

Denotational Semantics of Object Specification

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Abstract

From an arbitrary temporal logic institution we show how to set up the corresponding institution of objects. The main properties of the resulting institution are studied and used in establishing a categorical, denotational semantics of several basic constructs of object specification, namely aggregation (parallel composition), interconnection, abstraction (interfacing) and monotonic specialization. A duality is established between the category of theories and the category of objects, as a corollary of the Galois correspondence between these concrete categories. The special case of linear temporal logic is analysed in detail in order to show that categorical products do reflect interleaving and reducts may lead to internal non-determinism.

Key words:

object-orientation, system specification, temporal logic, institution, denotational semantics, duality.

1 Introduction

The advantages of object-orientation in software engineering in general and system specification in particular are by now well understood [35, 51]. We consider herein the specific problem of specifying communities of concurrent, interacting, reactive objects [47]. Therefore, it is natural to adopt temporal logic as a convenient way of specifying such objects in a declarative way [38, 43, 46, 34]. However, the state of the art in the latter domain is yet progressing towards the envisaged degree of compositionality, since the pioneering effort reported in [9]: how should we combine temporal specifications of isolated components in a meaningful specification of the whole system? This problem has been attracting much attention, at both semantic and verification levels [1, 2, 36, 29, 30, 21]. See also [4, 5, 3, 14], though not necessarily within the temporal logic setting.

Herein, by combining temporal logic and object-orientation we hope to contribute decisively to achieving higher degrees of compositionality. Undeniably, objects provide the tools for controlling what is private and what is public

and, therefore, the means for specifying and reasoning about open systems that can be easily embedded in arbitrary environments. To this end, we also found advantageous to work within the setting of the theory of institutions [26, 27]. Indeed, the notion of reduct and the satisfaction condition are essential to any combination mechanism across different alphabets.

We show how to build from any given temporal institution (satisfying a few reasonable requirements) a suitable semantic domain where it is possible to define categorially the (final) semantics of all basic object constructs (namely, aggregation or parallel composition, interconnection, abstraction or interfacing, and monotonic specialization). This semantic domain is also conveniently presented as an institution (canonically induced by the given temporal institution). To this end, we use a technique based on a cofibration for indexing categories (over signatures). The basic idea is to identify each object with a set of lives (set of life-cycles in the case of linear temporal logic or set of life-trees in the case of branching temporal logic). Relevant work in this direction is reported for special cases [23, 44, 22]. An interesting alternative not explored herein would be to identify each object with a sheaf [25].

Hence, the basic achievement of the paper is to build a semantic domain around temporal logics so that a duality can be established between temporal theories and the semantic units (the objects). This duality arises from a Galois correspondence. We associate with each specification the theory induced by it and define its semantics as an object. We show that the semantics is well behaved: the semantics of an aggregation (which is a coproduct of the theories) is the product of the objects (that define the semantics of the components); and the semantics of the theory generated by abstracting away some components of a specification is the reduct (obtained by cartesian lifting) of the semantics of the theory generated by the specification. In this way, we obtain a categorial, denotational semantics for any logic suitable for reactive object specification. We should emphasize that a similar methodology can be applied when using for specification another logic (not modal/temporal based): for instance, a many-sorted first order logic with a specific sort for dealing with states. For preliminary results in this direction see [40] within the situation calculus [42].

The importance and the usefulness of the semantics to reasoning is discussed in [48] where we present a temporal logic (OSL) for reasoning about objects, object identifiers, aggregation of objects and monotonic specialization. Among other relevant mechanisms, this logic includes inheritance of axioms from the components to the aggregation and specialization, as well as reflection of properties from the aggregation and the specialization to the components. In [49], the logic is further explored and special rules are given to verify safety properties namely involving combination of effects. This logic is extended in [16] with transactions for dealing with refinement.

For the sake of economy of presentation, herein we consider only the propositional fragment of the logic at hand. No problems appear when extending the results to full first order temporal logic, as long as the variable domains are rigid and object-independent.

We assume that the reader is conversant with the field of temporal logic specification for instance at the level of [28, 20] (for more details on modal

and temporal logic see [24]) and with institutions [27]. We also use a little bit of category theory assuming that the reader has a working knowledge of the basic notions, up to (co)limits, as found in the introductory chapters of a textbook on the subject, e.g., [7]. See also [39]. But we include in the Appendix a brief on (co)fibrations (following [8]) and Galois correspondences (following [6]). We feel that category theory provides the right abstraction to deal with compositionality issues at the core of this paper, following the steps of earlier pioneering work in specifications [27] and processes [53].

In section 2, we provide an intuitive road map for the rest of the paper built around an example specified using the propositional fragment of OSL, the linear temporal logic of objects presented in [48]. In section 3, we state the (few) requirements that we need to impose on the base temporal institution and we establish the most interesting properties of theories of that institution. We also extend the notion of satisfaction to sets of interpretation structures. In section 4, we introduce objects and their main properties, paving the way for introducing in section 5 the envisaged object institution. In section 6, we introduce and establish the main properties of the final semantics functor, including the Galois correspondence between theories and objects that we prove to be a duality. In section 7, we discuss the semantics of the main object constructs, and the special case of linear temporal logic is considered in detail. We leave some brief comments on future work, alternative approaches and limitations to section 8.

2 Outline of the approach

At first sight [47, 45, 48] an object is an entity with an internal state (reflected in the values of its slots or attributes) that may interact with other objects. While so interacting an object displays some behaviour. That is, depending on its internal state, the object may be ready or not to provide some service. Clearly, the basic unit of interaction is the event: the occurrence of some action. Thus, the behaviour of an object may be described as a state-dependent menu of enabled actions.

Therefore, the specification of an object should state the set of its (attribute) observations, the set of its actions, and the constraints on its behaviour. As an illustration consider a simple *flip-flop* as an object. The *flip-flop* has a unique attribute status with two possible observations: *on* and *off* (not *on*). It allows only two actions *flip* and *flop*. Its behaviour is easily specified using a suitable dialect of propositional linear temporal logic (X stands for “next”), as follows:

1. $(\Diamond flip \Leftrightarrow on)$;
2. $(\Diamond flop \Leftrightarrow (\neg on))$;
3. $(\nabla flip \Rightarrow (X(\neg on)))$;
4. $(\nabla flop \Rightarrow (X on))$;
5. $((\neg \nabla flip) \wedge (\neg \nabla flop)) \Rightarrow (on \Leftrightarrow (X on))$.

The propositional symbols are of the form ∇c and $\diamond c$, where c is an action, or b , where b is an observation. The symbol ∇c (c happens or c occurs) indicates the occurrence of action c . The symbol $\diamond c$ (c is enabled) indicates that action c is possible. Axioms 1 and 2 indicate when actions are enabled. Axioms 3 and 4 state the effects of actions on observations. Finally, axiom 5 indicates that observations do not change if no (local) action occurs.

Such a logic of behaviour with happening (∇) distinct from enabling (\diamond) is rich enough for describing in practice useful reactive systems. For further examples see [48], where a proof system is provided for dealing with object specialization and aggregation. Note that fairness assumptions are easily specified in this logic. For instance, if so specified, $((F(G \diamond c)) \Rightarrow (G(F \nabla c)))$ states that if action c is continuously enabled from a certain position on then it happens infinitely many times (F stands for “sometime” and G for “always”). Clearly, we expect the universal property $(\nabla c \Rightarrow \diamond c)$ since an action c may happen only when enabled. As we shall see, the proposed semantics ensures the universal validity of this assertion.

Linear temporal interpretation structures are sequences of valuations for the corresponding propositional symbols. Thus, for instance, the sequences

- $\{on, \diamond flip\}^\omega$
- $(\{on, \diamond flip, \nabla flip\}.\{\diamond flop, \nabla flop\})^\omega$
- $(\{\diamond flop, \nabla flop\}.\{on, \diamond flip, \nabla flip\})^\omega$

satisfy the *flip-flop* specification above. In other words, they represent possible executions (runs or lives) of an object meeting the *flip-flop* specification. This intuition clearly agrees with the assumption that an object should be represented by the set of its possible executions. In fact, due to either internal or external choices (the external ones depending on the environment), an object can in general display many distinct executions. In the sequel, the “behaviour” of an object will therefore be thought of as the set of its possible “lives”, a “life” being an interpretation structure satisfying the specification within the base temporal logic at hand.

The intended (maximal or final) behaviour of *flip-flop* is represented by the automaton in Figure 1. The circles represent states. The labels of the arrows indicate the set of actions that can occur when the *flip-flop* is in a certain state and the direction of the arrow indicates the new state. The label \emptyset reflects axiom 5. If nothing happens when the *flip-flop* is in a state it will remain in that state. The set of life-cycles of the automaton represents the set of all possible executions of an object satisfying the specification. The denotation of the specification is therefore meant to be the object consisting of all those “lives”.

The purpose of the present work is precisely to present a method for building the “power model” institution canonically induced by any base temporal institution fulfilling the few requirements presented in section 3. While the syntactic part of the institution does not change (we keep the specification language), interpretation structures are required to be admissible sets of interpretation structures of the base logic. These new interpretation structures will

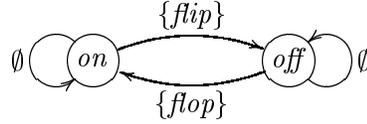


Figure 1: The intended *flip-flop* behaviour.

be called “objects”. Of course, satisfaction changes accordingly. The underlying intuition is very simple: an object satisfies a specification if all of its possible “lives” satisfy the specification in the given base institution. This summarizes the contents of sections 4 and 5.

This new institution is subsequently used for providing a categorical, denotational semantics of object specification. In section 6 we introduce the (maximal or final) denotation of a specification and establish the envisaged duality between specifications (theories) and objects. Theories and objects are related in such an exact way that, a coproduct of theories corresponds to the product of the component objects, the reduct of a theory corresponds to the reduct of the original object, and so on.

We now illustrate the application of the technique by giving semantics to interconnection (constrained parallel composition) and abstraction (hiding) constructs within the linear temporal logic at hand.

For the purpose of illustrating the specification of interconnections between objects, consider the clone of the *flip-flop* above with the alphabet on' , $flip'$ plus $flop'$ and fulfilling the axioms 1'-5' obtained in an obvious way from axioms 1-5. By imposing, for instance, the interconnection axiom ($\nabla flip \Leftrightarrow \nabla flop'$) we state that the happening of each *flip* in the first *flip-flop* must synchronize with the happening of each *flop'* in the clone. Clearly, a full fledged specification logic needs the ability to refer to the identities of the objects at hand: that is of course possible in the first order version of the logic OSL (see [48]).

Following the general approach of the theory of institutions [10], composition of specifications (theories) corresponds to taking colimits of adequate diagrams of component theories whose morphisms explain the desired interconnection. In more concrete terms, the axioms of the resulting theory of the composed system are the adequate translations of the axioms of the component theories. This approach goes hand in hand with the principle of “composition as conjunction” advocated in [1] for the parallel composition of reactive systems. On the semantic side, the study and classification of several categories of models of behaviour [53] also suggests the characterization of parallel composition as a product of the components and the use of restriction mechanisms via liftings from signature morphisms for interconnection. Thus, aggregation is to be captured by products, and interconnections in general can be obtained by restricting the product according to the desired interplay between the components [15].

Therefore, the specification of the above community of two interconnected *flip-flops* consists of the specification of their unconstrained aggregation (the union of the two specifications) plus the desired interconnection constraints (in

this case just $(\nabla flip \Leftrightarrow \nabla flop')$. Semantically, it is clear that, for instance, the sequence

- $(\{on, \diamond flip, \nabla flop', \nabla flop'\}. \{\diamond flop, \nabla flop, on', \diamond flip', \nabla flop'\})^\omega$

is a possible execution of the interconnection of the two objects. The projections of the sequence to the language of each of the *flip-flops* yield possible executions of each of them and, moreover, the interaction requirement is fulfilled. But it is clear that many others are allowed. The intended behaviour for the two interconnected *flip-flops* (the corresponding restricted product of their denotations) is represented in Figure 2. Now the circles indicate the states of the two *flip-flops* and the labels of the arrows indicate which actions from both *flip-flops* are allowed to occur. For example, the label $\{flip, flop'\}$ corresponds to the interaction axiom: it says that the two events can happen together. And there is no possible transition from state $on|on'$ to state $off|off'$ since the actions *flip* and *flip'* cannot occur at the same time because of the interconnection axiom.

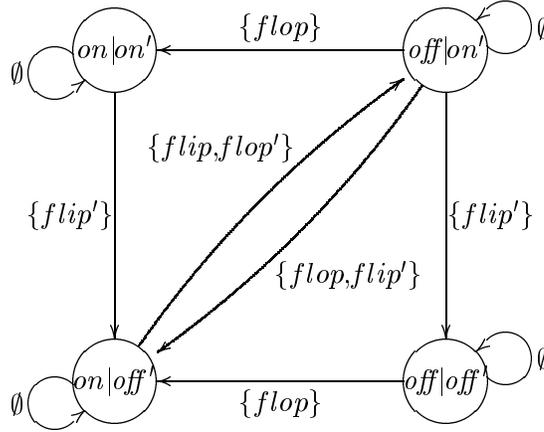


Figure 2: The interconnection of the *flip-flops*.

For illustrating the specification of abstractions (interfacing) consider the problem of specifying a “view” of the original *flip-flop* where *flop* is made invisible. Within the envisaged categorial setting, this desideratum is easily achieved by stating that the view is the “reduct” of that *flip-flop* along the obvious inclusion alphabet map (signature morphism).

Herein, we follow the general principle of using renaming for abstraction by means of suitable signature morphisms as successfully used in [53] for several models of behaviour. At the specification level, axioms of the abstraction are all the theorems of the original *flip-flop* which can be written without mentioning *flop*. On the semantic level, one just omits *flop* from the labels.

Obviously, “hiding” the corresponding *flop* in each execution of the *flip-flop* yields an execution of the abstraction, e.g.,

- $(\{on, \diamond flip, \nabla flop'\}. \emptyset)^\omega$

The intended behaviour of the abstraction (the corresponding view of the denotation of the *flip-flop*) is represented in Figure 3. Note that from state *off* the *flip-flop* can go to state *on* with no actions occurring. This means that there are “hidden” transitions and non-determinism involved.

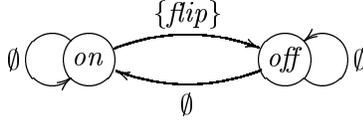


Figure 3: A *flip-flop* with hidden *flop*.

Monotonic specialization can also be very easily illustrated: more alphabet and/or axioms may be added to the original specification of the *flip-flop*. However, some care must be taken in order to avoid falling into contradiction (e.g., if overriding or side effects are envisaged).

Following these ideas, the *general method* we propose in this paper (for establishing a categorial, denotational semantics of object specification from any suitable temporal logic) is as follows:

- Check that the given base temporal logic fulfils the (rather weak) requirements stated in section 3.
- Set up the envisaged institution of objects by identifying each object with a suitable set of lives (models of the given base logic), as detailed in sections 4 and 5.
- Obtain the “final semantics” map (actually a functor) that associates to each specification the maximal (final) object satisfying it, as presented in section 6.
- Capitalize on the general properties of this final semantics functor (as established in section 6) to conclude that the resulting semantics is denotational (and categorial) for several relevant object constructions, as exemplified in section 7. For instance “conjunction” of specifications corresponds to “product” of objects.

3 Base temporal institution

In this section, we start by introducing the basic requirements that the base temporal institution must satisfy. Then, we introduce the concepts of specification and theory and study the main properties of theories namely how to compose theories and how to obtain canonical theories from signature morphisms. Finally, we discuss satisfaction.

3.1 Requirements

Recall that an *institution* is a quadruple $I = \langle Sig, Sen, Int, \Vdash \rangle$ where:

- Sig is a category whose elements are *signatures* (alphabets or vocabularies);
- $Sen : Sig \rightarrow Set$ is a functor giving for each signature the *set of formulae* for that signature;
- $Int : Sig \rightarrow Cat^{op}$ is a functor giving for each signature a (small) category of *interpretation structures* for that signature;
- $\models = \{\models_{\Sigma}\}_{\Sigma \in |Sig|}$ where $\models_{\Sigma} \subseteq |Int(\Sigma)| \times Sen(\Sigma)$ is the *satisfaction relation* for the signature Σ ;

such that the following *satisfaction condition* holds:

$$I' \models_{\Sigma'} Sen(\sigma)(\varphi) \text{ iff } Int(\sigma)(I') \models_{\Sigma} \varphi$$

for every signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, interpretation structure I' over Σ' and formula φ over Σ .

Most of the logics that are used in computer science are institutions. Therefore, in an institution we can work with several vocabularies and we can relate them (signature morphisms). Whenever there is a signature morphism we can (functor Sen) translate formulae from the source signature to formulae from the target signature. Whenever there is a signature morphism we can (functor Int) get an interpretation structure for the source signature from an interpretation structure for the target signature. Note that we are also able to relate interpretation structures over the same signature by homomorphisms (morphisms of $Int(\Sigma)$ for signature Σ). The satisfaction relation relates formulae and interpretation structures over the same signature. Hence, if $I \models_{\Sigma} \varphi$ we say that I satisfies φ . The satisfaction condition imposes a requirement on satisfaction over two signatures that are related by a signature morphism.

Herein, we assume that we are given a base temporal institution $B = \langle Sig, Sen, Int, \models \rangle$ with a set Υ of modal operators, satisfying the following *base requirements*:

- Sig is the category $Set \times Set$.
Given a signature $\langle \Sigma_{obs}, \Sigma_{act} \rangle$, each element $b \in \Sigma_{obs}$ is an *observation symbol* (we are abstracting away attributes and their values) and each element $c \in \Sigma_{act}$ is an *action symbol*. Each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ is of the form $\langle \sigma_{obs} : \Sigma_{obs} \rightarrow \Sigma'_{obs}, \sigma_{act} : \Sigma_{act} \rightarrow \Sigma'_{act} \rangle$.
- Let A_{Σ} be the least set satisfying the following rules:
 - $b \in A_{\Sigma}$, provided that $b \in \Sigma_{obs}$;
 - $\forall c, \diamond c \in A_{\Sigma}$, provided that $c \in \Sigma_{act}$.

Then, $Sen(\Sigma)$ (the set of *formulae*) is inductively defined on the set A_{Σ} (set of *atomic formulae*) as follows:

- $(\neg \gamma) \in Sen(\Sigma)$, provided that $\gamma \in Sen(\Sigma)$;
- $(\gamma_1 \Rightarrow \gamma_2) \in Sen(\Sigma)$, provided that $\gamma_1, \gamma_2 \in Sen(\Sigma)$;

- $(\Box\gamma) \in \text{Sen}(\Sigma)$, provided that $\gamma \in \text{Sen}(\Sigma)$ and $\Box \in \Upsilon$.
- The *translation map* $\text{Sen}(\sigma) : \text{Sen}(\Sigma) \rightarrow \text{Sen}(\Sigma')$ induced by the signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ satisfies the following rules:
 - $\text{Sen}(\sigma)(b) = \sigma_{\text{obs}}(b)$;
 - $\text{Sen}(\sigma)(\nabla c) = \nabla \sigma_{\text{act}}(c)$;
 - $\text{Sen}(\sigma)(\Diamond c) = \Diamond \sigma_{\text{act}}(c)$;
 - $\text{Sen}(\sigma)(\neg \gamma) = (\neg \text{Sen}(\sigma)(\gamma))$;
 - $\text{Sen}(\sigma)(\gamma_1 \Rightarrow \gamma_2) = (\text{Sen}(\sigma)(\gamma_1) \Rightarrow \text{Sen}(\sigma)(\gamma_2))$;
 - $\text{Sen}(\sigma)(\Box\gamma) = (\Box \text{Sen}(\sigma)(\gamma))$.
- Each interpretation structure λ of $\text{Int}(\Sigma)$ is a triple $\langle W, R, V \rangle$ where:
 - W is a non empty set (of *worlds*);
 - R is a Υ -indexed set of binary relations on W (the *accessibility relations*, one per each element of Υ);
 - $V : A_\Sigma \rightarrow \wp W$ (the *valuation map*) is such that $V(\nabla c) \subseteq V(\Diamond c)$ for each $c \in \Sigma_{\text{act}}$.
- For each interpretation structure $\langle W', R', V' \rangle$ and signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, $\text{Int}(\sigma)(\langle W', R', V' \rangle) = \langle W', R', V' \circ \text{Sen}(\sigma) \rangle$ (the *reduct* of $\langle W', R', V' \rangle$ with respect to σ).
- For each interpretation structure $\lambda = \langle W, R, V \rangle$ and each formula γ of $\text{Sen}(\Sigma)$, $\lambda \Vdash_\Sigma \gamma$ iff $\lambda \Vdash_w \gamma$ for every world $w \in W$, assuming:
 - $\lambda \Vdash_w \alpha$ iff $w \in V(\alpha)$, for each $\alpha \in A_\Sigma$;
 - $\lambda \Vdash_w (\neg \gamma)$ iff not $\lambda \Vdash_w \gamma$;
 - $\lambda \Vdash_w (\gamma_1 \Rightarrow \gamma_2)$ iff $\lambda \Vdash_w \gamma_2$ or not $\lambda \Vdash_w \gamma_1$;
 - $\lambda \Vdash_w (\Box\gamma)$ iff $\lambda \Vdash_{w_1} \gamma$ for every $w_1 \in W$ such that $wR_\Box w_1$.

The base institution allows different sets of modal operators as well as the corresponding accessibility relations (for example, in the case of linear temporal logic the set of modal operators Υ is composed by two operators X , next, and G , always in the future). It may be the case that in a particular institution we want to add more operators (like for example U , until, in linear temporal logic). That can be done by adding more clauses in the inductive construction of formulae and subsequent developments. We note that we could have more generic signatures. But it seems that there is a consensus that the state of an object can be changed by actions and its properties can be observed. Therefore, it would always be possible to extract from a signature the set of actions and the set of observations (via suitable forgetful functors).

Note that it may be the case that not every $\lambda = \langle W, R, V \rangle$ is in $\text{Int}(\Sigma)$. Given an interpretation structure $\langle W, R, V \rangle$, the pair $\langle W, R \rangle$ is called the *frame* of the interpretation structure. Linear temporal logic is uniframe since for all

the frames W is \mathbb{N} , the relation R_X is successor and the relation R_G is the transitive closure of R_X .

In the sequel, and as a matter of simplicity we just consider one modal operator \Box and one accessibility relation R .

Note that $\emptyset = \langle \emptyset, \emptyset \rangle$ is a possible signature (no observation symbols and no action symbols), called the *empty signature* and $Sen(\langle \emptyset, \emptyset \rangle) = \emptyset$. There is a signature morphism 0_Σ from the empty signature to Σ , for every signature Σ , composed by two empty maps and $Sen(0_\Sigma)$ is the empty map \emptyset . Moreover, $Int(0_\Sigma)(\langle W, R, V \rangle) = \langle W, R, V \circ \emptyset \rangle = \langle W, R, \emptyset \rangle$, for every interpretation structure $\langle W, R, V \rangle$.

For our purposes the homomorphisms between interpretation structures over Σ are not relevant (we could adopt the p -morphisms as in [28]).

In the sequel, we denote $Sen(\Sigma)$ by L_Σ , $Sen(\sigma)(\varphi)$ by $\sigma(\varphi)$, $Int(\Sigma)$ by Λ_Σ , $Int(\sigma)(\lambda')$ by $\sigma^{-1}(\lambda')$, and $V' \circ Sen(\sigma)$ by $\sigma^{-1}(V')$. Furthermore, provided that $\Phi \subseteq L_\Sigma$ and $\Phi' \subseteq L'_\Sigma$, we denote by:

- $\sigma(\Phi)$ the set $\{\sigma(\varphi) \in L_{\Sigma'} : \varphi \in \Phi\}$ (*image* of Φ by σ);
- $\sigma^{-1}(\Phi')$ the set $\{\varphi \in L_\Sigma : \sigma(\varphi) \in \Phi'\}$ (*reverse image* of Φ' by σ).

We use Φ^F for denoting the semantic closure of Φ : $\{\psi \in L_\Sigma : \Phi \models_\Sigma \psi\}$. And as usual, $\Phi \models_\Sigma \psi$ states that Φ *entails* ψ : for every interpretation structure λ over Σ , if $\lambda \Vdash_\Sigma \varphi$ for each $\varphi \in \Phi$ then $\lambda \models_\Sigma \psi$. Finally, $\lambda \Vdash_\Sigma \Phi$ iff $\lambda \models_\Sigma \varphi$ for every $\varphi \in \Phi$.

Each valuation map V induces the map $\underline{V} : W \rightarrow \wp A_\Sigma$ such that $\underline{V}(w) = \{\alpha \in A_\Sigma : w \in V(\alpha)\}$ and vice-versa. In the sequel, we shall use either of them as a matter of convenience. Clearly, $\sigma^{-1}(\underline{V}')$ maps each world w to the set $\{\alpha \in A_\Sigma : \sigma(\alpha) \in \underline{V}'(w)\}$.

Finally, recall that, according to the definition of institution, the *satisfaction condition* holds in B .

3.2 Specifications and theories

We recall in this paragraph the main concepts related to theories in B that do not depend on the base requirements, presenting the category of theories as a concrete category.

Definition 3.1 A *specification* over Σ is a pair $spec = \langle \Sigma, \Phi \rangle$ where $\Sigma = Si(spec)$ is a signature and $\Phi \subseteq L_\Sigma$. A *theory* over Σ is a specification $th = \langle \Sigma, \Phi \rangle$ such that $\Phi^F = \Phi$.

In a specification (or presentation) $spec = \langle \Sigma, \Phi \rangle$, the elements of Φ are called *axioms*. In the case of a theory they are called *theorems*. Clearly, every specification induces a theory by semantic closure. Theories can be related by theory morphisms.

Definition 3.2 A *theory morphism* $f : th \rightarrow th'$, with $th = \langle \Sigma, \Phi \rangle$ and $th' = \langle \Sigma', \Phi' \rangle$, is a signature morphism $Si(f) : Si(th) \rightarrow Si(th')$ such that $Si(f)(\Phi) \subseteq \Phi'$.

So there is a theory morphism between two theories provided that we are able to map the signature of the source to the signature of the target (signature morphism) and that the translation of each formula in the source language is a formula of the target language.

For instance there is a theory morphism from the theory generated by semantic closure of the specification of the *flip-flop* to the theory generated by semantic closure of the specification of the interconnection between the *flip-flop* and its clone.

Prop/Definition 3.3 Theories and their morphisms constitute a concrete category $\langle Th, Si \rangle$ over *Sig*.

Proof: The fact that *Th* is a category in any institution is well known. And it is trivial to verify that the functor $Si : Th \rightarrow Sig$ is faithful. QED

A concrete category indicates that the category of theories is built from the category of signatures and that there is a way (the functor) to obtain the signature of a theory (forgetting the formulae) and to obtain the signature morphism of a theory morphism (forgetting the condition on formulae). The recognition of this concrete theory is important because we are going to “lift” in a canonic way signatures to theories.

3.3 Main properties of theories

We now present some useful results concerning theories in *B*. We start by seeing that the category of signatures is cocartesian, that is to say that it has finite coproducts. Thus, we have to prove that initial objects exist (we will see the importance of empty signatures) and that binary coproducts exist as well.

Proposition 3.4 The category *Sig* is cocartesian.

Proof:

1. The empty signature $\langle \emptyset, \emptyset \rangle$ is initial.

Denote by 0_Σ the pair $\langle 0_{obs} : \emptyset \rightarrow \Sigma_{obs}, 0_{act} : \emptyset \rightarrow \Sigma_{act} \rangle$. Then, 0_Σ is the unique signature morphism from $\langle \emptyset, \emptyset \rangle$ to Σ .

2. A coproduct of signatures $\Sigma + \Sigma'$ is given by $\langle \Sigma_{obs} + \Sigma'_{obs}, \Sigma_{act} + \Sigma'_{act} \rangle$ with injections $inj_\Sigma = \langle i, j \rangle : \Sigma \rightarrow \Sigma + \Sigma'$ and $inj_{\Sigma'} = \langle i', j' \rangle : \Sigma' \rightarrow \Sigma + \Sigma'$, assuming that: $\Sigma_{obs} + \Sigma'_{obs}$ with injections $i : \Sigma_{obs} \rightarrow \Sigma_{obs} + \Sigma'_{obs}$ and $i' : \Sigma'_{obs} \rightarrow \Sigma_{obs} + \Sigma'_{obs}$ is the coproduct in *Set* of Σ_{obs} and Σ'_{obs} ; and $\Sigma_{act} + \Sigma'_{act}$ with injections $j : \Sigma_{act} \rightarrow \Sigma_{act} + \Sigma'_{act}$ and $j' : \Sigma'_{act} \rightarrow \Sigma_{act} + \Sigma'_{act}$ is the coproduct in *Set* of Σ_{act} and Σ'_{act} . QED

Now we know how to put together two signatures but still recognizing the components (the coproducts in *Set* are disjoint unions). Then, in $L_{\Sigma + \Sigma'}$ we have formulae that involve symbols from both signatures.

Example 3.5 Consider, for instance, the signatures Σ and Σ' of a *flip-flop* and its clone, respectively:

- $\Sigma = \langle \Sigma_{obs}, \Sigma_{act} \rangle$ with $\Sigma_{obs} = \{on\}$ and $\Sigma_{act} = \{flip, flop\}$;

- $\Sigma' = \langle \Sigma'_{obs}, \Sigma'_{act} \rangle$ with $\Sigma'_{obs} = \{on'\}$ and $\Sigma'_{act} = \{flip', flop'\}$.

The coproduct $\Sigma + \Sigma'$ is the signature $\langle \{on, on'\}, \{flip, flop, flip', flop'\} \rangle$ together with the injections $inj_{\Sigma} = \langle i, j \rangle : \Sigma \rightarrow \Sigma + \Sigma'$ and $inj_{\Sigma'} = \langle i', j' \rangle : \Sigma' \rightarrow \Sigma + \Sigma'$, respectively:

- $i : \{on\} \rightarrow \{on, on'\}$ such that $i(on) = on$;
 $j : \{flip, flop\} \rightarrow \{flip, flop, flip', flop'\}$ such that $j(flip) = flip$ and $j(flop) = flop$;
- $i' : \{on'\} \rightarrow \{on, on'\}$ such that $i'(on') = on'$;
 $j' : \{flip', flop'\} \rightarrow \{flip, flop, flip', flop'\}$ such that $j'(flip') = flip'$ and $j'(flop') = flop'$.

We also prove that Th is has finite coproducts.

Proposition 3.6 The category Th is cocartesian.

Proof:

1. The theory $\langle \langle \emptyset, \emptyset \rangle, \emptyset^{\mathbb{F}} \rangle$ is initial.

The signature morphism 0_{Σ} is the unique theory morphism from $\langle \langle \emptyset, \emptyset \rangle, \emptyset^{\mathbb{F}} \rangle$ to $\langle \Sigma, \Phi \rangle$.

2. A coproduct of theories $th + th'$ where $th = \langle \Sigma, \Phi \rangle$ and $th' = \langle \Sigma', \Phi' \rangle$ is given by $\langle \Sigma + \Sigma', (inj_{\Sigma}(\Phi) \cup inj_{\Sigma'}(\Phi'))^{\mathbb{F}} \rangle$ where $\Sigma + \Sigma'$ is a coproduct in Sig with injections inj_{Σ} and $inj_{\Sigma'}$. QED

The next result is trivial but it is needed for technical reasons later on. We have to show that if we have two theories over the same signature Σ and the identity id_{Σ} is a theory morphism in both directions then the theories are the same.

Proposition 3.7 The concrete category $\langle Th, Si \rangle$ is amnestic.

Proof: Let $th_1 = \langle \Sigma, \Phi_1 \rangle$ and $th_2 = \langle \Sigma, \Phi_2 \rangle$ be theories over the same signature Σ , $f : th_1 \rightarrow th_2$ and $g : th_2 \rightarrow th_1$ theory morphisms such that $Si(f) = Si(g) = id_{\Sigma}$. Then, $id_{\Sigma}(\Phi_1) = \Phi_1 \subseteq \Phi_2$ and $id_{\Sigma}(\Phi_2) = \Phi_2 \subseteq \Phi_1$ and so $\Phi_1 = \Phi_2$. QED

Now, the basic problem that we have is to provide the indexing of theories by signatures. That is to say, we want to associate with a signature the category of theories for that signature. There is a standard way to do so (see [8]) taking advantage of the fact that the category of theories is concrete. More precisely, we have to start by showing that the forgetful functor Si from the categories of theories to the category of signatures is a cofibration. This means that there is a canonical way (see the Appendix) to built a theory of signature Σ' giving a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and a theory over Σ as well as to promote σ to a theory morphism f (f is said to be cocartesian for Si for the signature morphism on the theory over Σ). In our case the theory over Σ' is unique (unique cocleavage). Then, we have to prove the splitting: namely that f is the identity when σ is the identity and that f is a composition when σ is $\sigma_2 \circ \sigma_1$.

In Figure 4, we present the construction that will make the proof easier to understand. The canonical theory is $\langle \Sigma', \sigma(\Phi)^{\mathbb{F}} \rangle$ built from σ and $th = \langle \Sigma, \Phi \rangle$

$$\begin{array}{ccc}
Th & \xrightarrow{Si} & Sig \\
\\
\langle \Sigma, \Phi \rangle & \xrightarrow{f} & \langle \Sigma', \sigma(\Phi)^{F'} \rangle \\
& \searrow g & \downarrow h \\
& & \langle \Sigma'', \Phi'' \rangle
\end{array}
\qquad
\begin{array}{ccc}
\Sigma & \xrightarrow{\sigma} & \Sigma' \\
& \searrow Si(g) & \downarrow \omega \\
& & \Sigma''
\end{array}$$

Figure 4: The morphism f with $Si(f) = \sigma$ is cocartesian by Si for σ on $\langle \Sigma, \Phi \rangle$.

and the cocartesian morphism is f . The couniversal property for f consists in showing that for every g and ω such that, in Sig , $\omega \circ \sigma = Si(g)$, there is a morphism h in Th such that forgetting the condition on theory morphisms will give ω and $h \circ f = g$.

The intuition that we have is that if we give the theory $\langle \Sigma, \Phi \rangle$ then the canonical theory must be $\langle \Sigma', \sigma(\Phi)^{F'} \rangle$. The next example shows that $\sigma(\Phi)$ may not be a theory.

Example 3.8 Assume that we have two signatures Σ where $\Sigma_{act} = \emptyset$ and $\Sigma_{obs} = \{b_1, b_2, b_3, b_4\}$ and Σ' where $\Sigma'_{act} = \emptyset$ and $\Sigma'_{obs} = \{b'_1, b'_2, b'_3\}$ and a signature morphism σ such that $\sigma_{obs}(b_1) = b'_1$, $\sigma_{obs}(b_2) = \sigma_{obs}(b_3) = b'_2$ and $\sigma_{obs}(b_4) = b'_3$. Consider the theory $\langle \Sigma, \{(b_1 \Rightarrow b_2), (b_3 \Rightarrow b_4)\}^F \rangle$. Then $\langle \Sigma', \sigma(\{(b_1 \Rightarrow b_2), (b_3 \Rightarrow b_2)\}^F) \rangle$ is not a theory since it does not include $(b'_1 \Rightarrow b'_3)$.

Proposition 3.9 The functor $Si : Th \rightarrow Sig$ is a split cofibration with a unique cocleavage.

Proof:

1. Cofibration.

For each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and theory $th = \langle \Sigma, \Phi \rangle$, the theory morphism $f : th \rightarrow th'$ where $th' = \langle \Sigma', \sigma(\Phi)^{F'} \rangle$ such that $Si(f) = \sigma$ is cocartesian by Si for σ on th . Indeed, f is a theory morphism because $\sigma(\Phi) \subseteq \sigma(\Phi)^{F'}$. Moreover, f has the envisaged couniversal property: assume that $g : th \rightarrow th''$ is a morphism in Th with $th'' = \langle \Sigma'', \Phi'' \rangle$ and $\omega : \Sigma' \rightarrow Si(th'')$ is a morphism in Sig such that $\omega \circ \sigma = Si(g)$; the problem is to find a unique $h : th' \rightarrow th''$ such that $Si(h) = \omega$ and $h \circ f = g$; we have only to verify that ω is a theory morphism: since $Si(g)(\Phi) \subseteq \Phi''$ then $\omega(\sigma(\Phi)) \subseteq \Phi''$ and so invoking the closure lemma we have $\omega(\sigma(\Phi)^{F'}) \subseteq \Phi''$.

2. Uniqueness of cocleavage.

We have to show that for each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and theory $th = \langle \Sigma, \Phi \rangle$, the theory morphism $f : th \rightarrow \langle \Sigma', \sigma(\Phi)^{F'} \rangle$ is the unique cocartesian morphism by Si for σ on th . Assume that there is another $g : th \rightarrow \langle \Sigma', \Psi \rangle$ cocartesian morphism by Si for σ on th . Then, necessarily, $Si(g) = \sigma$. Therefore, by considering $w = id_\Sigma$ and the universal property of each of the two cocartesian morphisms it easily follows that $\Psi = \sigma(\Phi)^{F'}$.

3. Splitting.

It is trivial to see that $id_{\langle \Sigma, \Phi \rangle}$ is cocartesian by S_i for id_Σ on $\langle \Sigma, \Phi \rangle$. On the other hand, let $\sigma_1 : \Sigma \rightarrow \Sigma_1$ and $\sigma_2 : \Sigma \rightarrow \Sigma_2$ be signature morphisms, $f_1 : th \rightarrow \langle \Sigma_1, \Phi_1 \rangle$ the cocartesian by S_i for σ_1 on $\langle \Sigma, \Phi \rangle$ and $f_2 : \langle \Sigma_1, \Phi_1 \rangle \rightarrow \langle \Sigma_2, \Phi_2 \rangle$ the cocartesian by S_i for σ_1 on $\langle \Sigma_1, \Phi_1 \rangle$: then, we know that $f_2 \circ f_1$ is cocartesian by S_i for $\sigma_2 \circ \sigma_1$ on $\langle \Sigma, \Phi \rangle$. QED

Namely, in order to understand the abstraction construction in object-orientation we need another property of S_i . There is a canonical way (see the Appendix) to built a theory of signature Σ giving a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and a theory over Σ' as well as to promote σ to a theory morphism f . In our *flip-flop* example where we want the *flop* invisible we can think of the theory over Σ' as being the semantic closure of the specification of the *flip-flop*, Σ as the signature of the *flip-flop* without the *flop*, and σ as an inclusion. Then the canonical theory will be the theory of the *flip-flop* without the *flop*. Not only we have a way to get it but the way is “minimal”.

$$\begin{array}{ccc}
 Th & \xrightarrow{S_i} & Sig \\
 \\
 \begin{array}{ccc}
 \langle \Sigma, \sigma^{-1}(\Phi') \rangle & \xrightarrow{f} & \langle \Sigma', \Phi' \rangle \\
 \vdots \uparrow h & \nearrow g & \\
 \langle \Sigma'', \Phi'' \rangle & &
 \end{array} & &
 \begin{array}{ccc}
 \Sigma & \xrightarrow{\sigma} & \Sigma' \\
 \omega \uparrow & \nearrow Si(g) & \\
 \Sigma'' & &
 \end{array}
 \end{array}$$

Figure 5: The morphism f with $S_i(f) = \sigma$ is cartesian by S_i for σ on $\langle \Sigma', \Phi' \rangle$.

In Figure 5, we present the construction that will make the proof easier to understand.

Proposition 3.10 The functor $S_i : Th \rightarrow Sig$ is a fibration.

Proof:

For each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and theory $th' = \langle \Sigma', \Phi' \rangle$, the theory morphism $f : th \rightarrow th'$ such that $S_i(f) = \sigma$ and $th = \langle \Sigma, \Phi \rangle$, where $\Phi = \sigma^{-1}(\Phi')$ is cartesian by S_i for σ on th' .

(a) th is a theory.

Indeed, th is a theory: let $\psi \in L_\Sigma$ be such that if $\lambda \Vdash_\Sigma \Phi$ then $\lambda \Vdash_\Sigma \psi$ for each $\lambda \in \Lambda_\Sigma$; we have to show that $\psi \in \Phi$; let $\lambda' \in \Lambda_{\Sigma'}$ be any interpretation structure such that $\lambda' \Vdash_{\Sigma'} \Phi'$; in particular, $\lambda' \Vdash_{\Sigma'} \sigma(\varphi)$ for every $\varphi \in \Phi$; then, $\sigma^{-1}(\lambda') \Vdash_\Sigma \varphi$ for every $\varphi \in \Phi$ and so $\sigma^{-1}(\lambda') \Vdash_\Sigma \psi$; hence, $\lambda' \Vdash_{\Sigma'} \sigma(\psi)$; since th' is a theory $\sigma(\psi) \in \Phi'$ and so $\psi \in \Phi$.

(b) f is a theory morphism: $\sigma(\Phi) \subseteq \Phi'$.

(c) Universal property: assume that $g : th'' \rightarrow th'$ with $th'' = \langle \Sigma'', \Phi'' \rangle$ is a morphism in Th and $\omega : Si(th'') \rightarrow \Sigma$ is a morphism in Sig such that $\sigma \circ \omega = Si(g)$; the problem is to find a unique $h : th'' \rightarrow th'$ such that $S_i(h) = \omega$ and $f \circ h = g$; we have only to verify that ω is a theory morphism: since

$\sigma(\omega(\Phi'')) \subseteq \Phi'$, $\sigma(\Phi) \subseteq \Phi'$ and $\sigma(\Phi)$ includes all formulae whose images belong to Φ' , then we have $\omega(\Phi'') \subseteq \Phi$. QED

The importance of the universal property for the abstraction construction in object-orientation is straightforward and we illustrate it for the *flip-flop* example: if there is a theory morphism from any theory to the theory generated by the *flip-flop* specification then there is a theory morphism from that theory to the theory of the *flip-flop* without the *flop*.

3.4 Indexing theories

We now extract from the flat category of theories its indexing on signatures (built upon the split cofibration with a unique cocleavage). To this end, we obtain the functor $TH : Sig \rightarrow Cat$ that maps each signature to the category of all theories over that signature. So we have three basic steps: definition of the fiber (category of theories over the same signature), definition of functors between fibers induced by signature morphisms and definition of the indexing functor. These constructions are general (see the Appendix) because we proved that Si is a cofibration with splitting.

Definition 3.11 Let Σ be a signature. The category Th_Σ is the fiber $Si^{-1}(\Sigma)$. The theories of Th_Σ are called *theories* over Σ . The morphisms of Th_Σ are called *homomorphisms* between theories over Σ .

In the fiber Th_Σ the elements are the theories over Σ and the morphisms are theory morphisms that are mapped by Si to the identity over Σ . The next step is the definition of the functor that must exist between the fibers Th_Σ and $Th_{\Sigma'}$ whenever there is a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$. We start by introducing the notion of image of a theory.

Definition 3.12 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism and th a theory over Σ . The *image* $\sigma(th) = \langle \Sigma', \{Sen(\sigma)(\varphi) : \varphi \in \Phi\}^{\Sigma'} \rangle$ of th with respect to σ is the target of the cocartesian morphism $\bar{\sigma}^{th}$ for σ on th .

Now we lift the image theory construction to a functor taking advantage of the couniversal property of the cocartesian morphism.

Prop/Definition 3.13 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism. The *image functor* $\sigma : Th_\Sigma \rightarrow Th_{\Sigma'}$ maps each theory th to the image $\sigma(th)$ and each homomorphism $h : th_1 \rightarrow th_2$ to the unique homomorphism $\sigma(h) : \sigma(th_1) \rightarrow \sigma(th_2)$ such that $\bar{\sigma}^{th_2} \circ h = \sigma(h) \circ \bar{\sigma}^{th_1}$.

Proof:

See Appendix. QED

Note that $\sigma(h)$ exists and is unique because $\bar{\sigma}^{th_1}$ is the cocartesian morphism by Si for σ on th_1 . Now the definition of the indexing functor TH is trivial. Note again (as pointed out in the Appendix) that the need for the splitting is essential for TH to be a functor.

Prop/Definition 3.14 The *theory functor* $TH : Sig \rightarrow Cat$ maps each signature Σ to the fiber Th_Σ and each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ to the image functor $\sigma : Th_\Sigma \rightarrow Th_{\Sigma'}$.

Proof:

See Appendix. QED

As expected, by an appropriate flattening construction we recover the category Th from the indexed category $TH^{op} : Sig^{op} \rightarrow Cat^{op}$ (c.f., [50]). Clearly, we might develop the counterparts to these functors for “reducts” instead of images provided that we verify that the fibration Si is split. But we refrain to do so because we do not need them in the sequel. However, it is useful to introduce:

Definition 3.15 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism and th' a theory over Σ' . The *reduct* $\sigma^{-1}(th')$ of th' with respect to σ is the source of the cartesian morphism for σ on th .

In the sequel, in order to characterize the (tight) relationship between theories and (some) objects it is also convenient to work with TH^{op} and $Si^{op} : Th^{op} \rightarrow Sig^{op}$. Clearly, Si^{op} is also both a cofibration and a fibration (cartesian morphisms become cocartesian morphisms and vice-versa).

3.5 Satisfaction vs sets of interpretation structures

In the sequel, we need some additional notation and results concerning sets of interpretation structures of the base institution. Provided that $\sigma : \Sigma \rightarrow \Sigma'$ is a signature morphism, $\Lambda \subseteq \Lambda_\Sigma$ and $\Lambda' \subseteq \Lambda_{\Sigma'}$, we use:

- $\Lambda \Vdash_\Sigma \varphi$ iff $\lambda \Vdash_\Sigma \varphi$ for each $\lambda \in \Lambda$;
- $\sigma^{-1}(\Lambda') = \{\sigma^{-1}(\lambda') : \lambda' \in \Lambda'\}$;
- $\sigma(\Lambda) = \{\lambda' : \sigma^{-1}(\lambda') \in \Lambda\}$.

As an illustration of the latter, consider the following example:

Example 3.16 Consider the signatures $\Sigma = \langle \emptyset, \{a, b\} \rangle$, $\Sigma' = \langle \emptyset, \{x\} \rangle$, the (unique) signature morphism σ , a set W , a relation R on W , $\underline{V}_1 : W \rightarrow \wp A_\Sigma$ such that $\underline{V}_1(w) = \{\Diamond a, \Diamond b, \nabla a, \nabla b\}$ for each $w \in W$, and $\underline{V}_2 : W \rightarrow \wp A_\Sigma$ such that $\underline{V}_2(w) = \{\Diamond a, \nabla a\}$. Then:

- $\sigma(\{(W, R, V_1)\}) = \{(W, R, V'_1)\}$
where $\underline{V}'_1(w) = \{\Diamond x, \nabla x\}$ for each $w \in W$;
- $\sigma(\{(W, R, V_2)\}) = \emptyset$.

Proposition 3.17 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, φ a formula in L_Σ and $\Lambda' \subseteq \Lambda_{\Sigma'}$. Then, $\Lambda' \Vdash_{\Sigma'} \sigma(\varphi)$ iff $\sigma^{-1}(\Lambda') \Vdash_\Sigma \varphi$.

Proof:

$\Lambda' \Vdash_{\Sigma'} \sigma(\varphi)$ iff $\lambda' \Vdash_{\Sigma'} \sigma(\varphi)$ for every $\lambda' \in \Lambda'$ iff $\sigma^{-1}(\lambda') \Vdash_{\Sigma} \varphi$ for every $\lambda' \in \Lambda'$ iff $\sigma^{-1}(\lambda') \Vdash_{\Sigma} \varphi$ for every $\sigma^{-1}(\lambda') \in \sigma^{-1}(\Lambda')$ iff $\sigma^{-1}(\Lambda') \Vdash_{\Sigma} \varphi$. QED

We may call this result the *set satisfaction condition* (of the base institution). Of course, it extends the usual satisfaction conditions because now we work with sets of interpretation structures instead of single interpretation structures.

Proposition 3.18 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, φ a formula in L_{Σ} and $\Lambda \subseteq \Lambda_{\Sigma}$. Then, if $\Lambda \Vdash_{\Sigma} \varphi$ then $\sigma(\Lambda) \Vdash_{\Sigma'} \sigma(\varphi)$.

Proof:

Assume that we do not have $\sigma(\Lambda) \Vdash_{\Sigma'} \sigma(\varphi)$; then, there is an interpretation structure $\lambda' \in \sigma(\Lambda)$ such that we do not have $\lambda' \Vdash_{\Sigma'} \sigma(\varphi)$; therefore, we do not have $\sigma^{-1}(\lambda') \Vdash_{\Sigma} \varphi$; and so, we do not have $\Lambda \Vdash_{\Sigma} \varphi$ since $\sigma^{-1}(\lambda') \in \Lambda$. QED

We may call this result the *set forward satisfaction condition* (of the base institution). We provide a counterexample showing that in some base institution if $\sigma(\Lambda) \Vdash_{\Sigma'} \sigma(\varphi)$ then it may be the case that we do not have $\Lambda \Vdash_{\Sigma} \varphi$.

Example 3.19 Consider the signatures $\Sigma = \langle \emptyset, \{a, b, c\} \rangle$, $\Sigma' = \langle \emptyset, \{x, y\} \rangle$, the signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ such that $\sigma_{act}(a) = \sigma_{act}(b) = x$ and $\sigma_{act}(c) = y$, and the set $\Lambda = \{ \langle W, R, V_1 \rangle, \langle W, R, V_2 \rangle \}$ where $W = \{w_1, w_2\}$, $R = \{ \langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle \}$ and $\underline{V}_1(w) = \{ \Diamond a, \nabla a \}$ and $\underline{V}_2(w) = \{ \Diamond c, \nabla c \}$ for each $w \in W$. Its image $\sigma(\Lambda)$ is $\{ \langle W, R, V' \rangle \}$ where $\underline{V}'(w) = \{ \Diamond y, \nabla y \}$ for each $w \in W$. We have that $\sigma(\Lambda) \Vdash_{\Sigma'} \sigma(\Box(\nabla c))$ but we do not have $\Lambda \Vdash_{\Sigma} (\Box(\nabla c))$.

From the example, we see that the problem appeared because σ_{act} is not an injective mapping.

4 Objects over the base institution

We are now ready to introduce the notions of object and object morphism within the setting of the given base institution. The basic idea is to identify an object with its behaviour, that is a set of lives (interpretation structures) and lift in the obvious way the notion of interpretation structure reducts to sets. However, not all sets of interpretation structures are acceptable as objects as we shall see, since the base logic may have limitations in distinguishing them.

We start by introducing the notion of object and then we study properties of objects.

4.1 Preobjects and objects

We introduced theories as “closed” specifications. Along the same line we introduce objects as “closed” preobjects.

Definition 4.1 A *preobject* over Σ is a pair $ob = \langle \Sigma, \Lambda \rangle$ where $\Sigma = Sg(ob)$ is a signature and $\Lambda \subseteq \Lambda_{\Sigma}$.

The elements in Λ are called the admissible *lives* (or *behaviours*). A particular case is when Λ is empty meaning that there is no admissible behaviour. In order to define “closed” preobjects we need the concept of entailment of a life by a set of lives.

Definition 4.2 Let Σ be a signature, $\lambda \in \Lambda_\Sigma$ and $\Lambda \subseteq \Lambda_\Sigma$. We say that Λ *entails* (written $\Lambda \vDash_\Sigma \lambda$) iff, for every $\varphi \in L_\Sigma$, if $\Lambda \Vdash_\Sigma \varphi$ then $\lambda \Vdash_\Sigma \varphi$.

Thus, λ satisfies at least all the formulae that are satisfied by all the elements in Λ . Of course, not every set of interpretation structures is closed as it is shown in the next example. We assume that Λ_Σ is theoretically closed for every signature Σ .

Example 4.3 Let $\Sigma = \langle \{a\}, \emptyset \rangle$ be a signature, $\Lambda = \{ \langle \{w_1, w_2\}, \{ \langle \{w_1, w_2\} \rangle, V \rangle \} \}$ where $\underline{V}(w_1) = \underline{V}(w_2) = \{a\}$ a singleton of interpretation structures and $\lambda = \langle \{w\}, \emptyset, U \rangle$ where $\underline{U}(w) = \{a\}$ an interpretation structure. Then, $\Lambda \vDash_\Sigma \lambda$ and $\lambda \notin \Lambda$. Note that $\lambda \Vdash_\Sigma (\Box a)$ and $\lambda \Vdash_\Sigma (\Box(\neg a))$.

This example motivates the following definition of closure.

Definition 4.4 Let Σ be a signature and $\Lambda \subseteq \Lambda_\Sigma$. The *theoretic closure* of Λ is the following set of interpretation structures: $\Lambda^\vDash = \{ \lambda \in \Lambda_\Sigma : \Lambda \vDash_\Sigma \lambda \}$.

Now we can define an object as being a “closed” preobject as follows:

Definition 4.5 An *object* over Σ is a preobject $ob = \langle \Sigma, \Lambda \rangle$ such that $\Lambda = \Lambda^\vDash$.

Objects can be related by object morphisms.

Definition 4.6 An *object morphism* over $f : ob \rightarrow ob'$ where $ob = \langle \Sigma, \Lambda \rangle$ and $ob' = \langle \Sigma', \Lambda' \rangle$ is a signature op-morphism $Sg(f) : Sg(ob) \rightarrow Sg(ob')$ such that $(Sg(f)^{op})^{-1}(\Lambda) \subseteq \Lambda'$.

Thus, $Sg(f)^{op}$ is a signature morphism. The condition states that every behaviour in the reduct of Λ is a behaviour in Λ' . Intuitively speaking, we can have more actions and observations in the signature $Sg(ob)$ but if we throw away in an interpretation structure for ob the things that do not correspond to actions and observations in $Sg(ob')$ we get an interpretation structure for ob' .

Example 4.7 Consider the objects $ob = \langle \Sigma, \Lambda \rangle$ and $ob' = \langle \Sigma', \Lambda' \rangle$ where:

- $\Sigma = \langle \emptyset, \{x, y\} \rangle$;
- $\Lambda = \{ \langle W, R, V \rangle \}$ where $\underline{V}(w) = \{ \Diamond x, \nabla x \}$ for every $w \in W$;
- $\Sigma' = \langle \emptyset, \{a, b, c\} \rangle$;
- $\Lambda' = \{ \langle W, R, V_1' \rangle, \langle W, R, V_2' \rangle \}$
where $\underline{V}_1'(w) = \{ \Diamond a, \Diamond b, \nabla a, \nabla b \}$ and $\underline{V}_2'(w) = \{ \Diamond c, \nabla c \}$ for every $w \in W$.

Let $\sigma : \Sigma' \rightarrow \Sigma$ be the signature morphism such that $\sigma_{act}(a) = \sigma_{act}(b) = x$, $\sigma_{act}(c) = y$. Since $\sigma^{-1}(\langle W, R, V \rangle) = \langle W, R, V_1' \rangle$, σ induces an object morphism from ob to ob' .

As expected we can get a signature out of an object by forgetting the admissible behaviours.

Prop/Definition 4.8 Objects and object morphisms constitute a concrete category $\langle Ob, Sg \rangle$ over Sig^{op} .

Proof:

We have to show that Ob is a category and that $Sg : Ob \rightarrow Sig^{op}$ is a faithful functor. We only prove that the composition of two object morphisms is an object morphism since the rest is (also) trivial. Let $ob = \langle \Sigma, \Lambda \rangle$, $ob' = \langle \Sigma', \Lambda' \rangle$, $ob'' = \langle \Sigma'', \Lambda'' \rangle$ be objects and $f : ob \rightarrow ob'$, $g : ob' \rightarrow ob''$ be object morphisms; the signature morphism $Sg(f)^{op} \circ Sg(g)^{op}$ is an object morphism since: $(Sg(f)^{op} \circ Sg(g)^{op})^{-1}(\Lambda) = (Sg(g)^{op})^{-1}((Sg(f)^{op})^{-1}(\Lambda)) \subseteq (Sg(g)^{op})^{-1}(\Lambda') \subseteq \Lambda''$. QED

In this case the base category is the opposite of Sig because object morphisms are contravariant with signature morphisms. Thus, Sg takes an object morphism to a morphism in Sig^{op} .

4.2 Main properties of objects

We show that finite products exist in Ob . Actually, Ob is (small) complete but we do not need to consider general limits in this paper. Furthermore, we prove that Ob is amnesic and that Sg is both a cofibration and a fibration.

We start with two lemmas about theoretic closures that are needed for proving the existence of finite products. The two lemmas are dedicated to the closure of an empty set of interpretation structures and to the closure of a set of interpretation structures for a coproduct of signatures.

Lemma 4.9 The empty set of interpretation structures over a nonempty signature is theoretically closed. And the closure of the empty set of interpretation structures over an empty signature is Λ_\emptyset .

Proof:

(a) Σ is not the empty signature.

Assume that $\emptyset \models_\Sigma \xi$ where $\xi = \langle W, R, V \rangle$. Since, $\emptyset \models_\Sigma \varphi$ for every $\varphi \in L_\Sigma$ then $\xi \models_\Sigma \varphi$ for every $\varphi \in L_\Sigma$ and so $\xi \models_w \varphi$ for every $\varphi \in L_\Sigma$ and $w \in W$. In particular, consider $\alpha \in A_\Sigma$. Two cases are possible:

1. $\alpha \in \underline{V}(w)$ and then we do not have $\xi \models_w (\neg \alpha)$;
2. $\alpha \notin \underline{V}(w)$ and then we do not have $\xi \models_w \alpha$.

We reach a contradiction and so no ξ is entailed by the \emptyset .

(b) Σ is the empty signature.

It is trivial since $L_\emptyset = \emptyset$ and every interpretation structure satisfies the empty set of formulae. QED

Lemma 4.10 Let Σ and Σ' be signatures, $\Lambda \subseteq \Lambda_\Sigma$ and $\Lambda' \subseteq \Lambda_{\Sigma'}$ theoretically closed sets of interpretation structures. Then, the set

$$\Theta = \{\theta \in \Lambda_{\Sigma+\Sigma'} : inj_{\Sigma}^{-1}(\theta) \in \Lambda, inj_{\Sigma'}^{-1}(\theta) \in \Lambda'\}$$

where $inj_{\Sigma} : \Sigma \rightarrow \Sigma + \Sigma'$ and where $inj_{\Sigma'} : \Sigma' \rightarrow \Sigma + \Sigma'$ are the injections is theoretically closed.

Proof:

Assume that $\Theta \models_{\Sigma+\Sigma'} \xi$. We have to show that $\xi \in \Theta$, i.e. $inj_{\Sigma}^{-1}(\xi) \in \Lambda$ and $inj_{\Sigma'}^{-1}(\xi) \in \Lambda'$. We only show that $inj_{\Sigma}^{-1}(\xi) \in \Lambda$ (the other part is similar). Since Λ is theoretically closed, it is enough to verify that $\Lambda \models_{\Sigma} inj_{\Sigma}^{-1}(\xi)$. Let $\varphi \in L_{\Sigma}$ be any formula such that $\Lambda \Vdash_{\Sigma} \varphi$. We show that $inj_{\Sigma}^{-1}(\xi) \Vdash_{\Sigma} \varphi$: $inj_{\Sigma}(\Lambda) \Vdash_{\Sigma+\Sigma'} inj_{\Sigma}(\varphi)$ (using the forward condition of the base institution); hence $\Theta \Vdash_{\Sigma+\Sigma'} inj_{\Sigma}(\varphi)$ (since $\Theta \subseteq inj_{\Sigma}(\Lambda)$); therefore, since $\Theta \models_{\Sigma+\Sigma'} \xi$ we have $\xi \Vdash_{\Sigma+\Sigma'} inj_{\Sigma}(\varphi)$; finally, $inj_{\Sigma}^{-1}(\xi) \Vdash_{\Sigma} \varphi$ (using the satisfaction condition of the base institution). QED

Now we can prove that Ob has finite products by showing that it has final (or terminal) object and binary products.

Proposition 4.11 The category Ob is cartesian.

Proof:

1. Ob has terminal object:

The obvious candidate is $\langle \langle \emptyset, \emptyset \rangle, \emptyset^{\mathbb{F}} \rangle$ and for each object $ob = \langle \Sigma, \Lambda \rangle$ the unique morphism $1_{ob} : ob \rightarrow \langle \langle \emptyset, \emptyset \rangle, \emptyset^{\mathbb{F}} \rangle$ is induced by the empty signature morphism $0_{\Sigma} : \langle \emptyset, \emptyset \rangle \rightarrow \Sigma$.

2. Ob has binary products:

Assume that $ob = \langle \Sigma, \Lambda \rangle$ and $ob' = \langle \Sigma', \Lambda' \rangle$ are objects, the binary product object is $ob \times ob' = \langle \Sigma + \Sigma', \{\theta \in \Lambda_{\Sigma+\Sigma'} : inj_{\Sigma}^{-1}(\theta) \in \Lambda, inj_{\Sigma'}^{-1}(\theta) \in \Lambda'\} \rangle$ endowed with the projections: $proj_{ob} : ob \times ob' \rightarrow ob$ and $proj_{ob'} : ob \times ob' \rightarrow ob'$ such that $Sg(proj_{ob})^{op} = inj_{\Sigma}$ and $Sg(proj_{ob'})^{op} = inj_{\Sigma'}$. Moreover, the candidate does have the envisaged universal property: assuming that $ob'' = \langle \Sigma'', \Lambda'' \rangle$ is an object and $g : ob'' \rightarrow ob$ and $g' : ob'' \rightarrow ob'$ are object morphisms, it is straightforward to verify that there is a unique object morphism $k : ob'' \rightarrow ob \times ob'$ such that $g = proj_{ob} \circ k$ and $g' = proj_{ob'} \circ k$; k is induced by the unique signature morphism $\sigma : \Sigma + \Sigma' \rightarrow \Sigma''$ such that $Sg(g)^{op} = \sigma \circ inj_{\Sigma}$ and $Sg(g')^{op} = \sigma \circ inj_{\Sigma'}$. QED

Example 4.12 Consider two objects $ob = \langle \Sigma, \Lambda \rangle$ and $ob' = \langle \Sigma', \Lambda' \rangle$, with $\Sigma = \langle \Sigma_{obs}, \Sigma_{act} \rangle$ such that $\Sigma_{obs} = \{on\}$ and $\Sigma_{act} = \{flip, flop\}$; and $\Sigma' = \langle \Sigma'_{obs}, \Sigma'_{act} \rangle$ such that $\Sigma'_{obs} = \{on'\}$ and $\Sigma'_{act} = \{flip', flop'\}$. Moreover, suppose that $\lambda = \langle \{w_1, w_2\}, \{\langle w_1, w_2 \rangle\}, V \rangle \in \Lambda$ and $\lambda' = \langle \{w_1, w_2\}, \{\langle w_1, w_2 \rangle\}, V' \rangle \in \Lambda'$ are such that:

- $\underline{V}(w_1) = \{on, \diamond flip, \nabla flip\}$
 $\underline{V}(w_2) = \{\diamond flop, \nabla flop\}$;
- $\underline{V}'(w_1) = \{\diamond flop', \nabla flop'\}$
 $\underline{V}'(w_2) = \{on', \diamond flip', \nabla flip'\}$.

Then, if $ob \times ob' = \langle \Sigma + \Sigma', \Xi \rangle$, we have that $\xi = \langle \{w_1, w_2\}, \{\{w_1, w_2\}\}, U \rangle \in \Xi$, with:

- $\underline{U}(w_1) = \{on, \diamond flip, \nabla flip, \diamond flop', \nabla flop'\}$
- $\underline{U}(w_2) = \{\diamond flop, \nabla flop, on', \diamond flip', \nabla flip'\}$.

Again for technical reasons we need the following property:

Proposition 4.13 The category Ob is amnesitic.

Proof:

Let $ob_1 = \langle \Sigma, \Lambda_1 \rangle$ and $ob_2 = \langle \Sigma, \Lambda_2 \rangle$ be objects over the same signature, $f : ob_1 \rightarrow ob_2$ and $g : ob_2 \rightarrow ob_1$ object morphisms such that $Sg(f) = Sg(g) = id_\Sigma$. Then $id_\Sigma(\Lambda_1) = \Lambda_1 \subseteq \Lambda_2$ and $id_\Sigma(\Lambda_2) = \Lambda_2 \subseteq \Lambda_1$ and so $\Lambda_1 = \Lambda_2$. QED

We have two problems: indexing of objects by signatures and semantics of an object obtained by abstraction from another object. We start by showing that the forgetful functor from the categories of objects to the opposite of the category of signatures is a cofibration. This means that there is a canonical way (see the Appendix) to built an object of signature Σ' giving a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and an object over Σ as well as to promote σ into an object morphism f . In our case the object over Σ' is unique (unique cocleavage). Then, we have to prove the splitting: namely that f is the identity when σ is the identity and that f is a composition when σ is $\sigma_2 \circ \sigma_1$. The intuition that we have is that if we give the object $\langle \Sigma', \Lambda' \rangle$ then the canonical object must be $\langle \Sigma, \sigma^{-1}(\Lambda')^F \rangle$. Before exploring this idea we have a long way to go to see if $\sigma^{-1}(\Lambda')$ is closed or not. We start by analysing the problem when the signature morphism is an inclusion and after that we consider the general case. We start with an auxiliary lemma.

Lemma 4.14 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, $\Lambda' \subseteq \Lambda_{\Sigma'}$, $\xi = \langle W, R, V \rangle \in \Lambda_\Sigma$ and $\Theta_{\Sigma'} = \{\lambda' \in \Lambda_{\Sigma'} : \lambda' = \langle W, R, V' \rangle\} \neq \emptyset$. If $\sigma^{-1}(\Lambda') \vDash_\Sigma \xi$ then there is $\xi' \in \Theta_{\Sigma'}$ such that $\xi = \sigma^{-1}(\xi')$.

Proof:

Assume there is no $\xi' \in \Theta_{\Sigma'}$ such that $\xi = \sigma^{-1}(\xi')$. Then, there is an atomic formula $\alpha \in A_\Sigma$ and a world $w \in W$ such that $\alpha \in \underline{V}(w)$ but $\beta \notin \underline{V}(w)$ for some $\beta \in A_\Sigma$ such that $\sigma(\alpha) = \sigma(\beta)$. Therefore, $\sigma^{-1}(\Lambda') \Vdash_\Sigma (\alpha \Leftrightarrow \beta)$, since $\Lambda' \Vdash_{\Sigma'} \sigma(\alpha \Leftrightarrow \beta)$, but not $\xi \Vdash_\Sigma (\alpha \Leftrightarrow \beta)$ and so we do not have $\sigma^{-1}(\Lambda') \vDash_\Sigma \xi$.

QED

Note that the frame of ξ' is the same as the frame of ξ . Now we introduce the concept of image of a signature given a signature morphism.

Definition 4.15 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism. The *range* of σ is the signature $\sigma(\Sigma)$ such that

- $\sigma(\Sigma)_{obs} = \{b' \in \Sigma'_{obs} : \text{there is } b \in \Sigma_{obs} \text{ such that } \sigma_{obs}(b) = b'\};$
- $\sigma(\Sigma)_{act} = \{c' \in \Sigma'_{act} : \text{there is } c \in \Sigma_{act} \text{ such that } \sigma_{act}(c) = c'\}.$

In the sequel, $i : \sigma(\Sigma) \rightarrow \Sigma'$ is the inclusion signature morphism from $\sigma(\Sigma)$ to Σ' and $j : (\Sigma' \setminus \sigma(\Sigma)) \rightarrow \Sigma'$ is the inclusion signature morphism from $(\Sigma' \setminus \sigma(\Sigma))$ to Σ' .

In the following lemma we use $(\diamond\varphi)$ as an abbreviation of $(\neg(\Box(\neg\varphi)))$, $(\Box^n\sigma(\varphi))$ as an abbreviation of applying n times \Box to φ and $(\diamond^n\sigma(\varphi))$ as an abbreviation of applying n times \diamond to φ . The operator \diamond is known in modal logic as the *possibility* operator.

Lemma 4.16 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, where Σ is not the empty signature, and $\Lambda' \subseteq \Lambda_{\Sigma'}$ a theoretically closed set of interpretation structures. Then, $i^{-1}(\Lambda')$ is theoretically closed.

Proof:

Assume that Λ' is a set of lives theoretically closed and that $i^{-1}(\Lambda') \models_{\sigma(\Sigma)} \theta'$ where $\theta' = \langle Y', S', U' \rangle$.

Let ξ' be a life such that $i^{-1}(\xi') = \theta'$ and $\xi' \Vdash_{\Sigma'} \varphi'$ iff $\Lambda' \Vdash_{\Sigma'} \varphi'$ for every φ' such that there is no $\varphi \in L_{\Sigma}$ such that $\sigma(\varphi) = \varphi'$. Then $\Lambda' \models_{\Sigma'} \xi'$ and so $\xi' \in \Lambda'$. It remains to show that such a ξ' exists. For this purpose we consider several cases.

1. Assume that $S' = \emptyset$. Then,

- $\theta' \Vdash_{\sigma(\Sigma)} (\Box^n\sigma(\varphi))$ and $\theta' \Vdash_{\sigma(\Sigma)} (\Box^n(\neg\sigma(\varphi)))$;
- not $\theta' \Vdash_{\sigma(\Sigma)} (\diamond^n\sigma(\varphi))$ and not $\theta' \Vdash_{\sigma(\Sigma)} (\diamond^n(\neg\sigma(\varphi)))$;

for every $n \geq 1$ and $\varphi \in L_{\Sigma}$. Moreover,

- $\xi' \Vdash_{\Sigma'} (\Box^n\varphi')$ and $\xi' \Vdash_{\Sigma'} (\Box^n(\neg\varphi'))$;
- not $\xi' \Vdash_{\Sigma'} (\diamond^n\varphi')$ and not $\xi' \Vdash_{\Sigma'} (\diamond^n(\neg\varphi'))$;

for every $n \geq 1$ and $\varphi' \in L_{\Sigma'}$. Assume that $\Lambda' \Vdash_{\Sigma'} (\diamond\varphi')$ for some $\varphi' \in L_{\Sigma'}$. Then, for every $\lambda' = \langle W', R', V' \rangle \in \Lambda'$ the accessibility relation R' is such that for every $w' \in W'$ there is $w'' \in W'$ such that $w'R'w''$. Therefore, for every $\alpha \in A_{\Sigma}$ we have: $\lambda' \Vdash_{\Sigma'} (\diamond\sigma(\alpha))$ or $\lambda' \Vdash_{\Sigma'} (\diamond\sigma(\neg\alpha))$ and so $\Lambda' \Vdash_{\Sigma'} ((\diamond\sigma(\alpha)) \vee (\diamond\sigma(\neg\alpha)))$. Then, $i^{-1}(\Lambda') \Vdash_{\sigma(\Sigma)} ((\diamond\sigma(\alpha)) \vee (\diamond\sigma(\neg\alpha)))$ and so $\theta' \Vdash_{\sigma(\Sigma)} ((\diamond\sigma(\alpha)) \vee (\diamond\sigma(\neg\alpha)))$ which is impossible.

2. Assume now that the accessibility relation in every life of Λ' is empty. Then,

- $\Lambda' \Vdash_{\Sigma'} (\Box^n\varphi')$ and $\Lambda' \Vdash_{\Sigma'} (\Box^n(\neg\varphi'))$;

for every $n \geq 1$ and $\varphi' \in L_{\Sigma'}$. Moreover,

- $i^{-1}(\Lambda') \Vdash_{\sigma(\Sigma)} (\Box^n\sigma(\varphi))$ and $i^{-1}(\Lambda') \Vdash_{\sigma(\Sigma)} (\Box^n(\neg\sigma(\varphi)))$;

for every $n \geq 1$ and $\varphi \in L_{\Sigma}$. Therefore $S' = \emptyset$ and so we can apply 1.

3. Assume now that the accessibility relation in every life of Λ' is nonempty and that S' is also nonempty and that $\Lambda' \Vdash_{\Sigma'} (\diamond^n\varphi')$ for some $\varphi' \in L_{\Sigma'}$. Following the same reasoning as in 1. we get that $\theta' \Vdash_{\sigma(\Sigma)} ((\diamond\sigma(\alpha)) \vee (\diamond\sigma(\neg\alpha)))$ meaning that S' is such that for every $s' \in S'$ there is $s'' \in S'$ such that $s'S's''$. QED

The following lemma states that the reduct of a closed set of interpretation structures over any signature with the exception of the empty one is closed as well.

Lemma 4.17 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, where Σ is not the empty signature, and $\Lambda' \subseteq \Lambda_{\Sigma'}$ a theoretically closed set of interpretation structures. Then $\sigma^{-1}(\Lambda')$ is theoretically closed.

Proof:

Assume that $\sigma^{-1}(\Lambda') \vDash_{\Sigma} \sigma^{-1}(\xi')$; hence, if $\sigma^{-1}(\Lambda') \Vdash_{\Sigma} \varphi$ then $\sigma^{-1}(\xi') \Vdash_{\Sigma} \varphi$ for every $\varphi \in L_{\Sigma}$. That is (using the set satisfaction condition and the satisfaction condition of the base institution), if $\Lambda' \Vdash_{\Sigma'} \sigma(\varphi)$ then $\xi' \Vdash_{\Sigma'} \sigma(\varphi)$ for every $\varphi \in L_{\Sigma}$. Hence, if $i^{-1}(\Lambda') \Vdash_{\sigma(\Sigma)} \sigma(\varphi)$ then $i^{-1}(\xi') \Vdash_{\sigma(\Sigma)} \sigma(\varphi)$ for every $\varphi \in L_{\Sigma}$. Therefore, $i^{-1}(\xi') \in i^{-1}(\Lambda')$. Finally, $\sigma^{-1}(\xi') \in \sigma^{-1}(\Lambda')$ because $\sigma^{-1}(\xi') = \sigma^{-1}(i^{-1}(\xi'))$ and $\sigma^{-1}(\Lambda') = \sigma^{-1}(i^{-1}(\Lambda'))$. QED

Now we can give the main result about obtaining canonical objects. The proof reflects the fact that the results depend whether or not our source signature is the empty one.

Proposition 4.18 The functor $Sg : Ob \rightarrow Sig^{op}$ is a split cofibration with a unique cocleavage.

Proof:

1. Cofibration.

(a) For the signature morphism $0_{\Sigma'}$, the object morphism $f : ob' \rightarrow ob$ where $ob = \langle \emptyset, \Lambda_{\emptyset} \rangle$, such that $Sg(f) = 0_{\Sigma'}^{op}$ is cocartesian for Sg for $0_{\Sigma'}^{op}$ on ob' . We show that f has the envisaged couniversal property: assume that $g : ob' \rightarrow ob''$, where $ob'' = \langle \Sigma'', \Lambda'' \rangle$ is an object morphism and $\omega : \Sigma'' \rightarrow 0_{\Sigma'}$ is a signature morphism. Then, necessarily Σ'' is the empty signature and so the result follows trivially.

(b) For each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, where Σ is not the empty signature, and object $ob' = \langle \Sigma', \Lambda' \rangle$, the object morphism $f : ob' \rightarrow ob$ where $ob = \langle \Sigma, \sigma^{-1}(\Lambda') \rangle$, such that $Sg(f) = \sigma^{op}$ is cocartesian for Sg for σ^{op} on ob' . Indeed, ob is an object and f is trivially a morphism. Moreover, f has the envisaged couniversal property: assume that $g : ob' \rightarrow ob''$, where $ob'' = \langle \Sigma'', \Lambda'' \rangle$ is an object morphism and $\omega : \Sigma'' \rightarrow \Sigma$ is a signature morphism such that $\omega^{op} \circ \sigma^{op} = Sg(g)$; the problem is to find a unique $h : ob \rightarrow ob''$ such that $Sg(h) = \omega^{op}$ and $h \circ f = g$; we only have to verify that ω induces an object morphism: since $(Sg(h)^{op})^{-1}(\Lambda') = \omega^{-1}(\sigma^{-1}(\Lambda'))$ and $(Sg(h)^{op})^{-1}(\Lambda') \subseteq \Lambda''$ then we have $\omega^{-1}(\sigma^{-1}(\Lambda')) \subseteq \Lambda''$.

2. Uniqueness of cocleavage.

(a) Clearly, the only object over the empty signature is $\langle \emptyset, \Lambda_{\emptyset} \rangle$.

(b) We show that for each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, where Σ is not the empty signature, and object $ob' = \langle \Sigma', \Lambda' \rangle$ that the object morphism $f : ob' \rightarrow \langle \Sigma, \sigma^{-1}(\Lambda') \rangle$ is the unique cocartesian morphism by Sg for σ on ob' . Assume that there is another cocartesian morphism $g : ob' \rightarrow \langle \Sigma, \Xi \rangle$ by Sg for σ on ob' . Then, there is an isomorphism $h : \langle \Sigma, \sigma^{-1}(\Lambda') \rangle \rightarrow \langle \Sigma, \Xi \rangle$ such that $\sigma \circ Sg(h)^{op} = \sigma$. Then, if $Sg^{op}(h)(a) = b$ then $\sigma(a) = \sigma(b)$, for all $a \in \Sigma_{obs} \cup \Sigma_{act}$ (note that it may be the case that $a \neq b$). We show that $\lambda \in \Xi$ whenever $\lambda \in \sigma^{-1}(\Lambda')$: let $\langle W, R, \overline{V} \rangle \in \sigma^{-1}(\Lambda')$; then, $\langle W, R, \overline{V} \rangle = (Sg(h)^{op})^{-1}(\langle W, R, \overline{V} \rangle)$ since, for each $\alpha, \beta \in A_{\Sigma}$, if $\sigma(\alpha) = \sigma(\beta)$ then $\alpha \in \overline{V}(w)$ iff $\beta \in \overline{V}(w)$; therefore, $\langle W, R, \overline{V} \rangle \in \Xi$ since h is an object

morphism. A similar reasoning on h^{-1} allow us to conclude that $\lambda \in \sigma^{-1}(\Lambda')$ whenever $\lambda \in \Xi$. Thus, $\sigma^{-1}(\Lambda') = \Xi$.

3. Splitting.

Similar to the proof that Si is a split cofibration. QED

In Figure 6, we indicate that $\langle \Sigma, \sigma^{-1}(\Lambda') \rangle$ is the canonical object and that f is the cocartesian morphism. Assume that Σ is the signature of the *flip-flop* with the action *flop* hidden, Σ' is the signature of the *flip-flop*, σ is the inclusion, and that Λ' is the closure of the singleton with the interpretation structure $\langle \mathbb{N}, \{\langle i, i+1 \rangle : i \in \mathbb{N}\}, V' \rangle$ where $V'(\diamond flop) = V'(\nabla flop)$ is the set of odd numbers, $V'(\diamond flip) = V'(\nabla flip) = V'(on)$ is the set of even numbers. Then, the canonical interpretation structure is the closure of the singleton with the interpretation structure $\langle \mathbb{N}, \{\langle i, i+1 \rangle : i \in \mathbb{N}\}, V \rangle$ where $V(\diamond flip) = V(\nabla flip) = V(on)$ is the set of even numbers.

$$\begin{array}{ccc}
 Ob & \xrightarrow{Sg} & Sig^{op} \\
 \\
 \langle \Sigma', \Lambda' \rangle & \xrightarrow{f} & \langle \Sigma, \sigma^{-1}(\Lambda') \rangle \\
 & \searrow g & \downarrow h \\
 & & \langle \Sigma'', \Lambda'' \rangle
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Sigma' & \xrightarrow{\sigma^{op}} & \Sigma \\
 & \searrow Sg(g) & \downarrow \omega^{op} \\
 & & \Sigma''
 \end{array}$$

Figure 6: The morphism f with $Sg(f) = \sigma^{op}$ is cocartesian by Sg for σ^{op} on $\langle \Sigma', \Lambda' \rangle$, provided that Σ is not the empty signature.

Finally, we prove there is a canonical way of getting an object for signature Σ' given an object for signature Σ and a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$. Again, we start by showing that $\sigma(\Lambda)$ is closed whenever Λ is closed.

Lemma 4.19 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism and $\Lambda \subseteq \Lambda_\Sigma$ a theoretically closed set of interpretation structures. Then $\sigma(\Lambda)$ is theoretically closed.

Proof:

1. $\sigma(\emptyset^\pm) = \Lambda_{\Sigma'}$ if Σ is the empty signature.
2. Let Σ be a non empty signature. Assume that $\sigma(\Lambda)$ is not theoretically closed. Then, there is $\xi' \in \Lambda_{\Sigma'}$ such that $\sigma(\Lambda) \vDash_{\Sigma'} \xi'$ but $\xi' \notin \sigma(\Lambda)$ and so $\sigma^{-1}(\xi') \notin \Lambda$. Therefore, there is $\varphi \in L_\Sigma$ such that $\Lambda \Vdash_\Sigma \varphi$ but not $\sigma^{-1}(\xi') \Vdash_\Sigma \varphi$. Then, $\sigma(\Lambda) \Vdash_{\Sigma'} \sigma(\varphi)$ using the forward satisfaction condition of the base institution but not $\xi' \Vdash_{\Sigma'} \sigma(\varphi)$. Finally, we do not have $\sigma(\Lambda) \vDash_{\Sigma'} \xi'$. QED

Proposition 4.20 The functor $Sg : Ob \rightarrow Sig^{op}$ is a fibration.

Proof:

For each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ and object $ob = \langle \Sigma, \Lambda \rangle$, the object morphism $f : ob' \rightarrow ob$ where $ob' = \langle \Sigma, \sigma(\Lambda) \rangle$, such that $Sg(f) = \sigma^{op}$ is cartesian

for Sg for σ^{op} on ob . Indeed, ob' is an object and $\sigma^{-1}(\sigma(\Lambda)) \subseteq \Lambda$. Moreover, f has the envisaged universal property: assume that $g : ob'' \rightarrow ob$ where $ob'' = \langle \Sigma, \Lambda'' \rangle$ is an object morphism and $\omega : \Sigma' \rightarrow \Sigma''$ is a signature morphism such that $\sigma^{op} \circ \omega^{op} = Sg(g)$; the problem is to find a unique $h : ob'' \rightarrow ob'$ such that $Sg(h) = \omega^{op}$ and $f \circ h = g$; we have only to verify that ω induces an object morphism: $\omega^{-1}(\Lambda'') \subseteq \sigma(\Lambda)$ because if $\lambda' \in \omega^{-1}(\Lambda'')$ then $\sigma^{-1}(\lambda') \in \Lambda$ (since $\sigma^{-1}(\omega^{-1}(\Lambda'')) \subseteq \Lambda$), and thus $\lambda' \in \{\lambda' \in \Lambda_{\Sigma'} : \sigma^{-1}(\lambda') \in \Lambda\} = \sigma(\Lambda)$. QED

$$\begin{array}{ccc}
 Ob & \xrightarrow{Sg} & Sig^{op} \\
 \\
 \begin{array}{ccc}
 \langle \Sigma', \sigma(\Lambda) \rangle & \xrightarrow{f} & \langle \Sigma, \Lambda \rangle \\
 \uparrow \text{dotted } h & \nearrow g & \\
 \langle \Sigma'', \Lambda'' \rangle & &
 \end{array} & &
 \begin{array}{ccc}
 \Sigma' & \xrightarrow{\sigma^{op}} & \Sigma \\
 \uparrow \omega^{op} & \nearrow Sg(g) & \\
 \Sigma'' & &
 \end{array}
 \end{array}$$

Figure 7: The morphism f with $Sg(f) = \sigma^{op}$ is cartesian by Sg for σ^{op} on $\langle \Sigma, \Lambda \rangle$.

5 Induced object institution

We are now ready to set up the envisaged institution of objects where objects play the role of models. The object institution induced by B is the quadruple $O_B = \langle Sig, Sen, Obj, \Vdash \rangle$ where Sig and Sen are as in B , and the functor $Obj : Sig \rightarrow Cat^{op}$ and the family \Vdash of satisfaction relations are defined below. The functor Obj is defined capitalizing on the fact that the functor Sg is a split cofibration (following a general construction as presented in [8]: see the Appendix): it maps each signature to the category of objects over that signature.

Since B and O_B share the category Sig , and the functor Sen , they also share the language and the category of signatures. Therefore, all the results and constructions of sections 3.3 and 3.4 are retained, since they do not depend on the semantic components of the institutions.

We start this section by building the institution of objects. Then, we discuss the relationship between theories and objects in both directions. For each theory we provide a loose semantics composed by all the objects that satisfy the theory and a final semantics provided by the “biggest” possible object. In a similar way, for each object we provide a loose theoretical view composed by all the theories and the “biggest” theory composed by all formulae satisfied by the object.

5.1 Interpretation over objects

So we have three basic steps: definition of the fiber (category of objects over the same signature), definition of functors between fibers induced by signature morphisms and definition of the indexing functor. These constructions are general (see the Appendix) because we proved that Sg is a cofibration with splitting.

Definition 5.1 Let Σ be a signature. The category Ob_Σ is the fiber $Sg^{-1}(\Sigma)$. The elements of Ob_Σ are called *interpretation structures* over Σ . The morphisms of Ob_Σ are called *homomorphisms* between interpretation structures over Σ .

In the fiber Ob_Σ the elements are the objects over Σ and the morphisms are object morphisms that are mapped by Sg to the opposite of the identity over Σ . The next step is the definition of the functor that must exist between the fibers $Ob_{\Sigma'}$ and Ob_Σ whenever there is a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$. We start by introducing the notion of reduct of an object.

Definition 5.2 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism and ob' an interpretation structure over Σ' . The *reduct object* of ob' with respect to σ (denoted by $\sigma^{-1}(ob')$) is the target of the cocartesian morphism $\underline{\sigma}_{ob'}^{op}$ for σ^{op} on ob' .

Clearly, as expected, the signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ leads to an object morphism from $ob' = \langle \Sigma', \Lambda' \rangle$ to $ob = \langle \Sigma, \Lambda \rangle$ iff the set of interpretation structures for the reduct $\sigma^{-1}(\Lambda')$ is a subset of Λ .

Prop/Definition 5.3 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism. The *reduct functor* $\sigma^{-1} : Ob_{\Sigma'} \rightarrow Ob_\Sigma$ maps each ob' of $Ob_{\Sigma'}$ to the reduct $\sigma^{-1}(ob')$ and each homomorphism $h' : ob'_1 \rightarrow ob'_2$ to the unique homomorphism $\sigma^{-1}(h') : \sigma^{-1}(ob'_1) \rightarrow \sigma^{-1}(ob'_2)$ such that $\sigma^{-1}(h') \circ \underline{\sigma}_{ob'_1}^{op} = \underline{\sigma}_{ob'_2}^{op} \circ h'$.

Proof:

Consider the contravariant version of A11 in the Appendix. QED

Note that $\sigma^{-1}(h')$ exists and is unique because $\underline{\sigma}_{ob'_1}^{op}$ is the cartesian morphism by Sg for σ on ob'_1 . Now the definition of the indexing functor Obj is trivial. Note again (as pointed out in the Appendix) that the need for the splitting is essential for Obj to be a functor.

Prop/Definition 5.4 The *object interpretation functor* $Obj : Sig \rightarrow Cat^{op}$ maps each signature Σ to the fiber Ob_Σ , and each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$ to the opposite of the reduct functor $\sigma^{-1} : Ob_{\Sigma'} \rightarrow Ob_\Sigma$.

Proof:

Consider the contravariant version of A11 in the Appendix. QED

In this way we establish an indexed category (c.f., [50]) $Ob^{op} : Sig^{op} \rightarrow Cat$. By an appropriate flattening construction we should recover the original category Ob . However we shall not dwell on this issue. Clearly, we might develop the counterparts to these functors for “images” instead of reducts provided that we verify that the fibration Sg is split. But we refrain to do so because we do not need them in the sequel. However, it is useful to introduce:

Definition 5.5 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism and ob an interpretation structure over Σ . The *image object* of ob with respect to σ (denoted by $\sigma(ob)$) is the source of the cartesian morphism for σ^{op} on ob .

5.2 Satisfaction by objects

Since now our semantic units are objects we must say when an object satisfies a formula.

Definition 5.6 Let Σ be a signature, φ a formula of L_Σ , and $ob = \langle \Sigma, \Lambda \rangle$ an object. We say that ob *satisfies* φ (written $ob \Vdash_\Sigma \varphi$) iff $\Lambda \Vdash_\Sigma \varphi$.

We also have the satisfaction condition relating satisfaction of formulae by reducts of objects with satisfaction of images of formulae by objects.

Proposition 5.7 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, φ a formula of L_Σ , and $ob' = \langle \Sigma', \Lambda' \rangle$ an object. Then, $ob' \Vdash_{\Sigma'} \sigma(\varphi)$ iff $\sigma^{-1}(ob') \Vdash_\Sigma \varphi$.

Proof:

Corollary of proposition 3.17.

QED

Finally, we can conclude:

Proposition 5.8 The quadruple $O_B = \langle Sig, Sen, Obj, \Vdash \rangle$ is an institution.

The forward satisfaction condition also holds:

Proposition 5.9 Let Σ be a signature, φ a formula of L_Σ , and $ob = \langle \Sigma, \Lambda \rangle$ an object. Then, if $ob \Vdash_\Sigma \varphi$ then $\sigma(ob) \Vdash_{\Sigma'} \sigma(\varphi)$.

Proof:

Corollary of proposition 3.18.

QED

5.3 Loose semantics

Although we envisage a final semantics mapping each theory to the “maximal” object that satisfies the theory, we start by introducing the traditional loose semantics of theories within the setting of the object institution, since we need in the sequel to be able to talk about all object models of a theory and all theories entailed by an object.

Prop/Definition 5.10 The *loose semantics functor* $Mod : Th \rightarrow Cat^{op}$ maps each theory th to the full subcategory $Mod(th)$ of $Obj(\Sigma)$ whose elements satisfy th , and each theory morphism $f : th \rightarrow th'$ to $Mod(f) : Mod(th) \rightarrow Mod(th')$ corresponding to the restriction of the op-functor $Obj(Si(f)) : Obj(Si(th)) \rightarrow Obj(Si(th'))$ to $Mod(th)$.

Proof:

Let $th = \langle \Sigma, \Phi \rangle$ and $th' = \langle \Sigma', \Phi' \rangle$ be theories, $f : th \rightarrow th'$ a theory morphism and $ob' = \langle \Sigma', \Lambda' \rangle$ an object. Recall that $Obj(Si(f)) : Obj(\Sigma) \rightarrow Obj(\Sigma')$ is the opposite of the reduct functor $(Si(f))^{-1} : Obj(\Sigma') \rightarrow Obj(\Sigma)$. We just verify that if $ob' \in Mod(th')$ then $(Obj(Si(f)))^{op}(ob') \in Mod(th)$. Indeed, if $ob' \in Mod(th')$ then: $\Lambda' \Vdash_{\Sigma'} \varphi'$ for each $\varphi' \in \Phi'$ and since $Si(f)(\varphi) \in \Phi'$ then $\Lambda' \Vdash_{\Sigma'} Si(f)(\varphi)$ for each $\varphi \in \Phi$. Hence, $(Si(f))^{-1}(\Lambda') \Vdash_{\Sigma} \varphi$ (using the set satisfaction condition of the base institution). Therefore, we obtain $(Obj(Si(f)))^{op}(ob') = (Si(f))^{-1}(ob') \in Mod(th)$. QED

Conversely, we might be interested in extracting from an object the category of all theories that are satisfied by it. Taking advantage of the indexing of theories by signatures introduced above it is straightforward to establish:

Prop/Definition 5.11 The *loose theory functor* $Lth : Ob \rightarrow Cat^{op}$ maps each object ob to the full subcategory $Lth(ob)$ of $TH(\Sigma)$ whose elements are satisfied by ob , and each object morphism $f : ob \rightarrow ob'$ to $Lth(f) : Lth(ob) \rightarrow Lth(ob')$ corresponding to the restriction of the op-functor $TH(Sg(f)^{op})^{op} : TH(Sg(ob)) \rightarrow TH(Sg(ob'))$ to $Lth(ob')$.

Proof:

Let $ob = \langle \Sigma, \Lambda \rangle$ and $ob' = \langle \Sigma', \Lambda' \rangle$ be objects, $f : ob \rightarrow ob'$ an object morphism and $th' = \langle \Sigma', \Phi' \rangle$ a theory. Recall that $TH(Sg^{op}) : TH(\Sigma') \rightarrow TH(\Sigma)$ is the image functor $(Sg(f)^{op}) : TH(\Sigma') \rightarrow TH(\Sigma)$. We just verify that if $th' \in Lth(ob')$ then $(TH(Sg(f)^{op})(th')) \in Lth(ob)$. Indeed, if $th' \in Lth(ob')$ then: $\Lambda' \Vdash_{\Sigma'} \varphi'$ for each $\varphi' \in \Phi'$ and since $(Sg(f)^{op})^{-1}(\Lambda) \subseteq \Lambda'$ then $(Sg(f)^{op})^{-1}(\Lambda) \Vdash_{\Sigma'} \varphi'$ for each $\varphi' \in \Phi'$. Hence, $\Lambda \Vdash_{\Sigma} (Sg(f)^{op})(\varphi')$ (using the set satisfaction condition of the base institution). Therefore, we obtain $(TH(Si(f)))^{op}(th') = (Sg(f)^{op})(th') \in Lth(ob)$. QED

6 Final semantics

We are now ready to present the envisaged final semantics functor that maps each theory to the object composed of all lives that satisfy (in the base institution) the theory. We also present the functor that maps each object to the theory composed of all formulae satisfied (in the base institution) by the object. As might be expected, these two functors are the reverse of each other and, therefore, establish a powerful isomorphism between the category of theories and the category of objects.

6.1 Maximal object model

Now we associate to each theory the “biggest” possible object. We prove that this semantics is canonical in the sense that this object is final among all the objects that can be associated with the theory (that is to say there is a unique morphism from each of those to the canonical object).

Proposition 6.1 Let $th = \langle \Sigma, \Phi \rangle$ be a theory and $\Omega_{th} = \{\lambda \in \Lambda_{\Sigma} : \lambda \Vdash_{\Sigma} \Phi\}$. Then, $\langle \sigma, \Omega_{th} \rangle$ is the final object in the category $Mod(th)$.

Proof:

1. Theoretical closure: we have to verify that $\Omega_{th} = \Omega_{th}^{\mathbb{F}}$; the left to right inclusion is trivial; the opposite inclusion is obtained as follows: if $\Omega_{th} \Vdash_{\Sigma} \xi$ where $\xi \in \Lambda_{\Sigma}$ then $\xi \Vdash_{\Sigma} \varphi$ for all $\varphi \in \Phi$ and, thus, $\xi \in \Omega_{th}$.
2. Finality: let $\langle \Sigma, \Lambda \rangle$ be any object in the category $Mod(th)$; we have a (unique) object homomorphism $f : \langle \Sigma, \Lambda \rangle \rightarrow \langle \Sigma, \Omega_{th} \rangle$ since $\Lambda \subseteq \Omega_{th}$. QED

Let $ob = \langle \Sigma, \Lambda \rangle \in Mod(th)$ for some theory $th = \langle \Sigma, \Phi \rangle$. We denote by $1_{ob} : ob \rightarrow \langle \Sigma, \Omega_{th} \rangle$ the unique homomorphism that exists in $Mod(th)$ from ob to the final object in $Mod(th)$.

We can lift our construction to a functor mapping each theory to the final object and also each theory morphism to a morphism between the respective final objects.

Prop/Definition 6.2 The final semantics functor $F : Th^{op} \rightarrow Ob$ maps each theory $th = \langle \Sigma, \Phi \rangle$ to the final object of $Mod(th)$ and each theory morphism $f : th \rightarrow th'$ to the morphism $F(f) = 1_{(Si(f))^{-1}(F(th'))} \circ \underline{Si(f)}_{F(th')}^{op} : F(th') \rightarrow F(th)$.

That is, $F(th)$ is the set of all interpretation structures that satisfy all axioms of th ; and $F(f)$ is given by composing the cocartesian morphism for $Si(f)^{op}$ on $F(th')$

$$\underline{Si(f)}_{F(th')}^{op} : F(th') \rightarrow (Si(f))^{-1}(F(th'))$$

with the unique homomorphism

$$1_{(Si(f))^{-1}(F(th'))} : (Si(f))^{-1}(F(th')) \rightarrow F(th)$$

from the reduct $(Si(f))^{-1}(F(th'))$ into the final object.

6.2 Maximal entailed theory

Conversely, we are also interested in associating to each object the theory containing all formulae satisfied by it, i.e. the largest theory satisfied by it.

Proposition 6.3 Let $ob = \langle \Sigma, \Lambda \rangle$ be an object and $\Omega_{ob} = \{\varphi \in L_{\Sigma} : \Lambda \Vdash_{\Sigma} \varphi\}$. Then, $\langle \sigma, \Omega_{ob} \rangle$ is the final theory in the category $Lth(ob)$.

Proof:

1. Semantic closure: we have to verify that $\Omega_{ob} = \Omega_{ob}^{\mathbb{F}}$; the left to right inclusion is trivial; the opposite inclusion is obtained as follows: if $\Omega_{ob} \models_{\Sigma} \psi$ where $\psi \in L_{\Sigma}$ then $\Lambda \Vdash_{\Sigma} \psi$ for all $\psi \in L_{\Sigma}$ and, thus, $\psi \in \Omega_{ob}$.
2. Finality: let $\langle \Sigma, \Phi \rangle$ be any theory in the category $Lth(ob)$; we have a (unique) theory homomorphism $f : \langle \Sigma, \Phi \rangle \rightarrow \langle \Sigma, \Omega_{ob} \rangle$ since $\Phi \subseteq \Omega_{ob}$. QED

Let $th = \langle \Sigma, \Phi \rangle \in Lth(ob)$ for some object $ob = \langle \Sigma, \Lambda \rangle$. We denote by $1_{th} : th \rightarrow \langle \Sigma, \Omega_{ob} \rangle$ the unique homomorphism that exists in $Lth(ob)$ from th to the final object in $Lth(ob)$.

We can lift our construction to a functor mapping each object to the final theory and also each object morphism to a morphism between the respective final theories.

Prop/Definition 6.4 The final theory functor $G : Ob \rightarrow Th^{op}$ maps each object $ob = \langle \Sigma, \Lambda \rangle$ to the final object of $Lth(ob)$ and each object morphism $f : ob \rightarrow ob'$ to the opposite of the morphism $G(f) = 1_{(Sg(f))^{op}(G(th'))} \circ \overline{Sg(f)^{op}}^{G(th')} : G(ob') \rightarrow G(ob)$.

That is, $G(ob)$ contains all formulae satisfied by ob ; and $G(f)$ is given by composing the cocartesian morphism for $Sg(f)^{op}$ on $G(ob')$ with the unique homomorphism from the image theory $Sg(f)^{op}(G(ob'))$ into the final theory.

6.3 Main properties

Since we have two functors $F : Th^{op} \rightarrow Ob$ and $G : Ob \rightarrow Th^{op}$ we now discuss the relationship between them. The things to keep in mind at this point are that both categories are concrete over the same base category Sig^{op} . Hence, we should try a special case of adjunction which is known as Galois correspondence. We start by showing that if we start with a theory and apply first F and then G we get the same theory.

Lemma 6.5 Let $th = \langle \Sigma, \Phi \rangle$ be a theory. Then $G(F(th)) = th$.

Proof:

Let $G(F(th)) = \langle \Sigma, \Psi \rangle$ and $F(th) = \langle \Sigma, \Omega_{th} \rangle$. Clearly, $\Phi \subseteq \Psi$. Moreover, $\Psi \subseteq \Phi$. Indeed, we show that $\Phi \models_{\Sigma} \psi$ for every $\psi \in \Psi$: assume that $\xi \Vdash_{\Sigma} \Phi$; then, $\xi \in \Omega_{th}$; on the other hand, $\Omega_{th} \Vdash_{\Sigma} \psi$ for every $\psi \in \Psi$ and so $\xi \Vdash_{\Sigma} \psi$ for every $\psi \in \Psi$; therefore, $\Phi \models_{\Sigma} \psi$ for every $\psi \in \Psi$. And so, $\psi \in \Phi$ for every $\psi \in \Psi$. QED

Also if we start with an object and apply first G and then F we get the same object.

Lemma 6.6 Let $ob = \langle \Sigma, \Lambda \rangle$ be an object. Then $F(G(ob)) = ob$.

Proof:

Let $F(G(ob)) = \langle \Sigma, \Xi \rangle$ and $G(ob) = \langle \Sigma, \Omega_{ob} \rangle$. Clearly, $\Lambda \subseteq \Xi$. Moreover, $\Xi \subseteq \Lambda$. Indeed, we show that $\Lambda \models_{\Sigma} \xi$ for every $\xi \in \Xi$: $\Lambda \Vdash_{\Sigma} \varphi$ iff $\varphi \in \Omega_{ob}$; $\xi \Vdash_{\Sigma} \varphi$ for every $\varphi \in \Omega_{ob}$; hence, $\Lambda \models_{\Sigma} \xi$. Therefore, $\xi \in \Lambda$ for every $\xi \in \Xi$. QED

Now we prove that F and G constitute a Galois correspondence. Basically, (see the Appendix), we have to show that a signature morphism induces a morphism between theories iff the op-signature morphism induces an object morphism.

In Figure 8, we present the basic steps of the proof. The triangle in the center depicts all the functors that were defined between the categories Sig , Th and Ob . We are given a signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, a theory th' and an object ob . We have to prove that σ induces a theory morphism g ($Si(g) = \sigma$) from th' to the image of ob by G iff σ induces an object morphism h ($Sg(h) = \sigma^{op}$) from the image of th' by F to ob .

Proposition 6.7 The pair of functors $\langle F, G \rangle$ is a Galois correspondence.

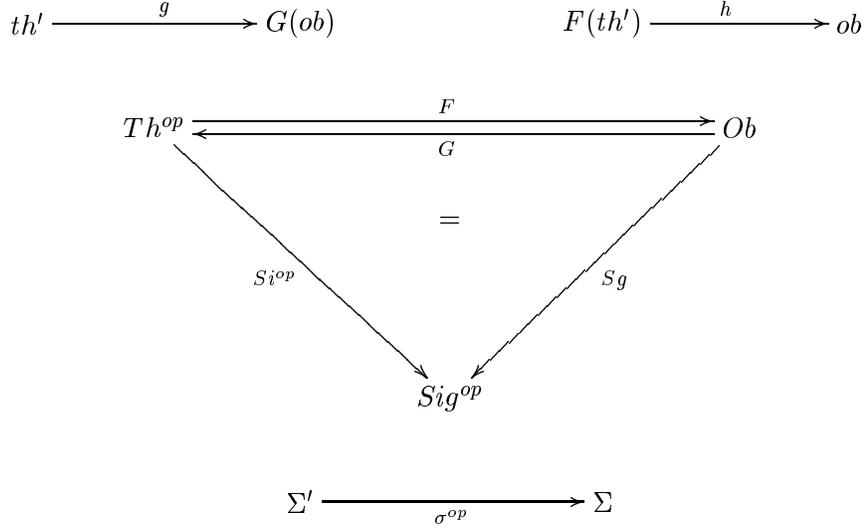


Figure 8: Galois correspondence $\langle F, G \rangle$ between $\langle Th^{op}, Si^{op} \rangle$ and $\langle Ob, Sg \rangle$.

Proof:

Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, $ob = \langle \Sigma, \Lambda \rangle$ be an object and $th' = \langle \Sigma', \Phi' \rangle$ a theory. We show that σ^{op} induces a theory op-morphism from th' to $G(ob)$ iff σ^{op} induces an object morphism from $F(th')$ to ob . That is, we have to prove that $\sigma(\Omega_{ob}) \subseteq \Phi'$ iff $\sigma^{-1}(\Omega_{th'}) \subseteq \Lambda$.

1. Assume that $\sigma(\Omega_{ob}) \subseteq \Phi'$. Then, since $\Omega_{th'} \Vdash_{\Sigma'} \varphi'$ for every $\varphi' \in \Phi'$, we conclude that $\Omega_{th'} \Vdash_{\Sigma'} \sigma(\varphi)$ for every $\varphi \in \Omega_{ob}$. Therefore, $\sigma^{-1}(\Omega_{th'}) \Vdash_{\Sigma} \varphi$ for every $\varphi \in \Omega_{ob}$. Hence, $\sigma^{-1}(\Omega_{th'}) \subseteq \Xi$, where $F(G(ob)) = \langle \Sigma, \Xi \rangle$. But $\Xi = \Lambda$ and so, $\sigma^{-1}(\Omega_{th'}) \subseteq \Lambda$.

2. Assume that $\sigma^{-1}(\Omega_{th'}) \subseteq \Lambda$. Let $\varphi \in \Omega_{ob}$. Then, $\Lambda \Vdash_{\Sigma} \varphi$ and so $\sigma^{-1}(\Omega_{th'}) \Vdash_{\Sigma} \varphi$. Thus, $\Omega_{th'} \Vdash_{\Sigma'} \sigma(\varphi)$ and so $\sigma(\varphi) \in \Psi'$, where $G(F(th')) = \langle \Sigma', \Psi' \rangle$. But $\Psi' = \Phi'$, therefore, $\sigma(\varphi) \in \Phi'$. QED

In terms of adjunctions we can say that F is left adjoint of G . A particular case is when the given object ob and the given theory th' have the same signature. Then, we get:

Corollary 6.8 Let $ob = \langle \Sigma, \Lambda \rangle$ be an object and $th = \langle \Sigma, \Phi \rangle$ a theory such that $G(ob) = \langle \Sigma, ? \rangle$ and $F(th) = \langle \Sigma, \Xi \rangle$. Then, $\Xi \subseteq \Lambda$ iff $? \subseteq \Phi$.

If we look into the Appendix we see that it would be nice that the pair of functors constitute a concrete isomorphism because of the preservation of products and coproducts as well as of cartesian and cocartesian morphisms. For showing that F and G constitute a concrete isomorphism with respect to Si^{op} and Sg we have to show that $Sg \circ F = Si^{op}$ and that $Si^{op} \circ G = Sg$. Since the categories of theories and objects are amnestic and there is the Galois correspondence $\langle F, G \rangle$ we can prove this in an easier way.

Proposition 6.9 The pair of functors $\langle F, G \rangle$ is a concrete isomorphism over Sig^{op} .

Proof:

Since $\langle Th, Si \rangle$ and $\langle Ob, Sg \rangle$ are amnesic, by proposition A16 of the Appendix, we can conclude that $F^* : F(|Th^{op}|) \rightarrow G(|Ob^{op}|)$ and $G : G(|Ob^{op}|) \rightarrow F(|Th^{op}|)$ constitute a concrete isomorphism over Sig^{op} . Moreover, $F(|Th^{op}|) = Ob$ and $G(|Ob^{op}|) = Th^{op}$ and thus $F^* = F$ and $G^* = G$. QED

Proposition 6.10 The functor $F : Th^{op} \rightarrow Ob$ preserves limits and colimits, as well as cartesian and cocartesian morphisms.

Proof:

1. Clearly, an isomorphism preserves (co)limits.
2. Concerning (co)cartesian morphisms see proposition A13 of the Appendix. QED

From now on we will use $\llbracket th \rrbracket$ instead of $F(th)$ so that we are closer to the usual notation for denotations.

Corollary 6.11 $\llbracket \sigma^{-1}(th') \rrbracket = \sigma^{-1}(\llbracket th' \rrbracket)$ where $\sigma : \Sigma \rightarrow \Sigma'$ is a signature morphism and th' a theory over Σ' .

This corollary states that the denotation of the theory corresponding to an abstraction of a theory is the reduct of the denotation of the theory.

Corollary 6.12 $\llbracket th + th' \rrbracket = \llbracket th \rrbracket \times \llbracket th' \rrbracket$ where th and th' are theories.

This corollary states that the denotation of an aggregation of two theories is the product of the denotations.

The importance of these results will be seen in the next section.

7 Application to denotational semantics

Herein we discuss the denotational semantics of the main constructions of object specifications namely: specifications of primitive objects, aggregation with and without interconnection, abstraction and monotonic specialization. The semantics is compositional: the semantics of the whole is expressed by the semantics of the parts. The cocartesian morphisms provide the semantics for abstraction.

7.1 Main object constructs

We provide the semantics of a specification in a two step process: first we obtain the theory generated by semantic closure of the specification and after we get the maximal object that satisfies the theory.

The specification $spec = \langle \Sigma, \Phi \rangle$ of a primitive object induces the theory $th = \langle \Sigma, \Phi^F \rangle$. The denotation of the theory is the final object $\llbracket th \rrbracket$ given by the functor F .

The semantics of the *flip-flop* specification is the maximal object associated by F to the theory generated by the specification.

The aggregation of two objects specified by $spec = \langle \Sigma, \Phi \rangle$ and $spec' = \langle \Sigma', \Phi' \rangle$, inducing the theories $th = \langle \Sigma, \Phi^F \rangle$ and $th' = \langle \Sigma', \Phi'^F \rangle$, is defined by the coproduct theory $th + th' = \langle \Sigma + \Sigma', (inj_\Sigma(\Phi) \cup inj_{\Sigma'}(\Phi'))^F \rangle$. Its denotation is the final object $\llbracket th + th' \rrbracket = \llbracket th \rrbracket \times \llbracket th' \rrbracket$.

The semantics of the specification of the *flip-flop* and its clone is the product of the maximal objects corresponding to the theory of induced by the specification of the *flip-flop* and the theory induced by the specification of the clone.

The interconnection of two objects specified by $spec = \langle \Sigma, \Phi \rangle$, $spec' = \langle \Sigma', \Phi' \rangle$ and a set Ψ of interaction axioms over $\Sigma + \Sigma'$ (e.g., $(\nabla c \Rightarrow \nabla c')$ for c calls c' and $(\nabla c \Rightarrow b')$ for c queries b') is defined by the theory $th + th' = \langle \Sigma + \Sigma', (inj_\Sigma(\Phi) \cup inj_{\Sigma'}(\Phi') \cup \Psi)^F \rangle$ corresponding to the coproduct of $th = \langle \Sigma, \Phi^F \rangle$ and $th' = \langle \Sigma', \Phi'^F \rangle$ enriched with the interaction axioms. Its denotation is the “constrained product” of $\llbracket th \rrbracket$ and $\llbracket th' \rrbracket$ that also satisfies the interaction axioms. That is to say, in the interconnection we have more axioms than in an aggregation but fewer lives satisfy those axioms.

The view (abstraction) given by an injective hiding signature morphism $\sigma : \Sigma' \rightarrow \Sigma$ of an object specified by $spec = \langle \Sigma, \Phi \rangle$ is defined by the theory $\sigma^{-1}(th) = \langle \Sigma', \sigma^{-1}(\Phi^F) \rangle$. Its denotation is, as expected, $\llbracket \sigma^{-1}(th) \rrbracket = \sigma^{-1}(\llbracket th \rrbracket)$.

The (monotonic) specialization of an object specified by $spec = \langle \Sigma, \Phi \rangle$ is specified by $spec = \langle \Sigma, \Phi \cup \Phi_1 \rangle$. The denotation of the resulting theory is the “constrained” object that also satisfies the additional axioms.

7.2 Working with linear temporal logic

We illustrate in some detail the proposed general construction when the base institution is (propositional) linear temporal logic institution L . This institution is established by the following specific requirements (besides the base requirements stated in section 3):

- The modal (temporal) operators considered are X (next) and G (always).
- The set L_Σ of formulae over a signature Σ is inductively defined on A_Σ as follows:
 - $\star \in L_\Sigma$;
 - $(\neg \gamma) \in L_\Sigma$, provided that $\gamma \in L_\Sigma$;
 - $(\gamma_1 \Rightarrow \gamma_2) \in L_\Sigma$, provided that $\gamma_1, \gamma_2 \in L_\Sigma$;
 - $(X \gamma) \in L_\Sigma$, provided that $\gamma \in L_\Sigma$;
 - $(G \gamma) \in L_\Sigma$, provided that $\gamma \in L_\Sigma$.
- Each interpretation structure λ , called a *life-cycle*, is of the form $\langle W, R, V \rangle$ where $W = \mathbb{N}$, $R_X = \{\langle i, i + 1 \rangle : i \in \mathbb{N}\}$ and $R_G = \{\langle i, j \rangle : i < j\}$ is its transitive closure. Note that this institution is “uniframe” in the sense that all interpretation structures are built over the same frame

$\langle \mathbb{N}, \{R_X, R_G\} \rangle$. Therefore, in the sequel, we identify each λ with the sequence \underline{V} and write $\lambda : \mathbb{N} \rightarrow \wp A_\Sigma$. For each n , $\lambda(n)$ is called the *situation* at position n .

- Let λ be a life-cycle and $n \in \mathbb{N}$. Then:
 - $\lambda \Vdash_n \star$ iff $n = 0$;
 - $\lambda \Vdash_n (X \gamma)$ iff $\lambda \Vdash_{n+1} \gamma$;
 - $\lambda \Vdash_n (G \gamma)$ iff $\lambda \Vdash_i \gamma$ for every $i > n$.

In the case of L each object ob is a pair $\langle \Sigma, \Lambda \rangle$ where Λ is a set of life-cycles. A morphism σ from ob to ob' states that each life-cycle of ob , when reduced by $\sigma : Sg(ob') \rightarrow Sg(ob)$ to the signature of ob' , must be a life-cycle of ob' .

7.3 Products for linear temporal logic

We examine now the *products* in the category Ob built upon L . We show that for “reasonable” objects the product reflects “fair interleaving” of the components. Such reasonable objects are those objects that are open to embedding in any environment. Namely, they do not assume that they will be able to execute actions before the environment does something else. In order to formalize this concept we need to develop the notion of (fair) delay.

For $k > 0$ we denote by $[\lambda]_k$ the prefix $\lambda_0, \dots, \lambda_{k-1}$ of length k of λ . By convention $[\lambda]_0$ is the empty sequence. Given $s \subseteq A_\Sigma$ we denote by $s \setminus \nabla$ the set given by $s \cap (\{b : b \in \Sigma_{obs}\} \cup \{\diamond c : c \in \Sigma_{act}\})$.

Definition 7.1 Let Σ be a signature and k an ordinal $\leq \omega$. The *one-step delay at k map* $\Delta^k : \Lambda_\Sigma \rightarrow \Lambda_\Sigma$ is defined as follows:

- for $k \in \mathbb{N}$, $\Delta^k(\lambda)_n = \lambda_n$ for every $n < k$, $\Delta^k(\lambda)_n = \lambda_{n-1}$ for every $n > k$ and $\Delta^k(\lambda)_k = (\lambda_k) \setminus \nabla$;
- $\Delta^\omega(\lambda) = \lambda$.

Thus, the one step delay map Δ^k when k is finite applied to a life-cycle λ gives the life-cycle $\Delta^k(\lambda)$ which is obtained from λ by inserting at position k the situation λ_k without the occurrences. When k is infinite then the map is just the identity.

Definition 7.2 A *delay-point sequence* η is a sequence of ordinals such that:

- for each $i \in \mathbb{N}$ $\eta_i \leq \omega$;
- for each $i, j \in \mathbb{N}$ such that $i < j$, either $\eta_i < \eta_j$ or $\eta_i = \eta_j = \omega$;
- for each $i \in \mathbb{N}$ there is $j \in \mathbb{N}$ such that either $\eta_j \in \mathbb{N}$ and $\eta_j - \eta_i > j - i$ or $\eta_j = \omega$.

The first condition is monotonicity: the sequence of ordinals is strictly increasing until it reaches ω . The last condition in the definition is called *the diligence condition* borrowing the terminology from [34]. “Diligent” objects allow delays of their actions but not forever.

Definition 7.3 Given a life-cycle λ over a signature Σ , a sequence η of delay points *induces* the sequence $\delta^\eta(\lambda)$ of life-cycles over Σ such that:

- $\delta^\eta(\lambda)^0 = \lambda$;
- $\delta^\eta(\lambda)^{i+1} = \Delta^{\eta_i}(\delta^\eta(\lambda)^i)$.

Example 7.4 Consider the signature $\Sigma = \langle \{b\}, \{a\} \rangle$ and let λ be the life-cycle over Σ such that $\lambda_n = \{b, \diamond a, \nabla a\}$ for n even and $\lambda_n = \{\diamond a, \nabla a\}$ for n odd. If we take the delay point sequence $\eta = 1.3.\omega \dots$, then sequence $\delta^\eta(\lambda)$ is as defined in Figure 9.

	0	1	2	3	4	5	...
$\delta^\eta(\lambda)^0 = \lambda$	b $\diamond a$ ∇a	$\diamond a$ ∇a	b $\diamond a$ ∇a	$\diamond a$ ∇a	b $\diamond a$ ∇a	$\diamond a$ ∇a	...
$\delta^\eta(\lambda)^1 = \Delta^1(\lambda)$	b $\diamond a$ ∇a	$\diamond a$	$\diamond a$ ∇a	b $\diamond a$ ∇a	$\diamond a$ ∇a	b $\diamond a$ ∇a	...
$\delta^\eta(\lambda)^2 = \Delta^3(\Delta^1(\lambda))$	b $\diamond a$ ∇a	$\diamond a$	$\diamond a$ ∇a	b $\diamond a$	b $\diamond a$ ∇a	$\diamond a$ ∇a	...
$\delta^\eta(\lambda)^{k+3} = \delta^\eta(\lambda)^2$	b $\diamond a$ ∇a	$\diamond a$	$\diamond a$ ∇a	b $\diamond a$	b $\diamond a$ ∇a	$\diamond a$ ∇a	...

Figure 9: The sequence $\delta^\eta(\lambda)$.

Definition 7.5 The *delay relation* $\infty \subseteq \Lambda_\Sigma \times \Lambda_\Sigma$ is defined as follows: $\lambda \infty \lambda'$ iff there is a delay-point sequence η such that for each $n \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $[\lambda']_n = [\delta^\eta(\lambda)]_n^i$.

Example 7.6 For instance, $\lambda \infty \lambda'$, assuming that $\lambda_n = \{\diamond c, \nabla c\}$ for $n \in \mathbb{N}$, and $\lambda'_n = \{\diamond c, \nabla c\}$ for n odd and $\lambda'_n = \{\diamond c\}$ for n even. Indeed, consider the sequence η such that $\eta_i = i \times 2$ for every $i \in \mathbb{N}$ which trivially satisfies the diligence condition. The sequence $\delta^\eta(\lambda)$ is as follows for $i > 0$:

- $\delta^\eta(\lambda)_n = \{\diamond c\}$ for $n < \eta_{i-1} + 1$ and n even;
- $\delta^\eta(\lambda)_n = \{\diamond c, \nabla c\}$ for $n < \eta_{i-1} + 1$ and n odd;
- $\delta^\eta(\lambda)_n = \{\diamond c, \nabla c\}$ for $n > \eta_{i-1} + 1$.

On the other hand, if $\lambda''_n = \{\diamond c\}$ for $n \in \mathbb{N}$ then we do not have $\lambda \infty \lambda''$ because the sequence $\eta = \{i\}_{i \in \mathbb{N}}$ violates the diligence condition. The life-cycle λ'' would be an “unfair” delay of λ since action c would never happen. Note that if $\lambda \infty \lambda'$ and the delay-point sequence is η such that there is $i \in \mathbb{N}$ such that $\eta_j = \omega$ for all $j \geq i$ then λ' has a finite number of delays with respect to λ .

Proposition 7.7 The delay relation is reflexive and transitive.

Proof:

1. Reflexivity: choose $\eta_i = \omega$ for every $i \in \mathbb{N}$.
2. Transitivity: assume that $\lambda \infty \lambda'$ and $\lambda' \infty \lambda''$ with delay-point sequences η and η' , respectively; consider the sequence $\eta; \eta'$ obtained by merging η' with the sequence η such that:

- $\nu_n = \eta_n + \max\{k + 1 : \eta'_k < (\eta_n + k + 1)\}$ if $\eta_n < \omega$;
- $\nu_n = \omega$, otherwise;

that is, each delay-point of η is moved to the right depending on the delays introduced by η' ; it is trivial to see that $\eta; \eta'$ satisfies the first two requirements on delay-point sequences; verifying that $\eta; \eta'$ satisfies the third requirement is more difficult; the thesis is obtained by showing that each point (of \mathbb{N}) where η does not introduce a delay is preserved by $\eta; \eta'$ as a no-delay-point, but suitably moved to the right (like a delay-point: a similar formula is applied). QED

Definition 7.8 Let Λ be a set of life-cycles over a signature Σ . The *closure for delays* of Λ is the following set of life-cycles over Σ :

$$\Lambda^\emptyset = \{\lambda' \in \Lambda_\Sigma : \lambda \infty \lambda' \text{ for some } \lambda \in \Lambda\}$$

Definition 7.9 An object $\langle \Sigma, \Lambda \rangle$ is said to be *closed for delays* iff $\Lambda = \Lambda^\emptyset$.

We assume that Λ_Σ is closed for delays for every signature Σ .

Example 7.10 Consider the object $\langle \langle \emptyset, \{a\} \rangle, \{\lambda\}^\emptyset \rangle$ where the life-cycle λ is such that $\lambda_n = \{\diamond a, \nabla a\}$ for every $n \in \mathbb{N}$. Then,

- $\langle \langle \emptyset, \{a\} \rangle, \{\lambda\}^\emptyset \rangle \vDash_{\langle \emptyset, \{a\} \rangle} \diamond a$;
- $\langle \langle \emptyset, \{a\} \rangle, \{\lambda\}^\emptyset \rangle \vDash_{\langle \emptyset, \{a\} \rangle} (G \diamond a)$;

- $\langle \langle \emptyset, \{a\} \rangle, \{\lambda\}^\emptyset \rangle \models_{\langle \emptyset, \{a\} \rangle} (F\nabla a)$;

Note that the diligence condition is essential for the satisfaction of $(F\nabla a)$ by $\langle \langle \emptyset, \{a\} \rangle, \{\lambda\}^\emptyset \rangle$.

The following results show that the relevant constructions (reducts, images and products) respect the closure for delays. We start by showing that the reduct of a set of life-cycles closed for delays is also closed for delays.

We start by showing that reducts of sets of life-cycles that are closed for delays are also closed for delays. This means that when defining an abstraction from an object closed for delays we get a “smaller” object also closed for delays.

Proposition 7.11 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, $\Lambda' \subseteq \Lambda_{\Sigma'}$ and $\Lambda' = (\Lambda')^\emptyset$. Then, $\sigma^{-1}(\Lambda')$ is closed for delays.

Proof:

Note first that if $\lambda = \sigma^{-1}(\lambda')$ for some $\lambda' \in \Lambda'$ then $\Delta^k(\lambda) = \sigma^{-1}(\Delta^k(\lambda'))$ and thus $\Delta^k(\lambda) \in \sigma^{-1}(\Lambda')$ since $\Delta^k(\lambda') \in \Lambda'$. Now assume that $\lambda \in \sigma^{-1}(\Lambda')$, $\theta \in \Lambda_\Sigma$ and $\lambda \infty \theta$. We have to show that $\theta \in \Lambda_\Sigma$. Since $\lambda \infty \theta$ there is a delay-point sequence η such that for each $n \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $[\theta]_n = [\delta^n(\lambda)]^i_n$. Moreover, there is $\lambda' \in \Lambda'$ such that $\lambda = \sigma^{-1}(\lambda')$. Thus, $\delta^n(\lambda)^i = \sigma^{-1}(\delta^n(\lambda')^i) \in \sigma^{-1}(\Lambda')$ for each $i \in \mathbb{N}$. Hence, there is $\theta' \in \Lambda_{\Sigma'}$ such that $\theta = \sigma^{-1}(\theta')$. Indeed, otherwise for some $n \in \mathbb{N}$ there would be no $s' \subseteq A_{\Sigma'}$ such that $\theta_n = \sigma^{-1}(s')$, which is absurd because for each $n \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $\theta_n = (\delta^n(\lambda)^i)_n$. Now we show that $\lambda' \infty \theta'$. Consider again the delay-point sequence η . It is straightforward to verify that for each $n \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $[\theta']_n = [\delta^n(\lambda')]^i_n$. Thus, $\theta' \in \Lambda'$ and so $\theta \in \sigma^{-1}(\Lambda')$. QED

Now we show that the image of a set of life-cycles closed for delays is also closed for delays.

Proposition 7.12 Let $\sigma : \Sigma \rightarrow \Sigma'$ be a signature morphism, $\Lambda \subseteq \Lambda_\Sigma$ and $\Lambda = \Lambda^\emptyset$. Then, $\sigma(\Lambda)$ is closed for delays.

Proof:

Assume that $\lambda' \in \sigma(\Lambda)$ and $\lambda' \infty \theta'$. We have to show that $\theta' \in \sigma(\Lambda)$. Since $\lambda' \infty \theta'$ there is a delay-point sequence η such that for each $n \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $[\theta']_n = [\delta^n(\lambda')]^i_n$. It is straightforward to verify that $\sigma^{-1}(\lambda') \in \Lambda$ since $\lambda' \in \sigma(\Lambda)$. Therefore, $\sigma^{-1}(\lambda') \in \Lambda$ and so $\sigma^{-1}(\lambda') \in \Lambda$. Moreover, $\sigma^{-1}(\lambda') \in \Lambda$ since $\sigma^{-1}(\lambda') \in \Lambda$. Therefore, $\theta' \in \sigma(\Lambda)$. QED

Finally, we show that the product of sets of life-cycles that are closed for delays is also closed for delays. This means, that when we aggregate two objects with “fairness” then the aggregation will keep the “fairness”.

Proposition 7.13 Let ob and ob' be objects closed for delays. Then $ob \times ob'$ is closed for delays.

Proof:

Let $ob = \langle \Sigma, \Lambda \rangle$ and $ob' = \langle \Sigma', \Lambda' \rangle$, $\theta \in \Lambda \times \Lambda'$ and $\theta \infty \theta'$. Then, using the same

delay-point sequence, we obtain $inj_{\Sigma}^{-1}(\theta) \infty inj_{\Sigma}^{-1}(\theta')$ and $inj_{\Sigma'}^{-1}(\theta) \infty inj_{\Sigma'}^{-1}(\theta')$.
 QED

Let ob and ob' be objects closed for delays. Then, the set of life-cycles of $ob \times ob'$ is the set of fair interleavings of life-cycles of ob and ob' .

Example 7.14 As an illustration consider the following objects and their product: $ob = \langle \langle \emptyset, \{u\} \rangle, \{\lambda\}^{\emptyset} \rangle$, $ob' = \langle \langle \emptyset, \{v\} \rangle, \{\xi\}^{\emptyset} \rangle$ where $\lambda_n = \{\Diamond u, \nabla u\}$ and $\xi_n = \{\Diamond v, \nabla v\}$ for each $n \in \mathbb{N}$. It is easy to verify that any fair interleaving of λ and ξ is a life-cycle of the product $ob \times ob'$. By fair interleaving we mean an interleaving of actions u and v where there is an infinite number of u 's and v 's. If we dropped the diligence condition, we would obtain all interleavings of u 's and v 's including those life-cycles where action u does not occur or occurs a finite number of times (and similarly for action v). Clearly, the diligence condition ensures fairness in the aggregation (parallel composition) of objects.

It should be obvious that the product of two objects that are not closed for delays may not include the envisaged interleavings. For instance the product of $ob = \langle \Sigma, \{\lambda\} \rangle$ and $ob' = \langle \Sigma', \{\xi\} \rangle$ contains a unique life-cycle: total synchronization.

7.4 Reducts for linear temporal logic

It is also worthwhile to examine the *reducts* in the category Ob built upon L . We explain how the reduct may be used for “hiding” observations/actions. Moreover, we show with an example that it may lead to “internal non-determinism”.

Example 7.15 Consider the object $ob = \langle \langle \{x\}, \{c\} \rangle, \{\lambda\}^{\emptyset} \rangle$ such that $\lambda_n = \{x, \Diamond c, \nabla c\}$ for n even and $\lambda_n = \{\Diamond c, \nabla c\}$ for n odd. Note that ob is “deterministic” in the sense that when a given event (set of actions) happens if we know the “value” of x (true or false) then we know the next situation: from true to false and from false to true if $\{c\}$ happens; and from true to true and from false to false if \emptyset happens.

Consider now the reduct induced by the hiding signature morphism $\sigma : \langle \{x\}, \emptyset \rangle \rightarrow \langle \{x\}, \{c\} \rangle$ that hides the action c . Then, $\sigma^{-1}(ob) = \langle \langle \{x\}, \emptyset \rangle, \{\xi\}^{\emptyset} \rangle$ where $\xi_n = \{x\}$ for n even and $\xi_n = \emptyset$ for n odd. The resulting object is “non-deterministic” since when the event \emptyset happens (no action occurs) the value of x may remain unchanged or it may change (for no apparent reason).

In conclusion: the reduct mechanism used for hiding may introduce internal “non-determinism”.

8 Concluding remarks

We started by showing how to set up the “power model” institution of objects canonically induced by any given base temporal institution fulfilling some rather weak requirements. Then, we established some strong general properties of a semantic functor mapping each specification to a final object. Finally, we

concluded that those properties were sufficient to obtain in every case a denotational (and categorial) semantics of several typical constructions in object specifications, such as aggregation, interconnection, abstraction and (monotonic) specialization.

We analysed in detail the special case of a base linear temporal logic. It is possible to adopt instead a multilinear temporal logic or a full branching temporal logic as the base logic. It is up to the specifier the choice of the most convenient base logic. In any case (as long as the requirements on the base logic are fulfilled), the envisaged categorial, denotational semantics of the object constructs is obtained as desired, according to the chosen base institution.

Such non-linear temporal logics are quite powerful. For instance, they provide the means for expressing a deep relationship between the enabling and the occurrence of an action. Beyond $(\nabla c \Rightarrow \diamond c)$ already expressible in the linear temporal language, we may want to impose $(\diamond c \Rightarrow (\mathbf{M}\nabla \mathbf{c}))$ stating (in the multilinear logic) that if action c is enabled in position n of a given trajectory t then there is a trajectory t' identical to t up to n and with the same observations and enablings at n where c occurs at n . In short, informally, if c is enabled then c happens in an alternative trajectory. In that case, an object is not any more a set of life-cycles, but instead a set of sets of life-cycles in the case of a multilinear base or a set of trees in the case of a branching base. In both these cases, the base institution is not any more “uniframe”: different interpretation structures may have different frames, that is, they may differ in the set of worlds and/or in the accessibility relations.

An even more interesting alternative is to consider a base institution that may lead to categories of objects with products reflecting “full concurrency”. Work on this direction is reported in [19] and [12], considering event structures [52, 53] as basic interpretation structures and their associated temporal logics [31, 32, 37]. Another possibility would be to consider the temporal logic proposed in [41].

Please note that, whichever is the base logic (institution) being considered, for each signature Σ , every interpretation structure $m \in |Int(\Sigma)|$ is a model of the initial theory $\langle \Sigma, \emptyset^F \rangle$. Therefore its semantics is defined to be the whole $|Int(\Sigma)|$. Clearly, it is a set since $Int(\Sigma)$ is a small category. Remarkably, this is usually the case whenever the base logic is uniframe, but we may get into trouble if that is not the case. For instance, tree structures form a proper class. Therefore, some care must be taken when setting up an institution for branching temporal logic (e.g., restricting to well-founded, countable-branching trees modulo renaming). The same applies to event structures [12]. Alternatively, we might work with “large” institutions over “large” categories and classes of models [33].

The proposed semantics of object specification constructs can of course be used when starting with a full-fledged temporal specification logic as the base institution. For examples of such specification logics see [48, 49], where techniques for reasoning about communities of objects are presented.

Clearly, the proposed construction and its properties do not depend on the fact that the base logic is modal. However, the investigation of the special case is well justified by the envisaged domain of application: denotational semantics of

temporal specifications of objects. Preliminary results show that the approach is still applicable when starting with a non modal logic of behaviour like the situation calculus [40].

Another important research issue concerns the semantics of dynamic reconfiguration constructs: objects are created, deleted, dynamically specialised, and their interconnections are changed. It is not clear at this stage how suitable are traditional temporal languages for this purpose. And in any case much work is needed in the semantic front. The reification of objects (for instance via action refinement) also deserves attention. Some preliminary ideas within the proposed setting can be found in [18, 16].

We are working in establishing a suitable categorical link (adjunction) between the proposed denotational semantics of objects and an operational semantics (such as the one described in [44]). Work on this subject is reported in [11, 13].

Finally, it is worthwhile to note that in the proposed approach each object appears as a set of models of the base institution. An interesting alternative would be to postulate that an object is a sheaf [25]. Relevant work in this direction is reported in [17].

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Appendix

Fiber

Definition A.1 Let $F : C \rightarrow X$ be a functor and x an element of X . The *fiber* C_x of C over x is the subcategory of C whose elements are mapped by F to x and whose arrows are mapped to the identity morphism at x (id_x).

Concrete category

Definition A.2 Let X be a category. A *concrete category* over X is a pair $\langle C, U \rangle$ where C is a category and $U : C \rightarrow X$ is a faithful functor.

Definition A.3 Let $\langle C, U \rangle$ be a concrete category over X . We say that a morphism $m : U(c) \rightarrow U(c')$ in X *induces* a morphism in C iff there is a morphism $h : c \rightarrow c'$ in C such that $U(h) = m$. If there is such a morphism it is unique.

$$\begin{array}{ccc}
C & \xrightarrow{U} & X \\
c & & U(c) \\
\vdots & & \downarrow m=U(h) \\
c' & & U(c')
\end{array}$$

Figure 10: The morphism h is induced by m .

Definition A.4 A concrete category $\langle C, U \rangle$ over X is said to be *amnesitic* iff C_x is a partial order for each element x of X .

Definition A.5 Let $\langle C, U \rangle$ and $\langle D, V \rangle$ be concrete categories over X . A *concrete functor* from $\langle C, U \rangle$ into $\langle D, V \rangle$ is a functor $F : C \rightarrow D$ such that $U = V \circ F$.

$$\begin{array}{ccc}
C & \xrightarrow{F} & D \\
U \searrow & & \swarrow V \\
& X &
\end{array}$$

Figure 11: A concrete functor $F : \langle C, U \rangle \rightarrow \langle D, V \rangle$.

Cofibration (and fibration)

Definition A.6 Let $F : C \rightarrow X$ be a functor, $m : x \rightarrow x'$ a morphism in X and c an element of C such that $F(c) = x$. A morphism $f : c \rightarrow c'$ in C is said to be *cocartesian* by F for m on c iff $F(f) = m$ and for each pair of morphisms $g : c \rightarrow c''$ in C and $n : x' \rightarrow x''$ with $x'' = F(c'')$ in X such that $n \circ m = F(g)$ there is a unique morphism $h : c' \rightarrow c''$ in C such that $F(h) = n$ and $h \circ f = g$. We say that c' is the *target* of the cocartesian morphism f .

Proposition A.7 Let $F : C \rightarrow X$ be a functor, $m : x \rightarrow x'$ a morphism in X and $F(c) = x$. If f_1 and f_2 are cocartesian morphisms by F for m on c then their targets are isomorphic.

Proof:

Straightforward: use the couniversal property of f_1 on f_2 and itself (and vice-versa). QED

$$\begin{array}{ccc}
C & \xrightarrow{F} & X \\
\\
\begin{array}{ccc}
c & \xrightarrow{f} & c' \\
\searrow g & \text{=} & \downarrow h \\
& & c''
\end{array} & & \begin{array}{ccc}
x & \xrightarrow{m} & x' \\
\searrow F(g) & \text{=} & \downarrow n \\
& & x''
\end{array}
\end{array}$$

Figure 12: The morphism f is cocartesian by F for m on c .

Proposition A.8 Let $F : C \rightarrow X$ be a functor, $m_1 : x \rightarrow x_1$ and $m_2 : x_1 \rightarrow x_2$ morphisms in X , and $F(c) = x$. If $f_1 : c \rightarrow c_1$ is cocartesian by F for m_1 on c , and $f_2 : c_1 \rightarrow c_2$ is cocartesian by F for m_2 on c_1 then $f_2 \circ f_1 : c \rightarrow c_2$ is cocartesian by F for $m_2 \circ m_1$ on c .

Proof:

Straightforward: assume that $m_1 : x \rightarrow x_1$ and $m_2 : x_1 \rightarrow x_2$ are morphisms in X , $f_1 : c \rightarrow c_1$ is cocartesian by F for m_1 on c , and $f_2 : c_1 \rightarrow c_2$ is cocartesian by F for m_2 on c_1 . We show that $f_2 \circ f_1$ is cocartesian by F for $m_2 \circ m_1$ on c . Let $g : c \rightarrow c_3$ be any morphism in C and $n_2 : x_2 \rightarrow x_3$ be any morphism in X such that $F(c_3) = x_3$ and $n_2 \circ (m_2 \circ m_1) = F(g)$; consider the morphism $n_1 : x_1 \rightarrow x_3$ such that $n_1 = n_2 \circ m_2$; thus, $F(g) = n_1 \circ m_1$; hence, there is a unique morphism $h_1 : c_1 \rightarrow c_3$ such that $F(h_1) = n_1$ and $h_1 \circ f_1 = g$; on the other hand, there is a unique morphism $h_2 : c_2 \rightarrow c_3$ such that $F(h_2) = n_2$ and $h_2 \circ f_2 = h_1$; finally we may conclude that h_2 is the unique morphism such that $F(h_2) = n_2$ and $h_2 \circ (f_2 \circ f_1) = g$. QED

Definition A.9 A functor $F : C \rightarrow X$ is said to be a *cofibration* iff there is a cocartesian morphism by F for m on c for every morphism $m : x \rightarrow x'$ in X and every c of C such that $F(c) = x$.

Definition A.10 Let $F : C \rightarrow X$ be a cofibration. A *cocleavage* k for F maps each pair $\langle m : F(c) \rightarrow x', c \rangle$ to a cocartesian morphism by F for m on c . A cocleavage is said to be a *splitting* of the cofibration iff

- $k(\langle id_{F(c)}, c \rangle) = id_c$;
- $k(\langle m' \circ m : F(c) \rightarrow x'', c \rangle) = k(\langle m' : F(c') \rightarrow x'', c' \rangle) \circ k(\langle m : F(c) \rightarrow F(c'), c \rangle)$.

Prop/Definition A.11 Let $F : C \rightarrow X$ be a cofibration. Any splitting cocleavage k for F induces the functors:

- for each $m : x \rightarrow x'$ in X , the *transformation functor* on m , $T_m : C_x \rightarrow C_{x'}$, such that $T_m(c)$ is the target of the cocartesian morphism $k(\langle m : F(c) \rightarrow x', c \rangle)$, and $T_m(h : c_1 \rightarrow c_2)$ is the unique morphism h' in $C_{x'}$ such that $h' \circ k(\langle m : F(c_1) \rightarrow x', c_1 \rangle) = k(\langle m : F(c_2) \rightarrow x', c_2 \rangle) \circ h$;

- the *indexing functor* $I : X \rightarrow \mathit{Cat}$ such that $I(x)$ is the fiber C_x and $I(m : x \rightarrow x')$ is T_m .

Proof:

See pages 255–256 of [8]. Note that the splitting property is essential only for establishing the indexing functor. QED

Note that if the cofibration functor is $F : C \rightarrow X^{op}$, then $T_m : C_{x'} \rightarrow C_x$ and $I : X \rightarrow \mathit{Cat}^{op}$.

By duality we introduce the notion of *fibration* and obtain the corresponding results.

Concrete isomorphism

Definition A.12 Assume that $H : C \rightarrow D$ and $K : D \rightarrow C$ establish an isomorphism in Cat and that $F : C \rightarrow X$ and $G : D \rightarrow X$ are functors. The isomorphism is said to be *concrete* with respect to F and G iff $G \circ H = F$ and $F \circ K = G$.

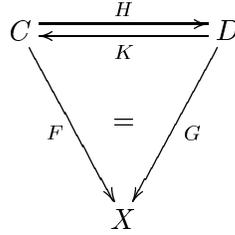


Figure 13: A concrete isomorphism.

Proposition A.13 Assume that $H : C \rightarrow D$ and $K : D \rightarrow C$ establish a concrete isomorphism in Cat with respect to $F : C \rightarrow X$ and $G : D \rightarrow X$. Then H preserves cocartesian and cartesian morphisms.

Proof:

- Let $m : x \rightarrow x'$ be a morphism in X and c an element of C such that $F(c) = x$. Assume that $f : c \rightarrow c'$ is cocartesian by F for m on c . We want to show that $H(f) : H(c) \rightarrow H(c')$ is cocartesian by G for m on $H(c)$. Clearly, $G(H(f)) = m$. We have to verify that this morphism satisfies the envisaged couniversal property. Let $g : H(c) \rightarrow h''$ be a morphism in D , and $n : x' \rightarrow G(d'')$ a morphism in X such that $n \circ m = G(g)$. Consider the morphism $K(g) : c \rightarrow K(d'')$ in C . Clearly, $n \circ m = F(K(g)) = G(g)$. By the couniversal property of $F : c \rightarrow c'$, there is a unique morphism $h : c' \rightarrow K(d'')$ such that $h \circ f = K(g)$. Consider the morphism $H(h) : H(c') \rightarrow d''$ in D . Then, $H(h) \circ H(f) = H(h \circ f) = H(K(g)) = g$ and, clearly, $G(H(h)) = n$. Moreover, $H8H$ is the unique morphism such that $G(H(h)) = n$ and $H(h) \circ H(f) = g$. Otherwise, we would be able to violate the couniversal property of f mapping back via K .

- Preservation of cartesian morphisms follows by duality.

QED

Galois correspondence

Definition A.14 Let $\langle C, U \rangle$ and $\langle D, V \rangle$ be concrete categories over X and $H : C \rightarrow D$ and $K : D \rightarrow C$ concrete functors over X . The pair $\langle H, K \rangle$ is said to be a *Galois correspondence* iff, for any c of C and d of D , a morphism $f : U(c) \rightarrow V(d)$ in X induces a morphism in C from c into $K(d)$ iff f induces a morphism in D from $H(c)$ into d .

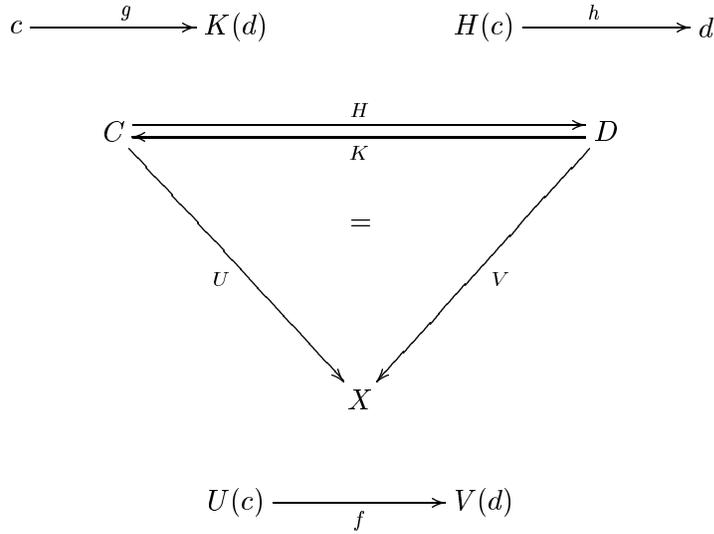


Figure 14: Galois correspondence $\langle H, K \rangle$ between $\langle C, U \rangle$ and $\langle D, V \rangle$.

Definition A.15 Let $H : C \rightarrow D$ be a functor. We denote by $H(|C|)$ the full subcategory of D containing all elements of D in the range of H .

Proposition A.16 Let $\langle C, U \rangle$ and $\langle D, V \rangle$ be amnestic concrete categories over X and $\langle H : C \rightarrow D, K : D \rightarrow C \rangle$ a Galois correspondence. The restrictions of H and K to $H(|C|)$ and $K(|D|)$, respectively, are concrete isomorphisms $H^* : H(|C|) \rightarrow K(|D|)$ and $K^* : K(|D|) \rightarrow H(|C|)$, inverse to each other.

Proof:

See page 83 of [6].

QED

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