On the logic of merging

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Abstract

This work proposes an axiomatic characterization of merging operators. It underlines the differences between arbitration operators and majority operators. A representation theorem is stated showing that each merging operator corresponds to a family of partial preorders on interpretations. Examples of operators are given. They show the consistency of the axiomatic characterization. A new merging operator $\triangle_{GMax}$ is provided. It is proved that it is actually an arbitration operator.

1 Introduction

In a growing number of applications, we face conflicting information coming from several sources. The problem is to reach a coherent piece of information from these contradicting ones. A lot of different merging methods have already been given [BI84, LMa, BKM91, BKMS92, Sub94]. Instead of giving one particular merging method we propose, in this paper, a characterization of such methods following the rationality of the postulates they satisfy. We shall call merging operators those methods that obey a minimal set of rational merging postulates. Then we shall investigate two subclasses of merging operators: arbitration operators and majority operators.

Merging operators are useful in a lot of applications: to find a coherent information in a distributed database system, to solve a conflict between several people or several agents, to find an answer in a decision-making committee, to take a decision when information given by some captors is contradictory, etc.

This work is related to the AGM (Alchourrón, Gärdenfors, Makinson) framework of revision theory [AGM85, Gär88, KM91]. Revision is the process of according a knowledge base in the view of a new evidence. One basic assumption of revision is that the new information is more reliable that the knowledge base, but it is not always the case. We can distinguish 3 cases:

- *The new piece of information is more reliable than the knowledge base:* it is the assumption made in the revision theory so we can revise our knowledge base by the new piece of information.
- *The new piece of information is less reliable than the knowledge base:* a drastic point of view could be to ignore this unreliable piece of information but if we want to be more constructive we can take this piece of information into account if it is consistent with the knowledge base and ignore it only if it is inconsistent with our belief. Another interesting way would be to reverse the revision, i.e. to revise the new piece of information by the knowledge base.
- *The new piece of information is as reliable as the knowledge base:* here we can't give the preference to one of the two items of knowledge, so we have to find something else. This is the aim of merging operators.

The intuitive difference between arbitration and majority operators is that arbitration operators reach a consensus between the protagonists' views by trying to satisfy as much as possible all the protagonists, whereas majority operators elect, in a sense, the result of the merging by taking the majority into account. In other words arbitration operators try to minimize individual dissatisfaction, whereas majority operators try to minimize global dissatisfaction. One of our main concerns in this work is to state these intuitions in a formal way.

Some operators quite close to merging operators have already been formally studied. Revesz defined in [Rev93, Rev97] model-fitting operators which can be considered as a generalization of revision for multiple knowledge bases. Revesz also defined arbitration operators from model-fitting operators. We make a criticism about Revesz's postulates: they do not distinguish between majority and arbitration.

Liberatore and Schaerf have proposed postulates to characterize arbitration [LS95, LS98]. Their definition has a strong connection with revision operators, but the major drawback, in our opinion, is that those operators arbitrate only two knowledge bases. Furthermore they select some interpretations in the two knowledge bases as the result of the arbitration. We consider that we can't ignore interpretations which do not belong to these knowledge bases, consider the following example:

Example 1 Suppose that we want to speculate on the stock exchange. We ask two financial experts about four shares A,B,C,D. We denote 1 if the share rises and 0 if it falls (we suppose that its value can't be stable). These agents have the same expert level and so they are both equally reliable. The first one says that all the shares will rise: $\varphi_1 = \{(1,1,1,1)\}$, the second one thinks that all the shares will fall: $\varphi_2 = \{(0,0,0,0)\}$. The Liberatore and Schaerf operators will arbitrate these opinions and give the following result: $R = \{(0,0,0,0),(1,1,1,1)\}$. So it means that either φ_1 is totally wrong or it's φ_2 who is completely mistaken. But intuitively, if the two experts are equally reliable, there is no reason to think that one of them has failed more than the other: they both have to be at the same "distance" of the truth. So they are certainly both wrong on two shares and the result has to be: $R^{'} = \{(0,0,1,1),(0,1,0,1),(0,1,1,0),(1,0,0,1),(1,0,1,0),(1,1,0,0)\}$. So two of the shares will rise and two will fall but we don't know which ones.

In our opinion Liberatore and Schaerf's operators have to be seen as selection operators and have to be used in applications which require the result be one of the possibilities given by the protagonists. For example, if the result of the arbitration is a medical treatment, we can't "merge" several therapies and so we have to use Liberatore and Schaerf operators. Liberatore and Schaerf's operators take, in a sense, the interpretation as unit of change, we propose to take the propositional variable as such a unit, as Dalal says in [Dal88]: *"Change in truth value of a single symbol can be considered as the smallest unit of change"*, we want to apply this to arbitration.

Lin and Mendelzon proposed a *theory merging by majority* operator [LMa, Lin96] which solves conflicts between knowledge bases by taking the majority into account. Their *theory merging operators* are what we call majority operators.

The paper is organized as follows: in section 2 we give some definitions and state some notations. In section 3 we propose postulates for merging operators, majority operators and arbitration operators and we study the relationships between some of the postulates. In section 4 we give a model-theoretic characterization of those operators. In section 5 we give some examples of merging operators, especially we show that an operator, called $\triangle_{GMax}$, is an arbitration operator. Finally, in section 6 we give some conclusions and discuss open problems.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional letters. An interpretation is a function from $\mathcal{P}$ to $\{0,1\}$. The set of all the interpretations is denoted $\mathcal{W}$. An interpretation I is a model of a formula if and only if it makes it true in the usual classical truth functional way. Let φ be a formula, $Mod(\varphi)$ denote the set of models of φ. And let M be a set of interpretations, $form(M)$ denote a formula which set of models is M. When $M = \{I\}$ we will use the notation $form(I)$ for reading convenience.

A *knowledge base* K is a finite set of propositional formulae which can be seen as the formula φ which is the conjunction of the formulae of K. By abuse, we will use K to denote the formula φ. We will note K_I a knowledge base the sole model is I.

Let $K_1, \ldots, K_n$ be n knowledge bases (not necessarily different). We call *knowledge set* the multi-set E consisting of those n knowledge bases: $E = \{K_1, \ldots, K_n\}$. We note $\bigwedge E$ the conjunction of the knowledge bases of E, i.e. $\bigwedge E = K_1 \wedge \cdots \wedge K_n$. The union of multi-sets will be noted $\sqcup$.

Remark 2 *Since an inconsistent knowledge base gives no information for the merging process, we'll suppose in the rest of the paper that the knowledge bases are consistent.*

$\mathcal{K}$ will denote the set of consistent knowledge bases and $\mathcal{E}$ will denote the set of non empty finite multi-sets with elements in $\mathcal{K}$.

Let's denote $\mathcal{S}$ the set of sets of interpretations without the empty set, *i.e.* $\mathcal{S} = \mathcal{P}(\mathcal{W}) \setminus \{\emptyset\}$; and let's denote $\mathcal{M}$ the set of finite non empty multi-sets with elements in $\mathcal{S}$. Elements of $\mathcal{S}$ and $\mathcal{M}$ will be denoted by the letters S and M respectively with possibly subscripts.

So a typical element $M \in \mathcal{M}$ will be of the shape $\{S_1, \ldots, S_n\}$. Let $M = \{S_1, \ldots, S_n\}$, we define $\bigcap M$ in the usual way: $I \in \bigcap M$ iff $\forall S_i \in M \quad I \in S_i$.

Definition 3 *A knowledge set E is consistent if and only if $\bigwedge E$ is consistent. We will use $Mod(E)$ to denote $Mod(\bigwedge E)$.*

Definition 4 *Let E_1, E_2 be two knowledge sets. E_1 and E_2 are equivalent, noted $E_1 \leftrightarrow E_2$, iff there exists a bijection f from $E_1 = \{K_1^1, \ldots, K_n^1\}$ to $E_2 = \{K_1^2, \ldots, K_n^2\}$ such that $\vdash f(K) \leftrightarrow K$.*

Note that the relation $\leftrightarrow$ is an equivalence relation on knowledge sets. As usual, we denote by $\mathcal{E}/\leftrightarrow$ the quotient of $\mathcal{E}$ by the relation $\leftrightarrow$. Thus the function $\imath : \mathcal{E}/\leftrightarrow \longrightarrow \mathcal{M}$, defined by $\imath([\{K_1, \ldots, K_n\}]_{\leftrightarrow}) = \{Mod(K_1), \ldots, Mod(K_n)\}$ is a bijection. By abuse we will write $\imath(E)$ instead of $\imath([E]_{\leftrightarrow})$.

A pre-order over $\mathcal{W}$ is a reflexive and transitive relation on $\mathcal{W}$. Let $\leq$ be a pre-order over $\mathcal{W}$, we define $<$ as follows: $I < J$ iff $I \leq J$ and $J \not\leq I$. And $\simeq$ as $I \simeq J$ iff $I \leq J$ and $J \leq I$. Let I be an interpretation, we wrote $I \in min(\leq)$ iff $\nexists J \in \mathcal{W}$ s.t. $J < I$.

By abuse if R is in $\mathcal{K}$ (respectively in $\mathcal{S}$) then R will denote also the multi-set $\{R\}$ which is in $\mathcal{E}$ (resp. in $\mathcal{M}$). For a positive integer n we will denote R^n the multi-set $\{\underbrace{R, \ldots, R}_{n}\}$. Thus $R^n = \underbrace{R \sqcup \ldots \sqcup R}_{n}$.

An operator $\triangle$ will be a function mapping knowledge sets into knowledge bases. In the rest of the paper we will distinguish between *operator* and *merging operator*: the former when no special properties are satisfied the later to indicate that the operator satisfies the postulates of definition 5. Let K, E and $\triangle$ be a knowledge base, a knowledge set and an operator respectively. We define the sequence $\langle \triangle^n(E, K) \rangle_{n \geq 1}$ by the following:

$$\triangle^1(E, K) = \triangle(E \sqcup K)$$
$$\text{and} \quad \triangle^{n+1} = \triangle(\triangle^n(E, K) \sqcup K)$$

3 Postulates

In this section, we are going to propose a characterization of merging operators, i.e. we give a minimal set of properties an operator has to satisfy in order to have a rational behaviour concerning the merging. Let E be a knowledge set, and let $\triangle$ be an operator which assigns to each knowledge set E a knowledge base $\triangle(E)$.

Definition 5 *$\triangle$ is a merging operator if and only if it satisfies the following postulates:*

(A1) $\triangle(E)$ *is consistent*

(A2) If E is consistent, then $\triangle(E) = \bigwedge E$

(A3) If $E_1 \leftrightarrow E_2$, then $\vdash \triangle(E_1) \leftrightarrow \triangle(E_2)$

(A4) If $K \wedge K'$ is not consistent, then $\triangle(K \sqcup K') \nvdash K$

(A5) $\triangle(E_1) \wedge \triangle(E_2) \vdash \triangle(E_1 \sqcup E_2)$

(A6) If $\triangle(E_1) \wedge \triangle(E_2)$ is consistent, then
$\triangle(E_1 \sqcup E_2) \vdash \triangle(E_1) \wedge \triangle(E_2)$

These six postulates are the basic properties a merging operator has to satisfy, the intuitive meaning of the postulates is easy to understand: we always want to extract a piece of information from the knowledge set, what is forced by $(A1)$ (Notice that, as assumed in remark 2, all the knowledge bases of the knowledge set are consistent). If all the knowledge bases agree on some alternatives, $(A2)$ assures that the result of the merging will be the conjunction of the knowledge bases. $(A3)$ states that the operator $\triangle$ obeys a principle of irrelevance of syntax, i.e. if two knowledge sets are equivalent in the sense of definition 4, then the two knowledge bases resulting from the merging will be logically equivalent. $(A4)$ is the fairness postulates, the point is that when we merge two knowledge bases, merging operators must not give preference to one of them. We will see (theorem 11) that (A4) is the clue for distinguishing arbitration operators from majority operators. $(A5)$ expresses the following idea: if a group E_1 compromises on a set of alternatives which I belongs to, and another group E_2 compromises on an another set of alternatives which contains I, so I has to be in the chosen alternatives if we join the two groups. $(A5)$ and $(A6)$ together state that if you could find two subgroups which agree on at least one alternative, then the result of the global arbitration will be exactly those alternatives the two groups agree on. The postulates $(A5)$ and $(A6)$ have been given in [Rev97] by Revesz for weighted model fitting operators.

Observation 6 *By definition, merging operators are commutative, i.e. the result of a merging does not depend on any order of elements of the knowledge set.*

Let's now turn our attention to the difference between majority and arbitration operators. We give here a postulate that renders the behaviour of majority operators, that is to say that if an opinion has a large audience, then it will be the opinion of the group:

$$(M7) \quad \forall K \; \exists n \;\; \triangle(E \sqcup K^n) \vdash K$$

Thus we define majority operators by the following:

Definition 7 *A merging operator is a majority operator if it satisfies* $(M7)$.

Besides, arbitration operators are those operators which are, in a large extent, majority insensitive. We first give a postulate which seems to be a good characterization of arbitration operator:

$$(A7') \quad \forall K\ \forall n\ \ \triangle(E \sqcup K^n) = \triangle(E \sqcup K)$$

This postulate states that the result of an arbitration is fully independent from the frequency of different views. Unfortunately the set of postulates $\{A1, \ldots, A6, A7'\}$ is not consistent. The proof of this result has been pointed out by P. Liberatore (personal communication):

Theorem 8 *There is no merging operator satisfying* $(A7')$.

Proof: Let $E_1 = \{K, \neg K\}$ and $E_2 = \{K\}$ be two knowledge sets. By $(A7')$ we have that $\triangle(E_1 \sqcup E_2) = \triangle(E_1)$. By $(A4)$ we have also that $\triangle(E_1) \nvdash K$ and $\triangle(E_1) \nvdash \neg K$. Furthermore by $(A2)$ we deduce $\triangle(E_2) = K$. So $\triangle(E_1) \wedge \triangle(E_2)$ is consistent and by $(A6)$ we have $\triangle(E_1 \sqcup E_2) \vdash \triangle(E_1) \wedge \triangle(E_2)$, it can be rewritten as $\triangle(E_1) \vdash \triangle(E_1) \wedge K$. Then $\triangle(E_1) \vdash K$, which contradicts $(A4)$. ∎

Thus if we want to have a postulate expressing majority insensitivity while being consistent with $(A1 - A6)$ we must weaken $(A7')$. We propose the following alternative:

$$(A7) \quad \forall K'\ \exists K\ K' \nvdash K\ \forall n\ \triangle(K' \sqcup K^n) = \triangle(K' \sqcup K)$$

$(A7)$ states that, to a large extent, the result of the arbitration is independent from the frequency of the different views.

And we define arbitration operator in the following way:

Definition 9 *A merging operator is an arbitration operator if it satisfies* $(A7)$.

Now we investigate some relations between the postulates.

Theorem 10 *If an operator satisfies* $(A1)$*, then it can't satisfy both* $(A7')$ *and* $(M7)$.

Proof: From $(A7')$ and $(M7)$ we deduce that for any arbitrary E

$$\forall\ K\ \ \triangle(E \sqcup K) \vdash K \quad (*)$$

Take K' such that $K \wedge K' \vdash -$. Now putting $E = K'$, by $(*)$, we have $\triangle(K' \sqcup K) \vdash K$. In a symmetrical way we have $\triangle(K \sqcup K') \vdash K'$ so $\triangle(K \sqcup K') \vdash K \wedge K'$ and then $\triangle(K \sqcup K') \vdash -$ which contradicts $(A1)$. ∎

A merging operator can't be an arbitration operator and a majority operator, more precisely we have the following:

Theorem 11 *If an operator satisfies* $(A4)$*, then it can't satisfy both* $(A7)$ *and* $(M7)$.

Proof: From $(A7)$ and $(M7)$ we deduce easily $\forall K'\ \exists K\ \ K' \nvdash K\ \ \triangle(K' \sqcup K) \vdash K$. Let's choose $K' = K_I = form(I)$, then $\exists K\ K_I \nvdash K\ \ \triangle(K_I \sqcup K) \vdash K$. But $K_I \nvdash K$ is equivalent to $K_I \wedge K \vdash -$ and so by $(A4)$ we have that $\triangle(K_I \sqcup K) \nvdash K$. Contradiction. ∎

So, although it seems very weak, the fairness postulate $(A4)$ play a very important role, since it allows us to differentiate arbitration operators and majority operators.

In addition to these basic postulates we can find various other properties, we investigate some of them below.

An interesting property for a merging operator is the following which we call the *iteration* property:

$$(A_{it}) \quad \exists n\ \triangle^n(E, K) \vdash K$$

The intuitive idea is that, since the merging operators give, in a sense, the average knowledge of a knowledge set, if we always take the result of a merging and iterate with the same knowledge base, we have to reach this knowledge base after enough iterations. But, even if it seems to be a reasonable requirement, we don't know if all merging operators obey (A_{it}), more exactly we suspect that those operators satisfying (A_{it}) are topological operators, *i.e.* operators defined from a distance.

Now let's turn our attention to the two properties of associativity and monotony. We claim that they are not desirable for merging operators and we show that merging operators do not satisfy any of them. First let's give a formal definition of associativity and monotony:

$$(Ass) \quad \triangle(E_1 \sqcup \triangle(E_2)) = \triangle(E_1 \sqcup E_2)$$

Associativity seems to be an interesting property since it would allow sub-merging within the knowledge set. So merging could be implemented more easily and more efficiently.

(Mon) If $K_1 \vdash K'_1, \ldots, K_n \vdash K'_n$ then $\triangle(K_1 \sqcup \ldots \sqcup K_n) \vdash \triangle(K'_1 \sqcup \ldots \sqcup K'_n)$

The monotony property expresses that if a knowledge set E_1 is "stronger" than a knowledge set E_2, then the merging of E_1 has to be logically stronger than the merging of E_2.

Theorem 12 *If an operator satisfies* $(A2)$ *and* $(A4)$*, then it doesn't satisfy* (Mon)*.*

Proof: Let I, J be two different interpretations. Let $K_1 = K'_1 = form(I)$, $K_2 = form(J)$, and $K'_2 = form(I, J)$, so we have $K_1 \vdash K'_1$ and $K_2 \vdash K'_2$. From $(A2)$ $\triangle(K'_1 \sqcup K'_2) = form(I)$ and from $(A4)$ $\triangle(K_1 \sqcup K_2) \nvdash form(I)$. So we have $\triangle(K_1 \sqcup K_2) \nvdash \triangle(K'_1 \sqcup K'_2)$. ∎

So it is clear that monotony is not satisfied by merging operators, it is not exactly the same with associativity, we show that it is not satisfied by majority operators and that it is not compatible with the iteration property:

Theorem 13 *If an operator satisfies* $(A2)$ $(A4)$ *and* $(M7)$*, then it can't satisfy* (Ass)*.*

Proof: Let's take K_I and K_J two different complete formulae, by $(M7)$ we have that $\exists n \triangle(K_I \sqcup K_J^n) \vdash K_J$. By (Ass) we have that $\triangle(K_I \sqcup K_J^n) = \triangle(K_I \sqcup \triangle(K_J^n))$. But by $(A2)$ we have $\triangle(K_J^n) = K_J$. So we obtain that $\triangle(K_I \sqcup K_J) \vdash K_J$. What contradicts $(A4)$. ∎

Theorem 14 *If an operator satisfies* $(A2)$ *and* $(A4)$*, then it can't satisfy both* (A_{it}) *and* (Ass)*.*

Proof: (A_{it}) $\exists n$ $\triangle^n(E, K) \vdash K$, but by (Ass) we find that $\triangle^n(E, K) = \triangle(E \sqcup K^n) = \triangle(E \sqcup \triangle(K^n))$ and by (A2) we have that $\triangle(E \sqcup \triangle(K^n)) = \triangle(E \sqcup K)$. So we have that $\triangle(E \sqcup K) \vdash K$, what, taking $E = K'$ with $K' \wedge K \vdash -$, contradicts (A4). ∎

So, if we want some additional property for a merging operator, we have to choose between iteration and associativity. We claim that iteration is a desirable property for merging operators, so associativity is not.

4 Semantical characterizations

In this section we give a model-theoretic characterization of merging operators first in terms of functions on sets of interpretations and then in terms of family of orders. More exactly we show that each merging operator corresponds to a function from multi-sets of sets of interpretations to sets of interpretations and then we show that each merging operator corresponds to a family of partial pre-orders on interpretations. The semantical characterization of the merging operators in terms of pre-orders is very close to the axiomatic characterization. This is due to the fact that we can't have a definition of the pre-order as subtle as in the case of belief revision. But this semantical characterization is very useful in the proofs and is a starting point for generalizing merging operators (*e.g.* when one considers the set of alternatives as a parameter).

First we define what is a merging function:

Definition 15 *A function* $\delta : \mathcal{M} \longrightarrow \mathcal{S}$ *is said to be a merging function if the following properties hold for any* $M, M_1, M_2 \in \mathcal{M}$ *and* $S, S' \in \mathcal{S}$*:*

1. *If* $I \in \bigcap M$*, then* $I \in \delta(M)$
2. *If* $\bigcap M \neq \emptyset$ *and* $I \notin \bigcap M$*, then* $I \notin \delta(M)$
3. *If* $S \cap S' = \emptyset$*, then* $\delta(S \sqcup S') \nsubseteq S$
4. *If* $I \in \delta(M_1)$ *and* $I \in \delta(M_2)$*, then* $I \in \delta(M_1 \sqcup M_2)$
5. *If* $\delta(M_1) \cap \delta(M_2) \neq \emptyset$ *and* $I \notin \delta(M_1)$*, then* $I \notin \delta(M_1 \sqcup M_2)$

A majority merging function is a merging function that satisfies the following:

6. $\forall M \in \mathcal{M}\ \forall S \in \mathcal{S}\ \exists n\ \delta(M \sqcup S^n) \subseteq S$

A fair merging function is a merging function that satisfies the following:

7. $\forall S' \in \mathcal{S}\ \exists S \in \mathcal{S}\ S' \nsubseteq S\ \forall n\ \delta(S' \sqcup S^n) = \delta(S' \sqcup S)$

It is easy to see, via the bijection $\imath$ of section 2 that the properties $1 - 5$ are the semantical counterparts of postulates $(A1 - A6)$ (notice that postulate (A_1) corresponds to the fact $\emptyset \notin \mathcal{S}$), property 6 corresponds to postulate $(M7)$ and property 7 corresponds to postulate $(A7)$. More precisely we have the following representation theorem which proof is straightforward:

Theorem 16 *An operator* $\triangle$ *is a merging operator (it satisfies* $(A1 - A6)$*) if and only if there exists a merging function* $\delta : \mathcal{M} \longrightarrow \mathcal{S}$ *such that*

$$Mod(\triangle(E)) = \delta(\imath(E)).$$

Furthermore $\triangle$ *is a majority merging operator iff* δ *is a majority merging function; and* $\triangle$ *is an arbitration operator iff* δ *is a fair merging function.*

As in the AGM framework for revision, we can suppose the existence of some relation which intuitively represents how credible each interpretation is for some given knowledge set. We will see that there is a close relationship between merging function and these relations on knowledge sets. First we define what a syncretic assignment is:

Definition 17 *A syncretic assignment is an assignment which maps each knowledge set E to a pre-order $\leq_E$ over interpretations such that for any $E, E_1, E_2 \in \mathcal{E}$ and for any $K, K' \in \mathcal{K}$:*

1. *If $I \in Mod(E)$ and $J \in Mod(E)$, then $I \simeq_E J$*
2. *If $I \in Mod(E)$ and $J \notin Mod(E)$, then $I <_E J$*
3. *If $E_1 \leftrightarrow E_2$, then $\leq_{E_1} = \leq_{E_2}$*
4. *If $Mod(K) \cap Mod(K') = \emptyset$, then $\min(\leq_{K \sqcup K'}) \nsubseteq Mod(K)$*
5. *If $I \in \min(\leq_{E_1})$ and $I \in \min(\leq_{E_2})$, then $I \in \min(\leq_{E_1 \sqcup E_2})$*
6. *If $\min(\leq_{E_1}) \cap \min(\leq_{E_2}) \neq \emptyset$ and $I \notin \min(\leq_{E_1})$, then $I \notin \min(\leq_{E_1 \sqcup E_2})$*

A majority syncretic assignment is a syncretic assignment which satisfies the following:

7. $\forall E \in \mathcal{E}\ \forall K \in \mathcal{K}\ \exists n\ \min(\leq_{E \sqcup K^n}) \subseteq Mod(K)$

A fair syncretic assignment is a syncretic assignment which satisfies the following:

8. $\forall K'\ \ \exists K$ *if* $Mod(K') \nsubseteq Mod(K)$, *then* $\forall n\ \min(\leq_{K' \sqcup K^n}) = \min(\leq_{K' \sqcup K})$

If we have an assignment that maps each knowledge set E to a pre-order $\leq_E$ on $\mathcal{W}$, then we can define a function $\delta : \mathcal{M} \longrightarrow \mathcal{S}$ by the following: let $M \in \mathcal{M}$ and let $E \in \mathcal{E}$ be such that $\imath(E) = M$, put

$$\delta(M) \quad = \quad \min(\leq_E) \tag{1}$$

If the assignment satisfies property 3 above then δ is well defined.

Conversely, if we have a function $\delta : \mathcal{M} \longrightarrow \mathcal{S}$ we can define a corresponding family of relations on interpretations as $\forall E \in \mathcal{E}$:

$$\leq_E = [\delta(\imath(E)) \times (\mathcal{W} \setminus \delta(\imath(E)))] \cup \{\{I, I\} \setminus I \in \mathcal{W}\} \tag{2}$$

It is easy to show that if we have a (majority, fair) syncretic assignment, then the merging function obtained by equation 1 is a (majority, fair) merging function. Conversely, if we have a (majority, fair) merging function, then the family of relations obtained by equation 2 is a (majority, fair) syncretic assignment. This observation together with theorem 16 gives us straightforwardly the following:

Theorem 18 *An operator is a merging operator (respectively majority merging operator or arbitration operator) if and only if there exists a syncretic assignment (respectively majority syncretic assignment or fair syncretic assignment) that maps each knowledge set E to a pre-order $\leq_E$ such that*

$$Mod(\triangle(E)) = \min(\leq_E).$$

As pointed out by D. Makinson (personal communication), this definition of merging operators from such assignments can be compared to the framework of social choice theory [Kel78, Arr63]. The aim of social choice theory is to aggregate individual choices into a social choice, *i.e.* to find, for a given set of agents (corresponding to our knowledge sets) with individual preference relations, a social preference relation which reflects the preferences of the set of agents. This allows the definition of a welfare function selecting from a set of alternatives those that best fit the social preference relation.

5 Some merging operators

In this section we show the consistency of our merging postulates by giving three examples of operators. The first one is not a merging operator but it illustrates an approach to arbitration operators. The second one is a majority merging operator and the last one is a true arbitration operator.

For the following operators we will use the Dalal's distance [Dal88] to calculate the distance between two interpretations: let I, J be interpretations, $dist(I, J)$ is the number of propositional letters the two interpretations differ.

We also define the distance between an interpretation and a knowledge base as the minimum distance between this interpretation and the models of the knowledge base, that is:

$$dist(I, \varphi) = \min_{J \in Mod(\varphi)} dist(I, J)$$

Finally we define the distance between two knowledge bases by the following:

$$dist(\varphi, \varphi') = \min_{I \in Mod(\varphi)\ J \in Mod(\varphi')} dist(I, J)$$

Table 1: Distances

	φ_1	φ_2	φ_3	$\mathbf{dist_{Max}}$	$\mathbf{dist_{\Sigma}}$	$\mathbf{dist_{GMax}}$
$(0,0,0)$	1	1	3	3	5	(3,1,1)
$(0,0,1)$	0	0	2	2	2	(2,0,0)
$(0,1,0)$	2	0	2	2	4	(2,2,2)
$(0,1,1)$	1	1	1	1	3	(1,1,1)
$(1,0,0)$	0	2	2	2	4	(2,2,0)
$(1,0,1)$	0	1	1	1	2	(1,1,0)
$(1,1,0)$	1	1	1	1	3	(1,1,1)
$(1,1,1)$	1	2	0	2	3	(2,1,0)

The first operator we consider is the $\triangle_{Max}$ operator. It comes from an example of model fitting operator given by Revesz in [Rev97]. It is close to the minimax rule used in decision theory [Sav71]. The idea is to find the closest information to the overall knowledge set. Therefore it seems to be a good arbitration operator. But, as we will see, it doesn't satisfy all the postulates.

Definition 19 *Let φ be a knowledge base and E be a knowledge set:*

$$dist_{Max}(I,E) = \max_{\varphi \in E} dist(I,\varphi)$$

So, we define the following order:

$$I \leq_E^{Max} J \quad \textit{iff } dist_{Max}(I,E) \leq dist_{Max}(J,E)$$

$$\textit{and} \qquad Mod(\triangle_{Max}(E)) = \min(\leq_E^{Max})$$

The second operator we consider is the $\triangle_{\Sigma}$ operator. This is a majority merging operator as we will see below. Lin and Mendelzon give it as an example of what they called operators of *theory merging by majority* in [LMa]. Independently Revesz gives it as an example of weighted model fitting in [Rev93]. The Σ operator comes from a natural idea: the distance between an interpretation and a knowledge set is the sum of the distances between this interpretation and the knowledge bases of the knowledge set.

Definition 20 *Let E be a knowledge set and let I be an interpretation we put:*

$$dist_{\Sigma}(I,E) = \sum_{\varphi \in E} dist(I,\varphi)$$

$$I \leq_E^{\Sigma} J \ \textit{iff } dist_{\Sigma}(I,E) \leq dist_{\Sigma}(J,E)$$

$$\textit{and} \qquad Mod(\triangle_{\Sigma}(E)) = min(\leq_E^{\Sigma})$$

Next we present a new merging operator: $\triangle_{GMax}$ (stands for *Generalized Max*). The operator $\triangle_{GMax}$ is an arbitration operator and is a refinement of the $\triangle_{Max}$ operator.

Definition 21 *Let E be a knowledge set. Suppose $E = \{\varphi_1, \ldots, \varphi_n\}$. For each interpretation I we build the list $(d_1^I \ldots d_n^I)$ of distances between this interpretation and the n knowledge bases in E,* i.e. *$d_j^I = dist(I,\varphi_j)$. Let L_I be the list obtained from $(d_1^I \ldots d_n^I)$ by sorting it in descending order. Define $dist_{GMax}(I,E) = L_I$. Let $\leq_{lex}$ be the lexicographical order between sequences of integers. Now we put:*

$$I \leq_E^{GMax} J \quad \textit{iff} \quad dist_{GMax}(I,E) \leq_{lex} dist_{GMax}(J,E)$$

$$\textit{and} \qquad Mod(\triangle_{GMax}(E)) = \min(\leq_E^{GMax})$$

We will illustrate the behaviour of these three operators on the database class example given by Revesz in [Rev93]:

Example 22 Consider a database class with three students: $E = \{\varphi_1, \varphi_2, \varphi_3\}$. The teacher can teach SQL, Datalog and O_2. He asks his students in turn to choose what to teach to satisfy the class best. The first student wants to learn SQL or O_2: $\varphi_1 = (S \vee O) \wedge \neg D$. The second wants to learn Datalog or O_2 but not both: $\varphi_2 = (\neg S \wedge D \wedge \neg O) \vee (\neg S \wedge \neg D \wedge O)$. The third wants to learn the three languages: $\varphi_3 = (S \wedge D \wedge O)$. Considering the propositional letters S, D and O in that order we have: $Mod(\varphi_1) = \{(1,0,0),(0,0,1),(1,0,1)\}$, $Mod(\varphi_2) = \{(0,1,0),(0,0,1)\}$, $Mod(\varphi_3) = \{(1,1,1)\}$.

Table 1 contains all distances relevant to computations in order to calculate $\triangle_{Max}(E)$, $\triangle_{\Sigma}(E)$ and $\triangle_{GMax}(E)$.

As the min in the column of $dist_{Max}$ is 1 we have $Mod(\triangle_{Max}(E)) = \{(0,1,1),\ (1,0,1),(1,1,0)\}$, thus the teacher has to teach two of the three languages to best satisfy the class when the criterion to solve conflicts is $\triangle_{Max}$. Similarly as the min in the column of $dist_\Sigma$ is 2 we have $Mod(\triangle_\Sigma(E)) = \{(0,0,1),(1,0,1)\}$, thus the teacher has to teach both SQL and O_2 or O_2 alone to best fit the class when the criterion to solve conflicts is $\triangle_\Sigma$. Finally as the min in the column of $dist_{\triangle_{GMax}}$ is $(1,1,0)$ we have $Mod(\triangle_{GMax}(E)) = \{(1,0,1)\}$, thus the teacher has to teach SQL and O_2 to best satisfy the class when the criterion to solve conflicts is $\triangle_{GMax}$.

As we can expect the result of the merging highly depends on the operator we choose. Note in particular that the $\triangle_{Max}$ operator has selected interpretations that satisfy as much as possible each student, whereas the $\triangle_\Sigma$ operator has selected interpretations that satisfy the majority of students. Notice also that in this example the $\triangle_{GMax}$ operator selects the interpretation chosen by both $\triangle_{Max}$ and $\triangle_\Sigma$ operators, showing its good behaviour.

We will see now the logical properties of these three operators.

We first show that $\triangle_{Max}$ is not a merging operator.

Theorem 23 *$\triangle_{Max}$ satisfies postulates $(A1 - A5)$, $(A7')$ and (A_{it}) but it doesn't satisfy $(A6)$.*

Proof: The proof of $(A1-A3)$ and $(A5)$ is straightforward. To prove that $(A4)$ is satisfied suppose $K \wedge K' \vdash -$. We consider two cases: $dist(K,K') = 1$ or $dist(K,K') > 1$ If $dist(K,K') = 1$ then $\exists I \in Mod(K), \exists J \in Mod(K')$ such that $dist(I,J) = 1$, so as $dist(I,J)$ is minimum $I \in Mod(\triangle(K \sqcup K'))$ and $J \in Mod(\triangle(K \sqcup K'))$, so $\triangle(K \sqcup K') \not\vdash K$. Otherwise $dist(K,K') > 1$, and then $\exists I \in Mod(K), \exists J \in Mod(K')\ \forall I' \in Mod(K), \forall J' \in Mod(K')\ dist(I,J) \le dist(I',J')$ and $dist(I,J) > 1$. But it is easy to see that if $dist(I,J) = a > 1$ then there exists $L \in \mathcal{W}$ such that $dist(L,I) < a$ and $dist(L,J) < a$, so $dist_{Max}(L, K \sqcup K') < a$. Therefore $L <^{Max}_{K \sqcup K'} I$ so $I \notin Mod(\triangle(K \sqcup K'))$, so $\triangle(K \sqcup K') \not\vdash K$. $(A7')$ is satisfied because $\max_{\varphi \in E \sqcup K^n} dist(I,\varphi) = \max_{\varphi \in E \sqcup K} dist(I,\varphi)$. So $\triangle(E \sqcup K^n) = \triangle(E,K)$. As $(A7')$ is satisfied, $(A7)$ is satisfied. In order to show that $(A6)$ is not satisfied consider the example 22 and observe that if we take $E_1 = \{\varphi_1\}$ and $E_2 = \{\varphi_2, \varphi_3\}$, then $\triangle(E_1) \wedge \triangle(E_2) = form(\{(1,0,1)\})$ is consistent, and $\triangle(E_1 \sqcup E_2) = form(\{(0,1,1),(1,0,1),(1,1,0)\})$, so $\triangle(E_1 \sqcup E_2) \not\vdash \triangle(E_1) \wedge \triangle(E_2)$.

It remains to show that (A_{it}) holds. First, by induction on $dist(K,K')$ we prove that

$$\exists n \text{ such that } \triangle^n_{Max}(K',K) \vdash K \quad (*)$$

If $dist(K,K') = 0$ the proof is straightforward. Suppose $dist(K,K') = 1$. Then $\exists I \in Mod(K)\ \exists J \in Mod(K')\ dist(I,J) = 1$. So $I \in \triangle_{Max}(K,K')$ and then, by (A2), $\triangle^2_{Max}(K',K) = \triangle_{Max}(\triangle_{Max}(K',K),K) = \triangle_{Max}(K',K) \wedge K$. So $\triangle^2_{Max}(K',K) \vdash K$. Suppose that $dist(K,K') > 1$. Put $a = dist(K',K)$, i.e. $\exists I \in Mod(K)\ \exists J \in Mod(K')\ dist(I,J) = a$. Let $a/2$ be the integer part of the quotient of a by 2. Since I and J disagree on a letters, we can find an interpretation I' such that I' agrees with I on the letters on which I and J agree, and I' agrees with J on $a/2$ letters on which I and J disagree and I' agrees with I for the $a/2$ remaining letters if a is even and for the $a/2+1$ remaining letters if a is odd. So we have $dist(I',K) \le a/2$ and $dist(I',K') \le a/2$ if a is even or $dist(I',K') \le a/2+1$ if a is odd.
If a is even then $dist_{Max}(I',\{K,K'\}) \le a/2$, so if $J' \in Mod(\triangle_{Max}(K,K'))$ then $dist_{Max}(J',\{K,K'\}) \le a/2$. So we have that if $dist(K,K') = a$ with $a > 1$ then $dist(K, \triangle_{Max}(K,K')) \le a/2$. By induction hypothesis there exists n such that $\triangle^n_{Max}(\triangle_{Max}(K,K'),K) \vdash K$ that is $\triangle^{n+1}_{Max}(K',K) \vdash K$. The case where a is odd is similar. Now (A_{it}) follows from $(*)$ by putting $K' = \triangle(E \sqcup K)$. ∎

The operator $\triangle_\Sigma$ is a majority merging operator as stated in the following theorem.

Theorem 24 *$\triangle_\Sigma$ satisfies postulates $(A1-A6)$, $(M7)$ and (A_{it}).*

Proof: We will prove that the assignment $E \mapsto \le^\Sigma_E$ is a majority syncretic assignment. Then by theorem 18 we conclude that $\triangle_\Sigma$ satisfies $(A1-A6)$ and $(M7)$. Let's verify the conditions of a majority syncretic assignment:

1. If $I \in Mod(E)$ and $J \in Mod(E)$, then $dist_\Sigma(I,E) = 0$ and $dist_\Sigma(J,E) = 0$, so $I \simeq_E J$.

2. If $I \in Mod(E)$ and $J \notin Mod(E)$, then $dist_\Sigma(I,E) = 0$ and $dist_\Sigma(J,E) > 0$, so $I <_E J$.

3. Straightforward.

4. Suppose $K \wedge K' \vdash -$, so $dist(K,K') > 0$. So $\exists I \in Mod(K), \exists J \in Mod(K')\ \forall I' \in Mod(K), \forall J' \in Mod(K')\ dist(I,J) \le dist(I',J')$ and $dist(I,J) = a > 1$. It is easy to see that $a = \min\{dist_\Sigma(L, K \sqcup K') : L \in \mathcal{W}\}$ thus $I \in Mod(\triangle(K \sqcup K'))$ and $J \in Mod(\triangle(K \sqcup K'))$, so $\triangle(K \sqcup K') \not\vdash K$.

5. If $I \in \min(\le_{E_1})$ and $I \in \min(\le_{E_2})$, then $\forall J\ dist_\Sigma(I,E_1) \le dist_\Sigma(J,E_1)$ and

$dist_\Sigma(I,E_2) \leq dist_\Sigma(J,E_2)$. So $\forall J$ $dist_\Sigma(I,E_1) + dist_\Sigma(I,E_2) \leq dist_\Sigma(J,E_1) + dist_\Sigma(J,E_2)$. By definition of $dist_\Sigma$ is easy to see that for any L,E,E', $dist_\Sigma(L,E \sqcup E') = dist_\Sigma(L,E) + dist_\Sigma(L,E')$. Then $\forall J$ $dist_\Sigma(I,E_1 \sqcup E_2) \leq dist_\Sigma(J,E_1 \sqcup E_2)$. So $I \in \min(\leq_{E_1 \sqcup E_2})$.

6. If $\min(\leq_{E_1}) \cap \min(\leq_{E_2}) \neq \emptyset$, then $\exists J$ s.t. $J \in \min(\leq_{E_1})$ and $J \in \min(\leq_{E_2})$. Suppose $I \notin \min(\leq_{E_1})$, then $dist_\Sigma(J,E_1) < dist_\Sigma(I,E_1)$ and $dist_\Sigma(J,E_2) \leq dist_\Sigma(I,E_2)$. So $dist_\Sigma(J,E_1) + dist_\Sigma(J,E_2) < dist_\Sigma(I,E_1) + dist_\Sigma(I,E_2)$. Then $dist_\Sigma(J,E_1 \sqcup E_2) < dist_\Sigma(I,E_1 \sqcup E_2)$. Then $I \notin \min(\leq_{E_1 \sqcup E_2})$.

7. We have to find a n such that $\min(\leq_{E \sqcup K^n}) \subseteq Mod(K)$. Consider $x = \max_{I \in \mathcal{W}} dist_\Sigma(I,E)$, i.e. x is the distance of the furthest interpretation from E. We choose $n = x+1$, it is easy to see that if $I \in Mod(K)$ then $dist_\Sigma(I, E \sqcup K^n) < n$. And if $I \notin Mod(K)$ then $dist_\Sigma(I, E \sqcup K^n) \geq n$. So if $I \in \min(\leq_{E \sqcup K^n})$ then $I \in Mod(K)$.

Now we prove that (A_{it}) holds. We want to show that $\exists n$ $\triangle_\Sigma^n(K',K) \vdash K$. Let a be the distance between K and K'. Take $I \in Mod(K)$ and $J \in Mod(K')$ such that $dist(I,J) = a$. It is easy to see that $a = \min\{dist_\Sigma(L, K \sqcup K') : L \in \mathcal{W}\}$ thus $I \in Mod(\triangle(K \sqcup K'))$ and then $\triangle_\Sigma(\triangle_\Sigma(K' \sqcup K), K) \vdash K$. Therefore $\exists n$ $\triangle_\Sigma^n(K',K) \vdash K$. And with $K' = \triangle_\Sigma(E \sqcup K)$ we have $\exists n$ $\triangle_\Sigma^n(E,K) \vdash K$. ∎

Now, we will state some lemmas in order to prove that $\triangle_{GMax}$ has desirable properties.

Definition 25 *Let L_1 and L_2 be two lists of n numbers sorted in descending order. We define $L_1 \odot L_2$ the list obtained by sorting in descending order the concatenation of L_1 with L_2.*

Lemma 26 *Let L_1, L_1', L_2, L_2' be 4 lists of integers sorted in descending order. If $L_1 \leq_{lex} L_1'$ and $L_2 \leq_{lex} L_2'$ then $L_1 \odot L_2 \leq_{lex} L_1' \odot L_2'$.*

Proof: Suppose that $L_1 \leq L_1'$ and $L_2 \leq L_2'$. It is easy to see that the two following inequalities hold: $L_1 \odot L_2 \leq_{lex} L_1' \odot L_2$ and $L_2 \odot L_1' \leq_{lex} L_2' \odot L_1'$. So by transitivity $L_1 \odot L_2 \leq_{lex} L_1' \odot L_2'$. ∎

Lemma 27 *Let L_1, L_1', L_2, L_2' be 4 lists of integers sorted in descending order. If $L_1 \leq_{lex} L_1'$ and $L_2 <_{lex} L_2'$ then $L_1 \odot L_2 <_{lex} L_1' \odot L_2'$.*

Proof: With the assumptions it is easy to see that $L_1 \odot L_2 \leq_{lex} L_1' \odot L_2$ and $L_2 \odot L_1' <_{lex} L_2' \odot L_1'$. We conclude by transitivity of $\leq_{lex}$. ∎

The operator $\triangle_{GMax}$ is a true arbitration operator as showed in the following theorem.

Theorem 28 *The operator $\triangle_{GMax}$ satisfies postulates $(A1-A6)$ and (A_{it}). Furthermore $\triangle_{GMax}$ satisfies $(A7)$ iff $card(\mathcal{P}) > 1$. But it doesn't satisfy $(A7')$.*

Proof: In order to show that $GMax$ satisfies $(A1-A7)$ we use the representation theorem and we show that the assignment $E \mapsto \leq_E^{Gmax}$ is a fair syncretic assignment.

1. If $I \in Mod(E)$ and $J \in Mod(E)$, then $\forall K_i \in E$ $I \in Mod(K_i)$ and $J \in Mod(K_i)$, then $L_I = (0,\ldots,0)$ and $L_J = (0,\ldots,0)$, so $I \simeq_E J$.

2. If $I \in Mod(E)$ and $J \notin Mod(E)$, then $L_I = (0,\ldots,0)$ and $L_J \neq (0,\ldots,0)$, so $I <_E J$.

3. If $E_1 \leftrightarrow E_2$, then is obvious that $\leq_{E_1} = \leq_{E_2}$.

4. This property is proved in a similar way as (A_4) for $\triangle_{Max}$ (theorem 23).

5. If $I \in \min(\leq_{E_1})$ and $I \in \min(\leq_{E_2})$, then $\forall J \in \mathcal{W}$ $L_I^{E_1} \leq_{lex} L_J^{E_1}$ and $L_I^{E_2} \leq_{lex} L_J^{E_2}$. So, by lemma 26, we have $\forall J$ $L_I^{E_1 \sqcup E_2} \leq_{lex} L_J^{E_1 \sqcup E_2}$. Then $I \in \min(\leq_{E_1 \sqcup E_2})$.

6. If $\min(\leq_{E_1}) \cap \min(\leq_{E_2}) \neq \emptyset$ and $I \notin \min(\leq_{E_1})$, let $J \in \min(\leq_{E_1}) \cap \min(\leq_{E_2})$, so $L_J^{E_1} <_{lex} L_I^{E_1}$ and $L_J^{E_2} \leq_{lex} L_I^{E_2}$, and by lemma 27 follows $L_J^{E_1 \sqcup E_2} <_{lex} L_I^{E_1 \sqcup E_2}$. Then $I \notin \min(\leq_{E_1 \sqcup E_2})$.

7. Consider a knowledge base K'. We will show that if there are 2 or more propositional variables then there exists a K s.t. $K' \nvdash K$ and $\forall n$ $\min(\leq_{K' \sqcup K^n}) = \min(\leq_{K' \sqcup K})$. We consider 2 cases, first if $card(Mod(K')) > 1$ then let $I \in Mod(K')$, we choose $K = form(I)$. So, by condition 1 and 2, $\min(\leq_{K' \sqcup K}) = \{I\}$ and $\forall n$ $\min(\leq_{K' \sqcup K^n}) = \{I\}$. Hence $\forall n$ $\min(\leq_{K' \sqcup K^n}) = \min(\leq_{K' \sqcup K})$. Second, if $card(Mod(K')) = 1$, let $Mod(K') = \{J\}$, we choose $K = form(I)$ s.t. $dist(I,J) = 2$, this is possible because there are at least two propositional variables. So there exists I' s.t. $dist(I',I) = 1$ and $dist(I',J) = 1$. So $\min(\leq_{K' \sqcup K^n}) = \{I' : dist(I',I) = 1 \text{ and } dist(I',J) = 1\}$ otherwise if $\exists J'$ such that $dist(J',K) = 0$ then $dist(J',K') \geq 2$ or if $dist(J',K') = 0$ then $dist(J',K) \geq 2$, and so $L_{I'} < L_{J'}$. So $\forall n$ $\min(\leq_{K' \sqcup K^n}) = \{I' : dist(I',I) = 1 \text{ and } dist(I',J) = 1\}$. Then $\forall n$ $\min(\leq_{K' \sqcup K^n}) = \min(\leq_{K' \sqcup K})$. Conversely suppose that $\mathcal{P} = \{p\}$. Put $K' = p$. Then the only consistent K (up to logical equivalence) such

Table 2: Summary Table

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**	**A7′**	**M7**	$\mathbf{A_{it}}$
Max	✓	✓	✓	✓	✓	−	✓	✓	−	✓
Σ	✓	✓	✓	✓	✓	✓	−	−	✓	✓
$GMax$	✓	✓	✓	✓	✓	✓	✓	−	−	✓

that $K' \not\vdash K$ is $K = \neg p$ but $\triangle_{GMax}(K' \sqcup K^n) = \neg p$ for any $n \geq 2$ whereas $\triangle_{GMax}(K' \sqcup K) = \neg p \vee p$.

To show that $\triangle_{GMax}$ doesn't satisfy $(A7')$ consider the following example: Suppose that $\mathcal{P} = \{p, q\}$ and that $K' = \neg p \wedge \neg q$ and $K = \neg p \wedge q$. It is easy to see that $\triangle_{GMax}(K \sqcup K') = \neg p$ whereas $\triangle_{GMax}(K' \sqcup K^n) = \neg p \wedge q$ for any $n \geq 2$.

Finally the proof that the postulate (A_{it}) holds for $\triangle_{GMax}$ goes exactly the same way that for $\triangle_{Max}$ (theorem 23). ∎

Actually $GMax$ operator is a refinement of the Max operator. More precisely we have the following observation the proof of which is straightforward:

Observation 29 $\triangle_{GMax}(E) \vdash \triangle_{Max}(E)$.

We end this section referring the reader to table 2 which sums up the properties of operators defined above. It is filled using the results of this section together with some results of section 3. The symbol ✓ (respectively −) in a square means that the corresponding operator satisfies (resp. does not satisfy) the corresponding postulate.

6 Conclusion and future work

We have proposed in this paper a set of postulates that a rational merging operator has to satisfy. We have made a distinction between arbitration operators striving to minimize individual dissatisfaction and majority operators striving to minimize global dissatisfaction. The fairness postulate is the key postulate in this distinction. We have shown that our characterization is equivalent to a family of pre-orders on interpretations. We show the consistency of the axiomatic characterization by giving examples of operators. In particular, we have proposed a new rational merging operator called $\triangle_{GMax}$ and shown that it is an arbitration operator.

Actually, in a committee, all the protagonists do not have the same weight on the final decision and so one needs to weight each knowledge base to reflect this. The idea behind weights is that the higher weight a knowledge base has, the more important it is. If the knowledge bases reflect the view of several people, weights could represent, for example, the cardinality of each group. We want to characterize logically the use of this weights. Majority operators are close to this idea of weighted operators since they allow to take cardinalities into account. But a more subtle treatment of weights in merging is still to do, in particular the notion of weighted arbitration operators is missing.

In this work the result of a merging is a subset of the set of all interpretations but a lot of systems have to conform to a set of integrity constraints, for that reason it is interesting to be able to merge some knowledge sets in the presence of these constraints [LMb]. And so one has to restrain the result of the merging to be a subset of the set of allowed interpretations. Suppose that these integrity constraints are denoted by the knowledge base IC. If we consider a weighted rational merging, a way to incorporate integrity constraints is to add IC to E with a weight "infinity". Thus we would ensure that the interpretations selected were models of IC. Intuitively, it amounts to consider a person in the committee whose view is unquestionable and therefore one has to choose among the alternatives given by that person.

But the best way to include integrity constraints seems to be to select the minimal models in the models of the IC base rather than in $\mathcal{W}$. Intuitively, we restrict the choices of interpretations to those which satisfy IC. It is in a sense what Revesz called model fitting operators [Rev97].

In that paper we use only the Dalal's distance to define the distance between two interpretations, it would be interesting to study operators defined with other distances, in particular distances which give partial orders.

Notice also that the three merging operators defined in the paper are based on the Dalal's distance. But if one chooses an other distance between interpretations and keeps the same definitions, then one obtains other merging operators. So, more exactly, we have defined in this paper three families of merging operators, function of the definition of the distance between interpretations. It would be interesting to find what the minimum conditions on that distance are to ensure that the operators satisfy the axiomatic characterization.

Two other points of interest are to study merging operators which are not defined from a distance and to study syntactic definition of merging operators.

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