

STABILITY AND ROBUSTNESS FOR HYBRID SYSTEMS

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Abstract

Stability and robustness issues for hybrid systems are considered in this paper. General stability results that are extensions of classical Lyapunov theory have recently been formulated. However, these results are in general not straightforward to apply due to the following reasons. First, a search for multiple Lyapunov functions must be performed. However, existing theory does not unveil how to find such functions. Secondly, if the most general stability result is applied, knowledge about the continuous trajectory is required, at least at some time instants. Because of these drawbacks stronger conditions for stability are suggested, in which case it is shown that the search for Lyapunov functions can be formulated as a linear matrix inequality (LMI) problem for hybrid systems consisting of linear subsystems. Additionally, it is shown how robustness properties can be achieved when the Lyapunov functions are given. Specifically, it is described how to determine permitted switch regions around the nominal but uncertain switch sets such that stability is preserved. Furthermore, for integrator hybrid systems it is also shown how acceptable uncertainties around the nominal vector fields can be obtained. The restricted stability result is also applied to fuzzy systems, which may be viewed as hybrid systems as well. Present stability theory for fuzzy systems typically requires the existence of one common quadratic Lyapunov function. However, such a function does not always exist. Stability may then be shown by applying multiple Lyapunov functions, which is demonstrated in the end of this paper.

Keywords: Hybrid systems, switched systems, fuzzy systems, stability, robustness, LMIs.

1 Introduction

This paper deals with stability and robustness for switched and hybrid systems. It is assumed that there are no input signals, which may be a reasonable assumption for a closed-loop hybrid system. For such systems the continuous evolution can be described by differential equations of the form

$$\dot{x}(t) = f(x(t), m(t)) \quad (1)$$

where $x \in \mathbb{R}^n$ and $m \in M = \{m_1, \dots, m_\ell\}$. The discrete state of the system $m(t)$ is restricted to be piecewise constant, switching only at a finite number of times in finite time for which it is continuous from the right. This usually results in abrupt changes of the vector field f in (1).

It should be noted that the system above sometimes is written as $\dot{x}(t) = \bar{f}_i(x(t))$ which is obtained if (1) is evaluated for $m_i \in M$, i.e. $\bar{f}_i(x) = f(x, m_i)$. One interpretation of m is the discrete marking vector in a discrete Petri net [8, 20]. Each $f(\cdot, m_i)$ is called a subsystem.

For a system described by (1) it is possible to separate between switched and hybrid systems. If, for each $x \in \mathbb{R}^n$, only one $m_i \in M$ is possible, then a *switched system* is obtained. Thus, for a switched system the discrete state m is redundant. Other names of switched systems are variable structure systems, multi modal systems and mode switching systems. However, if there are some $x \in \mathbb{R}^n$ for which several discrete states are possible, then the system is called a *hybrid system*. Despite the redundancy for switched systems, m may in some cases still be introduced in the model. In such case it is more common to write the system as $\dot{x}(t) = \bar{f}_{m(t)}(x(t))$ as mentioned above, where

$m(t)$ then is a scalar that explicitly indicates which of the N different subsystems that is active at time t . Note that a switched open-loop system may become a closed-loop hybrid system when a controller has been designed.

Lately, several stability results of switched and hybrid systems have been presented in the literature [10, 24, 9, 4, 32, 34, 27, 36, 16] to name a few. Some of the results are quite general and can be applied to many different nonlinear systems while others are restricted to certain kind of systems, for instance systems having linear subsystems [10, 32]. In certain cases the main goal of a stability theory is to generate a stable control law. The control objective may be to choose which one of several subsystems that is to be used at each time t [32, 16]. This situation occurs e.g. when there are several controllers for the same continuous process.

Most of the mentioned stability results require the existence of one or several auxiliary functions with certain properties, which can be interpreted as an abstract measure of the system energy and are thus extensions of classical Lyapunov theory. In some of the papers the restriction on differentiability of the Lyapunov function V is weakened to obtain a result where only continuity of V is required [24, 27].

In this paper a stability theorem is formulated which is only a minor extension of a theorem in [36]. Stability is shown by introducing several continuous auxiliary functions that are positive definite about the origin. As above, the auxiliary functions can be interpreted as an abstract measure of the system energy. The difference from the theorem in [36] is that we only require continuity of the auxiliary functions. Furthermore, we strongly point out that it is the behavior of the sequence of auxiliary functions that decides if a hybrid system is stable, not which particular auxiliary functions that are used for the different discrete states. Therefore it is not necessary to specify beforehand how the auxiliary functions are coupled to the continuous and discrete states. Thus, the stability result is quite general and requires in fact only the existence of the trajectories for the given initial states. Hence, it can be used not only for hybrid systems but also for other kind of system categories where the existence of the trajectories can be guaranteed.

Even though the stability theorem in this paper appears to be attractive for hybrid systems, it is not straightforward to apply in practise due to two reasons. First, as is typical also for conventional Lyapunov theory, there must be a search for the auxiliary functions and one other function introduced in the theorem. Secondly, one condition in the theorem requires knowledge of the continuous trajectory for the hybrid system, at least at time instants when there are switches from one auxiliary function to another.

By restricting the general stability theorem, a corollary is presented which has the advantage that it does not require knowledge of the continuous trajectories. The auxiliary functions in the corollary have to be positive definite about the origin and have continuous partial derivatives. Nor in this corollary is it necessary to specify beforehand how the auxiliary functions are coupled to the continuous and discrete states. Several auxiliary functions may have to be coupled to the same discrete state; otherwise stability may not be shown, although the hybrid system is stable. Furthermore, for a hybrid system with linear subsystems, it will be shown how to formulate the search for the auxiliary functions as a linear matrix inequality (LMI) problem [3]. All LMI problems can be solved very efficiently by numerical methods available in the mathematical literature. Thus, we have a method to find the auxiliary functions, at least for hybrid systems with linear subsystems.

When the auxiliary functions have been found such that stability can be determined, it is also shown how robustness properties can be obtained. Robustness issues for hybrid systems have been quantified in [13] where it is discussed how variations of the continuous parameters affect the performance. In this paper we are mainly concerned with stability robustness, and how much the stable nominal hybrid system can be altered without becoming unstable. It will be described how to obtain regions around the nominal switch sets which indicate when it is required to change discrete states. Furthermore, for integrator hybrid systems [22, 23] it is demonstrated how acceptable uncertainties in form of regions for the nominal vector fields can be achieved. These regions can also be used to obtain bounds on an external disturbance that affects the system.

The presented stability results are quite general and can be applied also to fuzzy systems. Fuzzy systems may be viewed as hybrid systems where the finite rule bases are related to the discrete states of the hybrid system [5, 15]. In this paper we will illustrate the flexibility of the stability result by applying it to a Takagi-Sugeno fuzzy system [28, 31], which consists of fuzzy blending of linear subsystems. The stability result for fuzzy systems presented in [28, 31] requires the existence of one common positive definite matrix P . However, in some cases such a matrix does not exist. Stability may then be shown by use of multiple Lyapunov functions.

The outline of the paper is as follows: The next section introduces the model of the hybrid system. In Section 3 the stability results are presented and we discuss some of their properties. It is shown how the auxiliary functions are joined in the hybrid state space. In Section 4 it is described how the search for the auxiliary functions for hybrid systems consisting of linear subsystems can be formulated as an LMI problem. In Section 5 it is shown how

robustness properties can be obtained when the auxiliary functions have been found. In Section 6 three examples are presented that illustrate the theory in this paper.

2 Hybrid model

Several propositions for modelling of hybrid systems have been suggested in the literature [20, 6, 2, 7, 17, 1, 19, 12, 18] to name a few. In this paper, stability is studied for autonomous hybrid systems, obtained for instance when a controller has been designed resulting in a closed-loop hybrid system [20]. Models for autonomous hybrid systems can be found in [33, 29, 5].

The evolution of an autonomous hybrid system can be described by (1) combined with a discrete event system whose output determines which vector field is active, that is

$$\dot{x}(t) = f(x(t), m(t)) \quad (2)$$

$$m(t) = \phi(x(t), m(t^-)) \quad (3)$$

where $x \in \mathfrak{R}^n$, $m \in M = \{m_1, \dots, m_\ell\}$ and each $m_i = [m_{i,1} \dots m_{i,d}]^T \in \mathfrak{R}^d$. Here, x is the continuous state and m is the discrete state. The hybrid state space consists of a combination of the continuous and the discrete state space. The notion t^- indicates that $m(t)$ is piecewise constant from the right. When m changes value this usually results in abrupt changes in the vector field f in (2).

It is possible to express the change of discrete states (3) by defining a number of switch sets as

$$S_{ij} = \{x \in \mathfrak{R}^n \mid m_j = \phi(x, m_i)\}. \quad (4)$$

The interpretation of these sets is that if the discrete state is m_i and the continuous state reaches some point in S_{ij} , then the next discrete state becomes m_j . The change of m is illustrated in Figure 1. Typically, the sets S_{ij} are given

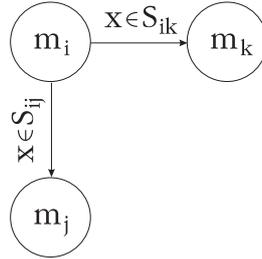


Figure 1: Changes of discrete states in the hybrid system.

by hypersurfaces $s_{ij}(x) = 0$, e.g. hyperplanes $s_{ij}(x) = \eta^T x = 0$, where η is the normal vector.

The hybrid system (2), (3) evolves from the initial conditions $(x_0, m_0) \in I_0 \subseteq X_0 \times M_0$, where I_0 represents the set of all possible initial conditions, and X_0 and M_0 are the sets of initial continuous and discrete states. The evolution can be described as follows. Starting at (x_0, m_i) at time t_0 , the continuous trajectory evolves according to $\dot{x} = f(x, m_i)$. If x reaches some $x_1 \in S_{ij}$ at time t_1 , then the state becomes (x_1, m_j) from which the process continues according to $\dot{x} = f(x, m_j)$, cf. Figure 2.

The evolution of the discrete states from an initial state $(x_0, m_0) \in I_0$ can be described by a switching sequence

$$\Delta_{(x_0, m_0)} = (\mu_0, t_0), (\mu_1, t_1), \dots \quad \mu_k \in M, k \in N \quad (5)$$

where $t_k < t_{k+1}$ and $\mu_k \neq \mu_{k+1} \forall k \in N$, and $\mu_0 = m_0$. This sequence may be finite or infinite. The notion (μ_k, t_k) means that $\dot{x}(t) = f(x(t), \mu_k)$ for $t_k \leq t < t_{k+1}$. Note that the continuous solution $x(t)$ to (2), (3) is at least once differentiable for all times except the times t_k in (5) when there are abrupt changes of f .

To show stability for the hybrid system (2), (3), starting from the initial conditions $(x_0, m_0) \in I_0$, restrictions are imposed on the functions f and ϕ . First, not surprising, it must be guaranteed that a solution exists for all initial conditions. Technically, this can be ensured by requiring continuity of $f(x, m_i) \forall m_i \in M$ (Cauchy's existence theorem [26]). If stronger conditions are imposed, for instance that every solution to (2) for every fixed $m_i \in M$

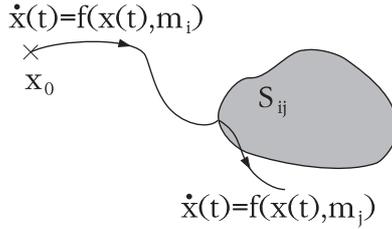


Figure 2: Change of vector field in the hybrid system

satisfies a Lipschitz condition, then it is guaranteed that the solution is unique for every fixed m_i [14]. For the uniqueness of the switch sequence (5) it is required that $S_{ij} \cap S_{i'j'} = \emptyset$, $j, j' \neq i, j \neq j'$, which means that there are no common points in these sets. However, uniqueness is not crucial for the stability results presented later on.

Secondly, it is required that there are finitely many switches of the discrete states in finite time. Thus, sliding modes are avoided, which otherwise would have complicated the stability analysis, cf. [11, 30]. When S_{ij} and $S_{j'i}$ are given by hypersurfaces such that $s_{ij}(x) = s_{j'i}(x) = 0$ for some points, sliding modes are avoided if the projections of the vectors $f(\cdot, m_i)$ and $f(\cdot, m_j)$ onto the normal of the surface $s_{ij}(x) = 0$ are not directed toward each other for any of the common points $s_{ij}(x) = s_{j'i}(x) = 0$ [11]. When a controller is designed to obtain the closed-loop hybrid system (2) and (3), sliding modes can be avoided by introducing hysteresis where it otherwise had occurred, cf. [25, 23].

Finally, it is assumed that the equilibrium point is the same for all discrete states and that it is located in the origin of the continuous state-space. Thus, $f(0, m_i) = 0, \forall m_i \in M$. There is no loss of generality in assuming that the origin is the equilibrium point since otherwise it can be shifted to the origin by a simple change of the continuous state variables.

Some additional comments about the model are in order. First it should be noted that if there is only one discrete state in M the system (2), (3) is simply a nonlinear system $\dot{x}(t) = f(x(t))$ where there are no abrupt changes in the vector field f . Secondly, there is a clear separation between the modelling procedure and stability analysis. This means that the auxiliary functions are not coupled to individual discrete states. In some cases stability may be shown by introducing several auxiliary functions for a specific discrete state, as in Example 6.1 in Section 6. On the other hand, it may also be possible to utilize the same auxiliary function for more than one discrete state, cf. Example 6.3. Finally, as mentioned in the introduction there is a separation between switched and hybrid systems. For hybrid systems it is usually known which discrete states that are possible for a given continuous state. This is the case for instance in systems containing hysteresis [5]. It may also be possible to conclude where a discrete state is possible by studying the direction of the vector fields at the switch sets, cf. Example 6.1. The reason for being concerned about where in the continuous state space the discrete states are possible is that this knowledge sometimes facilitates the stability analysis since stability will be shown by joining several auxiliary functions in the hybrid state space, and there is no meaning to couple such functions to hybrid states that will never occur.

3 Stability

3.1 Motivating example

It is interesting to note that a system consisting of two or several stable subsystems does not imply that the switched or hybrid system is stable, cf. the examples in [4, 16]. On the other hand, a system consisting of two or several unstable subsystems may be stable if they are appropriately switched, cf. [32]. This is also demonstrated by the following example which is shown to be stable in Example 6.1 in Section 6 applying the stability theory presented in this paper.

Example 1 Consider the following hybrid system:

$$\dot{x}(t) = A(m_i)x(t) = \begin{bmatrix} 1 & m_{i,1} \\ m_{i,2} & 1 \end{bmatrix} x(t) \quad (6)$$

The discrete set M contains two states, $M = \{m_1, m_2\}$, where $m_1 = [-100 \ 10]^T$ and $m_2 = [10 \ -100]^T$. Thus,

$$A(m_1) = \begin{bmatrix} 1 & -100 \\ 10 & 1 \end{bmatrix} \quad A(m_2) = \begin{bmatrix} 1 & 10 \\ -100 & 1 \end{bmatrix} \quad (7)$$

Both linear subsystems have eigenvalues $1 \pm j\sqrt{1000}$ and are thus unstable.

The sets S_{12} and S_{21} that determine the change of discrete states are defined by hyperplanes $s_{ij}(x) = \eta^T x = 0$ and are given by

- $S_{12} = \{x \in \mathbb{R}^2 \mid x_2 = 2.75x_1\}$
- $S_{21} = \{x \in \mathbb{R}^2 \mid x_2 = 0.36x_1\}$

In Figure 3, a simulation of the hybrid system is shown for $t = 0.17$ time units, with initial state $(x_0, m_0) = ([0.3 \ 0.826]^T, m_1)$. The system evolves similarly for the initial state $(x_0, m_0) = ([-0.3 \ -0.826]^T, m_1)$. As can be

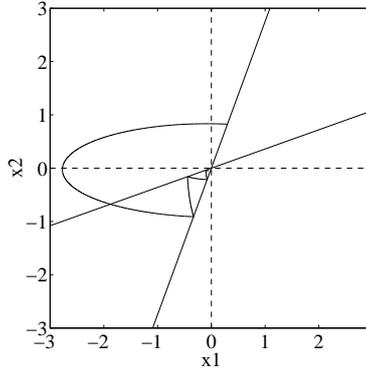


Figure 3: Simulation for $t = 0.17$ time units of the system with initial state $([0.3 \ 0.826]^T, m_1)$.

seen in the figure, the hybrid system appears to be (asymptotically) stable.

3.2 Preliminaries

Before a quite general stability result is presented it is necessary to introduce some preliminary definitions and concepts. We will start by defining the auxiliary functions that are used as measure of the system energy.

Definition 1 (Candidate Lyapunov function) *A function V_q is called a candidate Lyapunov function if it is positive definite about the origin and has continuous partial derivatives.*

Next, V is defined as a finite set of candidate Lyapunov functions, $V = \{V_1, V_2, \dots, V_p\}$, $p \geq 1$.

When the hybrid system (2), (3) evolves from an initial state $(x_0, m_0) \in I_0$, a sequence of candidate Lyapunov functions will be used according to

$$\Lambda_{(x_0, m_0)} = (\mathcal{V}_0, \tau_0), (\mathcal{V}_1, \tau_1), \dots \quad \mathcal{V}_k \in V, \quad k \in N \quad (8)$$

where (\mathcal{V}_k, τ_k) means that the system energy is measured by the candidate Lyapunov function \mathcal{V}_k for $\tau_k \leq t < \tau_{k+1}$. The sequence of candidate Lyapunov functions can be combined to obtain a function \bar{V} defined as

$$\bar{V} = \mathcal{V}_k \quad \tau_k \leq t < \tau_{k+1} \quad (9)$$

which is usually discontinuous at the times τ_k .

Note that up to this time it has not been specified how the candidate Lyapunov functions are coupled to the continuous and discrete states. Thus, the connection between the switching times τ_k in (8) and t_k in (5) is not yet specified. This is in fact not needed for the statement and proof of the general stability theorem.

To obtain the times exclusively in the sequence (8) the following projection is defined:

$$\pi(\Lambda_{(x_0, m_0)}) = \tau_0, \tau_1, \dots$$

To express the time intervals where each candidate Lyapunov function is used, the interval completion $\mathcal{I}(T)$ of a strictly increasing sequence of times $T = \tau_0, \tau_1, \dots$ is defined as:

$$\mathcal{I}(T) = \cup_{j \in \mathbb{N}} [\tau_{2j}, \tau_{2j+1}) \quad (10)$$

Furthermore, the even sequence of T is given by:

$$\mathcal{E}(T) = \tau_0, \tau_2, \tau_4, \dots \quad (11)$$

Finally, the notation $\Lambda_{(x_0, m_0)}|q$ is used to indicate the sequence of switching times in (8) when the candidate Lyapunov function V_q is switched on and off. These definitions are used in Theorem 1 and Corollary 1 and are illustrated in Figure 4.

3.3 General stability theorem

The following theorem may be used to show that the hybrid system (2), (3) is stable. The theorem is a minor extension of a theorem in [36]. However, compared to that theorem we strongly point out that it is the behavior of the sequence of candidate Lyapunov functions $\Lambda_{(x_0, m_0)}$ (8) that is used to show stability of the hybrid system, not the sequence of discrete states $\Delta_{(x_0, m_0)}$ (5). Therefore, it is not necessary to specify how the candidate Lyapunov functions are coupled to the continuous and discrete states. The theorem is very general and requires in fact only the existence of the trajectories from the given initial states. Hence the theorem may be used also for hybrid systems containing state jumps. In the same way as the authors of [36] point out, the proof can be carried out using similar arguments as in [4, 35]. A proof is included in the appendix for completeness.

Theorem 1 *Let a hybrid system be described by (2), (3) with $f(0, m_i) = 0 \forall m_i \in M$. If for all switching sequences $\Lambda_{(x_0, m_0)}$ in (8) occurring from $(x_0, m_0) \in I_0$*

1. $V_q(x(t)) \leq h(V_q(x(\tau_j))) \quad \forall t \in \mathcal{I}(\Lambda_{(x_0, m_0)}|q), \tau_j \in \mathcal{E}(\Lambda_{(x_0, m_0)}|q)$, where $h \in \mathcal{C}[\mathbb{R}^+, \mathbb{R}^+]$ with $h(0) = 0$
2. $V_q(x(\tau_{j+1})) \leq V_q(x(\tau_j)), \quad \tau_{j+1}, \tau_j \in \mathcal{E}(\Lambda_{(x_0, m_0)}|q)$

where each $V_q \in V$, then the origin is stable in the sense of Lyapunov.

The notion $\mathcal{C}[\mathbb{R}^+, \mathbb{R}^+]$ is the space of all continuous real valued functions from \mathbb{R}^+ to \mathbb{R}^+ . Note that this theorem in fact only requires continuity and not continuous partial derivatives as in [36], of the candidate Lyapunov functions.

To illustrate Theorem 1 consider Figure 4 where a sequence of Lyapunov functions is given that satisfies the conditions of the theorem. Note that the symbols in the figure (dots at τ_0, τ_2, τ_6 , triangles at τ_1, τ_4 and crosses at τ_3, τ_5) mark decreasing sequences at the corresponding times, in agreement to the second condition in Theorem 1.

Some remarks are in order:

- The theorem does not hold if there are infinitely many candidate Lyapunov functions in the set V . The reason for this is that it is then possible to change candidate Lyapunov function infinitely many times to functions not used before, thus satisfying the conditions in Theorem 1 even if the system may be unstable.
- If each candidate Lyapunov function is coupled to a discrete state, and the first condition in Theorem 1 is strengthened to $\dot{V}_q(x(t)) \leq 0 \forall t \in \mathcal{I}(\Lambda_{(x_0, m_0)}|q)$, then the result in [4] is obtained.
- If each candidate Lyapunov function is coupled to a discrete state, the hybrid system consists of linear subsystems, there is a switch to another $A(m_i)$ in an upper bounded time, and if the second condition in Theorem 1 is changed to $\mathcal{V}_{k+1}(x(\tau_{k+1})) \leq \mathcal{V}_k(x(\tau_k)) \quad \tau_k, \tau_{k+1} \in \pi(\Lambda_{(x_0, m_0)})$ that is a stronger condition, then the stability result in [32] is obtained.

Discussion

Even though Theorem 1 indeed is very attractive for hybrid systems it is not straightforward to use it to show

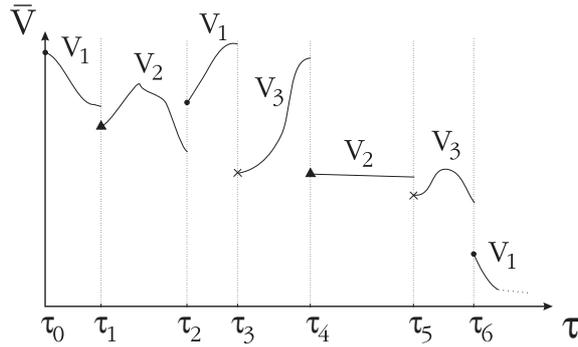


Figure 4: Illustration of a sequence of Lyapunov functions that satisfies the conditions in Theorem 1. In this case we have for instance $\Lambda_{(x_0, m_0)}|1 = \tau_0, \tau_1, \tau_2, \tau_3, \tau_6, \tau_7, \dots$, $\mathcal{I}(\Lambda_{(x_0, m_0)}|1) = [\tau_0, \tau_1) \cup [\tau_2, \tau_3) \cup [\tau_6, \tau_7) \cup \dots$ and $\mathcal{E}(\Lambda_{(x_0, m_0)}|1) = \tau_0, \tau_2, \tau_6, \dots$, cf. (10), (11).

stability due to two reasons. First, the theorem does not help in the practical search for the candidate Lyapunov functions or the function h . This is a typical problem of Lyapunov theory [14]. Secondly, the second condition in the theorem requires knowledge about the continuous trajectory of the hybrid system, at least at times when there are switches of candidate Lyapunov functions. However, these times are obtained as a result of the evolution of the hybrid trajectory, which is unknown. If the main goal of the stability result is to generate a stable control law, it is possible to use the theorem by first introducing a number of candidate Lyapunov functions and then switch between the different vector fields such that the conditions in the theorem are fulfilled. However, this is not the focus of this paper, since we are dealing with stability and robustness analysis for a given open-loop, or closed-loop hybrid system where perhaps a controller already has been designed.

3.4 Restricted stability result

By strengthen the conditions in Theorem 1 a restricted stability result is presented as a Corollary in this section. The purpose is to get rid of the requirement that the continuous trajectory of the hybrid system need to be at least partly known. In practise, the corollary requires only a local analysis where the switches of candidate Lyapunov functions occur. Furthermore, we will show that the candidate Lyapunov functions can be generated for hybrid systems consisting of linear subsystems by formulating the search of the functions as a linear matrix inequality (LMI) problem. Additionally, when the candidate Lyapunov functions are known, robustness properties can also be obtained.

Corollary 1 *Let a hybrid system be described by (2), (3) with $f(0, m_i) = 0 \forall m_i \in M$. If for all switching sequences $\Lambda_{(x_0, m_0)}$ in (8) occurring from $(x_0, m_0) \in I_0$*

1. $\dot{\mathcal{V}}_k(x(t)) \leq 0 \quad \forall t \in [\tau_k, \tau_{k+1}), \tau_k, \tau_{k+1} \in \pi(\Lambda_{(x_0, m_0)})$
2. $\mathcal{V}_{k+1}(x(\tau_k)) \leq \mathcal{V}_k(x(\tau_k^-)), \tau_k \in \pi(\Lambda_{(x_0, m_0)})$

where each $\mathcal{V}_k \in V$, then the origin is stable in the sense of Lyapunov.

Note that this corollary requires that the candidate Lyapunov functions have continuous partial derivatives since $\dot{\mathcal{V}}_q = \frac{\partial \mathcal{V}_q}{\partial x} f(x, m)$ is expected to be defined. It should be pointed out that if there is only one possible discrete state in M and only one Lyapunov function is used to show stability, then Corollary 1 is the usual theorem for Lyapunov stability [14, 26].

An illustration of a sequence that satisfies the conditions in Corollary 1 is given in Figure 5. As can be seen, the energy decreases at all times measured by some of the Lyapunov functions, regardless of if there is a switch of Lyapunov function or not. By comparing Figure 4 and Figure 5 the strengthen conditions in Theorem 1 are evident.

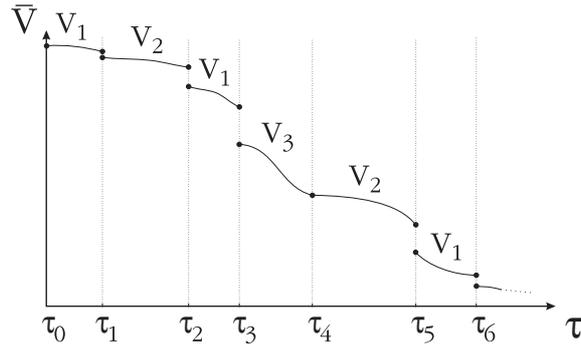


Figure 5: Illustration of a sequence of Lyapunov functions that satisfies the conditions in Corollary 1.

Asymptotic stability

Up to this point we have only discussed stability for hybrid systems. However, it is possible to strengthen the conditions in Theorem 1 and Corollary 1 to show asymptotic stability. For instance, if there are infinitely many candidate Lyapunov function switches and they strictly decrease an infinite number of these switching times, then the hybrid system is asymptotically stable. This can easily be proven by contradiction. Furthermore, if there are not infinitely many switches of the candidate Lyapunov functions, but the last one used is strictly decreasing, then the system is also asymptotically stable. This can also be proven easily by using the same arguments as for the usual Lyapunov proofs [26, 14].

3.5 How to apply the restricted stability result

The set V contains a number of candidate Lyapunov functions that are used as a measure of the hybrid system energy, $V = \{V_1, \dots, V_p\}$. Since the energy changes according to $\dot{V}_q = \frac{\partial V_q}{\partial x} f(x, m_i)$ for an arbitrary $V_q \in V$, this means that the change of energy depends on the vector field $f(x, m_i)$ and thus on the discrete state m_i . To express where in the continuous state space the energy measured by a candidate Lyapunov function V_q is decreasing for the discrete state m_i , and where in the continuous state space the energy decreases when there is a switch from candidate Lyapunov function V_q to V_r , the following sets are defined:

Definition 2

- $\Omega_i^q = \{x \in \mathbb{R}^n \mid \frac{\partial V_q}{\partial x} f(x, m_i) \leq 0\}$
- $\Omega^{qr} = \{x \in \mathbb{R}^n \mid V_q \geq V_r\}$

Next, the set $\bar{\Omega}_i^q$ is defined as all points in the continuous state space where the candidate Lyapunov function V_q is used as a measurement of the energy for the discrete state $m_i \in M$. Note that $\bar{\Omega}_i^q \subseteq \Omega_i^q$ to satisfy the first condition in Corollary 1.

Change of candidate Lyapunov function for the same discrete state

Assume that the candidate Lyapunov functions V_q and V_r are used as a measure of the energy for the same discrete state m_i . Furthermore, assume that the corresponding defined sets Ω_i^q , Ω_i^r and Ω^{qr} are those given in Figures 6a-c where $j = i$, $q \neq r$, and that $\bar{\Omega}_i^q$ and $\bar{\Omega}_i^r$ are given by Figure 6d. If the vector field $f(x, m_i)$ at the boundary between $\bar{\Omega}_i^q$ and $\bar{\Omega}_i^r$ is directed as in Figure 6d, then the conditions in Corollary 1 are satisfied when the trajectory moves from region $\bar{\Omega}_i^q$ to $\bar{\Omega}_i^r$. The reason for this is that when the trajectory is in region $\bar{\Omega}_i^q$ the energy measured by candidate Lyapunov function V_q is decreasing, and when the trajectory is in region $\bar{\Omega}_i^r$ the energy measured by candidate Lyapunov function V_r is decreasing. Furthermore, when the trajectory passes from region $\bar{\Omega}_i^q$ to $\bar{\Omega}_i^r$, $V_q \geq V_r$.

Change of candidate Lyapunov function for different discrete states

Assume now that the candidate Lyapunov functions V_q and V_r are used as a measure of the energy for the different

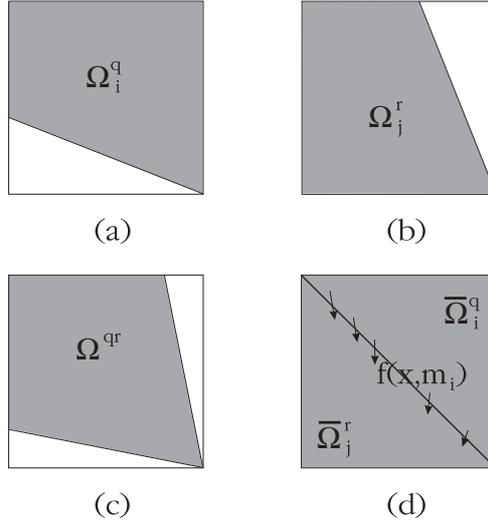


Figure 6: Change of candidate Lyapunov functions for the same discrete state. $j = i$, $q \neq r$. (a) Region Ω_i^q , cf. Definition 2. (b) Region Ω_j^r . (c) Region Ω^{qr} . (d) Regions $\bar{\Omega}_i^q$ and $\bar{\Omega}_j^r$ where the candidate Lyapunov functions V_q and V_r are used as a measurement of the energy in the hybrid system for the discrete state m_i . The vector field direction $f(x, m_i)$ at the boundary between the two sets $\bar{\Omega}_i^q$ and $\bar{\Omega}_j^r$ implies that the conditions in Corollary 1 are satisfied at this boundary.

discrete states m_i and m_j , and consider once more the Figures 6a-c, where $i \neq j$, $q \neq r$. If the discrete state is m_i and some point in S_{ij} is reached, then the discrete state becomes m_j , implying that the vector field is changed. Two possible situations may occur, which is illustrated in Figures 7a-b and Figures 7c-d. Either the vector fields have the same direction before and after the switch or opposite directions. In both cases the conditions in Corollary 1 are

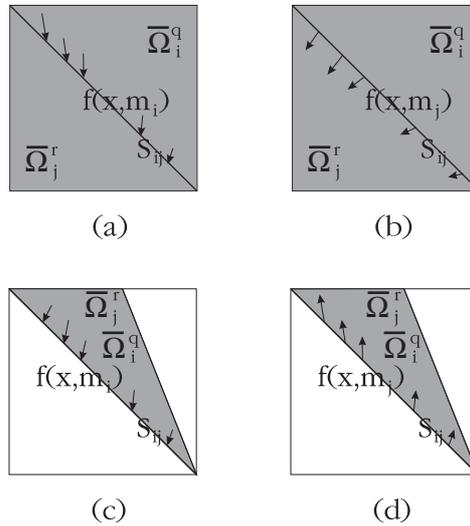


Figure 7: Switch set S_{ij} indicating the change of discrete state. (a) and (b) the same directions of the vector fields $f(x, m_i)$ and $f(x, m_j)$. (c) and (d) opposite directions of the vector fields $f(x, m_i)$ and $f(x, m_j)$.

satisfied when the trajectory moves from region $\bar{\Omega}_i^q$ to $\bar{\Omega}_j^r$, for the same reasons as above. In the first case the regions $\bar{\Omega}_i^q$ and $\bar{\Omega}_j^r$ are not overlapping. However, this is the case in the latter illustration. Note that S_{ij} only can be reached from $\bar{\Omega}_i^q$ because of the direction of the vector field $f(x, m_i)$.

Thus, to show stability by Corollary 1 the problem is to find candidate Lyapunov functions such that they are connected properly in the hybrid state space according to the cases just described. Note that in general it is not necessary to always use the same candidate Lyapunov function in a certain region of the hybrid state space. It may be possible to use different candidate Lyapunov functions as a measurement of the energy in the hybrid system at different times, if this is desirable.

It should be pointed out that it is not necessary to switch candidate Lyapunov functions when the discrete state is changed since the same candidate Lyapunov function can be used as a measure of the energy for several discrete states. For some hybrid systems it is possible to show stability by finding only one Lyapunov function, which is explained in the next section. The coupling between the different times in (8) and (5) is that some of the times coincide, namely when there is a change of both discrete state and candidate Lyapunov function at the same time. However, as mentioned above, it is possible to change discrete state without switching candidate Lyapunov function, or to switch candidate Lyapunov function for the same discrete state.

3.6 One Lyapunov function

If M contains several discrete states it is sometimes possible to use only one Lyapunov function to show stability. In [9] a stability result is presented with this assumption. However, as the authors of that paper point out, it is not always possible to find such a Lyapunov function even if the system is stable. Anyway, if only one Lyapunov function is used, it is possible to obtain sufficient conditions for a hybrid system (2), (3) to be Lyapunov stable.

Lemma 1 *If $x^T f(x, m_i) \leq 0 \forall m_i \in M$ and $\forall x \in \mathbb{R}^n$ then the system is stable in the sense of Lyapunov. Specifically, if each $f(\cdot, m_i)$ is linear then the system is stable if each $A(m_i) \leq 0$. This is equivalent to check if the eigenvalues for each $A(m_i) + A(m_i)^T$ is less or equal to zero.*

Proof Use the Lyapunov function $V(x) = \frac{1}{2}x^T x$. Then $\dot{V}(x) = x^T \dot{x} = x^T f(x, m_i)$, implying that if $x^T f(x, m_i) \leq 0 \forall m_i \in M$ and $\forall x \in \mathbb{R}^n$ then the system is stable. Specifically, if the system is linear then $\dot{V}(x) = x^T A(m_i)x$, implying that if each $A(m_i) \leq 0$ then the system is stable.

A matrix that is not symmetric can be written as $A(m_i) = \frac{A(m_i) + A(m_i)^T}{2} + \frac{A(m_i) - A(m_i)^T}{2}$ where the first part is symmetric and the second skew-symmetric [26]. Since a quadratic form of a skew symmetric matrix is zero [26], $x^T A(m_i)x = x^T \frac{A(m_i) + A(m_i)^T}{2}x$, implying that the system is stable if the eigenvalues for $A(m_i) + A(m_i)^T \forall m_i \in M$ is less or equal to zero. ■

The geometric interpretation of $x^T f(x, m_i) \leq 0$ is that the angle between x and $f(x, m_i)$ is greater or equal to 90° . This means that all trajectories are always approaching the origin regardless of the continuous and discrete states, and thus how the system is switched.

Example 2 *Consider the system with the following linear subsystems:*

$$A(m_1) = \begin{bmatrix} -1 & 4 \\ 0 & -5 \end{bmatrix} \quad A(m_2) = \begin{bmatrix} -1 & 10 \\ -10 & -1 \end{bmatrix} \quad (12)$$

$A(m_1) + A(m_1)^T$ has eigenvalues $\lambda = -6 \pm \sqrt{32}$ and $A(m_2) + A(m_2)^T$ has both eigenvalues at $\lambda = -2$. Thus, the system is stable no matter how it is switched.

4 Auxiliary function generation by solving linear matrix inequalities

When the hybrid system consists of linear subsystems and the switch sets S_{ij} satisfy certain properties, which are explained later on, the candidate Lyapunov functions can be found by formulating the search as a linear matrix inequality (LMI) problem [3]. A variety of problems arising in system and control theory can be reduced to problems involving LMIs. In fact, the first LMI used to analyse stability of a dynamical system was the Lyapunov inequality $A^T P + P A < 0$, which can be solved analytically by solving a set of linear equations. Most LMI problems do not have analytical solutions. However, all LMI problems can be solved numerically very efficiently, in polynomial-time, by recently developed interior-point methods [3].

Definition 3 [3] *A linear matrix inequality (LMI) has the form*

$$F(z) = F_0 + \sum_{i=1}^m z_i F_i > 0, \quad (13)$$

where $z \in \mathfrak{R}^m$ is the variable and $F_i \in \mathfrak{R}^{n \times n}$ are given symmetric matrices, $i = 1, \dots, m$.

Multiple LMIs $F^{(1)}(z) > 0, \dots, F^{(p)}(z) > 0$ can be expressed as the single LMI $\text{diag}(F^{(1)}(z), \dots, F^{(p)}(z)) > 0$. Often problems are encountered in which the variables are matrices, e.g. the Lyapunov inequality $A^T P + P A < 0$ where $P > 0$ is the variable. In this case the LMI is not written explicitly in the form $F(z) > 0$. In addition to save notation, this may also lead to more efficient computation [3]. Of course, the Lyapunov inequality can be put in the form of Definition 3 by letting P_1, \dots, P_m be a basis for the matrix P and then defining $F_0 = 0$ and $F_i = -A^T P_i - P_i A$.

Given an LMI $F(z) > 0$, the corresponding LMI problem is to find a feasible z^{feas} such that $F(z^{feas}) > 0$ or determine that the LMI is infeasible. This is a convex feasibility problem. Infeasibility of an LMI problem can be verified by a dual formulation.

Since we are dealing with linear subsystems, a quadratic Lyapunov function can always be found for those systems that are stable. Thus, a natural choice of candidate Lyapunov functions in the set V is the quadratic form $V_q = x^T P_q x$, where each P_q is a positive definite matrix. The sets Ω_i^q and Ω^{qr} of Definition 2 then become:

Definition 4

- $\Omega_i^q = \{x \in \mathfrak{R}^n \mid x^T (A(m_i)^T P_q + P_q A(m_i)) x \leq 0\}$
- $\Omega^{qr} = \{x \in \mathfrak{R}^n \mid x^T P_q x \geq x^T P_r x\}$

These sets are all of the quadratic form:

$$\Omega_Q = \{x \in \mathfrak{R}^n \mid x^T Q x \leq 0\} \quad (14)$$

If $x_0 \in \Omega_Q$ then also $\alpha x_0 \in \Omega_Q$ for an arbitrary α , which means that Ω_Q is a conic region. From the definition of positive definite, negative definite and indefinite matrices it is known that only $x = 0$ is contained in Ω_Q if Q is positive definite, all $x \in \mathfrak{R}^n$ are contained in Ω_Q if Q is negative definite, and there are $x_0, x_1 \in \mathfrak{R}^n$ such that $x_0 \in \Omega_Q$ and $x_1 \notin \Omega_Q$ if Q is indefinite.

Sets of the form Ω_Q^- will be used later on. These sets are the interior of the corresponding sets Ω_Q and are defined as:

$$\Omega_Q^- = \{x \in \mathfrak{R}^n \mid x^T Q x < 0\} \quad (15)$$

Also Ω_Q^- is a conic set.

In the next subsections a hierarchical procedure is given that is suggested as a logical way to conclude whether a hybrid system is stable or not.

4.1 One candidate Lyapunov function

The first thing to do is trying to find one matrix $P > 0$ such that $A(m_i)^T P + P A(m_i) \leq 0 (< 0) \forall m_i$. If such P exists the system is stable (asymptotically stable) no matter how the switches occur. Then it is not necessary to know where in the continuous state space the discrete states are possible. Of course there does not exist a $P > 0$ if any of the subsystems is unstable. The following theorem gives necessary conditions for ensuring the existence of a common P .

Theorem 2 *If there exists a $P > 0$ such that*

$$A(m_i)^T P + P A(m_i)^T \leq 0 (< 0) \quad \forall m_i \in M \quad (16)$$

then the system $\dot{x} = \sum_{i=1}^N \alpha_i A(m_i) x$ is stable (asymptotically stable) for arbitrary $\alpha_i \geq 0$.

Proof Assume that there exists a common $P > 0$. Then

$$\begin{aligned} A(m_1)^T P + P A(m_1)^T &\leq 0 (< 0) \\ &\vdots \\ A(m_\ell)^T P + P A(m_\ell)^T &\leq 0 (< 0) \end{aligned} \tag{17}$$

Multiplying each inequality with an arbitrary $\alpha_i \geq 0$ and summing them up gives:

$$\sum_{i=1}^{\ell} \alpha_i A(m_i)^T P + P \sum_{i=1}^{\ell} \alpha_i A(m_i) \leq 0 (< 0). \tag{18}$$

Thus, $\dot{x} = \sum_{i=1}^{\ell} \alpha_i A(m_i)x$ being stable (asymptotically stable), where $\alpha_i \geq 0$, is a necessary condition for the existence of a common $P > 0$. \blacksquare

Example 3 Consider the system in Example 2. Both $A(m_1)$ and $A(m_2)$ are stable, and so is any arbitrary weighted sum. Thus, there may exist a common $P > 0$. In fact, from Example 2 we know that P being the identity matrix is one such possibility.

Example 4 Consider the system with the linear subsystems:

$$A(m_1) = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix} \quad A(m_2) = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} \tag{19}$$

In this case:

$$A(m_1) + A(m_2) = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}, \tag{20}$$

which has the eigenvalues -1 and 5 , indicating that it is unstable. Thus, there does not exist a common $P > 0$. In fact, it is possible to switch the system in such a way that it becomes unstable.

The search for a common P can be formulated as an LMI problem [3].

LMI problem 1 Find $P > 0$ such that

- $A(m_i)^T P + P A(m_i) \leq 0 (< 0) \quad \forall m_i \in M$.

Thus, by solving this problem a common P is obtained if it exists.

4.2 One candidate Lyapunov function for each discrete state

If no common $P > 0$ exists, several candidate Lyapunov functions have to be introduced to, if possible, show that the hybrid system is stable. The next thing to do is trying to find one candidate Lyapunov function for each discrete state. The number of candidate Lyapunov functions in the set V is then the same as the number of discrete states, i.e. $p = \ell$. Candidate Lyapunov function V_i is used for the discrete state m_i . To begin we start by trying to find the different candidate Lyapunov functions without using any information where the different discrete states are possible. The conditions of Corollary 1 then gives the following problem to be solved:

Problem 1 Find $P_i > 0$, $i = 1, \dots, \ell$, such that

- $A(m_i)^T P_i + P_i A(m_i) \leq 0 (< 0)$, $i = 1, \dots, \ell$.
- $x^T (P_j - P_i)x \leq 0$ whenever $x \in S_{ij}$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$.

Note that if some of the switch sets are empty or only contains the origin, then the corresponding condition of the second item becomes redundant.

Switch sets in conic regions

As Problem 1 is stated it is not directly an LMI problem. However, if certain restrictions are imposed on the switch

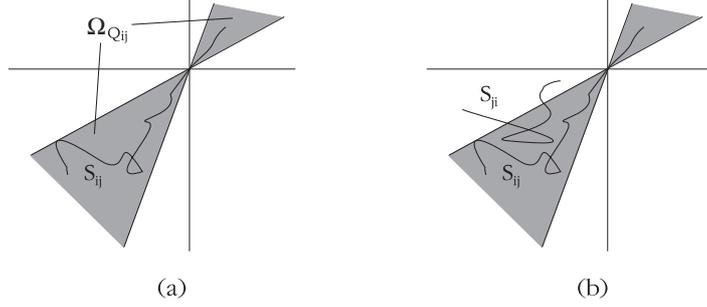


Figure 8: (a) All $x \in S_{ij}$ are contained in the conic shaded region $\Omega_{Q_{ij}}$. (b) Switch sets S_{ij} and S_{ji} for which it is not possible to find two different P_i and P_j .

sets S_{ij} , the problem can be reformulated as an LMI problem. As mentioned above, regions given by quadratic criteria are conic, cf. (14). Therefore, if a conic region is found where all points in the set S_{ij} also are included in the conic region, it will be shown that Problem 1 can be formulated as an LMI problem.

Hence, assume that all $x \in S_{ij}$ are contained in the conic region $\Omega_{Q_{ij}}$, cf. Figure 8a. It can always be assumed without loss of generality that $Q_{ij} = Q_{ij}^T$, cf. the discussion in the proof of Lemma 1. Problem 1 then becomes:

Problem 2 Find $P_i > 0$, $i = 1, \dots, \ell$, such that

- $A(m_i)^T P_i + P_i A(m_i) \leq 0$ (< 0), $i = 1, \dots, \ell$.
- $x^T (P_j - P_i)x \leq 0$ for all x such that $x^T Q_{ij}x \leq 0$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$.

Note that the second item now is stronger than in Problem 1 since $S_{ij} \subseteq \Omega_{Q_{ij}}$. This property will however be utilized in the robustness analysis in Section 5. Certain restrictions have to be imposed on the regions $\Omega_{Q_{ij}}$ and $\Omega_{Q_{ji}}$ if the second item of Problem 2 is to have a solution other than $P_i = P_j$.

Lemma 2 If $\Omega_{Q_{ij}}^- \cap \Omega_{Q_{ji}}^- \neq \{0\}$ then the only possible solution to the second item of Problem 2 is $P_i = P_j$.

Proof Assume that $\Omega_{Q_{ij}}^- \cap \Omega_{Q_{ji}}^- \neq \{0\}$. This means that $\exists x_0 \in \Omega_{Q_{ij}}^- \cap \Omega_{Q_{ji}}^-$ and an ϵ -neighborhood of x_0 , $B_\epsilon(x_0) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \epsilon\}$, such that $B_\epsilon(x_0) \subset \Omega_{Q_{ij}}^- \cap \Omega_{Q_{ji}}^-$ since $\Omega_{Q_{ij}}^-$ and $\Omega_{Q_{ji}}^- \neq \{0\}$ are open sets. The second item of Problem 2 then states that $x^T (P_j - P_i)x \leq 0$ and $x^T (P_j - P_i)x \geq 0 \forall x \in B_\epsilon(x_0)$, which only can be satisfied by $P_i = P_j$ since $x^T (P_j - P_i)x$ is a quadratic form, and hence $x^T (P_j - P_i)x = 0$ only for a finite number of normed x if $P_i \neq P_j$. ■

Note that the regions $\Omega_{Q_{ij}}$ and $\Omega_{Q_{ji}}$ may share boundary points. To fulfill the condition $\Omega_{Q_{ij}}^- \cap \Omega_{Q_{ji}}^- = \{0\}$, Q_{ij} and Q_{ji} cannot be negative definite, but they have to be indefinite. For switch sets that are empty or only contains the origin, the corresponding matrix can be chosen positive definite.

Hence, for Problem 1 to have a solution other than $P_i = P_j$, the switch sets S_{ij} and S_{ji} must be restricted to sets such that $S_{ij} \subseteq \Omega_{Q_{ij}}$, $S_{ji} \subseteq \Omega_{Q_{ji}}$ where Q_{ij} and Q_{ji} are indefinite matrices. Figure 8b exemplifies when it is not possible to find matrices Q_{ij} and Q_{ji} such that $\Omega_{Q_{ij}}^- \cap \Omega_{Q_{ji}}^- = \{0\}$. The only possible solution is then $P_i = P_j$. Hopefully, the system can be shown to be stable with $P_i = P_j$, cf. Section 4.3. Otherwise, non quadratic candidate Lyapunov functions must be searched for.

The S-procedure

Problem 2 requires quadratic forms to be negative whenever some other quadratic forms are negative. This problem has been studied by mathematicians for at least seventy years. For the solution of this kind of problems an S-procedure, which originally dates back to 1944, can be used [3]. In the S-procedure the problem is formulated as follows: Let F_0, \dots, F_s be quadratic functions of the variable $x \in \mathbb{R}^n$ of the form:

$$F_i(x) = x^T T_i x + 2u_i^T x + v_i, \quad i = 1, \dots, s, \quad (21)$$

where $T_i = T_i^T$. The conditions on F_0, \dots, F_s are:

$$F_0(x) \geq 0 \forall x \text{ such that } F_i(x) \geq 0, \quad i = 1, \dots, s. \quad (22)$$

If there exist $\alpha_1 \geq 0, \dots, \alpha_s \geq 0$ such that

$$F_0(x) - \sum_{i=1}^s \alpha_i F_i(x) \geq 0 \quad \forall x \quad (23)$$

then (22) holds. When $s = 1$ the converse is also true, i.e. (22) implies (23), provided that there is some x_0 such that $F_1(x_0) > 0$ [3].

Thus, for $s = 1$, $u_1 = v_1 = 0$, $F_0^{(ij)}(x) = -x^T(P_j - P_i)x$ and $F_1^{(ij)}(x) = -x^T Q_{ij}x$, Problem 2 can be reformulated as an LMI problem in the variables P_i and α_{ij} . Note that $F_1^{(ij)}(x) > 0$ when Q_{ij} is indefinite.

LMI problem 2 Find $P_i > 0$ and $\alpha_{ij} \geq 0$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$, such that

- $A(m_i)^T P_i + P_i A(m_i) \leq 0$ (< 0), $i = 1, \dots, \ell$.
- $(P_j - P_i) - \alpha_{ij} Q_{ij} \leq 0$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$.

Note that this is in fact a problem with multiple LMIs. However, as mentioned in the beginning of Section 4 multiple LMIs can be expressed as the single LMI $\text{diag}(F^{(1)}(z), \dots, F^{(p)}(z)) > 0$. Obviously, a solution to LMI problem 2 is also a solution to Problem 1. The converse is not true in general. However, if the matrices Q_{ij} are such that the regions $\Omega_{Q_{ij}}$ are as small as possible but still $S_{ij} \subseteq \Omega_{Q_{ij}}$, the converse is also true. The reason for this is that the candidate Lyapunov functions are given by positive definite matrices, implying that $x^T(P_j - P_i)x$ are conic regions, and so are the regions $\Omega_{Q_{ij}}$. Hence, the smallest region that contains S_{ij} , utilizing quadratic candidate Lyapunov functions, has to be a conic region. One reason for specifying the regions $\Omega_{Q_{ij}}$ larger than necessary is that the regions $\Omega_{Q_{ij}}$ can be used to give region of robustness, which is further discussed in Section 5.

Restricted regions using information about the discrete states

If there is no solution to LMI problem 2 the reason can be that no information is used concerning where in the continuous state space the different discrete states are possible. For example if one of the subsystems is unstable, there does not exist any solution since it is impossible to find a candidate Lyapunov function whose energy decreases throughout the entire continuous state space. However, if the unstable subsystem is not a positive scalar multiple of the identity matrix then there exist candidate Lyapunov functions whose energy decreases in certain regions of the state space [32]. Thus, it may be necessary to use knowledge of where in the continuous state space the different discrete states are possible in the search for the candidate Lyapunov functions. As mentioned in Section 2 it is usually known where in the continuous state space the discrete states are possible. However, if this is not the case a conservative estimation is necessary, i.e. the region where m_i may be possible has to be found. Assume that the regions where the different discrete states are possible, actual or estimated, are given by the sets:

$$R_{m_i} = \{x \in \mathfrak{R}^n \mid m_i \text{ is possible}\} \quad \forall m_i \in M.$$

The problem to be solved can then be formulated as:

Problem 3 Find $P_i > 0$, $i = 1, \dots, \ell$, such that

- $x^T(A(m_i)^T P_i + P_i A(m_i))x \leq 0$ whenever $x \in R_{m_i}$, $i = 1, \dots, \ell$.
- $x^T(P_j - P_i)x \leq 0$ whenever $x \in S_{ij}$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$.

This is not directly an LMI problem. However, in the same way as before, if matrices Q_i and Q_{ij} are chosen such that $R_{m_i} \subseteq \Omega_{Q_i}$ and $S_{ij} \subseteq \Omega_{Q_{ij}}$, this problem can be formulated as an LMI problem, by applying the \mathcal{S} -procedure:

LMI problem 3 Find $P_i > 0$, $\alpha_i \geq 0$ and $\alpha_{ij} \geq 0$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$, such that

- $(A(m_i)^T P_i + P_i A(m_i)) + \alpha_i Q_i \leq 0$, $i = 1, \dots, \ell$.
- $(P_j - P_i) - \alpha_{ij} Q_{ij} \leq 0$, $i = 1, \dots, \ell$, $j = 1, \dots, \ell$.

Note again that $Q_{ij} > 0$ implies $S_{ij} = \emptyset$, and that $Q_i > 0$ implies that $R_{m_i} = \mathfrak{R}^n$.

4.3 Several candidate Lyapunov functions for the same discrete state

If there is no solution to LMI problem 3, then several candidate Lyapunov functions may be tried for the same discrete state to, if possible, show stability. This is for instance the case when one of the subsystems is unstable and the corresponding discrete state is possible in almost the entire continuous state space, cf. Example 1. In this case there will be no solution to the first item in LMI problem 3.

When several candidate Lyapunov functions are used for the same discrete state, a further partitioning of the region R_{m_i} in several subregions is made, and each of these regions is coupled to different candidate Lyapunov functions which are given by quadratic matrices. There are infinitely many ways to make this partitioning. Of course, the goal is to make a partitioning such that it is possible to formulate the search for the corresponding candidate Lyapunov functions as an LMI problem. This can be done if each subregion is given by a conic region. It is sometimes possible that the hybrid system itself can give some useful insight how this partitioning is to be done. This can for instance be the case when a discrete state is possible in two or several disjoint subregions. Then it is natural to try if different candidate Lyapunov functions can be coupled to each of these subregions. Thus, if necessary a further partitioning of R_{m_i} is made in form of several conic subregions $\Omega_{Q_{qi}}$ given by the matrices Q_{qi} . The subscript i corresponds to the discrete state m_i , and q means that we couple a quadratic candidate Lyapunov function V_q to that region. The subregions must cover the entire set R_{m_i} . There is no restriction that the intersection of two regions has to be disjoint.

As mentioned in Section 3.5 it is important to know the direction of the vector field at the boundary between two neighbor subregions so that it can be guaranteed that the energy decreases when there are switches of candidate Lyapunov functions for the same discrete state. Therefore, when a further partitioning of R_{m_i} is made, conditions must also be stated that guarantee that the energy decreases when candidate Lyapunov functions are switched for the same discrete state. This can be expressed by conic regions, in the same way as before. Thus, there is no difference in the problem formulation whether there is a switch of candidate Lyapunov functions resulting from a change of the discrete states, or a switch of candidate Lyapunov functions for the same discrete state. The LMI problem to be solved can be formulated as:

LMI problem 4 Find $P_q > 0$, $\alpha_{qi} \geq 0$ and $\alpha_{qr} \geq 0$, $q = 1, \dots, p$, $r = 1, \dots, p$, $i = 1, \dots, \ell$, such that

- $(A(m_i)^T P_q + P_q A(m_i)) + \alpha_{qi} Q_{qi} \leq 0$, $i = 1, \dots, \ell$, $q = 1, \dots, p$.
- $(P_r - P_q) - \alpha_{qr} Q_{qr} \leq 0$, $q = 1, \dots, p$, $r = 1, \dots, p$.

Note that the conditions of the first item only has to be satisfied for all combinations of discrete states and Lyapunov functions that are possible. However, this can be formulated by letting Q_{qi} be negative definite for combinations that are not possible. As mentioned before, when $R_{m_i} = \mathbb{R}^n$ the corresponding matrix is positive definite. For combinations that are not possible in the second item, the corresponding matrices are positive definite.

Note that LMI problem 4 also supports the search of candidate Lyapunov functions that may be the same for different discrete states. In fact, LMI problem 4 is a quite general form of formulating the search of candidate Lyapunov functions and includes LMI problem 1-3. Thus, the search for the candidate Lyapunov functions can all be solved by LMI problem 4. If there is no solution to LMI problem 4 it may help to change the number of candidate Lyapunov functions or the different quadratic matrices Q_{qi} and Q_{qr} corresponding to the conic regions $\Omega_{Q_{qi}}$ and $\Omega_{Q_{qr}}$, cf. (14) and Definition 4.

Discussion

Up to this point, the conic regions that contain the switch sets and subregions are searched for by hand. A topic for future studies is to obtain a procedure that enables a computerized selection of the involved matrices.

5 Robustness

In [13] robustness issues for hybrid systems are discussed. Four possible variations of the continuous state are considered. One of them concerns uncertainties when the hybrid system is switched. However, the discussion is mainly focused on how the discrete states and times in (5) change when the switch sets S_{ij} are perturbed.

In this paper, the main focus is on stability robustness and how the stable nominal hybrid system can be changed without becoming unstable. Robustness properties can be obtained when the candidate Lyapunov functions have been found such that the conditions in Corollary 1 are satisfied. In this section we will explain how to obtain two

different kind of robustness issues. First, it is shown how to achieve uncertainty regions around the nominal switch sets. In a hybrid system there may be several reasons for the uncertainties of these switch sets:

- The locations of the switch surfaces (4) are not exactly known.
- Delays associated with the switches are not modelled.
- The vector field changes (2) are not abrupt but take some time.

We also explain how to obtain robustness results in form of acceptable uncertainties for the vector fields in the hybrid model.

Acceptable switch regions

To begin, assume that for a given hybrid system (2), (3), the candidate Lyapunov functions have been found such that Lyapunov stability of the system can be shown by using Corollary 1. Assume that the discrete state is m_i and an arbitrary value $x_k \in S_{ij}$ is reached at time τ_k , implying that discrete state is switched to m_j . Furthermore, assume that the candidate Lyapunov functions V_q and V_r are used as a measure of the system energy before and after the switch of discrete state. Before the switch $x(t) \in \bar{\Omega}_i^q$ for $t \in [\tau_{k-1}, \tau_k)$, where τ_{k-1} is the previous time the system changes candidate Lyapunov function and possibly discrete state, and $\bar{\Omega}_i^q$ is the region where V_q is used as a measure of the energy for the discrete state m_i . At the switch $x_k \in \Omega^{qr}$ since $V_q(x_k) \geq V_r(x_k)$. After the switch, $x(t) \in \bar{\Omega}_j^r$ for $t \in [\tau_k, \tau_{k+1})$, where τ_{k+1} , possibly infinite, is the next time the system changes candidate Lyapunov function and possibly discrete state. However, each x such that $x \in \Omega_{ij}^{qr}$, where $\Omega_{ij}^{qr} = \Omega_i^q \cap \Omega_j^r \cap \Omega^{qr}$, is also a possible state for the hybrid system to switch from m_i to m_j , because such a point also satisfies the conditions of Corollary 1. Thus, the *acceptable switch region*

$$\Omega_{ij}^{qr} = \Omega_i^q \cap \Omega_j^r \cap \Omega^{qr} \quad (24)$$

is a region in which the switches may occur and stability can be guaranteed. Note that the regions Ω_{ij}^{qr} are obtained independently of each other.

Figure 9 illustrates an example where it is shown how the set Ω_{ij}^{qr} is obtained. The set where the switches occur for the nominal hybrid system is S_{ij} . Figures 9a-c show the sets Ω_i^q , Ω_j^r and Ω^{qr} respectively. The intersection of these sets is given in Figure 9d. Thus, it is possible to switch anywhere in Ω_{ij}^{qr} and still guarantee that the system is stable.

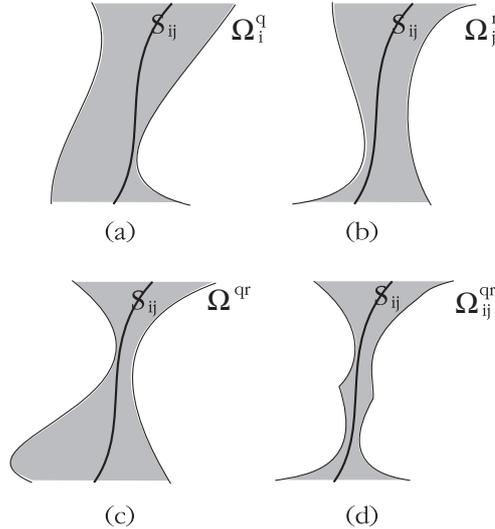


Figure 9: Switch set S_{ij} . (a) Ω_i^q . (b) Ω_j^r . (c) Ω^{qr} . (d) Ω_{ij}^{qr} .

It should be noted that the obtained regions Ω_{ij}^{qr} specify where the system must be switched to guarantee the stability by using Corollary 1. It may very well happen that the system is stable even if the system is switched outside these regions. Thus, the obtained sets Ω_{ij}^{qr} are conservative regions. This is not surprising since these sets

depend on the obtained candidate Lyapunov functions, which are not unique. Another choice of these functions may give different switch regions Ω_{ij}^{qr} .

Maximum robustness

If the candidate Lyapunov functions are generated by the procedure described in Section 4 it is possible to specify how large the sets Ω^{qr} , Ω_i^q and Ω_j^r will be by the different introduced matrices Q_{qi} and Q_{qr} . Thus, it is possible to beforehand determine a certain degree of robustness. However, if these regions are specified too large it is possible that there is no solution to the corresponding LMI problem. Thus, there is a trade off between maximum robustness and a possible solution to the corresponding LMI problem. Of course, it is always possible to start solving the problem with as small regions as possible and then enlarge them until there is no solution to the problem.

Acceptable uncertainties for the vector fields

The region where V_q is used as a measure of the system energy for the discrete state m_i is given by $\bar{\Omega}_i^q$. This means that if the continuous trajectory is in this region and the discrete state is m_i , then candidate Lyapunov function V_q is used as a measure of the system energy. Thus, this region is unchanged when there are changes of the vector field. However, the region Ω_i^q changes when there are perturbations of the vector field, since V_q depends on f according to $\dot{V}_q = \frac{\partial V_q}{\partial x} f(x, m_i)$.

Since most often $\bar{\Omega}_i^q \subset \Omega_i^q$, i.e. V_q is only used as a measure of the energy for the discrete state m_i in a part of the region where V_q decreases, this can be used to obtain robustness results in form of acceptable uncertainties for the vector fields in the hybrid model. Let $\tilde{\Omega}_i^q$ be the region where the energy decreases for the perturbed hybrid system. Then it is allowed to perturb the vector field such that $\bar{\Omega}_i^q \subseteq \tilde{\Omega}_i^q$ since V_q only is used as a measure of the energy in $\bar{\Omega}_i^q$, cf. Figure 10. We will use this fact to obtain uncertainty regions for the discrete states in Example 6.2 in the next section.

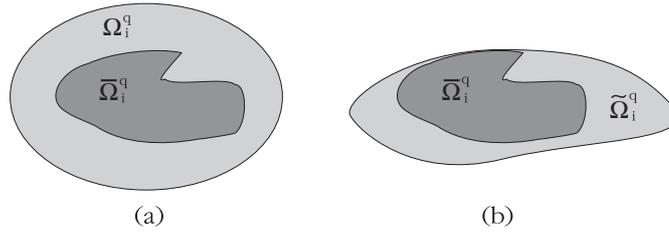


Figure 10: (a) Nominal hybrid system; the region Ω_i^q where the energy decreases and the region $\bar{\Omega}_i^q$ where V_q is used as a measure of the energy. (b) The perturbed hybrid system; the region $\bar{\Omega}_i^q$ and the region $\tilde{\Omega}_i^q$ where the energy of the perturbed hybrid system decreases.

6 Examples

In this section three examples are given that illustrate the theory presented in this paper.

6.1 Linear subsystems

Consider the hybrid system described in Example 1. The set of all initial conditions I_0 is

$$I_0 = \{(m_1, x) \mid x \in \mathbb{R}^2 \setminus S_{12}\}$$

where $x \in \mathbb{R}^2 \setminus S_{12}$ means all points in \mathbb{R}^2 except the ones in the set S_{12} .

Characteristics

The only equilibrium point of the specified hybrid system is the origin. By investigating the vector fields at S_{12} and S_{21} it can be concluded that the different regions where m_1 and m_2 are possible are the ones illustrated in Figure 11. Note that the system is hybrid and not switched, since in the shaded regions of the figure the discrete states depend

not only on the continuous state but also on which of the hyperplanes $s_{12}(x) = 0$ and $s_{21}(x) = 0$ that was last reached.

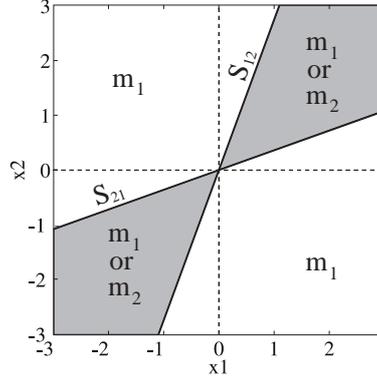


Figure 11: Regions where the discrete states m_1 and m_2 are possible, and the switch surfaces where the system changes vector field.

Stability

For this example it is not possible to use only one candidate Lyapunov function for the discrete state m_1 since $A(m_1)$ is unstable and is possible in the entire state space except for points in S_{12} . Therefore several candidate Lyapunov functions must be used for the discrete state m_1 .

To show that the hybrid system is stable (and asymptotically stable) four positive definite quadratic candidate Lyapunov functions, $V_q(x) = x^T P_q x$, are considered. The following positive definite matrices are used:

$$P_1 = \begin{bmatrix} 7.245 & 3.78 \\ 3.78 & 77.7 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.8298 & -7.41 \\ -7.41 & 39.78 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 4 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 4 & 0.5 \\ 0.5 & 2 \end{bmatrix}. \quad (25)$$

These matrices are obtained by solving LMI problem 4 given in Section 4.3. The different conic regions Ω_1^1 , Ω_1^2 , Ω_1^3 and Ω_2^4 are given in Figure 12. The regions Ω^{34} and Ω^{43} are shown in Figure 13.

The regions where the different candidate Lyapunov functions are used are given in Figure 14. Note that since m_2 only is possible in the region given in Figure 14b, $\bar{\Omega}_2^4$ is only used in this region.

At the boundary between the regions $\bar{\Omega}_1^1$ and $\bar{\Omega}_1^2$ the direction of the vector field $f(x, m_i)$ is from $\bar{\Omega}_1^1$ to $\bar{\Omega}_1^2$. In the same way, the direction of the vector field $f(x, m_i)$ at the boundary between $\bar{\Omega}_1^2$ and $\bar{\Omega}_1^3$ is from $\bar{\Omega}_1^2$ to $\bar{\Omega}_1^3$. Since the sets Ω^{12} and Ω^{23} are equal to \mathbb{R}^2 , the energy decreases when the trajectory passes from $\bar{\Omega}_1^1$ to $\bar{\Omega}_1^2$ and from $\bar{\Omega}_1^2$ to $\bar{\Omega}_1^3$.

As can be seen from the Figures 12-14, the defined candidate Lyapunov functions are joined in the state space such that the conditions in Corollary 1 are satisfied. Figure 15 illustrates how the joined candidate Lyapunov function \bar{V} decreases for the simulation in Figure 3.

Robustness

Given the above candidate Lyapunov functions, the regions where the system may switch and stability is guaranteed can be obtained. These regions, $\Omega_{12}^{34} = \Omega_1^3 \cap \Omega_2^4 \cap \Omega^{34}$ and $\Omega_{21}^{43} = \Omega_1^4 \cap \Omega_2^3 \cap \Omega^{43}$, are shown in Figure 16. Thus, the actual system may switch anywhere in these regions and still be stable. Note that another choice of the candidate Lyapunov functions may result in different regions Ω_{12}^{34} and Ω_{21}^{43} .

6.2 Integrator subsystems

Hybrid system

Consider the following hybrid system

$$\dot{x}(t) = m(t) + v(t) \quad (26)$$

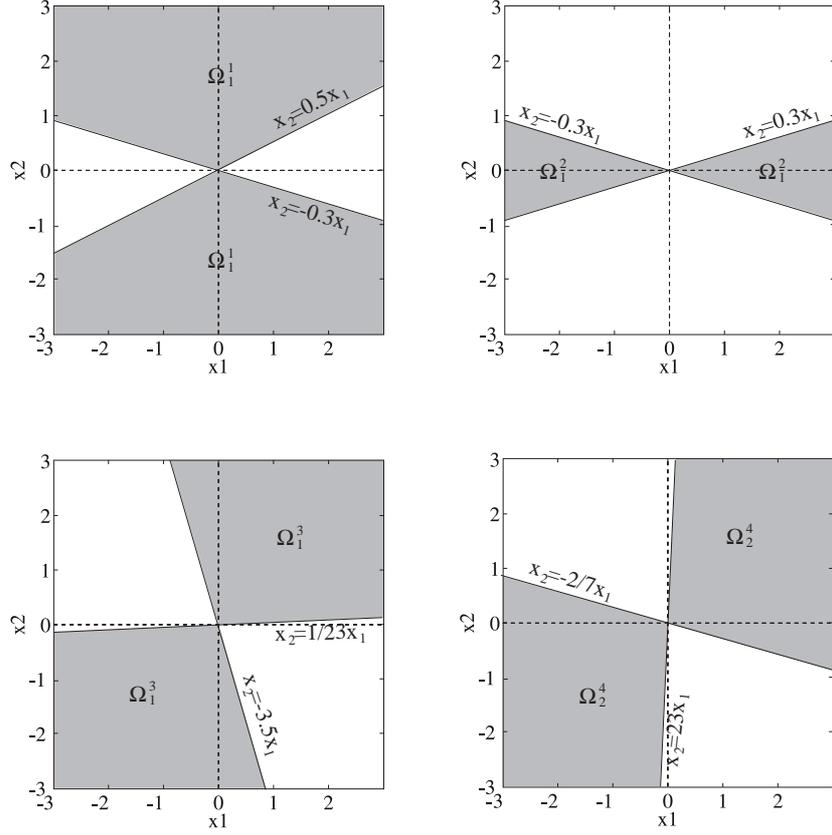


Figure 12: Conic regions indicating where the energy decreases for the candidate Lyapunov functions used for different discrete states.

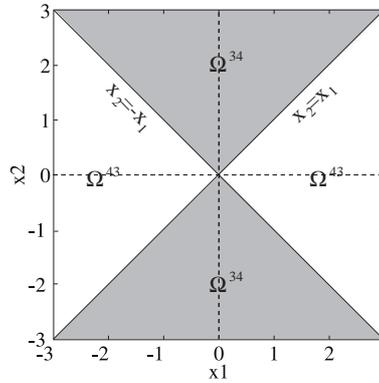


Figure 13: The region Ω^{34} where $V_3 \geq V_4$ shaded, and the region Ω^{43} where $V_4 \geq V_3$ white.

where the discrete set M contains several discrete states and $v(t)$ is an unknown disturbance affecting the system. The time-optimal controller can be obtained by the procedure given in [23]. In this specific example it is assumed that the time-optimal discrete vectors are given by

$$m_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad (27)$$

Furthermore, from the procedure, regions are obtained where the different discrete states may be used to reach a set

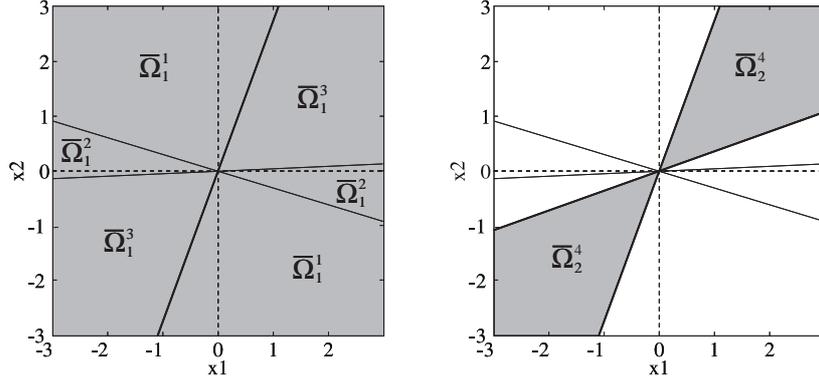


Figure 14: The regions where the candidate Lyapunov functions are used.

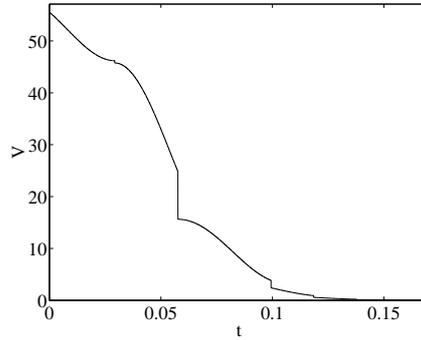


Figure 15: Values of the joined candidate Lyapunov functions for the simulation in Figure 3.

point in optimal time. These regions are given in Figure 17a where the origin is the set point. To obtain a unique solution, a switch strategy is introduced, which is shown in Figure 17b. The introduced switch sets S_{12} , S_{13} , S_{21} , S_{23} and S_{31} are thus given as a result of this specific switch strategy. Note that the system is hybrid since there is a hysteresis around the negative x_1 axis. Without disturbances the system is stable without the hysteresis. However, arbitrarily small disturbances introduce chattering around the negative x_1 axis and sliding mode occurs along this surface which results in an unstable solution [21]. The same phenomenon occurs in simulations due to computational numerical errors. The introduced switch strategy determines the initial conditions I_0 .

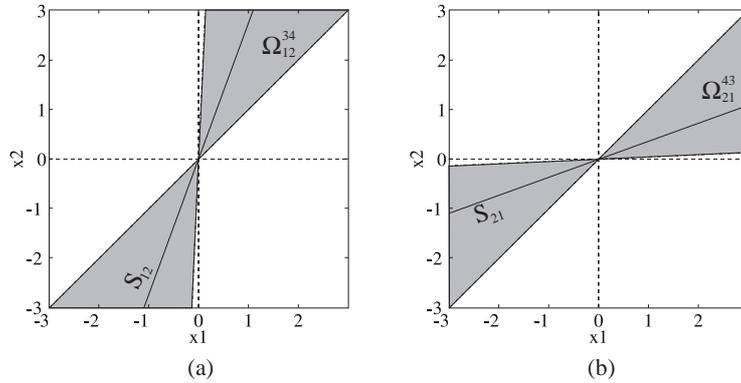


Figure 16: Robustness regions for the switch sets. (a) Regions around S_{12} . (b) Regions around S_{21} .

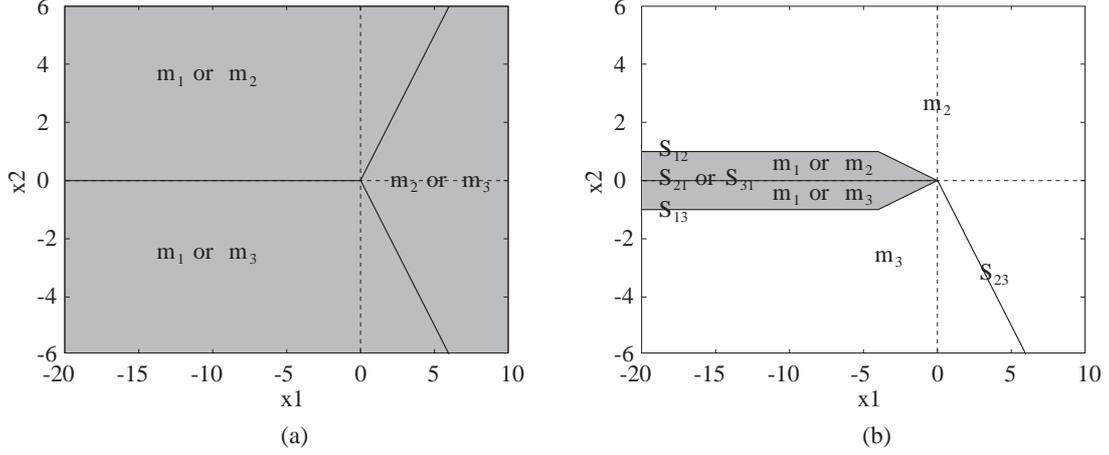


Figure 17: (a) Regions where the discrete states may be used to reach the origin in optimal time. (b) Regions where m_1 , m_2 and m_3 are used, and switch surfaces where the system change discrete states.

Stability

It is not hard to conclude that this system is stable since the origin is reached in finite time that depends on the distance from the initial state to the origin. However, we will use Corollary 1 to show stability. Also in this example positive definite quadratic candidate Lyapunov functions, $V_q(x) = x^T P_q x$ are considered. The following positive definite matrices are used:

$$P_1 = \begin{bmatrix} 1 & -3 \\ -3 & 10 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & -2/3 \\ -2/3 & 1/2 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}. \quad (28)$$

In Figure 18 it is illustrated where each candidate Lyapunov function is used. The candidate Lyapunov functions

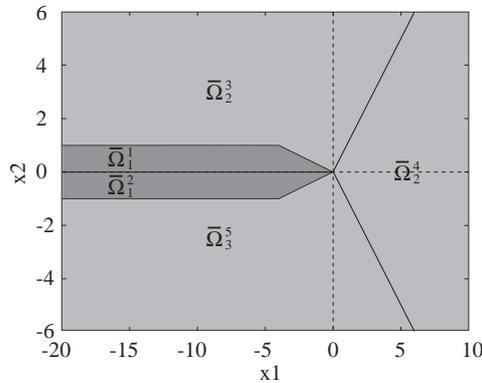


Figure 18: Regions where the candidate Lyapunov functions are used.

are joined such that the conditions in Corollary 1 are satisfied. Thus, the system is stable in sense of Lyapunov.

Robustness

The advantage of showing stability by joining several candidate Lyapunov functions is that several robustness properties are obtained. Specifically, regions around the switch sets S_{ij} where it is possible to switch and still guarantee the stability may be obtained, in the same way as in the previous example. Given these regions, the maximum time delays for the switches that are possible may be calculated. Furthermore, since most probably the modelled values of the discrete states m_i differ from the actual values, it is possible to obtain how large these uncertainties are allowed

to be. Thus, it is assumed that the discrete states are given by:

$$m_i^a = m_i + \Delta m_i, \quad (29)$$

where m_i is the nominal discrete value (27) and Δm_i is the uncertainty. By studying the regions Ω_i^q and Ω^{qr} , regions for the uncertainties of Δm_i 's are obtained. For the discrete states (27) and the obtained candidate Lyapunov functions (28), these regions are given by the shaded regions in Figure 19. The reason for the restrictive region for

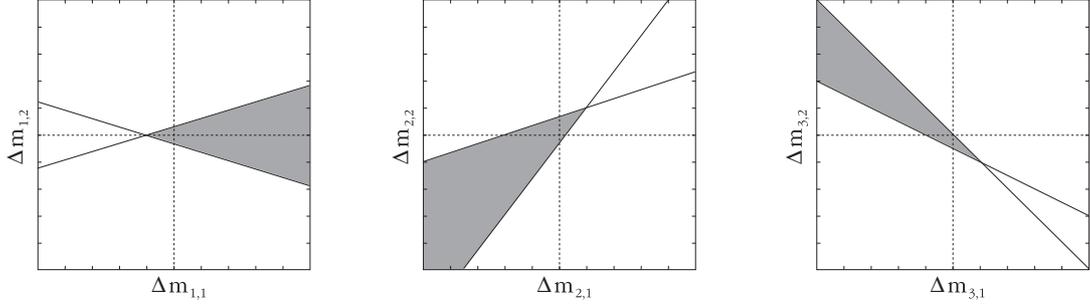


Figure 19: The shaded region is the allowed uncertainty Δm_i of the discrete state m_i .

the uncertainty of the discrete state m_3^a is that m_3^a is not allowed to be directed towards the switch surface S_{23} since the energy then will increase due to the switch of candidate Lyapunov functions from V_5 to V_4 .

The uncertainties of the nominal discrete states Δm_i can also be interpreted as a disturbance $v(t)$ affecting the system according to (26) since the uncertainties are not restricted to be time invariant. Note that $V_q(x) = \frac{\partial V_q}{\partial x} f(x, m)$ where the vector field may be time varying. Hence, the uncertainty can as well be time varying and therefore be regarded as a disturbance $v(t)$. Thus, the regions for the uncertainties can be used also to calculate the maximum amplitude for the disturbance $v(t)$.

6.3 Fuzzy system

To show the flexibility of the presented stability theory, it will be applied also to a fuzzy system. Fuzzy systems may be viewed as hybrid systems where the finite rule base is related to the discrete states of the hybrid system [5, 15]. We will consider an example given by a Takagi-Sugeno fuzzy system [28, 31], which consists of fuzzy blending of linear subsystems:

$$\dot{x}(t) = \frac{\sum_{i=1}^r w_i(t) \{A_i x(t)\}}{\sum_{i=1}^r w_i(t)}, \quad (30)$$

where

$$w_i(t) = \prod_{j=1}^n M_{ij}(x_j(t)). \quad (31)$$

$M_{ij}(x_j(t))$ is the grade of membership of $x_j(t)$ in M_{ij} . It is assumed that

$$\sum_{i=1}^r w_i(t) > 0 \quad \forall t \quad (32)$$

$$w_i(t) \geq 0 \quad i = 1, 2, \dots, r, \quad \forall t \quad (33)$$

The stability result for fuzzy systems presented in [28, 31] requires the existence of one common positive definite matrix P . In [28] only discrete-time systems are considered. Therefore we present the result for continuous time systems.

Lemma 3 *The origin of the fuzzy system (30) is (asymptotically) stable if there exists a common $P > 0$ such that:*

$$PA_i^T + A_i^T P \leq 0 \quad (< 0) \quad i = 1, 2, \dots, r. \quad (34)$$

Proof Assume that there exists a common $P > 0$ such that (34) is true and consider the Lyapunov function:

$$V = x^T P x$$

The time derivative is:

$$\begin{aligned} \dot{V} &= x^T P \dot{x} + \dot{x}^T P x = \\ &= \frac{1}{\sum_{i=1}^r w_i(t)} \sum_{i=1}^r w_i(t) x^T (P A_i + A_i^T P) x \end{aligned} \quad (35)$$

Since $w_i(t) \geq 0$ for $i = 1, 2, \dots, r$, $\sum_{i=1}^r w_i(t) > 0$ and (34) is true, $\dot{V} \leq 0$ (< 0), implying that the origin is stable. ■

Necessary conditions for ensuring the existence of a common P can be obtained from Theorem 2. The search for P can be formulated as LMI problem 1. In the following an example is presented where no common P exists. Then we apply Corollary 1 to find multiple candidate Lyapunov functions that are joined properly in the state space.

Consider a fuzzy system with two states and three membership functions given in Figure 20. Note that $M_{12}(\cdot) +$

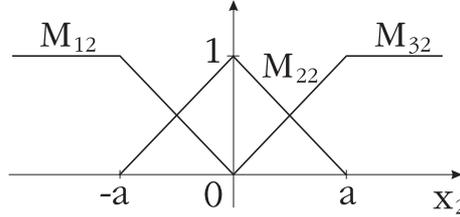


Figure 20: Membership functions for the Fuzzy example.

$M_{22}(\cdot) + M_{32}(\cdot) = 1$. The linear subsystems are given by:

$$A_1 = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix}. \quad (36)$$

Since

$$A_1 + A_3 = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \quad (37)$$

is unstable, cf. Example 4, there does not exist any common $P > 0$. To show that the fuzzy system is stable we will use two candidate Lyapunov functions $V_q(x) = x^T P_q x$, $q = 1, 2$, where the matrices are given by:

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix} \quad (38)$$

V_1 is used as a measure of the energy in the region $x_2 < 0$ and V_2 in the region $x_2 > 0$. On the line $x_2 = 0$ $V_1 = V_2$. Thus, the vector fields at the possible switches of candidate Lyapunov functions do not need to be investigated. By using the given candidate Lyapunov functions the conditions of Corollary 1 are satisfied and hence the system is stable. Note that this is an example where one candidate Lyapunov function is used in different discrete states.

7 Conclusions

In this paper we have motivated the application of a stability result requiring stronger conditions than recently have been suggested for stability analysis of hybrid systems. The advantage of this result is that the search for the necessary candidate Lyapunov functions for hybrid systems consisting of linear subsystems can be formulated as a linear matrix inequality (LMI) problem. These problems can be solved efficiently by numerical interior-point methods.

The stability conditions depend on properties of the sequences of candidate Lyapunov functions that are obtained when the hybrid system evolves from an initial point. This means that it is not necessary to couple each candidate Lyapunov function to a specific discrete state, which is often suggested in the literature. For instance, for unstable subsystems it may be necessary to use several candidate Lyapunov functions for the same discrete state, cf. Example 6.1 in Section 6. On the other hand, one candidate Lyapunov function can sometimes be used for several discrete states, cf. Example 6.3.

By applying the restricted stability result, robustness properties for hybrid systems can be obtained. We have discussed how to achieve permitted switch regions around the nominal switch sets. Furthermore, we have also explained how acceptable uncertainties for the vector fields in the hybrid model can be obtained, cf. Example 6.2.

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Appendix

The proof of Theorem 1.

Proof Let $R > 0$ be arbitrary. Since $h \in \mathcal{C}[\mathfrak{R}^+, \mathfrak{R}^+]$ and $h(0) = 0$, then for any $R > 0$ there exists $0 < \epsilon(R) \leq R$ such that $h(\epsilon(R)) \leq R$. Define $k_q(\alpha) = \min_{\|x\|=\alpha} V_q$, $\alpha > 0$. Since each $V_q(x)$ is continuous and each $V_q(0) = 0$, there exists $R_1^q > 0$ such that each $V_q(x) < k_q(\epsilon(R))$ whenever $\|x\| < R_1^q$. Let $R_1 = \min_q(R_1^q)$. Thus starting in $B(R_1)$ implies that we stay within $B(R)$, where $B(r) = \{x \in \mathfrak{R}^n \mid \|x\| < r\}$.

For the obtained R_1 this procedure is repeated and an R_2 is found such that if we start in $B(R_2)$ implies that we stay within $B(R_1)$. Therefore, whenever the other is first switched on we will have $V_q(x(t_1)) < k_q(\epsilon(R))$ so that we will stay within $B(R)$. This procedure is repeated p times, the same number as candidate functions in the set V . ■

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