On Y-compatible and strict Y-compatible functions *

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Abstract

Let $Y \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^k$ *Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z \leq Y$ if and only if $f(Z) \leq f(Y)$ and is *strict Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z < Y$ if and only if $f(Z) < f(Y)$. It is proved that for any $Y \in \mathbb{R}^n$, $n \geq 2$, there is no Y-compatible polynomial function $f : \mathbb{R}^n \to \mathbb{R}^k$, $1 \leq k < n$. It is also proved that for a strict Y-compatible map f, $J_f(Y) = 0$, where $J_f(Y)$ denote the Jacobian matrix of the mapping f in Y. These problems arose in studying data compression of analog signatures.

1 Introduction

This work was initiated by the problems of storage and processing of measured response data of analog circuits normally used by the fault dictionary techniques in fault localization [1], [2]. We explore the possibility of data compression of a series of real numbers representing given response data. In particular, we are looking for some data compression function that would enable us to determine for any two given responses y_1, y_2, ..., y_n and z_1, z_2,

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$\ldots, z_n$ whether $z_i \leq y_i$ holds for all i merely on the basis of their compressed response data (i.e., signatures). If such data compression function existed, regions that characterize the response of a circuit could be simply described by the signatures of their margins. Besides, it would also be possible to determine from the signature of the response if the operation of a circuit-under-test lies in the given region or not. More details on the state-of-the-art and practical background on this issue are given in [3].

The terminology and notions used in this note in general follow [4]. Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two vectors from $\mathbb{R}^n$. Then $X < Y$ means that $x_i < y_i$ holds for all i, and $X \leq Y$ means that $x_i \leq y_i$. Let $Y \in \mathbb{R}^n$. We call a function $f : \mathbb{R}^n \to \mathbb{R}^k$ *Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z \leq Y$ if and only if $f(Z) \leq f(Y)$. A function $f : \mathbb{R}^n \to \mathbb{R}^k$ is *strict Y-compatible*, if for any $Z \in \mathbb{R}^n$, $Z < Y$ if and only if $f(Z) < f(Y)$. Note that when we talk about Y-compatibility and strict Y-compatibility, Y is an arbitrary but fixed vector.

The rest of this note is organized as follows. In the next section we consider Y-compatible functions and prove that there are no such polynomial function when the dimension of the codomain is smaller than the one of the domain. In Section 3 we prove that for a strict Y-compatible map f, the Jacobian matrix of the mapping f in Y is equal to the zero matrix.

2 Compatible functions

Consider, for example, the following function $g : \mathbb{R}^n \to \mathbb{R}$:

$$g(X) = \begin{cases} (-1)^{n+1}(x_1 - y_1)(x_2 - y_2)\cdots(x_n - y_n); & X \leq Y, \\ \max\{x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\}; & \text{otherwise.} \end{cases}$$

It is easy to see that g is continuous and Y-compatible. Note that the definition of g involves Y and furthermore we would like that a Y-compatible function is a polynomial. In the rest of the section we are going to show that there are no such functions for $k < n$. For the proof we need the following simple observation.

Lemma 2.1 *Let p be a real polynomial of n variables. If there exists a neighbourhood $\mathcal{U} \subset \mathbb{R}^n$ such that $p(X) = c$ on $\mathcal{U}$ then $p \equiv c$.*

Proof. As p is a constant function on $\mathcal{U}$, all partial derivatives of p are equal to 0 on $\mathcal{U}$. Since the expansion of p into the Taylor series is finite, the result follows. □

Theorem 2.2 *For any $Y \in \mathbb{R}^n$, $n \geq 2$, there is no Y-compatible polynomial function $f : \mathbb{R}^n \to \mathbb{R}^k$, $1 \leq k < n$.*

Proof. For $m < n$ let $Y_m = (y_m, y_{m+1}, \ldots, y_n)$ and let $S = \{Z \in \mathbb{R}^{n-1};\ Z < Y_2\}$.

Suppose on the contrary that $f : \mathbb{R}^n \to \mathbb{R}^k$ is Y-compatible, $f = (f_1, f_2, \ldots, f_k)$. We claim that at least one of the following holds:

(i) there exist $j \in \{1, 2, \ldots, k\}$, $Z \in S$, and a neighbourhood $\mathcal{U} \subset S$ of Z in $\mathbb{R}^{n-1}$ such that

$$f_j(y_1, X) - f_j(Y) = 0, \quad \text{for every } X \in \mathcal{U}.$$

(ii) there exists $X \in \mathbb{R}^{n-1}$, $X < Y_2$, such that for all $i = 1, 2, \ldots, k$,

$$f_i(y_1, X) - f_i(Y) < 0.$$

Suppose that (i) is not true. Let $Z \in S$, $Z < Y_2$, and let $\mathcal{U}_1 \subset S$ be a neighbourhood of Z. As (i) does not hold, there is $X_1 \in \mathcal{U}_1$, such that $f_1(y_1, X_1) - f_1(Y) < 0$. As f is continuous, there is a neighbourhood $\mathcal{U}_2$ of X_1 such that $\mathcal{U}_2 \subset \mathcal{U}_1$ and $f_1(y_1, W) - f_1(Y) < 0$ for every $W \in \mathcal{U}_2$. Then in $\mathcal{U}_2$ we can choose a vector X_2 such that $f_2(y_1, X_2) - f_2(Y) < 0$. If we proceed in this manner we obtain nested neighbourhoods $\mathcal{U}_k \subset \mathcal{U}_{k-1} \subset \cdots \subset \mathcal{U}_1$. Finally we choose $X \in \mathcal{U}_k$ such that $f_k(y_1, X) - f_k(Y) < 0$. Since $\mathcal{U}_k \subset \mathcal{U}_i$ for $i = 1, 2, \ldots, k-1$, we also have $f_i(y_1, X_2) - f_i(Y) < 0$. Furthermore, as $X \in \mathcal{U}_1 \subset S$, it follows $X < Y$. Thus (ii) holds which proves the claim.

Suppose that (ii) holds. Then $f(y_1 + \delta, X) \leq f(Y)$ for $\delta > 0$ small enough, therefore f is not Y-compatible. Therefore (i) must hold. Thus there exists j such that $f_j(y_1, X) - f_j(Y) = 0$, for each X in some neighbourhood in $S \subset \mathbb{R}^{n-1}$. Consider $f_j(y_1, X) - f_j(Y)$ as a function of variables $x_2, x_3, \ldots, x_n$. By Lemma 2.1 we have

$$f_j(y_1, X) - f_j(Y) = 0, \quad X \in \mathbb{R}^{n-1}. \tag{1}$$

Set $g_i(Z) = f_i(y_1, Z)$, for $Z \in \mathbb{R}^{n-1}$ and $i = 1, 2, \ldots, k$. Note that for all i, $g_i(Y_2) = f_i(Y)$, and by (1), $g_j(Z) = g_j(Y_2)$. We next want to show that Y-compatibility of f implies Y_2-compatibility of $g : \mathbb{R}^{n-1} \to \mathbb{R}^{k-1}$, $g = (g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_k)$. Let $Z \in \mathbb{R}^{n-1}$ and let $Z \leq Y_2$. Then $(y_1, Z) \leq Y$ and $f(y_1, Z) \leq f(Y)$. We conclude $g(Z) \leq g(Y_2)$. Conversely, assume that $g(Z) \leq g(Y_2)$, i.e., $f_i(y_1, Z) \leq f_i(Y)$ for $i \neq j$. Since by (1)

for $i = j$, $f_j(y_1, Z) = f_j(Y)$, we have $f(y_1, Z) \leq f(Y)$ which in turn implies $(y_1, Z) \leq Y$ and therefore $Z \leq Y_2$.

By the above argument, an existence of Y-compatible function $f : \mathbb{R}^n \to \mathbb{R}^k$ implies an existence of Y_2-compatible function $g : \mathbb{R}^{n-1} \to \mathbb{R}^{k-1}$. We continue this procedure and finally end up with a $Y_k = (y_k, y_{k+1}, \ldots, y_n)$-compatible real mapping h. Again, at least one of (i) and (ii) holds for h. If (ii) holds then as above h is not Y_k-compatible. And if $h(y_k, X) - h(Y_k) = 0$ holds for $X \in \mathbb{R}^{n-k}$, we can easily choose X, which violates the compatibility. This final contradiction completes the proof. $\square$

Note that the theorem is clearly no longer true for $k = n$, as can be seen by considering the identity map. We also point out that by similar, but a little more careful, argument one can obtain the same result also for rational functions.

3 Strict-compatible functions

The function g defined in Section 2 can be made strict Y-compatible simply by changing $X \leq Y$ to $X < Y$. For another example consider the following function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(X) \;=\; \begin{cases} -\prod_{i=1}^n (x_i - y_i)^2; & X < Y, \\ \prod_{i=1}^n (x_i - y_i)^2; & \text{otherwise.} \end{cases}$$

It is easy to see that h is continuous, differentiable and strict Y-compatible. However, h can not be made Y-compatible. Note also that $\dfrac{\partial h}{\partial x_i}(Y) = 0$ for all i.

In the rest of this section we assume that all functions are differentiable. Let $J_f(Y)$ be the Jacobian matrix of the mapping f in Y.

Lemma 3.3 *Let $Y \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^k$ be strict Y-compatible. Then for $D = J_f(Y)$ we have:*

(i) for all $H \in \mathbb{R}^n$, $H < 0$ implies $DH \leq 0$,

(ii) for all $H \in \mathbb{R}^n$, $DH < 0$ implies $H < 0$.

Proof. As f is differentiable, for any $H \in \mathbb{R}^n$ we can write

$$f(Y + H) = f(Y) + DH + ||H||\, G(H),$$

where

$$\lim_{||H||\to 0} G(H) = 0.$$

For any $t > 0$ we thus have

$$f(Y + tH) = f(Y) + t(DH + ||H||\, G(tH)). \tag{2}$$

Since f is strict Y-compatible, for any $H < 0$ (2) gives

$$0 > f(Y + tH) - f(Y) = t(DH + ||H||\, G(tH)).$$

Now if (i) would not hold, say $(DH)_i > 0$ for some i, then for t small enough, the i-th component of $DH + ||H||\, G(tH)$ would still be positive, a contradiction. Thus (i) holds.

Suppose next $DH < 0$. Then by (2) $f(Y + tH) - f(Y) < 0$ holds for t small enough. Since f is strict Y-compatible, $tH < 0$. But this implies $H < 0$. □

Lemma 3.4 *Let D be a $k \times n$ matrix, $k < n$, and let $K \in \mathbb{R}^n$ be a vector such that $DK < 0$. Then there exists $H \in \mathbb{R}^n$, such that $DH < 0$ and $H \not< 0$.*

Proof. Let $DK = W < 0$. Then $DH = W$ can be interpreted as a consistent system of k linear equations with n variables $h_1, h_2, \ldots, h_n$. Let $m = \mathrm{rank}(D)$. Since $m \leq k < n$, there exists an $m \times m$ submatrix with nontrivial determinant. We may without loss of generality suppose that this is the upper left submatrix. Then the system $DH = W$ is equivalent to the system

$$h_i = g_i(h_{m+1}, h_{m+2}, \ldots, h_n, w_1, w_2, \ldots, w_k), \quad i = 1, 2, \ldots, m.$$

Thus we can arbitrarily choose components $h_{m+1}, h_{m+2}, \ldots, h_n$ and then calculate components $h_1, h_2, \ldots, h_m$. In this way we can obtain H such that $DH < 0$ but $H \not< 0$. □

Theorem 3.5 *For a strict Y-compatible map f, $J_f(Y) = 0$.*

Proof. By Lemma 3.3 (i), $DH \leq 0$ for any $H < 0$. If for some H, $DH < 0$, then Lemma 3.4 contradicts 3.3 (ii). Thus $DH = 0$ for all $H < 0$. Let now H be an arbitrary vector and define vectors H' and H'' in the following way:

$$H'_i = \begin{cases} -H_i; & H_i > 0, \\ 2H_i; & H_i < 0, \\ -1; & H_i = 0. \end{cases}$$

and

$$H_i'' = \begin{cases} H_i; & H_i < 0, \\ -2H_i; & H_i > 0, \\ 1; & H_i = 0. \end{cases}$$

Then $H' < 0$ and $H'' < 0$. Note also that $H = H' - H''$. Therefore $DH = D(H' - H'') = D(H') - D(H'') = 0$. Hence $DH = 0$ for any $H \in \mathbb{R}^n$ which is only possible if $J_f(Y) = 0$. $\square$

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