Warranty Pricing with Product Failures and Forward Looking Consumers: An Empirical Approach

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October 2012

Abstract

This paper considers warranty pricing for durable goods, taking possible product failures into account. Consumers are forward looking and solve a dynamic program to make purchase decisions. Due to the large number of products and remaining warranty lengths, the consumers' problem and the corresponding estimation problem are high dimensional. Therefore, we introduce the so-called inclusive value processes to capture the consumers' "value-to-go", which reduces the dimension of the estimation problem. We estimate consumers' preferences using the resulting structural model and individual level transaction data on product and warranty purchases from an electronics retail chain. By conducting counterfactual experiments, we find that extended warranties are generally underpriced in our data set; and the optimal price differs by brand. Specifically, extended warranties are overpriced for high-end brand Sony and underpriced for low-end brand RCA. Lastly, we explore the impact of product reliability on the retailer's profit.

Keywords: Warranty, Durable goods, Forward looking consumers, Structural estimation

1 Introduction

An extended warranty is an insurance product that prolongs the basic warranty provided by the manufacturer to consumers. Extended warranties are usually offered by retailers, manufacturers or warranty administrators. Consumers pay an extra premium usually during the purchase of the product to have an extended warranty against possible problems or breakdowns in the future.

The extended warranties were introduced by major electronics retail chains in late 80's and have been sold aggressively ever since. Marketing warranties costs almost nothing, and the products they cover usually do not need repairs during the warranty period. Therefore, extended warranties are virtually pure profit for electronics retailers such as Best Buy. The profit margins on extended warranty contracts are around 50% to 60%, nearly 18 times the margin on products themselves. In 2003, warranty sales accounted for only 3-4% of total revenues but warranty profits covered almost half of Best Buy's operating income, according to Berner (2004).

Given that the nation's largest electronics retailers count on extended warranty sales, it is crucial for them to understand how to price these insurance products accounting for the product and consumer characteristics in a dynamic environment. Therefore, our focus is to build a dynamic model of the extended warranty pricing with forward-looking consumers.

The product and extended warranty purchasing data comes from INFORMS Society for Marketing Science (ISMS) Durable Goods Dataset 1. This is a panel data set containing the transactions of 19,936 households made between December 1998 and November 2004 at a major U.S. consumer electronics retailer. There are a total of 173,262 transactions, including purchases and returns of products as well as extended warranties. Among 292 product categories, ranging from big ticket items such as computers to small ticket items such as CDs and batteries, we focused on the purchase records of televisions and related extended warranties. There are 6,627 individual transaction records of television purchases in total. We observe characteristics of televisions such as their brand and size, as well as 1850 combined product and extended warranty purchases in the data set.

We provide a structural estimation framework of consumers' dynamic purchasing behavior, and estimate the parameters of the model. Consumers are divided into discrete demographic groups; parameters of each group are estimated separately. We also account for different product characteristics in our estimation, and take a quarter as the period length.

In the model, consumers decide whether to buy a television to maximize their expected discounted utilities. If the consumer owns a functioning product, she gains utility from the product during its lifetime. The product may break down in each period. When it fails, it is repaired instantaneously at no cost to the consumer if it is still under warranty. However, if the warranty has expired, then the consumer buys a new product to replace the broken one. We assume that consumers do not repair broken televisions if they are out of warranty. Instead, they buy new televisions immediately to replace the broken ones. We only consider major problems as failures. For televisions with major problems, the cost of repair can be quite high. Moreover, in our estimation, we only include traditional cathode ray tube (CRT) televisions, whose prices were relatively low. Therefore, it makes more sense for a household to buy a new television than to repair a failed one. Additionally, the television (the product we focus on) is an essential electronic product for modern households. For households who watch TV, it is hardly the case that they will wait several months after their televisions break down before they buy new ones. Thus, for simplicity, we assume that they act immediately in our model.

A consumer may choose to buy a new product even if she has a functioning one. Her decision depends on her current product and its remaining warranty as well as the new products available for purchase (in the current and future periods) and their prices. If the consumer decides to buy a new product, she first chooses the product and then decides whether to purchase the extended warranty.

Because each consumer considers all available products in the current period as well as potential products in future periods (and their prices) repeatedly, this leads to a high-dimensional estimation problem, which is computationally intractable. Therefore, we introduce a one-dimensional inclusive value process to capture a consumer's future expected benefit. We show that the lower dimensional estimation problem is equivalent to the original one for modeling the consumer's decisions. The lower dimensional problem allows us to estimate the structural parameters. Using these parameters, we perform several counterfactual studies.

The counterfactual studies explore the impact of changing warranty prices and product reliability on the retailer's revenue and profit. Surprisingly, the extended warranties seem to be underpriced in practice. We observe that the current warranty prices maximize the revenue but not the profit from warranty sales. Increasing warranty prices causes the retailer to sell fewer warranties, but also leads to cost savings. The net effect is an increase in the profit from warranty sales. However, as the extended warranty prices increase, the retailer loses product revenue and profit. Interestingly though, the gain in warranty profits outweighs the loss in product profit. Indeed, increasing the warranty prices by 60% across the board leads to an 8% increase in total profit. We also find that the optimal extended warranty prices vary across different brands. In particular, the extended warranties are overpriced for the high-end brands, e.g., Sony, whereas they are underpriced for the low-end brands, e.g., RCA. Lastly, we find that the impact of improving product reliability on the retailer's profit depends on the brand. In particular, the retailer finds it more profitable to have higher failure rates for the high-end brands, e.g., Sony, but lower failure rates for the low-end brands, e.g., RCA, than the current failure rates.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model. Section 4 discusses the data set used in our estimation. Section 5 presents the estimation and its results. Counterfactual experiments are presented in Section 6. Section 7 concludes. Proofs are relegated to Appendix A throughout.

2 Literature Review

Warranty related problems have been studied in several fields. In the early operations research and management literature, Glickman and Berger (1976) propose a model to solve for the optimal product price and free manufacturer warranty length. Mamer (1987) studies discounted and per unit costs of product warranty from both the producers and the consumers perspective, and analyzes the trade-off between warranty and quality control. Kao and Smith (1993) extend Mamers model to phase-type product lifetimes and simplify the computation. These papers provide mathematical tools for firms to make optimal warranty related decisions, which require various parameters as inputs. Chun and Tang (1995) study a model with free-replacement, fixed-period warranty policy that determines the optimal warranty price for a given warranty period. Ding, Rusmevichientong and Topaloglu (2012) model a firm selling products with warranties and accumulating information about product reliability over time. They formulate a dynamic program with Bayesian learning to help the firm decide if and when to stop selling a product. Thomas and Rao (1999) provide an overview of the economic decision models for warranties extending the earlier reviews on mathematical models for warranties by Blischke (1990), Murthy and Blischke (1992), and Blischke and Murthy (1993). More recently, Murthy and Djamaludin (2002) overview papers that examine new product warranties in engineering, marketing and logistics fields. Our paper complements this stream of literature and provides a framework to estimate consumer demand from observed consumer purchases, which then enable counterfactual experiments help optimize the firms profit.

There are also papers using game-theoretic models to study warranty related problems. DeCroix (1999) uses a game-theoretic model for durable goods manufacturers to decide product warranty, reliability and price, and show that optimal warranties and reliability of products are complementary. Balachander (2001) explores the reason why a product with lower quality may offer a longer warranty. His explanation hinges on differences in consumers' perception about the reliability of existing and new products. The aforementioned papers study the free warranty offered by manufacturers. Recently, Jiang and Zhang (2011) focus their attention on extended warranties offered by a retailer. They analyze the impact of a retailer's extended warranty on a manufacturer's free warranty under various conditions.

In the marketing literature, Boulding and Kirmani (1993) study if firms can use warranties as signals of quality. Through an experiment, they show that customers' responses to warranties are consistent with the behavioral conclusions of signaling theory. Warranties with better terms are beneficial for a high-credibility firm, compatible with the existence of a separating equilibrium, whereas a long warranty is not viewed as a signal of quality for a low-credibility firm, consistent with a pooling equilibrium. On the other hand, Blair and Innis (1996) explore how experts and nonexperts perceive the product quality of well-known and not well-known brands when the warranty length is increased from 2 to 20 years. They find that the warranty length is the most indicative signal of product quality if consumers are not experts and the brand name is unknown. However, Agrawal et al. (1996) show that warranty terms poorly predict the brand reliability for a sample of household appliances and electronic products. Warranty terms become better predictors of brand reliability as the market penetration, the age of the product, or the variance of reliability among different brands increases. Chu and Chintagunta (2009, 2011) perform empirical tests in the PC server and automobile industry and show that warranties provide insurance to consumers against future failure risk, and let the seller observe consumer classes with different risk preferences; however, signaling and incentive roles of warranties are not verified. Chen, Kalra and Sun (2009) study in a static model how product and consumer characteristics and retailer actions such as promotions influence purchases of extended service contracts (ESC). They test their predictions using panel data of purchases across several electronic product categories, and find that consumers are more likely to purchase ESCs for products that have relatively higher hedonic value than utilitarian value. In parallel to our observations, they show that, as compared to high-income consumers, low-income consumers are more likely to purchase ESCs due to their sensitivity to the replacement cost in the event of a product failure. Chen and Sun (2009) build a dynamic structural model to examine the consumers' purchasing decisions of durable products and extended service plans over time in a market with fast declining product prices. They find that the consumers tend to buy ESCs during the early stages of product shelf life. However, the consumers' tendency to buy ESCs decreases with the declining product prices. Furthermore, consumers tend to delay their product purchases if ESC prices decrease over time. Therefore, increasing ESC prices encourage the consumers to adopt the product earlier and buy ESCs. However, in their model, consumers are assumed to make only one purchase and can not either upgrade to a better product or buy a new product to replace a failed one. In contrast, we allow upgrading and replacing purchases in our model, which are affected by consumers' purchases of extended warranties and are also an important component of the retailer's total revenue and profit.

The economics literature on warranty is largely concerned with the role of warranties for consumers. Heal (1977) emphasizes the economic role of warranties as consumers' insurance against the risk of product failure. There are several papers that advance the idea that the warranties are signals of reliability: as the reliability of the product increases, the length of the warranty increases (Spence 1977; Grossman 1981; Courville and Hausman, 1979; Gal-Or, 1989). Some early empirical papers such as Wiener (1985) show a positive correlation between warranty length and product quality, while Douglas et al. (1993) show a negative correlation. There are several papers that emphasize the role of warranties as sorting mechanisms, as mechanisms for the seller to screen the risk preferences of consumers (Holmes 1984; Kubo 1986; Matthews and More 1987; Emmons 1989). Cooper and Ross (1985) are concerned with the role of partial warranties as incentive contracts. Given the losses due to a product failure are shared by the seller and the buyer, these contracts encourage the seller to increase product quality and the buyer to care for the product.

From a methodology point of view, our paper adopts the discrete choice modeling framework[1] and builds on the structural estimation literature, see Rust (1987) and Berry, Levinsohn, and Pakes (1995). Rust (1987) focuses on the estimation of a regenerative optimal stopping problem, i.e., when to replace, whereas Berry, Levinsohn, and Pakes (1995) focus on static consumer choices for multiple products, i.e., what to buy. To understand consumers' purchasing behavior for durable goods, it is necessary to model both the intertemporal and the substitution effects. Several recent papers (for example, Nair 2007; Gowrisankaran and Rysman 2011; Goettler and Gordon 2011) consider both effects and study the dynamics of consumer choice for durable goods with aggregate data. A paper closely related to ours is Gowrisankaran and Rysman (2011). The authors build a structural dynamic model of consumer purchasing behavior for new durable goods. However, they estimate the model using aggregate data on digital camcorders. Our paper differs from the aforementioned papers in two important aspects: First, we use individual level data instead of aggregate data to estimate consumer demand. Second, we incorporate possible product failures and formally model consumers' purchase decisions of extended warranties, which affect consumers' current and future purchase decisions.

3 Model

This section introduces a dynamic model of consumers' purchase decisions. In each period, the retailer offers multiple products (e.g. different models/brands of televisions) at different prices. Let J_t denote the set of products offered in period t ($t \geq 0$). Also let p_{jt} denote the price of product j in that period ($j \in J_t$). Each product comes with a manufacturer's warranty of $\underline{\tau}$ periods at no additional charge to the consumers.[2] In addition, the retailer offers to extend the warranty to $\bar{\tau} > \underline{\tau}$ periods. The extended warranty price for product j is w_{jt}.[3] Let J denote the set of all products offered by the retailer during the selling horizon, i.e., $J = \cup J_t$. Then we denote $y_t = \{(p_{jt}, w_{jt}), j \in J\}$ as the exogenous state in period t and let S_Y be the state space for y_t.

Consumers make purchase decisions to maximize their expected discounted utilities. If consumer i owns a functioning product, say product j, then she enjoys a flow utility $\beta_i' q_j$ in each period, where β_i and q_j are N-dimensional column vectors. The n^{th} entry of β_i denotes how much utility consumer i enjoys from the n^{th} characteristic of product j, which is captured by the n^{th} entry of q_j. We assume that q_j is bounded for technical convenience.[4] Each product may break down; and the failure rate depends on the product, but it is independent of the product's age. More specifically, product j breaks down with probability λ_j in each period. For convenience, we also assume that all purchases happen at the beginning of a period, whereas products may fail only at the end of a period. When a product fails, it is repaired instantaneously at no cost to the consumer if it is still under warranty. However, if the product is out of warranty, then the consumer buys a new product to replace it.

We assume that owners do not fix broken products that are out of warranty. Rather, consumers buy new products immediately to replace broken ones. This is a reasonable assumption in our setting because we only consider major problems as failures. For televisions with major problems, the cost of repair can be quite high. Moreover, in our estimation, we only include traditional CRT televisions, whose prices were relatively low. Therefore, it makes more sense for a household to buy a new television than to repair a failed one. Additionally, the television (the product we focus on) is an essential electronic product for the modern households. For households who watch TV, it is hardly the case that they will wait several months after their televisions break down before they buy new ones. Thus, for simplicity, we assume that they act immediately in our model.

The product currently owned by a consumer affects her purchase decision. To model this, let $x_{it} = (j_{it}, \tau_{it})$ denote consumer i's state of ownership, where j_{it} and τ_{it} denote the product owned and its remaining warranty at the beginning of period t before she makes the purchase decision. The state space for x_{it} is $S_X = J \times \{0, 1, \dots, \bar{\tau}\} \cup \{(\phi, 0)\}$, where $x_{it} \in J \times \{0, ..., \bar{\tau}\}$ indicates that consumer i has the functioning product j_{it} with remaining warranty τ_{it}, whereas $x_{it} = (\phi, 0)$ means that consumer i does not have a functioning product.

At the beginning of each period consumer i makes purchase decisions depending on her state of ownership x_{it} and the exogenous state y_t. First, she decides which product to buy (if any). Second, if she purchases a new product, she then decides whether to purchase the extended warranty. Her product purchase decision is denoted by $a_1 \in \mathcal{A}_1(x_{it}, y_t)$, where $\mathcal{A}_1(x_{it}, y_t)$ denotes the set of actions (i.e., to purchase a product available in period t or not to purchase) as a function of (x_{it}, y_t). Her warranty purchase decision is denoted by $a_2 \in \mathcal{A}_2(a_1)$, where $\mathcal{A}_2(a_1)$ denotes the set of actions available to her as a function of her purchase decision a_1. On the one hand, we assume that if consumer i has no product at the beginning of period t, then she will purchase a new one. That is, $\mathcal{A}_1((\phi, 0), y_t) = J_t$. On the other hand, if she already has a functioning product, then she may not purchase a new one. Therefore, we have $\mathcal{A}_1((j, \tau), y_t) = J_t \cup \{\phi\}$, where ϕ denotes the no purchase action. In the first case, i.e., $a_1 \in J_t$, she has the option to buy an extended warranty or just get the manufacturer's warranty. That is, $\mathcal{A}_2(j) = \{\underline{\tau}, \bar{\tau}\}$ for $j \in J_t$. In contrast, if she does not purchase a new product, she cannot buy an extended warranty either, which we denote by $a_2 = 0$. In other words, $\mathcal{A}_2(\phi) = \{0\}$. Let $a = (a_1, a_2)$ denote consumer i's actions and $\mathcal{A}(x_{it}, y_t)$ the corresponding choice set. Then combining these cases gives

$$\mathcal{A}(x_{it}, y_t) = \begin{cases} J_t \times \{\underline{\tau}, \bar{\tau}\} & \text{if } x_{it} = \{(\phi, 0)\}, \\ \\ J_t \times \{\underline{\tau}, \bar{\tau}\} \cup \{(\phi, 0)\} & \text{if } x_{it} \in J \times \{0, 1, \dots, \bar{\tau}\}. \end{cases} \tag{1}$$

Consumer i receives an immediate nominal reward, denoted by $r_i(a, x_{it}; y_t)$, associated with each action $a \in \mathcal{A}(x_{it}, y_t)$. This reward can be decomposed into two with respect to the two stages of purchase: the product purchase and the warranty purchase. That is,

$$r_i(a, x_{it}; y_t) = r_{i1}(a_1, x_{it}; y_t) + r_{i2}(a, x_{it}; y_t), \tag{2}$$

where

$$r_{i1}(a_1, x_{it}; y_t) = \begin{cases} \alpha_i p_{jt} + \beta_i q_j & \text{if } a_1 \in J_t, \\ \\ \beta_i q_{j_{it}} & \text{if } a_1 = \phi, \end{cases} \tag{3}$$

and

$$r_{i2}(a, x_{it}; y_t) = \begin{cases} \alpha_i w_{jt} & \text{if } a_2 = \bar{\tau}, \\ \\ 0 & \text{if } a_2 \in \{\underline{\tau}, 0\}, \end{cases} \tag{4}$$

where α_i is consumer i's valuation of money.

Let $\pi(x_{i,t+1}, y_{t+1} | x_{it}, y_t, a)$ be the transition probability of the state (x_{it}, y_t) as a function of the action $a \in \mathcal{A}(x_{it}, y_t)$. We assume that consumer i's action impacts only the evolution of her ownership state, and that the exogenous state y_t and the ownership state x_{it} evolves independently. That is,

$$\pi(x_{i,t+1}, y_{t+1} | x_{it}, y_t, a) = \pi_X(x_{i,t+1} | x_{it}, a) \pi_Y(y_{t+1} | y_t), \tag{5}$$

where $\pi_X(x_{i,t+1} | x_{it}, a)$ is the transition probability of the ownership state x_{it}, and $\pi_Y(y_{t+1} | y_t)$ is the transition probability of the exogenous state y_t.

Consumers may not know the real data generating process of y_t and thus their belief of the evolution of exogenous state can be rational, but may not be perfect. Thus, $\pi_Y(y_{t+1}|y_t)$ represents consumers' belief of the exogenous state in the next period, which will be described in Section 3.1. The ownership state x_{it} evolves as follows:

- If consumer i has a product within warranty at the beginning of period t, i.e., $x_{it} \in J \times \{1, ..., \overline{\tau}\}$, and chooses not to purchase, i.e., $a = (\phi, 0)$, then in the next period, she will have the same product and the remaining warranty will be one period shorter, i.e., $x_{i,t+1} = (j_{it}, \tau_{it} - 1)$ with probability 1;

- If consumer i chooses to buy a new product, i.e., $a \in J_t \times \{\underline{\tau}, \overline{\tau}\}$, then in the next period, she will have the new product with the remaining warranty, i.e., $x_{i,t+1} = (a_1, a_2 - 1)$ with probability 1;

- If consumer i has a functioning product out of warranty at the beginning of period t, i.e., $x_{it} \in J \times \{0\}$, and chooses not to purchase, i.e., $a = (\phi, 0)$, then her product does not break down with probability $1 - \lambda_{j_{it}}$. Therefore, in the next period, $x_{i,t+1} = (j_{it}, 0)$ with probability $1 - \lambda_{j_{it}}$;

- If consumer i has a functioning product out of warranty at the beginning of period t and chooses not to purchase, then her product may break down with probability $\lambda_{j_{it}}$. Therefore, in the next period, $x_{i,t+1} = (\phi, 0)$ with probability $\lambda_{j_{it}}$;

- In all cases other than the above four cases, the transition probability is zero.

The above discussions are summarized in the following equation:

$$\pi_X(x_{i,t+1}|x_{it},a) = \begin{cases} 1 & \text{if } x_{it} \in J \times \{1,...,\overline{\tau}\}, a = (\phi, 0) \text{ and } x_{i,t+1} = (j_{it}, \tau_{it} - 1), \\ 1 & \text{if } a \in J_t \times \{\underline{\tau}, \overline{\tau}\} \text{ and } x_{i,t+1} = (a_1, a_2 - 1), \\ 1 - \lambda_{j_{it}} & \text{if } x_{it} \in J \times \{0\}, a = (\phi, 0) \text{ and } x_{i,t+1} = (j_{it}, 0), \\ \lambda_{j_{it}} & \text{if } x_{it} \in J \times \{0\}, a = (\phi, 0) \text{ and } x_{i,t+1} = (\phi, 0), \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Consumer i's utility is also subject to an idiosyncratic random shock, which depends on her decision. In particular, it has two components: a product related component $\epsilon_{it}(a_1)$ and a warranty related one $\tilde{\epsilon}_{it}(a_2)$. Since consumers make the product and warranty purchase decisions sequentially, consumer i first observes the random shock related to the product and decides which product to buy. And only after she determines the product, she observes warranty related shocks $\tilde{\epsilon}_{it}(a_2)$ and decides whether to buy an extended warranty. This reflects the reality: Consumers usually first decide which product to buy, and then choose whether to purchase extended warranties. There are also additional random factors that affect a consumer's decision on whether to buy an extended warranty after she decides which product to buy. For example, a sales person may try hard to sell an extended warranty after seeing the consumer has made the decision to buy a product. Therefore, when making a product purchase decision, the consumer only knows the expected value of a corresponding warranty purchase decision. Consumer i's expected utility of choosing a_1 given her ownership state x_{it}, the exogenous state y_t and ϵ_{it} is

$$\begin{aligned} u_i(a_1, x_{it}, y_t, \epsilon_{it}) =& r_{i1}(a_1, x_{it}; y_t) + \epsilon_{it}(a_1) + \mathop{\mathbb{E}}_{\tilde{\epsilon}_{it}} \max_{a_2 \in \mathcal{A}_2(a_1)} \Bigg\{ r_{i2}(a, x_{it}; y_t) + \tilde{\epsilon}_{it}(a_2) \\ &+ \gamma \mathop{\mathbb{E}}_{x_{i,t+1}, y_{t+1}} V_i(x_{i,t+1}, y_{t+1}) \Bigg\}, \end{aligned} \tag{7}$$

where γ is the discount factor, $a = (a_1, a_2)$, and $V_i(x_{it}, y_t)$ denotes consumer i's expected value upon entering state (x_{it}, y_t). That is,

$$V_i(x_{it}, y_t) = \mathop{\mathbb{E}}_{\epsilon_{it}} \left\{ \max_{a_1 \in \mathcal{A}_1(x_{it}, y_t)} u_i(a_1, x_{it}, y_t, \epsilon_{it}) \right\}. \tag{8}$$

Also let $O_i(a_1, x_{it}, y_t)$ denote the expected value of choosing whether to buy an extended warranty given that consumer i already makes the product decision a_1, that is,

$$O_i(a_1, x_{it}, y_t) = \mathop{\mathbb{E}}_{\tilde{\epsilon}_{it}} \max_{a_2 \in \mathcal{A}_2(a_1)} \left\{ r_{i2}(a, x_{it}; y_t) + \tilde{\epsilon}_{it}(a_2) + \gamma \mathop{\mathbb{E}}_{x_{i,t+1}, y_{t+1}} V_i(x_{i,t+1}, y_{t+1}) \right\}. \tag{9}$$

Then consumer i's utility $u_i(a_1, x_{it}, y_t, \epsilon_{it})$ from action a_1 can be simplified as follows:

$$u_i(a_1, x_{it}, y_t, \epsilon_{it}) = r_{i1}(a_1, x_{it}; y_t) + O_i(a_1, x_{it}, y_t) + \epsilon_{it}(a_1). \tag{10}$$

Assuming the error terms $\epsilon_{it}(a_1)$ and $\tilde{\epsilon}_{it}(a_2)$ are i.i.d. type-I extreme values, we can derive $V_i(x_{it}, y_t)$ and choice probabilities analytically, as presented in the following proposition.

Proposition 1 *Suppose that the idiosyncratic shocks $\epsilon_{it}(a_1)$ and $\tilde{\epsilon}_{it}(a_2)$ are random variables with type-I extreme values distributions. Then the probability that consumer i makes the warranty purchase decision a_2 given the state (x_{it}, y_t) and that she already made the product purchase decision a_1 is given by*

$$P_i(a_2 | a_1, x_{it}, y_t) = \frac{\exp\left\{ r_{i2}(a, x_{it}; y_t) + \gamma \, \mathbb{E}_{x_{i,t+1}, y_{t+1}} V_i(x_{i,t+1}, y_{t+1}) \right\}}{\exp\{O_i(a_1, x_{it}, y_t)\}}, \tag{11}$$

Similarly, her probability of making the product purchase decision a_1 given the state (x_{it}, y_t) is given by

$$P_i(a_1 | x_{it}, y_t) = \frac{\exp\{r_{i1}(a_1, x_{it}; y_t) + O_i(a_1, x_{it}, y_t)\}}{\exp\{V_i(x_{it}, y_t)\}}, \tag{12}$$

Combining these, the probability of consumer i choosing action a in period t is given by

$$P_i(a|x_{it}, y_t) = \frac{\exp\left\{r_i(a, x_{it}; y_t) + \gamma\, \mathbb{E}_{x_{i,t+1}, y_{t+1}}\, V_i(x_{i,t+1}, y_{t+1})\right\}}{\exp\left\{V_i(x_{it}, y_t)\right\}}. \tag{13}$$

Furthermore, the value function can be written recursively as

$$V_i(x_{it}, y_t) = \ln \sum_{a_1 \in \mathcal{A}_1(x_{it}, y_t)} \exp\left\{r_{i1}(a_1, x_{it}; y_t) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\{r_{i2}(a, x_{it}; y_t) + \gamma \mathop{\mathbb{E}}_{\substack{x_{i,t+1}\\ y_{t+1}}} V_i(x_{i,t+1}, y_{t+1})\}\right\}. \tag{14}$$

In the next subsection, we introduce the inclusive value processes, which reduce the problem's dimension and simplify it significantly.

3.1 Reducing the Dimension of Consumers' Problem

In order to make purchase decisions, forward-looking consumers need to form a belief on how the exogenous state evolves in the future. Consumers need to form an expectation not only on future prices, but also on future availabilities of all products. As the number of products increases, the dimension of consumers' problem also increases, which makes the problem intractable. To ameliorate this complexity, we develop a parsimonious model of consumers' beliefs about the future exogenous sates, which significantly reduces the dimension of consumers' problem. To this end, for every consumer i and exogenous state y, define

$$\delta_i(y) = V_i((\phi, 0), y), \tag{15}$$

which corresponds to consumer i's expected value of entering the exogenous state y with no functioning product. We will refer to $\delta_i(y)$ as the inclusive value of the exogenous state y for consumer i, which is also used by Gowrisankaran and Rysman (2011).

We make the following assumption under which it suffices to know $\delta_i(y)$ for the purpose of predicting $\delta_i(y')$, where y' is the exogenous state in the next period.

Assumption 1 *Inclusive Value Sufficiency (IVS): If $\delta_i(\hat{y}) = \delta_i(\tilde{y})$, then for all y*

$$\sum_{\delta_i(y')=\delta_i(y)} \pi_Y(y'|\hat{y}) = \sum_{\delta_i(y')=\delta_i(y)} \pi_Y(y'|\tilde{y}). \tag{16}$$

The IVS assumption says that as long as two states have the same inclusive value, the distribution of the next period's inclusive value is the same conditioning on the state of the current period.[5]

In what follows, we argue that $\delta_i(y)$ is a sufficient statistic for y in the sense that all the information in y that consumer i needs for decision making (i.e. for calculating $V_i(x, y)$) is captured by the inclusive value $\delta_i(y)$. This is shown next.

Proposition 2 *Under the IVS assumption, for all $\hat{y}, \tilde{y} \in S_Y$ such that $V_i((\phi, 0), \hat{y}) = V_i((\phi, 0), \tilde{y})$, we have that $V_i(x, \hat{y}) = V_i(x, \tilde{y})$ for all $x \in S_X$.*

Under the IVS assumption, instead of predicting the evolution of the whole exogenous state y_t, consumer i only needs to form an expectation of how δ_{it} evolves over time. We assume that the inclusive values are stable over time, i.e., they do not exhibit trends. This is consistent with what we observe in our data set: Although prices vary over time, there is no clear trend. Moreover, in our estimation results, the realized inclusive values do not have any significant trend over time either. Therefore, we model consumers' belief about the next period's inclusive value as follows:

$$\delta_{i,t+1} = \mu_{i0} + \nu_{i,t+1}, \tag{17}$$

where ν_{it} is normally distributed random noise. Furthermore, to simplify the computation, we assume that consumers use the expected future inclusive value μ_{i0} in making decisions.

To facilitate our analysis, we further assume that for any $\delta \in \mathbb{R}$, there exist an $y \in S_Y$ such that $\delta_i(y) = \delta$, and define an auxiliary value function

$$W_i(x, \delta) = V_i(x, y_i(\delta)), \quad x \in S_X, \delta \in \{\delta_i(y) : y \in S_Y\}, \tag{18}$$

where $y_i(\delta)$ is an exogenous state such that $\delta_i(y) = \delta$.[6] The following proposition characterizes a recursive relationship satisfied by the auxiliary value function.

Proposition 3 *The auxiliary value function satisfies the following:*

$$W_i(x_{it}, \delta_{it}) = \begin{cases} \delta_{it} & \text{if } x_{it} = (\phi, 0), \\ \ln\left\{\exp\left(\beta_i q_{j_{it}} + \gamma\, \mathbb{E}_{x_{i,t+1},\delta_{i,t+1}} W_i(x_{i,t+1}, \delta_{i,t+1})\right) + \exp \delta_{it}\right\} & \text{otherwise,} \end{cases} \tag{19}$$

and

$$\delta_i(y_t) = \ln \sum_{a \in J_t \times \{\underline{\tau}, \overline{\tau}\}} \exp\left\{r_i(a, x_{it}; y_t) + \gamma \mathop{\mathbb{E}}_{\substack{x_{i,t+1} \\ \delta_{i,t+1}}} W_i(x_{i,t+1} \delta_{i,t+1})\right\}. \tag{20}$$

The use of the auxiliary value function W instead of the original value function V leads to a dramatic reduction in the problem dimension. Replacing $V_i(x_{it}, y_t)$ with $W_i(x_{it}, \delta_{it})$ in equation (13), we have consumer i's choice probability as presented in Corollary 2.

Corollary 2 *Consumer i's choice probability can be written as*

$$P_i(a|x_{it}, y_t) = \frac{\exp\left\{r_i(a, x_{it}; y_t) + \gamma\, \mathbb{E}_{x_{i,t+1},\delta_{i,t+1}} W_i(x_{i,t+1}, \delta_{i,t+1})\right\}}{\exp\{W_i(x_{it}, \delta_i(y_t))\}}. \tag{21}$$

Having characterized the choice probabilities and simplified the consumers' problem, we next describe our data set as a preliminary to present the estimation framework.

4 Data

Our product and extended warranty purchasing data is from ISMS Durable Dataset 1. It contains 173,262 transactions made by 19,936 households in a major U.S. electronics chain between December 1998 and November 2004. The transactions involve purchases of durable goods in 292 product categories and their related services such as installation and extended service contracts.

We use transaction records of televisions and related extended warranty purchases in our estimation. The data set contains individual level transaction records, which link each purchase with the household that made it. This enables us to track what televisions households have after their first purchases. There are 6,627 television purchases (not including those were eventually returned) in total. We observe characteristics of televisions such as their brands, size ranges (for CRTs only) and types (LCD or CRT). We observe 1850 extended warranty purchases for television in the data set. The overall attach rate (warranty purchases as a percentage of television purchases) is 28%. Prices for extended warranties vary by different models but are positively correlated with product prices. The price ratios (extended warranty price over product price) are negatively correlated with product prices. Table 1 shows statistics of televisions and extended warranties by product categories.

Out of the 19,936 households, 5,275 (26% of total observed households) purchased at least one television from this retailer in the selling horizon. Summary statistics of repeat purchases are presented in Table 2. It is worth noting that of those who purchased televisions, about 42% made repeat purchases.

Households are categorized into nine groups, based on their income level. Households in an income group with a higher number have higher income than those in a group with a lower number. Table 3 presents some statistics by income groups. It can be seen that wealthier households tend to buy more expensive televisions and also upgrade their televisions more frequently, while the attach rate is negatively correlated with income level.

Category	Quantity sold	Warranty Sold	Attach Rate	TV Price	Warranty Price	Price Ratio
9-16" Color TV	568	63	11%	126.99	35.41	28%
19-20" Color TV	1181	189	16%	175.79	39.90	23%
25" TV	553	139	25%	243.76	52.18	21%
27" TV	1616	448	28%	353.01	68.68	19%
30" and larger TV	1427	449	31%	811.41	166.06	20%
LCD TV	60	18	30%	932.62	224.12	24%
TV projection	898	481	54%	1937.58	301.77	16%
Advanced tech TV	24	12	50%	3500.28	547.59	16%
Specialty TV	188	20	11%	97.66	18.11	19%

Table 1: Statistics of Televisions and Extended Warranties by Categories

Total Televisions Purchased	0	1	2	3	4	More than 5	Total
Number of Households	14,661	3,039	1,683	310	154	89	19,936

Table 2: Households Making Repeated Purchases

Income Group	Total Households	TV Quantity	TV Price	Attach Rate	Repeat Purchases
1	1361	395	449.10	33%	0.29
2	622	219	423.42	29%	0.35
3	1415	438	515.61	30%	0.31
4	1579	528	610.61	30%	0.33
5	1571	545	597.20	31%	0.35
6	3745	1278	653.26	28%	0.34
7	2591	911	667.33	27%	0.35
8	1450	532	724.12	29%	0.37
9	2477	991	709.72	24%	0.40

Table 3: Statistics by Income Levels

In our estimation, we focus on consumers' purchasing behavior for CRT televisions. Television projectors are not good substitutes for traditional CRT televisions. And the sales of LCD televisions and advanced technology televisions are small enough to be negligible in our data set. Furthermore, there is no size information for specialty televisions and the total quantity sold is also relatively small, so we drop them in our estimation.

Product characteristics have two parts, size and brand. We do not have precise size information for all television. For example, in the "9-16 Inch Color TV" category, we only know a television in this category has a diagonal between 9 inches and 16 inches. In such a case, we use the median size in the category. Therefore, for any television in category "9-16 Inch Color TV", we assume its size is 22.5 inches. For the category "30 Inch and Larger TV", we use a size of 35 inches. The data set has detailed brand information for televisions. There are 25 brands in total. Some brands have really few sales quantities. We consolidate all brands with a market share smaller than 1% into one group and name it as "others". Table 4 shows the market shares (in quantity) of televisions with different brands and different sizes. Average prices of different models are shown in Table 5.

We use failure rate data from Consumer Reports (2004). It contains the percentage of conventional 25- to 36-inch TV sets purchased new from 1999 to 2004 that were ever repaired or had a serious problem that was not resolved. To compute the quarterly failure rates, we assume constant failure rates and sales rates over time. For products that are not included in the report, we use failure rates of a similar product (a similar brand in the same size category or the same brand in the closest size category). The resulting quarterly failure rates are displayed in Table 6.

Brand	9-16”	19-20”	25”	27”	30” and larger	Total
SON	1.07%	2.76%	1.30%	7.25%	8.63%	21.01%
PAN	0.62%	3.03%	0.24%	6.35%	5.49%	15.73%
GE	1.24%	1.39%	2.63%	2.31%	2.05%	9.62%
RCA	1.62%	1.77%	1.13%	1.28%	3.40%	9.21%
ZEN	0.53%	1.75%	2.12%	3.44%	1.33%	9.17%
JVC	0.51%	1.20%	0.00%	3.61%	3.65%	8.98%
APX	0.88%	2.93%	0.75%	2.46%	0.47%	7.48%
MAG	1.62%	2.46%	1.18%	1.41%	0.73%	7.39%
BRK	1.39%	2.56%	0.00%	0.00%	0.00%	3.95%
PHL	0.51%	0.43%	0.75%	0.79%	0.32%	2.80%
OTHERS	0.21%	0.32%	0.02%	0.43%	0.71%	1.69%
SHA	0.00%	0.81%	0.00%	0.83%	0.00%	1.65%
SAM	0.51%	0.11%	0.00%	0.41%	0.30%	1.33%
Total	10.73%	21.52%	10.11%	30.56%	27.08%	100.00%

Table 4: Market Share in Quantity

Brand	9-16”	19-20”	25”	27”	30” and larger
SON	220.29	329.28	482.96	518.85	1235.79
PAN	173.51	204.93	232.29	335.36	667.42
GE	100.57	143.27	191.30	223.50	396.69
RCA	128.34	160.09	223.23	267.42	569.75
ZEN	101.65	151.49	235.92	321.23	605.25
JVC	179.18	244.34	-	358.84	671.23
APX	67.25	109.49	159.11	190.17	337.69
MAG	117.75	144.58	202.35	256.75	475.49
BRK	85.11	116.90	-	-	-
PHL	157.29	176.20	234.21	354.35	816.17
OTHERS	91.99	117.99	149.97	361.27	963.20
SHA	-	135.86	-	255.67	-
SAM	167.29	221.65	-	457.57	856.24

Table 5: Average Prices

	9-16”	19-20”	25”	27”	30” and Larger
SON	0.328%	0.328%	0.328%	0.328%	0.370%
PAN	0.387%	0.387%	0.387%	0.387%	0.430%
GE	0.494%	0.494%	0.494%	0.494%	0.662%
RCA	0.638%	0.638%	0.638%	0.638%	0.927%
ZEN	0.464%	0.464%	0.464%	0.464%	0.808%
JVC	0.328%	0.328%	0.328%	0.328%	0.328%
APX	0.494%	0.494%	0.494%	0.494%	0.662%
MAG	0.494%	0.494%	0.494%	0.494%	0.662%
BRK	0.494%	0.494%	0.494%	0.494%	0.662%
PHL	0.456%	0.456%	0.456%	0.456%	0.520%
OTHERS	0.494%	0.494%	0.494%	0.494%	0.662%
SHA	0.286%	0.286%	0.286%	0.286%	0.245%
SAM	0.585%	0.585%	0.585%	0.585%	0.662%

Table 6: Quarterly Failure Rates

5 Estimation Framework and Results

We use maximum likelihood estimation to estimate parameters of the model. There are nine discrete groups of consumers, depending on their income levels. We observe consumers' decisions for T periods in total. We pick the length of a period to be a quarter and let γ be 0.987 in our estimation, which corresponds to an annual discount rate of 0.95. We take the length of free manufacturer warranty $\underline{\tau}$ to be one year, which is consistent with the common practice in the industry. And based on the ratio between extended warranty prices and product prices, we assume that the length of extended warranties $\bar{\tau}$ is four years.

Consumers in each income group have the same parameters, i.e., all consumers i in group k have parameters $\theta_k = (\alpha_k, \beta_k, \mu_{k0})$. We estimate the parameters for each income group separately. Our model is identified, that is, θ_k are identified using our data. Specifically, if we observe more purchases of cheaper televisions, then we can infer that α_k is higher, which helps identify α_k. Similarly, more purchases of products with a certain brand indicates a higher β_k for this brand. And more purchases of larger televisions suggest a higher β_k for the size characteristic, which then helps identify β_k. Lastly, fewer extended warranty purchases mean that product failures are less painful for households, which indicates that μ_{k0} is higher and helps identify μ_{k0}. In summary, our data set is rich enough to identify the structural parameters, with a few exceptions. We do not observe purchases of some brands by households in certain income groups (SHA for income group 1, and SAM for income group 3 and 4) and thus we can not identify the corresponding β_k.

We observe the size and brand of televisions, which serve as product characteristics in our estimation. Furthermore, the exogenous state y_t and the actual action of consumer i in period t, denoted by a_{it}, are observed in the data. We do not observe consumer i's ownership x_{it} directly because we do not observe product failures. However, consumer i's latest purchased product and its remaining warranty at the beginning of t, denoted by $\tilde{x}_{it} = (\tilde{j}_{it}, \tilde{\tau}_{it})$, are observed. Then x_{it} can only be either $\tilde{x}_{it}$ or $(\phi, 0)$. Let $Q_{it}(x)$ be the probability that $x_{it} = x$, then in periods t, $(t = 2, \ldots, T)$, we have $Q_{it}(x) = 0$ if $x \notin \{\tilde{x}_{it}, (\phi, 0)\}$, therefore,

$$Q_{it}(\tilde{x}_{it}) = \begin{cases} 1 & \text{if } a_{i,t-1} \in J_{t-1} \times \{\underline{\tau}, \bar{\tau}\} \text{ or } a_{it} = (\phi, 0) \text{ or } \tilde{\tau}_{i,t-1} > 0, \\ 1 - \lambda_{\tilde{j}_{i,t-1}} & \text{if } a_{i,t-1} = (\phi, 0),\ a_{it} \in J_t \times \{\underline{\tau}, \bar{\tau}\},\ \tilde{\tau}_{i,t-1} = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

and

$$Q_{it}((\phi, 0)) = \begin{cases} \lambda_{\tilde{j}_{it}} & \text{if } a_{i,t-1} = (\phi, 0),\ a_{it} \in J_t \times \{\underline{\tau}, \bar{\tau}\},\ \tilde{\tau}_{i,t-1} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

In words, if consumer i's previously purchased product did not fail in period $t-1$ (this is the case if $a_{it} = (\phi, 0)$ or $\tilde{\tau}_{i,t-1} > 0$ or $a_{i,t-1} \in J_{t-1} \times \{\underline{\tau}, \bar{\tau}\}$), then $x_{it} = \tilde{x}_{it}$ with probability one.

In her purchasing periods t ($a_{it} \neq (\phi, 0)$), if the warranty of her product already expired in period $t-1$, and she did not buy any new product in period $t-1$, then the product may break down in period $t-1$ with probability $\lambda_{\tilde{j}_{i,t-1}}$. If the product breaks down in period $t-1$, then $x_{it} = (\phi, 0)$, otherwise, $x_{it} = \tilde{x}_{it}$.

We do not know what products households have until we observe their first purchases. Therefore, in our estimation, we only include periods from the first purchase by each household.

Given parameters θ_k, the likelihood of observing actions of consumers in the group $\{a_{it} : i \in k, t \in \{1, \dots, T\}, \tilde{j}_{it} \neq 0\}$[7] is $\prod_{i \in k, t \in \{1,\dots,T\}: \tilde{j}_{it} \neq 0} \sum_{x \in S_X} (P_i(a_{it}|x, y_t) Q_{it}(x))$. As a result the log-likelihood function is

$$\begin{aligned}
&l_k(\theta_k | \{(a_{it}, \tilde{x}_{it}, y_t) : i \in k, t = 1, \dots, T\}, \{q_j : j \in J\}) \\
= &\sum_{i \in k, t \in \{1,\dots,T\}: \tilde{j}_{it} \neq 0} \ln \sum_{x \in S_X} (P_i(a_{it}|x, y_t) Q_{it}(x)) \\
= &\sum_{\substack{i \in k \\ t \in \{1,\dots,T\} \\ \tilde{j}_{it} \neq 0}} \ln \left\{ \frac{\exp\left\{ r(a_{it}, \tilde{x}_{it}; y_t) + \gamma \sum_x W(x, \mu_0) \pi_X(x | \tilde{x}_{it}, a_{it}) \right\}}{\exp\{W(\tilde{x}_{it}, \delta_t)\}} Q_{it}(\tilde{x}_{it}) \right. \\
&\left. + \frac{\exp\left\{ r(a_{it}, (\phi, 0); y_t) + \gamma \sum_x W(x, \mu_0) \pi_X(x | (\phi, 0), a_{it}) \right\}}{\exp\{W((\phi, 0), \delta_t)\}} Q_{it}\big((\phi, 0)\big) \right\}.
\end{aligned} \tag{24}$$

The estimation problem for group k is to choose parameters θ_k, the auxiliary value function W_k and δ_{kt} to maximize the log-likelihood function. The constraints include equations (19), (20) and a rational expectation constraint, which requires consumers' expectation of future inclusive values to be consistent with the reality. That is, $\mu_{k0} = \frac{1}{T} \sum_{t=1,\dots,T} \delta_{kt}$. To summarize, the maximization problem (the group subscript k is omitted) is given next:

$$
\begin{aligned}
\max_{\substack{\theta, W(\cdot,\cdot) \\ \{\delta_t:, t=1,\ldots,T\}}} \quad & l(\theta|\{(a_{it}, \tilde{x}_{it}, y_t) : i \in k, t = 1, \ldots, T\}, \{q_j : j \in J\}) \\
\text{s.t.} \quad W(z, \delta) &= \delta, \text{ for all } z = (\phi, 0), \delta \in \{\mu_0\} \cup \{\delta_t, t = 1, \ldots, T\}, \\
W(z, \delta) &= \ln \left\{ \exp \left(\beta q_{z_1} + \gamma \sum_x W(x, \mu_0) \pi_X(x|z, (0,0)) \right) + \exp \delta \right\}, \\
& \quad \text{for all } z \in J \times \{0, \ldots, \bar{\tau}\}, \delta \in \{\mu_0\} \cup \{\delta_t, t = 1, \ldots, T\}, \\
\delta_t &= \ln \sum_{a \in J_t \times \{\underline{\tau}, \bar{\tau}\}} \exp \left\{ r(a, (\phi, 0); y_t) + \gamma W\big((a_1, a_2 - 1), \mu_0\big) \right\}, \\
& \quad \text{for all } t = 1, \ldots, T, \\
\mu_0 &= \frac{1}{T} \sum_{t=1,\ldots,T} \delta_t.
\end{aligned}
$$

Table 7 shows consumers' valuation on size and different brands in dollar amount, that is, $-\beta_k/\alpha_k$. The detailed estimation results are in Appendix B. As mentioned above, we cannot identify a few parameters (for example, SHA for income group 1) because we do not observe any household in that income group purchasing any product with this brand. Notice that consumers value Sony brand the most and are willing to pay a premium between ten to twenty dollars per quarter for Sony products. This is consistent with the general impression that Sony is a premium brand. Additionally, consumers' quarterly willingness to pay for a larger size is below two dollars per inch, which is reasonable. And the coefficient on size is positively correlated with the income levels, which suggests that richer households are willing to pay more for larger televisions.

6 Counterfactual Analysis

Having estimated the structural parameters, we perform counterfactual experiments about firms' decisions related to extended warranties and examine the effects on the retailer's revenue and profit. Based on Berner (2004), profit margins for extended warranties are between 50% and 60%, which is about 18 times the margins for products. In our experiments, we use 55% as the margin for extended warranties and 3.06% as the margin for products. We assume that the costs of extended warranties are proportional to product failure rates. However, the costs of extended warranties are invariant to their prices.

	Income Group								
	1	2	3	4	5	6	7	8	9
Size	1.47	0.98	1.65	1.69	1.74	1.76	1.76	1.88	1.74
APX	-0.75	-3.79	0.64	-15.06	-20.27	-19.30	-22.82	-12.74	-24.48
BRK	4.47	-7.08	13.88	8.19	4.60	4.49	7.54	5.15	-3.89
GE	0.20	-0.01	1.40	0.31	0.30	0.96	-1.68	-7.31	-3.46
JVC	2.38	1.15	3.49	3.20	3.33	4.51	5.36	3.08	2.74
MAG	-6.21	0.44	0.95	-0.47	-3.30	0.50	-14.99	0.04	-1.41
PAN	2.36	1.86	3.81	3.38	3.33	4.76	5.53	3.12	2.82
PHL	-5.12	-3.49	-7.20	-8.40	-10.18	-15.96	-4.52	-8.89	-16.02
RCA	3.47	2.94	4.51	4.10	3.99	4.99	6.09	3.15	3.01
SAM	8.04	6.57	-	-	8.88	9.01	11.23	7.08	8.26
SHA	-	0.42	1.13	-20.10	-5.92	-44.73	-40.28	-22.07	-37.15
SON	10.80	7.59	13.12	12.73	13.25	14.48	15.62	13.38	12.88
ZEN	3.83	3.96	5.78	4.76	4.90	5.91	7.15	3.74	3.55

Table 7: Consumers' Valuation on Size and Brands in Dollar Amount

6.1 Warranty Prices

We first investigate how the overall extended warranty price level affects the retailer's revenue. We compare the retailer's revenues under different extended warranty price levels. Increasing extended warranty prices has two opposite effects on the retailer's product revenue. On the one hand, increasing extended warranty prices decreases the option value of purchasing products and thus decreases the current product revenue. On the other hand, as extended warranty prices increase, consumers buy fewer extended warranties, which will lead to more product purchases due to product failures in the future. Figure 1 shows how changing the warranty price level affects the retailer's revenue and profit. The product revenue is monotonically decreasing in extended warranty price level, which suggests that the first effect described immediately above dominates. The warranty revenue is concave in extended warranty prices, which is intuitive. Moveover, the current price level roughly maximizes the warranty revenue. The total revenue is decreasing in extended warranty prices. That is, the effect of changing the warranty price level on product revenue dominates. However, the effects on profits are quite different than those on revenues. Both the warranty profit and the total profit are maximized at about 160% of the actual extended warranty prices. As the warranty price level increases, the retailer loses warranty revenue, but saves the cost of extended warranties. Furthermore, the effect of changing warranty prices on warranty profit dominates that on product profit. This suggests that the extended warranties are generally underpriced. As a matter of fact, if the retailer increases all extended warranty prices by 60%, then it can increase its profit by 8%.

However, products are different and thus the optimal pricing strategies for extended warranties of different products may be different. To better understand these differences, we consider changing the warranty prices for Sony, a high-end brand and for RCA, a low-end brand. The resulting revenues and profits are presented in Figure 2.

The optimal warranty price levels are different for Sony and RCA. The actual extended warranties are overpriced for Sony products but underpriced for RCA products.

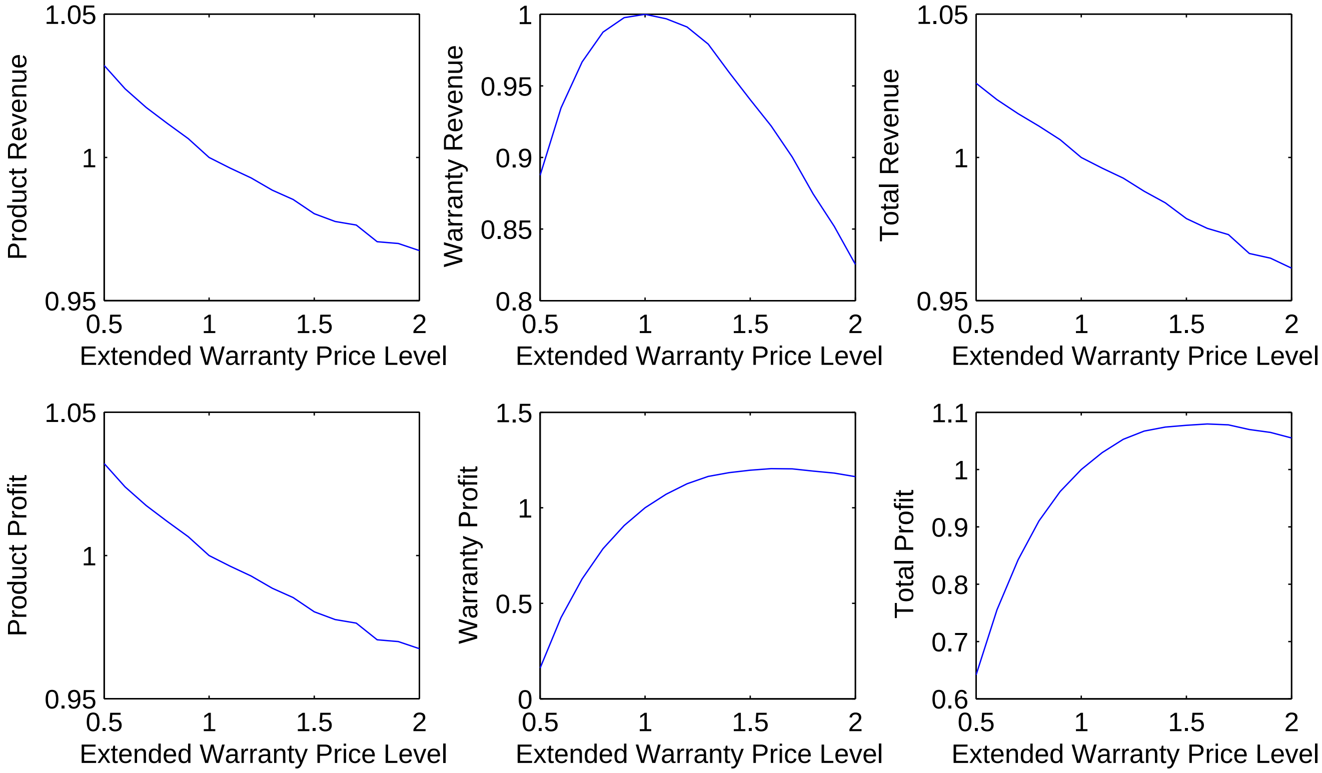

Figure 1: Relative Revenues and Profits under Different Extended Warranty Price Levels. The extended warranty price level indicates the ratio between experimented and actual extended warranty prices. For example, 1 indicates the actual extended warranty prices, and 1.5 means the all extended warranty prices are 50% higher than the actual ones. The revenues and profits are all normalized relatively to those under the actual price level.

6.2 Product Reliability

The more reliable a product is, the more consumers value it. However, more reliable products may not be desired from a retailer's point of view. Because the increased reliability decreases both repeat purchases and extended warranty purchases. We study the impact of changing failure rates for products with a certain brand on the retailer's revenue and profit. We vary the failure rates of a high failure rate brand, RCA, and a low failure rate brand, Sony, and compare the retailer's revenues and profits. The results are presented in Figure 3.

As failure rates of a brand's products increase, its products become less attractive to

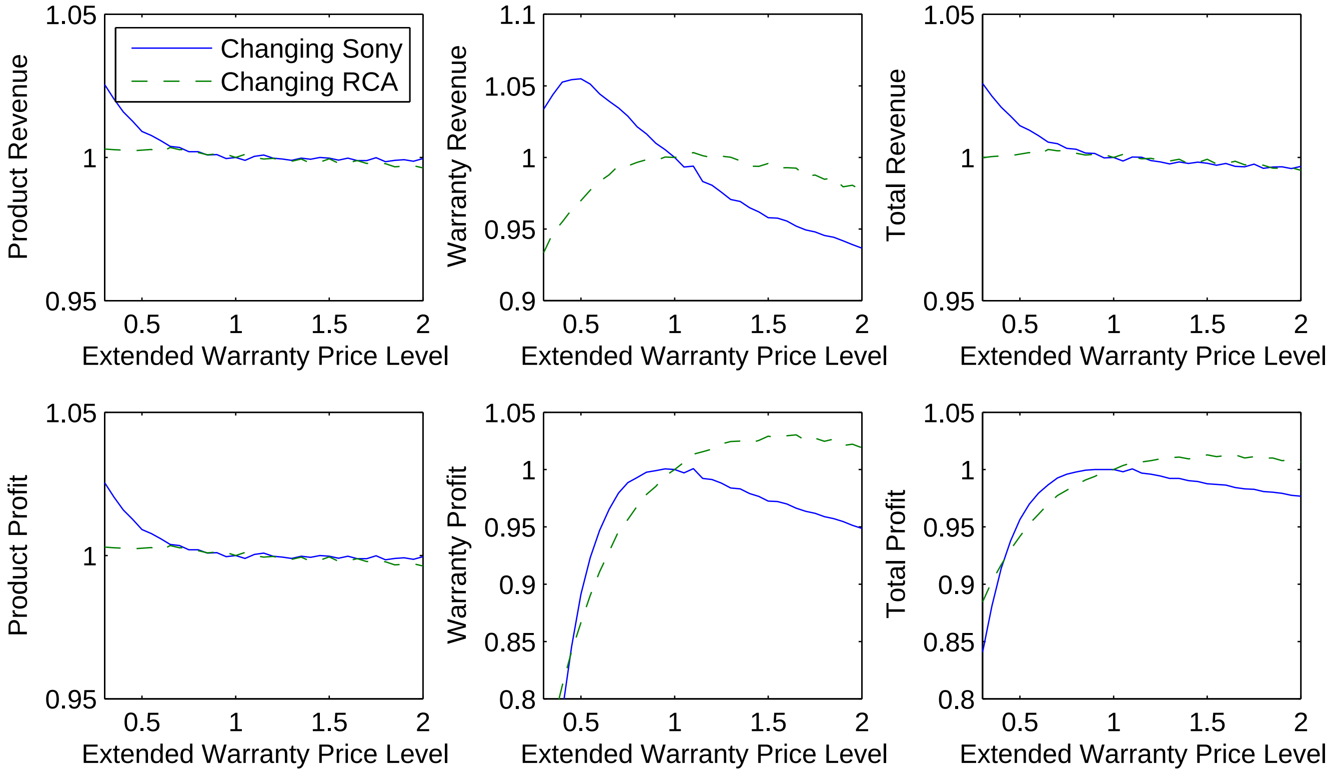

Figure 2: Relative Revenues and Profits under Different Brand Extended Warranty Price Levels

consumers and some consumers will switch to other brands. But among those who continue to buy this brand, the values of extended warranties increase and the attach rates also increase. The results suggest that the retailer would prefer less reliable Sony products and slightly more reliable RCA products. As failure rates of Sony products increase, consumers switch to cheaper brands and the retailer loses product profit. However, as consumers buy more extended warranties, the retailer gains more warranty profit. We show that the gain in the warranty profit is bigger than the loss in the product profit. If Sony doubles failure rates of its products, then the retailer's profit will increase by 3.42%. In contrast, if RCA increases failure rates of its products, the retailer gains product profit and loses warranty profit. And the warranty profit effect dominates. So the retailer loses profit if failure rates of RCA's products increase.

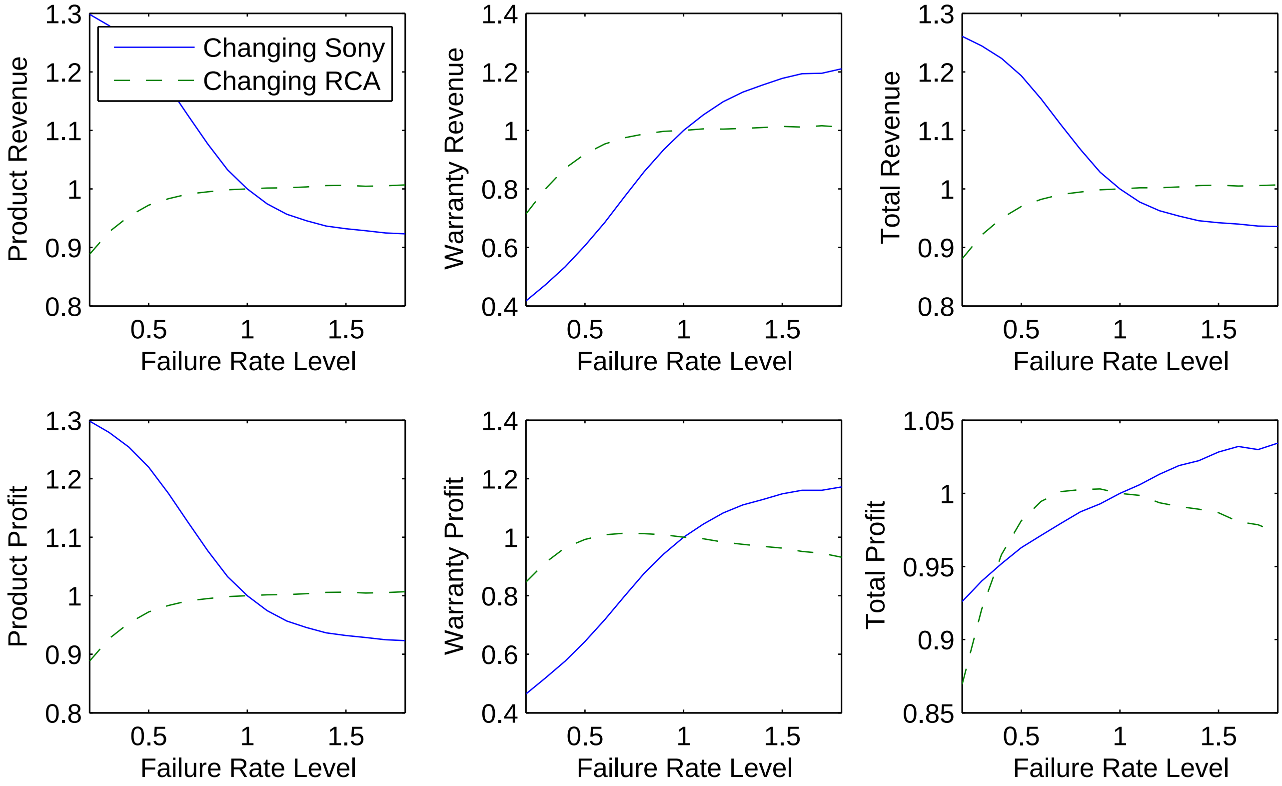

Figure 3: Relative Revenues and Profits under Different Product Failure Rates

The above analysis shows the impact of changing product reliability on the retailer's profit. However, it is the manufacturer who determines the product reliability. The retailer should design proper incentive schemes to motivate manufacturers to choose the right failure rates. Our analysis helps the retailer understand the benefit of changing product reliability, which is useful in negotiating with manufacturers.

7 Concluding Remarks

We model consumers' purchase decisions on both products and extended warranties as dynamic programming problems and use inclusive values to decrease the dimension of consumers' problem. Using this model, we empirically estimate parameters of consumer preferences. Counterfactual experiments show that in our data set, extended warranties are generally underpriced. But the optimal extended warranty price level differs by brand. For example, extended warranties for a high-end brand Sony are overpriced, whereas underpriced for a low-end brand RCA. We also examine the impact of changing product failure rates on the retailer's profit, and find that the retailer prefers a high-end brand Sony to increase failure rates of its products but prefers a low-end brand RCA to decrease failure rates.

Our work provides a basic framework to model the dynamics of consumers' product and extended warranty purchases. With more detailed cost data, our model can be used to optimize more complicated pricing strategies as well as to compare different pricing schemes. One limitation of our model is the lack of competitive effects between retailers. Including competing retailers requires both extensive data from competing retailers and a more complicated model capturing consumers' choices on retailers, products, and extended warranties, which we leave for future research. In addition, with detailed data on product failures and repair costs, our model can be extended to include consumers' decision on repair or replace when they encounter product failures. Lastly, one can easily extend our model to the case where multiple extended warranty options with different warranty lengths are offered.

Endnotes

1. See Ben-Akiva and Lerman (1985) for an overview of discrete choice models, and Natarajan et al. (2009) and the references therein for an alternative approach.

2. We pick the period length such that $\underline{\tau} \geq 1$.

3. For notational convenience, let $p_{jt} = w_{jt} = \infty$ whenever $j \notin J_t$.

4. The bounds of q_j can be considered as the technological limits of the product.

5. For technical convenience, we also assume that π_Y has the Feller property.

6. Note that this definition has no ambiguity because of Proposition 2.

7. We only use periods after the first purchase by each household. For notation purposes, we denote $\tilde{j}_{it} = 0$ if period t is before household i's first purchase and exclude these periods.

References

Agrawal, J. S., P. Richardson, P. E. Grimm. 1996. The Relationship between Warranty and Product Reliability. *The Journal of Consumer Affairs* **30** 421-443.

Balachander, S. 2001. Warranty Signalling and Reputation. *Management Science* **47**(9) 1282-1289.

Ben-Akiva, M., S. Lerman. 1985. Discrete Choice Analysis: Theory and Application to Travel Demand. MIT Press, Cambridge, MA.

Berner, R. 2004. The Warranty Windfall. *Bloomberg BusinessWeek.* Dec. 19, 2004. Accessed on August 16th, 2012. `http://www.businessweek.com/stories/2004-12-19/the-warranty-windfall`

Berry, S., J. Levinsohn, A. Pakes. 1995. Automobile Prices in Market Equilibrium. *Econometrica* **63**(4) 841-890.

Blair, M. E., D. E. Innis. 1996. The Effects of Product Knowledge on the Evaluation of Warranted Brands. *Psychology and Marketing* **13** 445-456.

Blischke, W. R. 1990. Mathematical Models for Analysis of warranty Policies. *Mathematical Comput. Modeling* **13**(7) 1-16.

Blischke, W. R., D. N. P. Murthy. 1993. Product Warranty Management-I: A Taxonomy of Warranty Policies. *European Journal of Operations Research* **62** 127-148.

Boulding, W., A. Kirmani. 1993. A Consumer-Side Experimental Examination of Signalling Theory. *The Journal of Consumer Research* **20**(1) 111-123.

Chen, T., A. Kalra, B. Sun. 2009. Why Do Consumers Buy Extended Service Contracts? *Journal of Consumer Research* **36** 611-623.

Chen, T., B. Sun. 2009. An Empirical Investigation of Consumer Purchases and Intertemporal Pricing of Retailers' Extended Service Contracts. *Carnegie Mellon University.* Working Paper.

Chu, J., P. K. Chintagunta. 2009. Quantifying the Economic Value of Warranties in the U.S. Server Market. *Marketing Science* **28**(1) 99-121.

Chu, J., P. K. Chintagunta. 2011. An Empirical Test of Warranty Theories in the U.S. Computer Server and Automobile Markets. *Journal of Marketing* **75** 75-92.

Chun, Y. H., K. Tang. 1995. Determining the Optimal Warranty Price Based on the

Producer's and Customer's Risk Preferences. *European Journal of Operational Research* **85** 97-110.

Consumer Reports. 2004. Conventional TVs: Prime Choices. *Consumer Reports*. December 2004.

Cooper, R., T. W. Ross. 1985. Product Warranties and Double Moral Hazard. *The RAND Journal of Economics* **16**(1) 103-113.

Courville, L., W. H. Hausman. 1979. Warranty Scope and Reliability Under Imperfect Information and Alternative Market Structures. *The Journal of Business* **52**(3) 361-378.

DeCroix, G. A. 1999. Optimal Warranties, Reliabilities and Prices for Durable Goods in an Oligopoly. *European Journal of Operational Research*. **112** 554-569.

Ding, C., P. Rusmevichientong, H. Topaloglu 2012. Balancing Revenues and Repair Costs under Partial Information about Product Reliability. *Working Paper*.

Douglas, E. J., D. C. Glennon, J. I. Lane. 1993. Warranty Quality and Price in the US Automobile Market. *Applied Economics* **25** 135-141.

Emmons, W. 1989. The Theory of Warranty Contracts. *Journal of Economic Surveys* **3**(1) 43-57.

Gal-Or, E. 1989. Warranties as a Signal of Quality. *The Canadian Journal of Economics* **22**(1) 50-61.

Glickman, T. S., P. D. Berger. 1976. Optimal Price and Protection Period Decisions for a Product under Warranty. *Management Science* **22**(12) 1381-1390.

Goettler, R., B. R. Gordon. 2011. Does AMD Spur Intel to Innovate More? *Journal of Political Economy* **119**(6) 1141-1200.

Gowrisankaran, G., M. Rysman. 2011. Dynamics of Consumer Demand for New Durable Goods. *NBER Working Paper*.

Grossman, S. T. 1981. The Informational Role of Warranties and Private Disclosure about Product Quality. *Journal of Law and Economics* **24**(3) 461-483.

Heal, G. 1977. Guarantees and Risk-sharing. *The Review of Economic Studies* **44**(3) 549-560.

Holmes, T. J. 1984. Monopoly Bundling of Warranty and Quality when Quality is Unobservable. *Northwestern University Discussion Papers.*

Jiang, B., X. Zhang. 2011. How Does a Retailers Service Plan Affect a Manufacturers Warranty? *Management Science* **57**(4) 727-740.

Kao, E. P. C., M. S. Smith. 1993. Discounted and Per Unit Net Revenues and Costs of Product Warranty: The Case of Phase-Type Lifetimes. *Management Science* **39**(10) 1246-1254.

Kubo, Y. 1986. Quality Uncertainty and Guarantee. *European Economic Review* **30** 1063-1079.

Mamer, J. W. 1987. Quantifying the Economic Value of Warranties in the U.S. Server Market. *Management Science* **33**(7) 916-930.

Matthews, S., J. Moore. 1987. Monopoly Provision of Quality and Warranties: An Exploration in the Theory of Multidimensional Screening. *Econometrica* **55**(2) 441-467.

Murthy D. N. P., W. H. Blischke. 1992. Product Warranty Management-III: A Review of Mathematical Models. *European Journal of Operations Research* **63** 134.

Murthy D. N. P., I. Djamaludin. 2002. New Product Warranty: A Literature Review. *Int. J. Production Economics* **79** 231-260.

Nair, H. 2007. Intertemporal Price Discrimination with Forward-looking Consumers: Application to the US Market for Console Video-games. *Quantitative Marketing and Economics* **5**(3) 239-292.

Natarajan, K., M. Song, C. P. Teo. 2009. Persistency Model and Its Applications in Choice Modeling. *Management Science* **55**(3) 453-469.

Rust, J. 1987. Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher. *Management Science* **55**(5) 999-1033.

Spence, M. 1977. Consumer Misperceptions, Product Failure and Producer Liability. *The Review of Economic Studies* **44**(3) 561-572.

Thomas, M. U., S. S. Rao. 1999. Warranty Economic Decision Models: A Summary and Some Suggested Directions for Future Research. *OR Chronicles* **47**(6) 807-820.

Wiener, J. L. 1985. Are Warranties Accurate Signals of Product Reliability? *Journal of Consumer Research* **12**(2) 245-250.

Appendix

A Proofs

Proof of Proposition 1:

Proof. We start by proving the following lemma:

Lemma A.1 *Suppose there are N random variables, $\{z_i,\ i = 1, \ldots, N\}$. If $z_i = c_i + \eta_i$, where c_i are constants and η_i are i.i.d. random variables with type-I extreme value distributions, then*

$$P(z_i' \geq \max_{i=1,\ldots,N} z_i) = \frac{\exp c_i'}{\sum_i \exp c_i}, \tag{A.1}$$

$$\mathbb{E} \max_i z_i = \ln \sum_i \exp c_i, \tag{A.2}$$

and

$$\max_i z_i = \ln \sum_i \exp c_i + \zeta_i, \tag{A.3}$$

where ζ_i is type-I extreme value distributed.

Proof. We first show that the results hold if $N = 2$. When $N = 2$, we have

$$\begin{aligned} P(z_1 \geq \max_{i=1,2} z_i) &= P(c_1 + \eta_1 \geq c_2 + \eta_2) \\ &= P(\eta_2 \leq c_1 - c_2 + \eta_1) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{c_1-c_2+\eta_1} de^{-e^{-\eta_2}} de^{-e^{-\eta_1}} \\ &= \int_{-\infty}^{\infty} e^{-e^{-(c_1-c_2+\eta_1)}} e^{-e^{-\eta_1}} e^{-\eta_1} d\eta_1 \\ &= \int_{-\infty}^{\infty} e^{-e^{-(c_1-c_2+\eta_1)}} e^{-e^{-(c_1-c_1+\eta_1)}} e^{-\eta_1} d\eta_1 \\ &= \int_{-\infty}^{\infty} \exp\left(-e^{-(c_1-c_2+\eta_1)} - e^{-(c_1-c_1+\eta_1)}\right) e^{-\eta_1} d\eta_1 \\ &= \int_{-\infty}^{\infty} \exp\left(-e^{-\eta_1} \textstyle\sum_{i=1,2} e^{-(c_1-c_i)}\right) e^{-\eta_1} d\eta_1. \end{aligned} \tag{A.4}$$

Let $l = e^{-\eta_1}$, then $dl = e^{-\eta_1} d\eta_1$ and $l \in (0, \infty)$. Now, integrate with respect to l we have,

$$\begin{aligned} P(z_1 \geq \max_{i=1,2} z_i) &= \int_0^{\infty} \exp\left(-l \sum_{i=1,2} e^{-(c_1-c_i)}\right) dl \\ &= -\frac{\exp\left(-l \sum_{i=1,2} e^{-(c_1-c_i)}\right)}{\sum_{i=1,2} e^{-(c_1-c_i)}}\Big|_0^{\infty} \\ &= \frac{1}{\sum_{i=1,2} e^{-(c_1-c_i)}} \\ &= \frac{\exp c_1}{\sum_{i=1,2} \exp c_i} \end{aligned} \tag{A.5}$$

Therefore, $P(z_2 \geq \max_i z_i) = 1 - P(z_1 \geq \max_i z_i) = \frac{\exp c_2}{\sum_{i=1,2} \exp c_i}$. Furthermore, the probability that $\max_i z_i \leq x$ is,

$$\begin{aligned} P(\max_{i=1,2} z_i \leq x) &= P(z_1 \leq x) P(z_2 \leq x) \\ &= P(\eta_1 \leq x - c_1) P(\eta_2 \leq x - c_2) \\ &= e^{-e^{-(x-c_1)}} e^{-e^{-(x-c_2)}} \\ &= \exp(-e^{-x}(e^{c_1} + e^{c_2})) \end{aligned} \tag{A.6}$$

Let $C_i = \ln \sum_i \exp c_i$, then

$$P(\max_{i=1,2} z_i \leq x) = \exp(-e^{-x} e^{C_2}) = e^{-e^{-(x-C_2)}} \tag{A.7}$$

Let $\zeta_i = \sum_i \exp c_i - C_i$, then $\sum_i \exp c_i = C_i + \zeta_i$ and ζ_2 is a random variable with type-I extreme value distribution.
Suppose that the results hold for $N-1$, that is, $P(z_{i'} \geq \max_{i=1,\ldots,N-1} z_i) = \frac{\exp c_{i'}}{\sum_{i=1,\ldots,N-1} \exp c_i}$ and $\max_{i=1,\ldots,N-1} z_i = C_{N-1} + \zeta_{N-1}$, where $C_{N-1} = \ln \sum_{i=1,\ldots,N-1} \exp c_i$ and ζ_{N-1} is type-I extreme value distributed. As $\{\eta_i,\ i = 1, \ldots, N\}$ are independent, ζ_{N+1} and η_N are independent of each other. Therefore,

$$\begin{aligned} P(z_N \geq \max_{i=1,\ldots,N} z_i) &= P(z_N \geq \max_{i=1,\ldots,N-1} z_i) \\ &= P(z_N \geq C_{N-1} + \zeta_{N-1}) \\ &= \frac{\exp c_N}{\exp c_N + \exp C_{N-1}} \\ &= \frac{\exp c_N}{\sum_{i=1,\ldots,N} \exp c_i}. \end{aligned} \tag{A.8}$$

and for any $i' = 1, \ldots, N-1$,

$$\begin{aligned} P(z_i' \geq \max_{i=1,\ldots,N} z_i) &= P(z_i' \geq \max_{i=1,\ldots,N-1} z_i) P(z_N \leq \max_{i=1,\ldots,N-1} z_i) \\ &= \frac{\exp c_{i'}}{\sum_{i=1,\ldots,N-1} \exp c_i} \left(1 - \frac{\exp c_N}{\sum_{i=1,\ldots,N} \exp c_i}\right) \\ &= \frac{\exp c_{i'}}{\sum_{i=1,\ldots,N} \exp c_i}. \end{aligned} \tag{A.9}$$

Furthermore,

$$\begin{aligned} \mathbb{E} \max_{i=1,\ldots,N} z_i &= \mathbb{E} \max\{z_N, \max_{i=1,\ldots,N-1} z_i\} \\ &= \mathbb{E} \max\{z_N, C_{N-1} + \zeta_{N-1}\} \\ &= \ln\{\exp c_N + \exp C_{N-1}\} \\ &= \ln \sum_{i=1,\ldots,N} \exp c_i, \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} \max_{i=1,\ldots,N} z_i &= \max\{z_N, \max_{i=1,\ldots,N-1} z_i\} \\ &= \max\{z_N, C_{N-1} + \zeta_{N-1}\} \\ &= \ln\{\exp c_N + \exp C_{N-1}\} + \zeta_N \\ &= \ln \sum_i \exp c_i + \zeta_N, \end{aligned} \tag{A.11}$$

where ζ_N is type-I extreme value distributed. So the results hold for N as well. ■

Equations (12), (11) and (14) follow immediately by replacing c_i in the above lemma with appropriate expressions. Then the probability of consumer i choosing action a in period t is

$$\begin{aligned} P_i(a|x_{it}, y_t) &= P_i(a_2|a_1, x_{it}, y_t) P_i(a_1|x_{it}, y_t) \\ &= \frac{\exp\left\{r_{i1}(a_1, x_{it}; y_t) + O_i(a_1, x_{it}, y_t)\right\}}{\exp\left\{V_i(x_{it}, y_t)\right\}} \frac{\exp\left\{r_{i2}(a, x_{it}; y_t) + \gamma\, \mathbb{E}_{x_{i,t+1}, y_{t+1}} V_i(x_{i,t+1}, y_{t+1})\right\}}{\exp\left\{O_i(a_1, x_{it}, y_t)\right\}} \\ &= \frac{\exp\left\{r_i(a, x_{it}; y_t) + \gamma\, \mathbb{E}_{x_{i,t+1}, y_{t+1}} V_i(x_{i,t+1}, y_{t+1})\right\}}{\exp\left\{V_i(x_{it}, y_t)\right\}}. \end{aligned}$$

■

Proof for Proposition 2:

Proof. Let $B(S)$ be the set of continuous bounded functions on S and $B_0(S)$ be the set of continuous bounded functions that satisfy Proposition 2 on S, where $S = S_X \times S_Y$. Note that $r_{i1}(a_1, x; y)$ and $r_{i2}(a, x; y)$ are both bounded and the transition probability has the Feller property, then we can define operator $T_i : B(S) \to B(S)$ as follows,

$$T_i v(x, y) = \ln \sum_{a_1 \in \mathcal{A}_1(x,y)} \exp\left\{r_{i1}(a_1, x; y) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\{r_{i2}(a, x; y) + \gamma \mathop{\mathbb{E}}_{x', y'} v(x', y')\}\right\}, \qquad \text{(A.12)}$$

where x' denotes the consumer's ownership state in the next period, and y' denotes the exogenous state in the next period.

In what follows, we will first show that T_i is a contraction mapping. Then we show that T_i maps any element in $B_0(S)$ into itself. Lastly, we show that V_i must be a fixed point of operator T_i in $B_0(S)$.

We use Blackwell's sufficient condition to prove that T_i is a contraction mapping. If suffices to show that T_i satisfies both monotonicity and sub-additivity. If $v(x, y) \geq w(x, y)$ for all $(x, y) \in S_X \times S_Y$, then $T_i v(x, y) \geq T_i w(x, y)$. Additionally, for any $c \in \mathbb{R}$, we have,

$$\begin{aligned} T_i(v + c)(x, y) &= \ln \sum_{a_1 \in \mathcal{A}_1(x,y)} \exp\left\{r_{i1}(a_1, x; y) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\{r_{i2}(a, x; y) + \gamma \mathop{\mathbb{E}}_{x', y'} (v(x', y') + c)\}\right\} \\ &= \ln \sum_{a_1 \in \mathcal{A}_1(x,y)} \exp\left\{r_{i1}(a_1, x; y) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\{r_{i2}(a, x; y) + \gamma \mathop{\mathbb{E}}_{x', y'} v(x', y')\} + \gamma c\right\} \\ &= \ln \sum_{a_1 \in \mathcal{A}_1(x,y)} \exp\left\{r_{i1}(a_1, x; y) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\{r_{i2}(a, x; y) + \gamma \mathop{\mathbb{E}}_{x', y'} v(x', y')\}\right\} + \gamma c \\ &= T_i(v)(x, y) + \gamma c. \end{aligned} \qquad \text{(A.13)}$$

Therefore, operator T_i satisfies Blackwell's sufficient conditions and is a contraction mapping with module γ.

Let v be an element in $B_0(S)$. That is, for all $\hat{y}, \tilde{y}$ such that $v((\phi, 0), \hat{y}) = v((\phi, 0), \tilde{y})$, we have that $v(x, \hat{y}) = v(x, \tilde{y})$. And we want to show that for any $\hat{y}, \tilde{y}$ such that $T_i v((\phi, 0), \hat{y}) =$

$T_i v((\phi,0),\tilde{y})$, we have $T_i v(x,\hat{y}) = T_i v(x,\tilde{y})$.
If $x = (\phi, 0)$, then $T_i v((\phi,0),\hat{y}) = T_i v((\phi,0),\tilde{y})$ as given in the condition.
If $x \in J \times \{0,\ldots,\bar{\tau}\}$, then $\mathcal{A}_1(x,\hat{y}) = \{0\} \cup J_t$, using equation (A.12) we have

$$\begin{aligned} T_i v(x,\hat{y}) =& \ln \Bigg(\exp\big\{\beta_i q_{j_{it}} + \gamma \sum_{x'} \big(\pi_X(x'|x,(\phi,0)) \sum_{y'} \pi_Y(y'|\hat{y}) v(x',y')\big)\big\} \\ &+ \sum_{a_1 \in J_t} \exp\big\{r_{i1}(a_1,x;\hat{y}) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\big(r_{i2}(a,x;\hat{y}) + \gamma \sum_{y'} \pi_Y(y'|\hat{y}) v(x',y')\big)\big\}\Bigg). \end{aligned} \tag{A.14}$$

Note that for all $y \in S_Y$,

$$T_i v((\phi,0),y) = \ln \sum_{a_1 \in J_t} \exp\Big\{r_{i1}(a_1,x;y) + \ln \sum_{a_2 \in \mathcal{A}_2(a_1)} \exp\big(r_{i2}(a,x;y) + \gamma \sum_{y'} \pi_Y(y'|y) v(x',y')\big)\Big\}. \tag{A.15}$$

Furthermore, as IVS holds, we have

$$\begin{aligned} \sum_{y'} \pi_Y(y'|\hat{y}) v(x',y') &= \sum_{\delta} \sum_{y':\delta_i(y')=\delta} (\pi_Y(y'|\hat{y}) v(x',y')) \\ &= \sum_{\delta} \sum_{y':\delta_i(y')=\delta} (\pi_Y(y'|\tilde{y}) v(x',y')) \\ &= \sum_{y'} \pi_Y(y'|\tilde{y}) v(x',y'). \end{aligned} \tag{A.16}$$

Plugging equation (A.15) and (A.16) into (A.14) we have,

$$\begin{aligned} T_i v(x,\hat{y}) =& \ln\big\{ \exp\{\beta_i q_{j_{it}} + \gamma \sum_{x'} \big(\pi_X(x'|x,(\phi,0)) \sum_{y'} \pi_Y(y'|\hat{y}) v(x',y')\big)\} \\ &+ \exp T_i v((\phi,0),\hat{y})\big\} \\ =& \ln\big\{ \exp\{\beta_i q_{j_{it}} + \gamma \sum_{x'} \big(\pi_X(x'|x,(\phi,0)) \sum_{y'} \pi_Y(y'|\tilde{y}) v(x',y')\}\big) \\ &+ \exp T_i v((\phi,0),\tilde{y})\big\} \\ =& T_i v(x,\tilde{y}). \end{aligned} \tag{A.17}$$

Therefore, $T_i v$ also satisfies Proposition 2.
Lastly, since $B(S)$ is a complete metric spaces, the value function V_i is the unique fixed point of T_i in $B(S)$. Similarly, function V_i is the unique fixed point of T_i in $B_0(S)$. As $B_0(S) \subseteq B(S)$, the value function must be in $B_0(S)$ and thus must satisfy Proposition 2. ■

Proof for Proposition 3:

Proof. If $x_{it} = (\phi,0)$, then $W_i(x_{it},\delta_{it}) = V_i((\phi,0),y(\delta_{it})) = \delta_{it}$.

If $x_{it} \in J \times \{0, \ldots, \bar{\tau}\}$, by equation of (14), we have

$$\begin{aligned} W_i(x_{it}, \delta_{it}) =& V_i(x_{it}, y_i(\delta_{it})) \\ =& \ln\Bigg(\exp\left\{\beta_i q_{j_{it}} + \gamma \sum_{x_{i,t+1}} \left(\pi_X(x_{i,t+1}|x_{it}, (\phi, 0)) \sum_{y_{t+1}} \pi_Y(y_{t+1}|y(\delta_{it})) V_i(x_{i,t+1}, y_{t+1})\right)\right\} \\ & + \exp \delta_{it} \Bigg). \end{aligned} \tag{A.18}$$

Using the IVS assumption and Proposition 2, replacing $V_i(x_{i,t+1}, y_{t+1})$ with $W_i(x_{i,t+1}, \delta_{i,t+1})$ in equation (A.18), we have

$$\begin{aligned} W_i(x_{it}, \delta_{it}) &= \ln\left(\exp\left\{\beta_i q_{j_{it}} + \gamma \textstyle\sum_{x_{i,t+1}} \left(\pi_X(x_{i,t+1}|x_{it}, (\phi, 0)) \sum_{\delta_{i,t+1}} W_i(x_{i,t+1}, \delta_{i,t+1})\right)\right\} + \exp \delta_{it} \right) \\ &= \ln\left\{\exp\left(\beta_i q_{j_{it}} + \gamma\, \mathbb{E}_{x_{i,t+1}, \delta_{i,t+1}} W_i(x_{i,t+1}, \delta_{i,t+1})\right) + \exp \delta_{it}\right\}, \end{aligned} \tag{A.19}$$

Plugging equation (14)into (15), and replacing $V_i(x_{i,t+1}, y_{t+1})$ with $W_i(x_{i,t+1}, \delta_{i,t+1})$, we have

$$\begin{aligned} \delta_i(y_t) &= \ln \sum_{a_1 \in J_t} \exp\left\{r_{i1}(a_1, (\phi, 0); y_t) + \ln \sum_{a_2 \in \{\underline{\tau}, \bar{\tau}\}} \exp\{r_{i2}(a, (\phi, 0); y_t) + \gamma \mathop{\mathbb{E}}_{\substack{x_{i,t+1} \\ \delta_{t+1}}} W_i(x_{i,t+1}, \delta_{t+1})\}\right\} \\ &= \ln \sum_{a \in J_t \times \{\underline{\tau}, \bar{\tau}\}} \exp\left\{r_i(a, x_{it}; y_t) + \gamma \mathop{\mathbb{E}}_{\substack{x_{i,t+1} \\ \delta_{i,t+1}}} W_i(x_{i,t+1} \delta_{i,t+1})\right\}. \end{aligned}$$

This completes the proof. ■

B Estimation Results

Table 8 and 9 present the estimated values and standard deviations of structural parameters.

	Income Group								
	1	2	3	4	5	6	7	8	9
α	-0.03	-0.03	-0.03	-0.03	-0.03	-0.03	-0.02	-0.03	-0.03
μ_0	96.35	69.14	113.03	107.77	109.86	111.95	111.57	116.11	106.17
Size	0.04	0.03	0.05	0.04	0.05	0.04	0.04	0.05	0.04
APX	-0.02	-0.12	0.02	-0.40	-0.53	-0.49	-0.57	-0.32	-0.62
BRK	0.12	-0.23	0.39	0.22	0.12	0.11	0.19	0.13	-0.10
GE	0.01	0.00	0.04	0.01	0.01	0.02	-0.04	-0.19	-0.09
JVC	0.07	0.04	0.10	0.08	0.09	0.12	0.13	0.08	0.07
MAG	-0.17	0.01	0.03	-0.01	-0.09	0.01	-0.37	0.00	-0.04
PAN	0.07	0.06	0.11	0.09	0.09	0.12	0.14	0.08	0.07
PHL	-0.14	-0.11	-0.20	-0.22	-0.26	-0.41	-0.11	-0.23	-0.41
RCA	0.10	0.09	0.13	0.11	0.10	0.13	0.15	0.08	0.08
SAM	0.22	0.21	-	-	0.23	0.23	0.28	0.18	0.21
SHA	-	0.01	0.03	-0.53	-0.15	-1.14	-1.01	-0.56	-0.94
SON	0.30	0.24	0.37	0.34	0.34	0.37	0.39	0.34	0.33
ZEN	0.11	0.13	0.16	0.13	0.13	0.15	0.18	0.10	0.09

Table 8: Estimation Results

	Income Group								
	1	2	3	4	5	6	7	8	9
α	0.0002	0.0004	0.0002	0.0003	0.0002	0.0001	0.0002	0.0002	0.0003
μ_0	1.4233	2.0267	2.1292	1.5943	1.1597	0.8505	1.3945	1.4853	1.0447
Size	0.0005	0.0008	0.0005	0.0006	0.0004	0.0003	0.0004	0.0006	0.0004
APX	0.0123	0.0691	0.0278	0.0970	0.0997	0.0757	0.0735	0.0974	0.0936
BRK	0.0882	0.3878	0.0447	0.0725	0.0924	0.0635	0.0700	0.0951	0.0942
GE	0.0100	0.0075	0.0253	0.0111	0.0126	0.0087	0.0494	0.0957	0.0514
JVC	0.0073	0.0105	0.0249	0.0103	0.0069	0.0075	0.0144	0.0068	0.0036
MAG	0.1038	0.0135	0.0464	0.0346	0.0886	0.0235	0.0925	0.0285	0.0504
PAN	0.0079	0.0066	0.0248	0.0100	0.0072	0.0075	0.0136	0.0069	0.0041
PHL	0.0361	0.0348	0.0833	0.0563	0.0953	0.0849	0.0276	0.1032	0.1168
RCA	0.0084	0.0081	0.0247	0.0105	0.0078	0.0078	0.0143	0.0085	0.0053
SAM	0.0177	0.0091	-	-	0.0128	0.0096	0.0145	0.0364	0.0067
SHA	-	0.1715	0.1352	0.3545	0.1740	0.3254	0.3916	0.3420	0.4171
SON	0.0080	0.0074	0.0247	0.0101	0.0075	0.0078	0.0141	0.0069	0.0039
ZEN	0.0098	0.0069	0.0245	0.0105	0.0080	0.0082	0.0158	0.0111	0.0066

Table 9: Standard Deviations