

On the Stability of the Kuramoto Model of Coupled Nonlinear Oscillators

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Abstract

We provide a complete analysis of the Kuramoto model of coupled nonlinear oscillators with uncertain natural frequencies and arbitrary interconnection topology. Our work extends and supersedes existing, partial results for the case of an all-to-all connected network. Using tools from spectral graph theory and control theory, we prove that for couplings above a critical value all the oscillators synchronize, resulting in convergence of all phase differences to a constant value, both in the case of identical natural frequencies as well as uncertain ones. We further explain the behavior of the system as the number of oscillators grows to infinity.

1 Background and Introduction

Over the past decade, considerable attention has been devoted to the problem of coordinated motion of multiple autonomous agents. In a variety of disciplines (ranging from ecology and evolutionary biology to the social sciences, and from systems and control theory to complexity theory, statistical physics and computer graphics), researchers have been developing an understanding of how a group of moving objects (such as flocks of birds, schools of fish, crowds of people [16, 29], or collections of autonomous robots or unmanned vehicles [27, 28]) can reach a consensus and move in formation without centralized coordination. This has coincided with a surge of interest in the area of network dynamics, which focusses on the relationship between graph structure and dynamical behavior of large networks of diverse origin—for two excellent reviews of this topic, see [23] and [19].

A classic example of distributed coordination in physics, engineering and biology is the synchronization of arrays of coupled nonlinear oscillators [22, 24–26, 37]. Building on long-standing experiments (from Huyghens to van der Pol), the problem of collective synchronization was posed mathematically by the Russian school of Andronov. Norbert Wiener [33, 34] recognized its ubiquity in the natural world, and even speculated about its relevance to the generation of characteristic rhythms in the brain [24, 26]. In collective synchronization, a large system of nonlinear oscillators spontaneously locks to a common frequency, despite the variations in the natural frequencies of the individual oscillators [25, 36, 37]. This is a pervasive phenomenon in biology: from pacemaker cells in the heart [17, 21] and circadian centers in the the brain [15] to metabolic synchrony in yeast cell suspensions [1, 6] and congregations of synchronously flashing fireflies [3] or chirping crickets [31]. There are also several examples in physics and engineering, including arrays of lasers [10], phase locking in microwave and RF oscillators [5, 38], computer clock synchronization, and superconducting Josephson junctions [35].

Following on key insights by Winfree [37], Kuramoto [12–14] proposed in the 1970s a tractable model for oscillator synchronization that has become archetypal in the physics and dynamical systems literatures. (See [22, 23] for an excellent review of the state-of-the-art on the analysis of this model.) Interestingly, a few researchers in the control community have recently started to look into the problem of synchronization, coordination and reaching consensus among multi-agent systems [9, 20] as well as synchronization phenomena among nonlinear oscillators [11, 18, 32].

2 Model Description

The Kuramoto model describes the dynamics of a set of N phase oscillators θ_i with natural frequencies ω_i . The time evolution of the i -th oscillator is given by:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad (1)$$

where K is the coupling strength, a parameter in the problem. One of Kuramoto’s results was to show numerically that when the ω_i ’s are randomly chosen from a Cauchy probability distribution in the infinite N limit, there is a critical value of the coupling above which all phase differences remain constant, resulting in *synchronization* of the oscillators [12, 13].

It is helpful to think of the N oscillators as points on a circle moving with different angular velocities. In the synchronized regime, the points move around the circle in sync, meaning that the phase differences remain constant. In his analysis, Kuramoto used the magnitude r of the centroid of the points as a ‘natural’ measure of synchronization:

$$r e^{j\psi} = \frac{1}{N} \sum_{i=1}^N e^{j\theta_i}. \quad (2)$$

Clearly, if all the ω_i ’s are the same then $r = 1$ when all agents are in sync. When the natural frequencies are not identical, r converges to a constant less than 1 when the oscillators synchronize. On the other hand, when all agents are completely out of phase with respect to each other the value of r remains close to 0 most of the time. This is why in the physics literature r is referred to as the *order parameter*.

In his analysis, Kuramoto used simple trigonometric identities to rewrite the state equation (1) in terms of the order parameter r . After switching to a rotating frame (which is equivalent to translating each phase from θ to $[\theta - (\sum_{i=1}^N \omega_i/N)t]$, effectively centering the natural frequencies) Kuramoto’s analysis revealed that

$$\dot{\theta}_i = \omega_i - \frac{K}{N} r \sin(\theta_i - \psi). \quad (3)$$

In other words, each phase is modulated by the magnitude r and phase ψ of the *average* phasor. In physics notation, this constitutes a *mean field* or “all-to-all” model.

Sporting some brilliant intuition, Kuramoto showed that when the number of oscillators N is infinite and for a coupling strength K beyond a critical value K_c , at least two oscillators start to synchronize. Beyond K_c , there is an additional critical coupling value K_L beyond which all oscillators remain synchronized, the order parameter r grows exponentially in time and then saturates at a value $r_\infty(k) < 1$. He also calculated analytically the value for K_c and r_∞ for a few well-known distributions. The branch of r corresponding to $K > K_c$ is called the synchronized state, while $K < K_c$ corresponds to the unsynchronized state.

Despite its success, Kuramoto’s analysis raised a lot of questions. For instance, what does it mean that r stays close to zero in the unsynchronized state $K < K_c$? This is certainly not true at all times: when $K = 0$ and the ω_i ’s are irrational with respect to each other, the trajectories are dense on the N -torus resulting in an r which will almost surely visit any number between 0 and 1. However, simulations indicate that it is true most of the time. Since its introduction in mathematical physics, the Kuramoto model still remains a puzzle even in the $N \rightarrow \infty$ limit with all-to-all connections. On the other extreme, the case of few oscillators has been tackled in the dynamical systems literature with rigorous bifurcation analysis. However, even basic results are not available for the finite N case. Indeed, as in many similar problems, the regime when there are a large but finite number of oscillators is of utmost interest. For a complete (and beautiful) examination of the Kuramoto model we refer the reader to Steve Strogatz’s recent review [22].

Our goal here is to analyze the model from a system theoretic point of view in the finite N case with arbitrary connectivity. In order to do so, we rewrite the model in terms of the incidence matrix of the undirected graph that describes the interconnection topology, and then consider the all-to-all connected case (i.e., the case of a complete graph) as a specific example. We also provide several necessary as well as sufficient lower bounds for the critical value of the coupling. These include a bound for K below which there is no fixed-point, and a value of K above which there is a unique fixed-point. We also show that contrary to the case of infinite oscillators, there is no partial synchronization phenomena. Moreover, we show that the critical value of the order parameter is *not* close to 0 as shown in Kuramoto’s original analysis. In other words, the generic branching of r at the critical value does not happen when N is finite. We remark that this was also observed by [8] in the case of 2 oscillators with a finite possible values for the natural frequencies.

3 Graph Theory Preliminaries

First, we introduce some graph theory terminology [7] which will be used in the following. An (undirected) graph \mathcal{G} consists of a n -dimensional vertex set, \mathcal{V} , and an e -dimensional edge set \mathcal{E} , where an edge is an unordered pair of distinct vertices in \mathcal{G} . If $x, y \in \mathcal{V}$, and $(x, y) \in \mathcal{E}$, then x and y are said to be adjacent, or neighbors, and we denote this by writing $x \sim y$. A graph is called complete if any two vertices are neighbors. The number of neighbors of each vertex is its valency d_i . A path of length ℓ from vertex x to vertex y is a sequence of $(\ell + 1)$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph \mathcal{G} , then \mathcal{G} is said to be connected.

The adjacency matrix $A(\mathcal{G}) = [a_{ij}]$ of an (undirected) graph \mathcal{G} is a $n \times n$ symmetric matrix with rows and columns indexed by the vertices of \mathcal{G} , such that $a_{ij} = 1$ if vertex i and vertex j are neighbors and $a_{ij} = 0$, otherwise. The valency matrix $D(\mathcal{G})$ of a graph \mathcal{G} is a diagonal matrix with rows and columns indexed by \mathcal{V} in which the (i, i) -entry is the valency of vertex i , $D(\mathcal{G}) = \text{diag}(d_i)$. An orientation of a graph \mathcal{G} is the assignment of a direction to each edge, so that the edge (i, j) is now an arc from vertex i to vertex j . We denote by \mathcal{G}^σ the graph \mathcal{G} with orientation σ . The incidence matrix $B(\mathcal{G}^\sigma)$ of an oriented graph \mathcal{G}^σ is the $n \times e$ matrix whose rows and columns are indexed by the vertices and edges of \mathcal{G} respectively, such that the i, j entry of $B(\mathcal{G}^\sigma)$ is equal to 1 if the edge j is incoming to vertex i , -1 if edge j is outgoing from vertex i , and 0 otherwise.

The symmetric matrix defined as:

$$L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G}) = B(\mathcal{G}^\sigma)B(\mathcal{G}^\sigma)^T$$

is called the Laplacian of \mathcal{G} and is independent of the choice of orientation σ . The Laplacian has several important properties. Among those: L is always positive semidefinite with a zero eigenvalue;

the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph; and the n -dimensional eigenvector associated with the zero eigenvalue is the vector of ones, $\mathbf{1}_n$. It is known that the spectrum of the Laplacian matrix $\{\lambda_i(L)\}$ captures many topological properties of the graph. Specifically, the first non-zero eigenvalue $\lambda_2(L)$, denoted the algebraic connectivity, gives a measure of connectedness of the graph. It is important to note the extreme simplicity of the Laplacian of the complete (all-to-all) graph:

$$L_c = B_c B_c^T = nI - \frac{\mathbf{1}_n \mathbf{1}_n^T}{n}. \quad (4)$$

It is the very special symmetry of this Laplacian that allows the analytical simplifications observed in the well-studied mean field case of the Kuramoto model.

If we associate a positive number W_i to each edge and form the diagonal matrix $W := \text{diag}(W_i)$, then the matrix

$$L_w(\mathcal{G}) = B(\mathcal{G}^\sigma) W B(\mathcal{G}^\sigma)$$

is a weighted Laplacian of \mathcal{G} . The weighted Laplacian also enjoys the above properties.

Using the above terminology, we can extend the Kuramoto model to any general interconnection topology. If \mathcal{N}_i is the set of adjacent vertices of node i , the general interconnected case of Eq. (1) becomes:

$$\dot{\theta}_i = \omega_i - \frac{K}{N} \sum_{j \in \mathcal{N}_i} \sin(\theta_i - \theta_j),$$

which can be rewritten in matrix form as:

$$\dot{\theta} = \omega - \frac{K}{N} B \sin(B^T \theta). \quad (5)$$

Here B is the incidence matrix of an arbitrary orientation of the unweighted graph, and θ and ω are $N \times 1$ vectors. It is also helpful to define the e -dimensional vector of phase differences $\phi := B^T \theta$.

Remark 1 *In the limit of small angles, the general Kuramoto model (5) gives the continuous-time Vicsek flocking model [30] which was analyzed in [9]:*

$$\dot{\theta} = \omega - \frac{K}{N} B \sin(B^T \theta) \approx \omega - \frac{K}{N} (BB^T) \theta.$$

Conversely, the classic Kuramoto model (1) could be thought of as a nonlinear extension of the Vicsek model for a complete graph.

4 Synchronization of identical coupled oscillators

We start by considering the general Kuramoto model (5) in its unperturbed version, i.e., when all the natural frequencies ω_i are zero or identical:

$$\dot{\theta} = -\frac{K}{N} B \sin(B^T \theta). \quad (6)$$

(Note that by switching to a rotating frame if necessary, we can assume that the natural frequencies are all zero without loss of generality.)

Theorem 1 Consider the unperturbed Kuramoto model (6) defined over an arbitrary connected graph with incidence matrix B . For any given $\theta_0 \in \mathbb{R}$ and any value of the coupling $K > 0$, the vector $(\theta_0 \mathbf{1}_N)$ is an asymptotically stable equilibrium solution, i.e., the synchronized state is globally asymptotically stable over any compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^N$. Moreover, the rate of approach to equilibrium is no worse than $2\frac{K}{N}\lambda_2(L)/\pi$.

Proof: Consider the Lyapunov function candidate

$$U(\theta) = \frac{N^2 - N - 2\mathbf{1}_e^T \cos(B^T \theta)}{N^2} := 1 - R^2. \quad (7)$$

A simple calculation reveals that $\nabla U(\theta) = (2/N^2)B \sin(B^T \theta)$ which leads to

$$\dot{U}(\theta) = \nabla U(\theta) \dot{\theta} = -\frac{2}{KN} \dot{\theta}^T \dot{\theta} \leq 0.$$

Therefore, U is a non-increasing function along the trajectories of the system. Moreover $U \geq 0$. It is easy to show that $0 \leq U \leq 1$. By using LaSalle's invariance principle over any compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^N$, we can conclude that U is a Lyapunov function for the system, and that all trajectories converge to the set where $\dot{\theta}$ is zero.

Define the $e \times e$ diagonal matrix $W(\theta) := \text{diag}(\text{sinc}(\phi_i))$, where e is the number of edges in the graph, and $\text{sinc}(\phi) = \sin(\phi)/\phi$ is positive for $\phi \in (-\pi, \pi)^e$. Thus, for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $W(\phi) > 0$ can be thought of as phase-dependent weight functions on the graph. As a result all trajectories converge to the fixed-points, which are the solutions of

$$BW(\phi)B^T \theta = 0.$$

The fact that $\theta_0 \mathbf{1}_N$ is the only equilibrium solution follows easily by noting that for any connected graph, the null space of the weighted Laplacian contains only the vector of all ones, i.e., $B^T \mathbf{1}_N = \mathbf{0}$.

We also point out that an alternative strategy is to consider consider the quadratic Lyapunov function candidate

$$V(\theta) = \frac{1}{2} \theta^T \theta.$$

A simple calculation reveals that

$$\dot{V}(\theta) = \nabla V(\theta) \dot{\theta} = -\frac{K}{N} \theta^T B \sin(B^T \theta).$$

It then follows that

$$\dot{V}(\theta) = -\frac{K}{N} \theta^T BW(\phi)B^T \theta \leq 0.$$

We now invoke LaSalle's invariance principle over any compact subset of $(-\pi, \pi)^e$, with the zero case occurring only when $\phi = B^T \theta$ is zero. When the graph is connected, this only happens at $\theta = \theta_0 \mathbf{1}_N$, which is the only invariant set for $\dot{V} = 0$. We could further conclude that over any compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$, the convergence is exponential with the rate determined by the algebraic connectivity of the weighted graph, the second smallest eigenvalue of the weighted Laplacian. While the first Lyapunov function provides a stronger decrease, it is harder to get an estimate on the rate of convergence. With the quadratic function $V(\theta)$ however, over the aforementioned region we have

$$\dot{V} \leq -\frac{K}{N} \lambda_2(BW(\phi)B^T) \|\theta_{\mathbf{1}^\perp}\|^2.$$

Where $\theta_{\mathbf{1}^\perp}$ is the centered component of θ . As the algebraic connectivity of the weighted graph can be bounded by the minimum weight and $\lambda_2(BB')$, the convergence rate will be no worse than $\frac{2K}{\pi N} \lambda_2(BB')$. \square

Corollary 1 *In the special case of the complete graph,*

$$\dot{V} \leq -\theta^T B_c W (B_c^T \theta) B_c^T \theta$$

with the synchronization rate being no worse than $2K/\pi$, since $\lambda_2(L_c) = N$.

Remark 2 *We note that similar results also hold even if the topology of the graph changes in time. Using arguments similar to those in [9], one can extend the result to general notions of connectivity, i.e., when the interconnection graph is not connected at all times but there is a path between any two nodes over contiguous, non-overlapping, and uniformly bounded time intervals. It is also possible to generalize to the case of directed graphs by introducing notions of weak connectivity [18].*

Remark 3 *The synchronization argument could be readily extended to the case of a more complicated coupling function $f(\cdot)$ other than the $\sin(\cdot)$ function, so long as $\phi^T f(\phi) \geq 0$.*

Theorem 1 states that as long as the graph is connected and the value of the coupling is positive, all of the oscillators asymptotically synchronize, suggesting that the order parameter should grow exponentially and asymptotically reach the value 1. In the next proposition, we show that an appropriate order parameter can be defined for the case of any connected graph, and show that it does indeed increase exponentially. This is a generalization of the synchronization order parameter introduced for the all-to-all case.

Proposition 2 *Consider the Kuramoto model in the case of an all-to all (complete) graph. Then, starting from any initial condition from any compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^N$, the order parameter grows exponentially. Moreover, $N^2(1 - r^2)$ is a valid Lyapunov function for the Kuramoto model.*

Proof: Through some algebra and trigonometrical identities, the square of the order parameter (2) can be written as:

$$r^2 = \frac{1}{N^2} \left[\left(\sum_{i=1}^N \cos \theta_i \right)^2 + \left(\sum_{i=1}^N \sin \theta_i \right)^2 \right] = \frac{1}{N^2} [N + 2 \mathbf{1}_e^T \cos(B_c^T \theta)].$$

It can be shown that the derivative of this function along the trajectories of the Kuramoto model

$$\dot{\theta} = -\frac{K}{N} B_c \sin(B_c^T \theta)$$

is always non-negative, and it is zero only at the equilibrium solution $\theta_0 \mathbf{1}_N$ for any component of θ_0 in a closed subset of $(-\frac{\pi}{2}, \frac{\pi}{2})$.

To see this, evaluate the derivative of r^2 , from which we obtain

$$\frac{dr^2}{dt} = \frac{1}{N^2} \frac{K}{N} [(\sin B_c^T \theta)^T (B_c^T B_c) (\sin B_c^T \theta)] = -\frac{K}{N} \dot{\theta}^T \dot{\theta} \leq 0.$$

We again invoke LaSalle's invariance principle and note that

$$\|\dot{\theta}\|^2 = \|L_w(\theta)\theta\|^2$$

which for a connected graph is zero only if $\theta = \theta_0 \mathbf{1}_N$. We also note that this is the only equilibrium solution. This ensures that away from equilibrium, r^2 (and therefore r) is an increasing function, and being upper bounded, it follows that at the limit, $r(t)$ converges to its upper bound value 1. From this, it is clear that $N^2 - \mathbf{1}_e^T \cos(B_c^T \theta)$ qualifies as a Lyapunov function for the Kuramoto model, where $\mathbf{1}_e$ is the vector of all ones in the edge space. \square

Remark 4 The function $\mathbf{1}_e^T \cos(B^T \theta)$ is an energy function for the XY-model in statistical physics. It was considered as a Lyapunov function for the Kuramoto model by Van Hemmen and Wreszinsky [8], as well as in [11].

Remark 5 The above proposition indicates that a more appropriate notion of the order parameter is indeed $N^2 r^2 = [N^2 - 2e + 2 \mathbf{1}_e^T \cos(B^T \theta)]$. While inspired by the notion of the magnitude of the average phasor, this definition does extend to the case of any connected graph, whereas the centroid does not.

5 The case of non-identical oscillators

In this section we treat the more complicated case of non-identical oscillators, i.e., when the vector of natural frequencies is non-zero. (Again, without loss of generality we can assume that the natural frequencies are centered around zero, i.e., that the mean has been extracted.) Although there is an extensive literature for the $N \rightarrow \infty$ case with all-to-all connectivity, we will focus here on the case of finite N and arbitrary topology given by Eq. (5). We remark that we consider the frequencies as random perturbations which, albeit drawn from a probability distribution, remain constant in time, i.e., the dynamics (5) is deterministic though uncertain. This is in contrast with some of the treatments in the physics literature, which transform the problem into a Fokker-Planck equation, effectively recasting it as a *stochastic* differential equation.

Synchronization is best defined in a *grounded* system, where the phases are defined with respect to a variable which is taken as reference (or ‘ground’). This is achieved by any projection $V_{N \times (N-1)}$ such that

$$V^T V = I, \quad V V^T = I - \frac{\mathbf{1}_N \mathbf{1}_N^T}{N}, \quad V^T \mathbf{1}_N = \mathbf{0}.$$

Thus, V is a matrix consisting of $N - 1$ orthonormal vectors which are orthogonal to the vector $\mathbf{1}_N$.

If we define the set of grounded coordinates $\bar{\theta} := V^T \theta$ and the set of grounded (or centered) frequencies $\bar{\omega} = V^T \omega$, we can write the generalized Kuramoto model as:

$$\dot{\bar{\theta}} = \bar{\omega} - \frac{K}{N} V^T B \sin(B^T V \bar{\theta}). \quad (8)$$

In this grounded system (8), the synchronized state becomes a fixed point.

This system can be rewritten as

$$\dot{\bar{\theta}} = \bar{\omega} - \frac{K}{N} V^T B W(\bar{\theta}) B^T V \bar{\theta}. \quad (9)$$

where W is defined as

$$W(\bar{\theta}) := \begin{pmatrix} (\text{sinc}(B^T V \bar{\theta}))_1 & \cdots & 0 \\ 0 & (\text{sinc}(B^T V \bar{\theta}))_2 & \cdots \\ \vdots & \cdots & \vdots \\ 0 & \cdots & (\text{sinc}(B^T V \bar{\theta}))_e \end{pmatrix}.$$

The term $BW(\bar{\theta})B^T$ is a *weighted Laplacian* of the graph with incidence matrix B and edge weight W , and is denoted by $L_W(\bar{\theta})$. Note that since the ungrounded state space is a compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^N$, the weights are always positive. The multiplication by V^T and V from left and

right, projects this matrix away from the span of $\mathbf{1}$, and is therefore positive definite. As a result, $V^T L_W(\bar{\theta})V$ is an $N - 1$ dimensional positive definite matrix.

A simple analysis reveals that away from some ellipse-like neighborhood of the origin, the derivative of the positive definite function

$$U(\bar{\theta}) = \frac{1}{2}\bar{\theta}^T\bar{\theta}.$$

is negative. The derivative of this Lyapunov function candidate along the trajectory of the state equation is

$$\dot{U} = \bar{\theta}^T\bar{\omega} - \frac{K}{N}\bar{\theta}^T V^T L_{W(\bar{\theta})} V \bar{\theta}.$$

The set over which this derivative is zero is an ellipse-like shape $\mathcal{E}_{\bar{\theta}}$ given by the equation¹

$$\bar{\theta}^T\bar{\omega} - \frac{K}{N}\bar{\theta}^T V^T L_{W(\bar{\theta})} V \bar{\theta} = 0,$$

which contains both the origin and the fixed points of the system.

We also note that as K is decreased, there is a value below which no fixed point exists, resulting in a running solution of the system of differential equations. We call this value K_c .

We can therefore think of the value of the function V as a measure of disagreement, or a disorder parameter.

As K is increased beyond K_c , fixed points emerge. A sufficient condition for the fixed point θ^* to be stable is for $\phi^* = B^T\theta^*$ to be contained in any closed subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^e$, which implies that $|\theta^*| < \frac{\pi}{4}$. This can be demonstrated by taking the Jacobian of $V^T B \sin B^T\theta$, and noting that this is equal to $V^T B \text{diag}[\cos(B^T\theta^*)]B^T V$, which is positive definite over that set.

Inside the ellipse, $\mathcal{E}_{\bar{\theta}}$, the vector field points outwards, and outside, it points towards the ellipse. However, as the ellipse-like set is not a level set of the Lyapunov function, it is not invariant. As a result, we can only guarantee ultimate boundedness of all trajectories.

The ellipse is centered at the point $\frac{N\bar{\omega}}{2K}$, and it passes through the origin. The algebraic connectivity of the weighted graph (i.e., the first nonzero eigenvalue of $L_{W(\bar{\theta})}$) and the maximum eigenvalue of the weighted Laplacian play a crucial role in the size of the ellipse. Although the derivative of the Lyapunov function is not negative every where, we use an ultimate boundedness argument to show that for $\|\bar{\theta}\|_2 \geq \frac{N}{K}\|\bar{\omega}\|_2$, i.e., outside the smallest sub-level set of the Lyapunov function containing the ellipse, the derivative is decreasing. As a result, $\|\theta\|_2$ is *ultimately bounded* by $\frac{N}{K}\|\omega\|_2$.

This analysis also holds even if the sin function is replaced by any other odd nonlinearity F satisfying $F(\theta)^T\theta > 0$.

An interesting, but slightly different type of analysis could be performed by considering a Lyapunov function candidate based on the square of the order parameter. Let

$$r^2 = \frac{1}{N^2} \left| \sum_{i=1}^N e^{j\theta_i} \right|^2.$$

The derivative of this function along the trajectories can be written as

$$\dot{r}^2 = \frac{1}{N^2} \left[\frac{K}{N} (\sin B^T\theta)^T B^T B (\sin B^T\theta) - \omega^T B \sin B^T\theta \right].$$

¹It is not quite an ellipse since the axes are changing with the state.

which is again an elliptical type region in the $\sin(B^T\theta)$ coordinate centered at $\frac{N\omega}{K}$. Similar to the previous discussion, outside a neighborhood of the origin given by

$$\|B \sin(B^T\theta)\|_2 > \frac{N}{K} \|\omega\|_2 \quad (10)$$

the derivative is positive, resulting in growth of the order parameter. The boundary of this region contains the equilibria. By using a similar ultimate boundedness argument, the trajectories are confined to the smallest sublevel-set of r containing the set 10.

We now use (10) to obtain an estimate of the asymptotic value of the order parameter. The vector $\sin(B^T\theta)$ can be decomposed into two orthogonal components: $y_1(\theta)$, in the null space of B , and $y_2(\theta)$ in the range space of B^T . The first component is annihilated when it is multiplied by B on the left. As a result, the region over which \dot{r}^2 is positive can be characterized as

$$\|y_2(\theta)\|_2 > \frac{N}{K\sqrt{\lambda_2(L)}} \|\omega\|_2.$$

where $\lambda_2(L)$ is the algebraic connectivity of the un weighted graph. We now try to bound the value of U over the region where \dot{U} is negative.

A simple bounding reveals that

$$2\mathbf{1}^T \cos(B^T\theta) \leq \|\mathbf{1}\|^2 + \|\cos(B^T\theta)\|^2 = 2\|\mathbf{1}\|^2 - \|\sin(B^T\theta)\|^2,$$

from which

$$r^2 \leq \frac{N^2 - \|\sin B^T\theta\|^2}{N^2} \leq \frac{N^2 - \|y_2(\theta)\|^2}{N^2} \leq \frac{N^2 - \frac{N^2\|\omega\|^2}{K^2\lambda_2(L)}}{N^2}.$$

We can immediately observe that the asymptotic behavior of the order parameter is inversely proportional to the algebraic connectivity of the graph. Of course, because of the over-bounding, the bound is conservative—its asymptotic value is 1 as opposed to the actual less-than-one value. Nevertheless, this gives us a bound on the growth rate of U , and, as a result, the growth rate on r is bounded by $\frac{1}{\sqrt{\lambda_2(L)}}$.

This means that asymptotically

$$r \leq \sqrt{1 - \frac{\|\omega\|^2}{K^2\lambda_2(L)}}$$

which would result in an increase rate of $\mathcal{O}(\frac{1}{\sqrt{N}})$ when the graph is *complete*.

Remark 6 *When the natural frequencies are independent random variables chosen from a Normal distribution with zero mean and standard deviation equal to σ , i.e., $\omega_i \sim \mathcal{N}(0, \sigma)$, then $\|\omega\|_2$ scales as $\sqrt{N}\sigma$, which results in a value of r that is independent of N .*

Remark 7 *In [8], the authors have added a linear term $\omega^T\theta$ to the Lyapunov function candidate to guarantee negativity of the derivative every where except at the fixed-points, reducing the perturbed model into a gradient system. The linear term, however, makes the Lyapunov function indefinite.*

We will see in the next section that if K is large enough to guarantee the existence of a unique fixed point (via a contraction argument), this condition (10) will be trivially satisfied. This means that if K is large enough the derivative of the order parameter will be positive, resulting in the asymptotic stability of the synchronized state.

5.1 Critical value of coupling for complete graphs

Our results generalize those of Van Hemmen *et al.* [8] in the case of a complete graph. Specifically, it can be shown that the critical value of the coupling is determined by the value of K for which the fixed point disappears. This can be easily explained by looking at the fixed point equation:

$$B \sin(B^T \theta^*) = \frac{N\omega}{K}.$$

Let $\omega_{max} = \|\omega\|_\infty$ and note that the induced infinity norm of a matrix is the maximum absolute row sum, i.e., $\|B\|_\infty = d_{max}$, where d_{max} is the maximum degree of the graph. In the case of a complete graph, $d_{max} = N - 1$. Then,

$$\frac{N\omega_{max}}{K} \leq d_{max}$$

resulting in the following lower bound for K_c , the coupling above which a fixed point exists

$$K_c > \frac{N\omega_{max}}{d_{max}}.$$

This bound can be tightened by using the generalized inverse of $V^T B$ and bounding the component of the $\sin(B^T \theta)$ in the range of B^T . The generalized inverse, denoted by $(V^T B)^\#$, is equal to $B^T V \Lambda^{-1}$, where Λ is the $N - 1$ diagonal matrix of the eigenvalues of the unweighted Laplacian. We therefore have the following expression

$$(\sin(B^T \theta))_{R(B^T)} = B^T V \Lambda^{-1} V^T \frac{N\omega}{K}.$$

Noting that $L^\# = V \Lambda^{-1} V^T$, we have

$$(\sin(B^T \theta))_{R(B^T)} = B^T L^\# B \sin(B^T \theta) = B^T L^\# \frac{N\omega}{K}.$$

The generalized inverse of the Laplacian, in the case of a complete graph can be written as $L_c^\# = \frac{1}{N}(I - \frac{\mathbf{1}\mathbf{1}^T}{N})$. Noting that the infinity norm of the sin vector is less than or equal to 1, and that $B^T L^\# B = \frac{B^T B}{N}$, we have

$$\frac{\|B^T \omega\|_\infty}{K} \leq \frac{\|B^T B\|_\infty}{N},$$

which gives us the bound

$$K_c \geq \|B^T \omega\|_\infty \frac{N}{2(N-1)}.$$

This is in excellent agreement with that of Van Hemmen *et al.* [8] which they obtained for the simplest case of two oscillators. We remark that if the graph is a tree, $V^T B$ has full row rank and $\sin(B\theta)$ does not have a component in the null space of L . In that case $K_c > \|B^T L^\# \omega\|_\infty$ is a tight bound, meaning that it is necessary and sufficient for synchronization. In the general case, however, this bound is just necessary. We will now provide sufficient conditions for the existence of a stable fixed-point.

6 Existence and uniqueness of stable fixed points

The fixed point equation can be written as

$$\theta = (BW(B^T\theta)B^T)^\# \frac{N\omega}{K} = L_W^\#(B^T\theta) \frac{N\omega}{K}.$$

Using Brouwer's fixed point theorem², we can develop conditions which guarantee the existence (but not uniqueness) of the fixed point. If a fixed-point exists in any compact subset of $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, it is stable, since this will ensure that $B^T\theta$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We therefore have to ensure that

$$K > \frac{4}{\pi} N \max_{|\theta_i| < \frac{\pi}{4}} \|L_W^\#(B^T\theta)\|_\infty \|\omega\|_\infty.$$

In the case of a complete graph, the infinity norm of the matrix $L_W^\#$ scales as $\mathcal{O}(\frac{1}{N})$. In general, however, it is not clear how the infinity norm of $L_W^\#$ scales with N . It is worth mentioning that the norm of $L_W^\#$ is a well studied object in the theory of Markov chains. The infinity norm of $L_w^\#$ is a measure of the sensitivity of the stationary distribution of the chain associated with L with respect to additive perturbations [4].

If the uncertain natural frequencies are 2-norm bounded, a better strategy would be to impose the boundedness condition with respect to the Euclidean norm. A sufficient condition for local stability of the fixed-point is for θ_i to belong to $(-\frac{\pi}{4}, \frac{\pi}{4})$, this amounts to having the Euclidean norm of θ be less than $\frac{\pi}{4}\sqrt{N}$. Again, using Brouwer's sufficient condition for existence of fixed-points we have:

$$\|BW(B^T\theta)B^T)^\# \|_2 \frac{N\|\omega\|_2}{K} \leq \frac{\pi}{4}\sqrt{N}.$$

Hence, a sufficient condition for synchronization of all oscillators can be determined in terms of a lower bound for K as follows:

$$K \geq \frac{4}{\pi} \frac{\sqrt{N}\|w\|_2}{\min_{|\theta_i| \leq \frac{\pi}{4}} \lambda_2(L_W(\theta))},$$

where we used the fact that $\|(BW(B^T\theta)B^T)^\# \|_2 = \frac{1}{\lambda_2(L_W)}$, and λ_2 is the algebraic connectivity of the (weighted) graph. A lower bound on the minimum value of λ_2 occurs for the minimum value of the weight which is $\frac{2}{\pi}$. As a result

$$K \geq K_c := 2 \frac{\sqrt{N}\|w\|_2}{\lambda_2(L)} \tag{11}$$

Remark 8 *Using the upper bound provided for the order parameter earlier, we can derive an upper bound for the asymptotic value of r at K_c : $r_\infty(K_c) \leq \frac{\sqrt{3}}{2}$. Furthermore, if the stable fixed-point is in $(-\pi/4, \pi/4)^N$, then the order parameter is lower bounded by $\sqrt{16 - \pi^2}/4$. This means that the contrary to the infinite N case, r is not close to zero at the critical point. Therefore the intuition that r has to be small before K_c is misleading.*

²Brouwer's fixed-point theorem states that a continuous function that maps a non-empty compact, convex set X into itself has at least one fixed-point

6.1 Existence of a unique fixed-point

In order to guarantee the existence of a unique fixed point we use Banach's contraction principle and ensure that the right hand side is a contraction. By noting that the Lipschitz constant for the $\text{sinc}(\cdot)$ function is $\alpha_s = \frac{1}{2}$, we provide a sufficient condition for contractivity (and therefore uniqueness of the fixed-point).

We impose the contractivity condition on the $N - 1$ dimensional grounded system. In the grounded case, we have $\bar{\theta} = V^T \theta$, and

$$\bar{\theta} = (V^T B W (B^T \theta) B^T V)^{-1} \frac{N V^T \omega}{K}.$$

After some algebra, the contraction requirement amounts to

$$K \geq \frac{\pi^2}{4} \frac{N \lambda_{max}(L) \|w\|_2}{\lambda_2(L)^2}, \quad (12)$$

where λ_{max} is the largest eigenvalue of the Laplacian of the graph.

Interestingly, this value of K also ensures that the derivative of r^2 is increasing, i.e., inequality (10) is satisfied, which means that the order parameter is increasing. Of course this is probably stronger than what is necessary for uniqueness, as the contraction argument is only sufficient. Nevertheless, we see that there is a large enough but finite value of the coupling which guarantees the existence and uniqueness of fixed points. We therefore have the following theorem whose proof is given in the appendix:

Theorem 3 *Consider the Kuramoto model for non-identical coupled oscillators with different natural frequencies ω_i . For $K \geq K_c := 2 \frac{\sqrt{N} \|w\|_2}{\lambda_2(L)}$, there exist at least one fixed-point for $|\theta_i| < \frac{\pi}{4}$ or $|(B^T \theta)_i| < \frac{\pi}{2}$. Moreover, for $K \geq \frac{\pi^2}{4} \frac{N \lambda_{max}(L) \|w\|_2}{\lambda_2(L)^2}$ there is only one stable fixed-point (modulo a vector in the span of $\mathbf{1}_N$), and the order parameter is strictly increasing.*

7 Concluding remarks

In this paper we provided a complete stability analysis for the Kuramoto model of coupled nonlinear oscillators. We showed that when the oscillators are identical, there are at least two Lyapunov functions which prove asymptotic stability of the synchronized state, when all the phase differences are bounded by $\frac{\pi}{2}$. We also showed that when the natural frequencies are not the same, there is a critical value of the coupling below which a fixed-point does not exist. Several bounds for this critical value was developed, based on norm bounded uncertain natural frequencies. these bounds are in excellent agreement with the existing bounds in the physics literature, for the case of all to all connected graph. We also point out that contrary to the infinite N case, there is no partially synchronized state, i.e., for values of the coupling below the critical value, the system of differential equations has a running solution. Furthermore, we showed that there is always a large enough finite value of the coupling which results in synchronization of oscillators and convergence of the angles to a unique fixed-point. Another important result of this paper is that contrary to the common belief, the value of the order parameter is not zero for the critical value of the coupling. In fact, at least when the fixed-point is in the $(-\pi/2, \pi/2)$ region, a rough estimate indicates that the value of r is bounded between $\frac{\sqrt{16-\pi^2}}{4} \approx 0.62$ and $\frac{\sqrt{3}}{2}$. Future research in this direction is needed to determine the bound for K when the natural frequencies are not just norm bounded quantities but uncertain numbers chosen from a probability distribution. Finally we mention that our value

for the upper bound of the order parameter is actually quite close to the actual numbers as seen in simulations.

Lastly, we would like to mention that there is a lot to be gained by the marriage of systems and control theory and graph theory, when one wants to study dynamical systems over networks [2].

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9 Appendix

Proof of Theorem 3 : As we see before fixed point equation can be written as

$$V^T B W(\bar{\theta}) B^T V \bar{\theta} = \frac{K}{N} \bar{\omega} \quad (13)$$

or

$$\bar{\theta} = (V^T B W(\bar{\theta}) B^T V)^{-1} \frac{K}{N} \bar{\omega} \quad (14)$$

$$= L_W(\bar{\theta})^{-1} \frac{K}{N} \bar{\omega} \quad (15)$$

We will Banach's contraction principle to show that (15) has a unique solution when $V\bar{\theta}$ is in any compact subset of $(-\frac{\pi}{2}, \frac{\pi}{2})^N$. We therefore need to show that

$$\left\| \frac{N}{K} (L_W(\bar{\theta}_1)^{-1} - L_W(\bar{\theta}_2)^{-1}) \bar{\omega} \right\| \leq \alpha \|\bar{\theta}_1 - \bar{\theta}_2\| \quad (16)$$

holds for some $0 \leq \alpha < 1$ and some norm. Using the 2-norm, we have

$$\begin{aligned} \left\| \frac{N}{K} (L_W(\bar{\theta}_1)^{-1} - L_W(\bar{\theta}_2)^{-1}) \bar{\omega} \right\|_2 &= \left\| \frac{N}{K} L_W(\bar{\theta}_1)^{-1} (L_W(\bar{\theta}_2) - L_W(\bar{\theta}_1)) L_W(\bar{\theta}_2)^{-1} \bar{\omega} \right\|_2 \\ &\leq \frac{N}{K} \|L_W(\bar{\theta}_1)^{-1}\|_2 \|L_W(\bar{\theta}_2)^{-1}\|_2 \|L_W(\bar{\theta}_2) - L_W(\bar{\theta}_1)\|_2 \|\bar{\omega}\|_2 \\ &\leq \frac{N}{K} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_1))} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_2))} \|V^T B (W(\bar{\theta}_1) - W(\bar{\theta}_2)) B^T V\|_2 \|\omega\|_2 \\ &\leq \frac{N}{K} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_1))} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_2))} \|V^T B\|_2 \|W(\bar{\theta}_1) - W(\bar{\theta}_2)\|_\infty \|B^T V\|_2 \|\omega\|_2 \\ &\leq \frac{N}{K} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_1))} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_2))} \lambda_{\max}(L) \|W(\bar{\theta}_1) - W(\bar{\theta}_2)\|_\infty \|\omega\|_2 \\ &\leq \frac{N}{K} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_1))} \frac{1}{\lambda_{\min}(L_W(\bar{\theta}_2))} \lambda_{\max}(L) \alpha_s \|B^T\|_\infty \|\bar{\theta}_1 - \bar{\theta}_2\|_\infty \|\omega\|_2 \quad (17) \end{aligned}$$

α_s is the Lipschitz constant of $\text{sinc}(\cdot)$ which is 0.5. For a weighted graph with non-negative weights $W = [w_{ij}]$, the second smallest non-zero eigenvalue is given by [F. Chung]

$$\lambda_{\min}(L_W(\bar{\theta})) = \lambda_2(L_W(\theta)) = (N-1) \inf_{f \perp D \mathbf{1}} \frac{\sum_{i \sim j} (f_i - f_j)^2 w_{ij}}{\sum_v f_i^2 d_i} \quad (18)$$

with $d_i = \sum_j w_{ij}$ and D denotes the diagonal matrix with the (i, i) -th entry having value d_i . Since $B^T \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})^e$, we have $\frac{2}{\pi} \leq w_{ij} \leq 1$. Therefore,

$$\lambda_2(L_W(\theta)) \geq \frac{2}{\pi} \lambda_2(L) \quad (19)$$

Applying (19) to (17), we get

$$\left\| \frac{N}{K} (L_W(\bar{\theta}_1)^{-1} - L_W(\bar{\theta}_2)^{-1}) \bar{\omega} \right\|_2 \leq \frac{N}{K} \left(\frac{\pi}{2} \right)^2 \frac{\lambda_N(L)}{\lambda_2(L)^2} \|\bar{\theta}_1 - \bar{\theta}_2\|_\infty \|\omega\|_2 \quad (20)$$

From here, we can find value of K 's that make this mapping contractive as follows

$$\frac{N}{K} \left(\frac{\pi}{2} \right)^2 \frac{\lambda_N(L)}{\lambda_2(L)^2} \|\omega\|_2 < 1 \quad (21)$$

or

$$K > \left(\frac{\pi}{2} \right)^2 \frac{N \lambda_N(L)}{\lambda_2(L)^2} \|\omega\|_2 \quad (22)$$

As a result for $K > K_c$, where $K_c = \left(\frac{\pi}{2} \right)^2 \frac{N \lambda_N(L)}{\lambda_2(L)^2} \|\omega\|_2$, the fixed-point equation (15) has a unique and stable solution. \square