 3.2. The results of the measurement-theoretic investigation impose a constraint on the elicitation method: the method should not be *intrusive* in the sense that it should not require arithmetic manipulations on the raw membership evaluation.

Furthermore, *automatic* elicitation methods like fuzzy clustering methods and fuzzy-neural techniques stem from a practical need of eliciting membership functions from a set of *given data*. These are considered more *objective* ways of elicitation when compared to others.

Notes

1. This is the definition given in Zadeh (1965). Various other formalisms also appear in the literature and many of those will be reviewed in this chapter.

2. Although this is the original notation, the recent trend is to use $F(.)$ instead of μ_F to avoid heavy mathematical notation. Since we are not going to use this notation extensively in formulas in this chapter, we stick with the older notation.

3. A many-valued (actually infinite-valued) logic based on fuzzy set theory exists and is called *fuzzy logic* (Goguen 1969, Zadeh 1975, Giles 1976). This is analogous to the one between two-valued set theory and first order predicate logic. Hence, the fuzzy set intersection and union correspond to connectives AND and OR of fuzzy logic, respectively. In this study, the terms union or disjunction of fuzzy sets and intersection or conjunction of fuzzy sets are used interchangeably.

4. Gaines (1978) also espouses a similar view in the context of the discussion "Are membership functions probabilities?". This debate is still alive (Laviolette & J. W. Seaman 1994).

5. For a different perspective see Henry E. Kyburg (1992) where errors of measurement are treated probabilistically in a Bayesian framework.

6. The term "nested" is used in the sense that whenever $\alpha \geq \alpha'$ then $F_\alpha \subseteq F_{\alpha'}$.

7. See also Klir (1994) in this connection.

8. The approach to measurement theory in this section is purely *representational and formal* as is exemplified and widely accepted in the references mentioned. However, see Section 3.4 for different views on measurement theory.

9. This "formality" of the measurement-theoretic approach is somewhat disliked by experimental psychologists (Smithson 1987, pp. 78–79).

10. These measurement problems should not be confused with a totally different measurement problem: measurement of *fuzzy measures*. That problem is studied elsewhere (Suppes 1974, Dubois 1986, Dubois 1988). There the aim is to consider an *algebra* of subsets of a set and their representation. That type of a measurement problem is more akin to the measurement theoretic representations of probabilities and highlights the formal differences between fuzzy set theory, probability theory and fuzzy measure theory (Dubois & Prade 1989).

11. Saaty used the same technique in his Analytic Hierarchy Process (AHP) Saaty (1977, 1980). However, the validity of the scale for AHP remained problematic to date (Triantaphyllou & Mann 1990, Lambert 1992, Murphy 1993, Saaty 1994). Particularly, Lambert (1992) concludes that (although based on ratio scales) the methods with which AHP come up with membership functions (priorities) are only "rough cuts".

12. In the Appendix, we show another representation theorem in which ratio scale formally arises and discuss its applicability to fuzzy set theory in Sections 3.2.5 and 3.2.5.

13. This is done mainly by following each item in Theorem 1 of the Appendix.

14. One might argue that the qualitative relation to be defined here is a "crisp" one. Then the sentence "John is cleverer than he is tall" simply means "John is clever but not tall" without showing any sign of *degrees* of tallness or cleverness. We do not subscribe to this point of view and assume that when one utters single subject sentences of the sort to be described, one has a degree of belonging in mind which we attempt to measure (Bolinger 1972).

15. Actually, one might consider a richer structure in which there is room for conjunction, negation, and implication as well. But we do not do that here for the following reasons: *(i)* conjunction and implication can easily be derived from a disjunction via the negation, and *(ii)* the representation that we have for negation functions is only *ordinal* (Trillas 1979). See also MV-algebras and their representations in this connection (Chang 1958, Chan 1959, Nola 1993).

16. Consider John, who is a basketball player, and consider his tallness. Among all the other attributes of John, let his tallness be the least that one can associate with him. Therefore, $\nu_{John}(tall) = 0$. But when compared to other people $\mu_{tall}(John) > 0$.

17. See also (de Cooman 1991, de Cooman & Kerre 1994) in this connection where the aim is not guided by measurement theory but similar ideas arise within their characterization of fuzzy sets.

18. Notice that in Figure 3.4 there is no mention of an *interval scale*. An interval scale requires a different algebraic structure $\langle A_1 \times A_2, \succsim, \oplus \rangle$ (positive difference structure) with slightly different axioms (Krantz *et al.* 1971, p. 147).

19. Another representation in which the membership functions are point-valued but the connectives are vague linguistic operators also generates interval-valued representations. These representations turn out to be ordinal as developed in Bilgiç (1995) based-on the idea of an "interval-valued fuzzy set" (Türkşen 1986).

20. See also Labov (1973) for the same approach.

21. This approach is in tune with the claim that the vagueness in verbal concepts is an integral part of the concept rather than a result of summing of variable responses over individuals (McCloskey & Glucksberg 1978).

22. This finding enables Zimmermann & Zysno (1980) to come up with a "compensatory AND" operator which is a parametric operator that covers minimum and product. Relationship of this operator to interval-valued fuzzy sets is shown in (Türkşen 1992).

23. There has been a recent interest in using trapezoidal membership functions in fuzzy set theory mainly because their special structure yields more efficient computations. By definition, triangular membership functions are special cases of trapezoidal membership functions. However, trapezoidal functions are not as general as continuous spline functions. See e.g. (Pedrycz 1994) for a motivation to use triangular membership functions in fuzzy control.

24. A membership function, μ, (over the real numbers) is called *convex* if and only if for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$, $\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}$.

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Appendix: Ordered algebraic structures and their representations

In this section basic definitions, and representation and uniqueness results for algebraic structures, called *ordered semigroups*, are given. These results form the basis of discussions on measurement-theoretic semantics that are discussed in Sections 3.2.5 and 3.2.5.

Mainly, the results of Fuchs (1963) and Schweizer & Sklar (1983) are translated in terms of ordered algebraic structures as is customary in the measurement theory literature.

Definition 1 *The algebraic structure $\langle A, \oplus \rangle$ where A is a nonempty set and $\oplus$ is a binary operation on A is called a* semigroup *if and only if $\oplus$ is associative (i.e., for all $a, b, c \in A$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$). If there exists $e \in A$ such that for all $a \in A$, $e \oplus a = a \oplus e = a$ the structure $\langle A, \oplus, e \rangle$, is called a* semigroup with identity e *or a* monoid*. Finally, $\langle A, \oplus, e \rangle$ is a* group *if and only if it is a semigroup with identity e and any element of A has an inverse:for all $a \in A$, there exists $b \in A$ such that $a \oplus b = b \oplus a = e$.*

When the algebraic structure is also endowed with an ordering, $\succsim$, we obtain *ordered algebraic structures*.

Definition 2 *Let A be a non-empty set, $\succsim$ a binary relation on A and $\oplus$ a binary operation on A. $\langle A, \succsim, \oplus \rangle$ is an* ordered structure *if and only if the following axioms are satisfied:*

(weak ordering) $\succsim$ is connected and transitive,

(monotonicity) for all $a, b, c, d \in A$, $a \succsim c$ and $b \succsim d$ imply $a \oplus b \succsim c \oplus d$.

The asymmetric part ($\succ$) and the symmetric complement ($\sim$) of any relation $\succsim$ are defined as usual: $a \succ b$ if and only if $a \succsim b$ and not $b \succsim a$ and $a \sim b$ if and only if $a \succsim b$ and $b \succsim a$.

Adding more properties to an ordered algebraic structure results in specializations of the concept. In this paper, we only consider ordered *semigroups* (where the concatenation is associative). These are summarized in the following definition:

Definition 3 *Let $\mathcal{A} = \langle A, \succsim, \oplus \rangle$ be an ordered algebraic structure such that $\langle A, \oplus \rangle$ is a semigroup. Then $\mathcal{A}$ is called an* ordered semigroup*. Furthermore, it is said to be:*

Weakly Associative *(WA) $a \oplus (b \oplus c) \sim (a \oplus b) \oplus c$.*

Solvable *(Sv) iff whenever $a \succ b$ then there exists $c \in A$ such that $a \succsim b \oplus c$.*

Strongly Monotonic *(SM) iff whenever $a \succsim b$ then $a \oplus c \succsim b \oplus c$ then $c \oplus a \succsim c \oplus b$.*

Homogeneous *(H) iff whenever $a \succsim b$ if and only if $a \oplus c \succsim b \oplus c$ if and only if $c \oplus a \succsim c \oplus b$.*

Idempotent *(Ip) iff for all $a \in A$, $a \oplus a \sim a$.*

Bounded *(B) iff there exist* u *and* e *in* A *such that: for all* $a \in A$, $u \succ e$, $u \succsim a$ *and* $a \succsim e$.

Archimedean *(Ar) iff for any* $a, b \in A$ *there exists a positive integer* m *such that* $a^{(m)} \succ b$ *where* $a^{(m)}$ *is recursively defined as* $a^1 = a$, $a^{(m)} = a \oplus a^{(m-1)}$.

Continuous *iff* $\oplus$ *is continuous as a function of two variables, using the order topology on its range and the relative product topology on its domain.*

By a representation of an ordered algebraic structure, we mean a real valued function that maps the ordered algebraic structure, $\langle A, \succsim, \oplus \rangle$ to a numerical structure, $\langle X, \geq, S \rangle$, where X is a subset of $\mathbb{R}$, $\geq$ is the natural ordering of real numbers and $S : X \times X \to X$ is a function. Since we focus on ordered semigroups, in the resulting representation, S is necessarily associative.

The boundary condition, asserts the existence of a minimal and a maximal element in set A. Hence, given the weak ordering and the boundaries, one can replace the set A by the familiar interval notation $[e, u]$

The following lemma demonstrates some of the consequences of axioms imposed on a bounded ordered semigroup (Schweizer & Sklar 1983).

Lemma 1 *Let* $\mathcal{A} = \langle A, \succsim, \oplus \rangle$ *be a bounded ordered semigroup with bounds* e *and* u*.Then* $\mathcal{A}$ *also satisfies the following conditions for all* $a, b \in A$*:*

(i) $a \oplus b \succsim sup(a, b)$,

(ii) $u \oplus a \sim a \oplus u \sim u$,

(iii) $a \oplus a \succsim a$.

Schweizer & Sklar (1983, Section 5.3) define a *function* on a closed real interval endowed with the natural ordering, $\geq$. Here, a more abstract structure is considered but their results carry over to our setting without modification since our relation, $\succsim$, is transitive and connected and hence $\sim$ is an equivalence.

Representation theorems with varying uniqueness characteristics can be given for ordered semigroups. These are summarized in the following:

Theorem 1 *The algebraic structure* $\langle A, \succsim, \oplus \rangle$ *is:*

(i) a bounded ordered semigroup *if and only if there exists* $\gamma : [e, u] \to X \triangleq [\underline{x}, \overline{x}]$ *such that,* $a \succsim b \iff \gamma(a) \geq \gamma(b)$, $\gamma(e) = \underline{x}$, $\gamma(u) = \overline{x}$, *and* $\gamma(a \oplus b) = S(\gamma(a), \gamma(b))$ *where* $X \triangleq [\underline{x}, \overline{x}]$ *is a closed subset of* $\mathbb{R}$ *and* S *is an associative, monotonic function such that* $S : [\underline{x}, \overline{x}] \times [\underline{x}, \overline{x}] \to [\underline{x}, \overline{x}]$ *which has* $\underline{x}$ *as its identity. Furthermore,* γ' *is another representation if and only if there exists a strictly increasing function* $\phi : [\underline{x}, \overline{x}] \to [\underline{x}', \overline{x}']$ *with* $\phi(\underline{x}) = \underline{x}'$ *and* $\phi(\overline{x}) = \overline{x}'$ *and such that for all* $x \in [\underline{x}, \overline{x}]$, $\gamma'(x) = \phi(\gamma(x))$ *(ordinal scale).*

(ii) a bounded idempotent semigroup *if and only if all the conditions in (i) are satisfied and* $S = \max$.

(iii) a continuous Archimedean bounded ordered semigroup *if and only if the conditions of (i) are satisfied with* $X = [0, 1]$*, and there exists a strictly increasing continuous function,* $g : [0, 1] \to \bar{\mathbb{R}}^+ = [0, \infty]$ *with* $g(0) = 0$*, such that for all* $x, y \in [0, 1]$*,* $S(x, y) = g^{[-1]}(g(x) + g(y))$*, where* $g^{[-1]}$ *is the pseudo-inverse of* g *given by:* $g^{[-1]}(\alpha) = g^{-1}(\min\{\alpha, g(1)\})$*. Furthermore,* g *is unique up to a positive constant (ratio scale).*

(iv) a solvable homogeneous Archimedean strongly monotonic ordered semigroup *if and only if it is isomorphic to a sub semigroup of* $\langle \mathbb{R}^+, \geq, + \rangle$*. Moreover, two such isomorphisms are unique up to a positive constant (ratio scale).*

(v) a solvable Archimedean strongly monotonic ordered semigroup *if and only if it is isomorphic to a sub semigroup,* $\langle [0, 1], \geq, S_W \rangle$ *where* $S_W(x, y) = \min\{x + y, 1\}$ *for all* $x, y \in [0, 1]$ *and two such isomorphisms are necessarily equivalent (absolute scale).*

First two parts of Theorem 1 can easily be proven (see Bilgiç & Türkşen (1995a) and Bilgiç (1995) for details), part three is Ling's (1965) representation theorem for a continuous Archimedean triangular norm (see Schweizer & Sklar (1983) for historical comments on this representation and Aczél (1966) for a proof) and parts four and five are from Fuchs (1963).