A Basic Extended Simple Type Theory

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Abstract

This paper presents an extended version of Church's simple type theory called *Basic Extended Simple Type Theory* (BESTT). By adding type variables and support for reasoning with tuples, lists, and sets to simple type theory, it is intended to be a practical logic for formalized mathematics.

1 Introduction

B. Russell introduced a logic in 1908 called the *theory of types* to serve as a foundation for mathematics. It included a hierarchy of types to avoid set-theoretic paradoxes like Russell's Paradox and was employed as the logic of Whitehead and Russell's monumental *Principia Mathematica* [12]. Being an overly complex system, R. Carnap, L. Chwistek, F. Ramsey, and other logicians suggested a simplified formulation of the theory of types called the *simple theory of types* or, more briefly, *simple type theory*.

A. Church presented in 1940 [4] a version of simple type theory that included lambda-notation. Church's simple type theory, which we will denote as CSTT, can be viewed as a "function theory": functions are the basic objects and reasoning with functions can be done with the help of full quantification over functions, types, and lambda-notation. Many other mathematical objects—such as tuples, sequences, and sets—can be represented in CSTT as certain kinds of functions. The primitive basis of CSTT can be made exceptionally small. The full machinery of predicate calculus can be developed in CSTT from just function application, function abstraction, and equality (first shown by L. Henkin in [9] and improved by P. Andrews in [1]).

CSTT has been widely influential in formalized mathematics. Many people have argued (e.g., see Andrews' remarks in [2]) that CSTT, with its strong

support for reasoning with functions, is a more practical reasoning system than traditional Zermelo-Fraenkel set theory. CSTT is the basis of the logics used in several computer theorem proving systems including HOL [8], IMPS [6, 7], PVS [10], and TPS [3].

This paper presents an extended version of CSTT called *Basic Extended Simple Type Theory* (BESTT). It adds the following facilities to CSTT:

(1) Type variables for forming polymorphic types and expressions as in the HOL logic [8].

(2) New type constructors, expression constants, and expression constructors for reasoning with tuples, lists (finite sequences), and sets.

BESTT is intended to be a practical logic for formalized mathematics that can be used either informally by hand or within a mechanized mathematics system that implements it. BESTT is essentially the same as the background theory of the ML programming language [11]. Therefore, it is an ideal logic in which to write and reason about specifications of ML programs.

The paper gives a full presentation of the syntax of BESTT, but only a few remarks are made about the semantics of BESTT. The presence of type variables in BESTT makes a full presentation of the semantics both lengthy and complicated. Moreover, there are two very natural semantics for BESTT, a *traditional semantics* in which all functions are total and all terms are defined and a *partial semantics* in which functions may be partial and terms may undefined (but formulas are always either true or false). We leave it as an exercise for the BESTT enthusiast to write down either the traditional or partial semantics for BESTT using previous work (see section 5 for references) as a guide.

2 Type Languages

A *type language* of BESTT is a pair $K = (\mathcal{A}, \mathcal{B})$ such that:

(1) $\mathcal{A}$ is an infinite set of symbols called the *type variables* of K.

(2) $\mathcal{B}$ is a set of symbols called the *type constants* of K. $\mathcal{B}$ contains one built-in type constant $*$, the type of truth values, as well as possibly other type constants.

(3) $\mathcal{A}$ and $\mathcal{B}$ are disjoint.

A *type* of K is a string of symbols defined by the rules below. $\mathbf{type}_K(\alpha)$ asserts that α is a type of K.

$$\mathbf{T1}\ \frac{\alpha \in \mathcal{A} \cup \mathcal{B}}{\mathbf{type}_K(\alpha)}\ \textbf{(Atomic type)}$$

$$\mathbf{T2}\ \frac{\mathbf{type}_K(\alpha),\ \mathbf{type}_K(\beta)}{\mathbf{type}_K((\alpha \to \beta))}\ \textbf{(Function type)}$$

$$\mathbf{T3}\ \frac{\mathbf{type}_K(\alpha),\ \mathbf{type}_K(\beta)}{\mathbf{type}_K((\alpha \times \beta))}\ \textbf{(Product type)}$$

$$\mathbf{T4}\ \frac{\mathbf{type}_K(\alpha)}{\mathbf{type}_K(\mathsf{lists}[\alpha])}\ \textbf{(List type)}$$

$$\mathbf{T5}\ \frac{\mathbf{type}_K(\alpha)}{\mathbf{type}_K(\mathsf{sets}[\alpha])}\ \textbf{(Set type)}$$

Let $[K]$ denote the set of types of K. For $\alpha, \beta \in [K]$, α is an *instance* of β if α can be obtained from β by simultaneously substituting types of K for the type variables in β.

Let the *kernel type language* of BESTT be a type language $K_0 = (\mathcal{A}_0, \mathcal{B}_0)$ of BESTT such that $\mathcal{B}_0$ contains only the built-in type constant $*$. Let $K_i = (\mathcal{A}, \mathcal{B}_i)$ be a type language for $i = 1, 2$. K_1 is a *sub type language* of K_2, and K_2 is a *super type language* or an *extension* of K_1, written $K_1 \leq K_2$, if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$. Notice that, by renaming type variables, every type language of BESTT can transformed into an extension of the kernel type language.

3 Languages

A *language* of BESTT is a tuple $L = (K, \mathcal{V}, \mathcal{C}, \tau)$ such that:

(1) $K = (\mathcal{A}, \mathcal{B})$ is a type language of BESTT.

(2) $\mathcal{V}$ is an infinite set of symbols called the *variable symbols* of L.

(3) $\mathcal{C}$ is a set of symbols called the *constant symbols* of L. $\mathcal{C}$ contains the built-in constant symbols in Table 1 as well as possibly other constant symbols.

(4) $\mathcal{V}$ and $\mathcal{C}$ are disjoint.

c	$\tau(c)$ (Note: $\alpha, \beta \in \mathcal{A}$.)
$=$	$((\alpha \times \alpha) \to *)$
fst	$((\alpha \times \beta) \to \alpha)$
snd	$((\alpha \times \beta) \to \beta)$
nil	$\mathsf{lists}[\alpha]$
cons	$((\alpha \times \mathsf{lists}[\alpha]) \to \mathsf{lists}[\alpha])$
hd	$(\mathsf{lists}[\alpha] \to \alpha)$
tl	$(\mathsf{lists}[\alpha] \to \mathsf{lists}[\alpha])$
$\in$	$((\alpha \times \mathsf{sets}[\alpha]) \to *)$

Table 1: The Built-In Constant Symbols of a Language

(5) $\tau : \mathcal{C} \to [K]$ is a total function. The definition of τ on the built-in constant symbols is given in Table 1.

In BESTT terms and formulas are merged together and called "expressions". An *expression of type* α of L is a string of symbols defined by the rules below. $\mathbf{expr}_L(E, \alpha)$ asserts that E is an expression of type α of L.

E1 $\dfrac{x \in \mathcal{V},\ \mathbf{type}_K(\alpha)}{\mathbf{expr}_L((x : \alpha), \alpha)}$ **(Variable)**

E2 $\dfrac{c \in \mathcal{C},\ \alpha \text{ is an instance of } \tau(c)}{\mathbf{expr}_L((c : \alpha), \alpha)}$ **(Constant)**

E3 $\dfrac{\mathbf{expr}_L(A, \alpha),\ \mathbf{expr}_L(F, (\alpha \to \beta))}{\mathbf{expr}_L(F(A), \beta)}$ **(Function application)**

E4 $\dfrac{x \in \mathcal{V},\ \mathbf{type}_K(\alpha),\ \mathbf{expr}_L(B, \beta)}{\mathbf{expr}_L((\lambda\, x : \alpha\ .\ B), (\alpha \to \beta))}$ **(Function abstraction)**

E5 $\dfrac{\mathbf{expr}_L(A, \alpha),\ \mathbf{expr}_L(B, \beta)}{\mathbf{expr}_L((A, B), (\alpha \times \beta))}$ **(Pair abstraction)**

E6 $\dfrac{x \in \mathcal{V},\ \mathbf{type}_K(\alpha),\ \mathbf{expr}_L(A, *)}{\mathbf{expr}_L((\mathrm{I}\, x : \alpha\ .\ A), \alpha)}$ **(Definite description)**

E7 $\dfrac{x \in \mathcal{V},\ \mathbf{type}_K(\alpha),\ \mathbf{expr}_L(A, *)}{\mathbf{expr}_L((\forall\, x : \alpha\ .\ A), *)}$ **(Universal quantification)**

E8 $\dfrac{x \in \mathcal{V},\ \mathbf{type}_K(\alpha),\ \mathbf{expr}_L(A, *)}{\mathbf{expr}_L((\mathrm{S}\, x : \alpha\ .\ A), \mathsf{sets}[\alpha])}$ **(Set abstraction)**

Let $[L]$ denote the set of expressions of L. By induction on the structure of expressions, for each expression $E \in [L]$, there is a *unique* type α such that $\mathbf{expr}_L(E, \alpha)$. A *formula* of L is an expression of L of type $*$. "Free variable", "closed", and similar notions are defined in the obvious way. A *sentence* is a closed formula.

Let the *kernel language* of BESTT be a language $L_0 = (K_0, \mathcal{V}_0, \mathcal{C}_0, \tau_0)$ of BESTT such that K_0 is the kernel type language of BESTT and $\mathcal{C}_0$ contains only the built-in constant symbols in Table 1. Let $L_i = (K_i, \mathcal{V}_i, \mathcal{C}_i, \tau_i)$ be a language for $i = 1, 2$. L_1 is a *sublanguage* of L_2, and L_2 is a *super language* or an *extension* of L_1, written $L_1 \leq L_2$, if $K_1 \leq K_2$, $\mathcal{V}_1 \subseteq \mathcal{V}_2$, $\mathcal{C}_1 \subseteq \mathcal{C}_2$, and τ_1 is a subfunction of τ_2. Notice that, by renaming type variables and variable symbols, every language of BESTT can transformed into an extension of the kernel language.

In the rest of the paper, let $L = (K, \mathcal{V}, \mathcal{C}, \tau)$ where $K = (\mathcal{A}, \mathcal{B})$ is a type language of BESTT. Let α, β, etc. denote types of K, and let A_α, B_α, etc. denote expressions of type α of L.

4 Definitions and Abbreviations

The following definitions introduce additional notation and vocabulary:

Trivial product type:

(α)	denotes	α.

Extended product type:

$(\alpha_1 \times \cdots \times \alpha_n)$	denotes	$(\alpha_1 \times (\alpha_2 \times \cdots \times \alpha_n))$ where $n \geq 2$.

Tuple formation:

$(A^1_{\alpha_1}, \ldots, A^n_{\alpha_n})$	denotes	$(A^1_{\alpha_1}, (A^2_{\alpha_2}, \ldots, A^n_{\alpha_n}))$ where $n \geq 3$.

First projection:

$\#1(A^1_{\alpha_1}, \ldots, A^n_{\alpha_n})$	denotes	$(\mathsf{fst} : \beta)(A^1_{\alpha_1}, \ldots, A^n_{\alpha_n})$ where $n \geq 2$ and $\beta = ((\alpha_1 \times \cdots \times \alpha_n) \to \alpha_1)$.

Higher projection:

$\#m(A^1_{\alpha_1}, \ldots, A^n_{\alpha_n})$	denotes	$\#(m-1)(\mathsf{snd} : \beta)(A^1_{\alpha_1}, \ldots, A^n_{\alpha_n})$ where $2 \leq m \leq n$, $n \geq 2$, and $\beta = ((\alpha_1 \times \cdots \times \alpha_n) \to (\alpha_2 \times \cdots \times \alpha_n))$.

Multivariate function application:

$F_\beta(A^1_{\alpha_1}, \ldots, A^n_{\alpha_n})$ denotes $F_\beta((A^1_{\alpha_1}, \ldots, A^n_{\alpha_n}))$
where $n \geq 2$ and
$\beta = ((\alpha_1 \times \cdots \times \alpha_n) \to \gamma)$.

Equality:

$(A_\alpha = B_\alpha)$ denotes $(=:\beta)(A_\alpha, B_\alpha)$
where $\beta = ((\alpha \times \alpha) \to *)$.

True:

T denotes $((\mathsf{nil} : \mathsf{lists}[*]) = (\mathsf{nil} : \mathsf{lists}[*]))$.

False:

F denotes $(\forall\, x : * \,.\, \mathsf{T})$.

Negation:

$\neg A_*$ denotes $(A_* = \mathsf{F})$.

Inequality:

$(A_* \neq B_*)$ denotes $\neg(A_* = B_*)$.

Conjunction:

$(A_* \wedge B_*)$ denotes $(\mathsf{T}, \mathsf{T}) = (A_*, B_*)$.

Implication:

$(A_* \supset B_*)$ denotes $(A_* = (A_* \wedge B_*))$

Disjunction:

$(A_* \vee B_*)$ denotes $\neg(\neg A_* \wedge \neg B_*)$.

Existential quantification:

$(\exists\, x : \alpha \,.\, A_*)$ denotes $\neg(\forall\, x : \alpha \,.\, \neg A_*)$.

Multivariate binding:

$(\Box\, x_1 : \alpha_1, \ldots, x_n : \alpha_n \,.\, A_\alpha)$
denotes $(\Box\, x_1 : \alpha_1 \,.\, (\Box\, x_2 : \alpha_2, \ldots, x_n : \alpha_n \,.\, A_\alpha))$
where $\Box \in \{\lambda, \forall, \exists, \mathrm{S}\}$ and $n \geq 2$.

Definedness:

$(A_\alpha{\downarrow})$ denotes $(\exists\, x : \alpha \,.\, ((x : \alpha) = A_\alpha))$
where $(x : \alpha)$ does not occur in A_α.

Undefinedness:

$(A_\alpha\!\uparrow)$ denotes $\neg(A_\alpha\!\downarrow)$.

Equivalence:

$(A_\alpha \simeq B_\alpha)$ denotes $(((A_\alpha\!\downarrow) \vee (B_\alpha\!\downarrow)) \supset (A_\alpha = B_\alpha))$.

Conditional expression:

$\mathsf{if}(A_*, B_\alpha, C_\alpha)$ denotes $(\mathrm{I}\, x : \alpha \,.\, ((A_* \supset ((x : \alpha) = B_\alpha)) \wedge (\neg A_* \supset ((x : \alpha) = C_\alpha))))$.
where $(x : \alpha)$ does not occur in A_*, B_α, or C_α.

Canonical undefined expression:

$\bot_\alpha$ denotes $(\mathrm{I}\, x : \alpha \,.\, ((x : \alpha) \neq (x : \alpha)))$.

Empty list formation:

$[\,]_\alpha$ denotes $(\mathsf{nil} : \mathsf{lists}[\alpha])$.

Single member list formation:

$[A_\alpha]$ denotes $(\mathsf{cons} : \beta)(A_\alpha, [\,]_\alpha)$
where $\beta = ((\alpha \times \mathsf{lists}[\alpha]) \to \mathsf{lists}[\alpha])$.

Multimember list formation:

$[A^1_\alpha, \ldots, A^n_\alpha]$ denotes $(\mathsf{cons} : \beta)(A^1_\alpha, [A^2_\alpha \ldots, A^n_\alpha])$
where $n \geq 2$ and
$\beta = ((\alpha \times \mathsf{lists}[\alpha]) \to \mathsf{lists}[\alpha])$.

Set membership:

$(A_\alpha \in B_\beta)$ denotes $(\in : \gamma)(A_\alpha, B_\beta)$
where $\beta = \mathsf{sets}[\alpha]$ and
$\gamma = ((\alpha \times \mathsf{sets}[\alpha]) \to *)$.

Traditional set abstraction:

$\{(x : \alpha) \mid A_*\}$ denotes $(\mathrm{S}\, x : \alpha \,.\, A_*)$.

The following abbreviation rules can be used to write expressions in a more compact form:

A1 A variable $(x : \alpha)$ occurring in the body B of $(\Box\, x : \alpha \,.\, B)$ where $\Box \in \{\lambda, \mathrm{I}, \forall, \exists, \mathrm{S}\}$ may be written as x if there is no resulting ambiguity.

A2 A variable $(x : \alpha)$ occurring freely in an expression E may be written as x if α can be determined from the rest of the expression.

A3 A constant $(c : \alpha)$ occurring in an expression E may be written as c if $\alpha = \tau(c)$ and contains no type variables or if α can be determined from the rest of the expression.

A4 A matching pair of parentheses in an expression may be dropped if there is no resulting ambiguity.

A5 An application

$$(c : ((\alpha \times \beta) \to \gamma))(A_\alpha, B_\beta)$$

will sometimes be written in infix notation as

$$(A_\alpha\ (c : ((\alpha \times \beta) \to \gamma))\ B_\beta)$$

or

$$(A_\alpha\ c\ B_\beta).$$

With the help of the definitions and abbreviation rules given in this section, expressions can usually be written in a natural and easy-to-read form. The definitions and abbreviation rules are used in the rest of the paper.

5 Semantics

As we mentioned in the introduction, there are two natural semantics for BESTT. The traditional semantics can be directly adapted from the semantics for the HOL logic given in [8]. According to the traditional semantics, every expression is defined, i.e., for every expression A_α of L, $(A_\alpha\downarrow)$ is true in every model for L. A value of a definite description $(\mathrm{I}\, x : \alpha\ .\ A_*)$ is the unique value x of type α satisfying A_* if it exists and is an unspecified member of type α otherwise.

The partial semantics for BESTT is based on the partial semantics for CSTT given in [5]. This semantics can be straightforwardly extended to handle type variables (following the semantics in [8]) and the machinery in BESTT for tuples, lists, and sets. The definedness of expressions is determined by the following principles:

P1 (Partial functions) Expressions may denote partial functions.

P2 (Formulas) Formulas (i.e., expressions of type $*$) are always defined and denote either true or false.

P3 (Universally defined expressions) Variables, constants, function abstractions, and set abstractions are always defined.

P4 (Functions applications) A function application is defined iff the function and argument expressions are defined, denoting f and a, respectively, and f is defined at a.

P5 (Predicate applications) A predicate application (i.e., a function application of type $*$) is false if the argument expression is undefined.

P6 (Pair abstractions) A pair abstraction is defined iff the argument expressions are defined.

P7 (Definite descriptions) The value of a definite description $(\mathrm{I}\, x : \alpha \;.\; A_*)$ is the unique value x of type α satisfying A_* if it exists and $(\mathrm{I}\, x : \alpha \;.\; A_*)$ is undefined otherwise.

P8 (Built-in constants) Constants of the form $(\mathsf{fst} : \alpha)$, $(\mathsf{snd} : \alpha)$, and $(\mathsf{cons} : \alpha)$ denote total functions, while $(\mathsf{hd} : \alpha)$ and $(\mathsf{tl} : \alpha)$ denote functions that are defined at all values except the empty list.

Since formulas are always true or false by **P2**, **P4** and **P7** must be modified for expressions of type $*$, $(\alpha \to *)$, $(\alpha \to (\beta \to *))$, etc. as in [5].

6 Theories

A *theory* of BESTT is a pair $T = (L, \Gamma)$ where L is a language of BESTT and Γ is a set of sentences of L called the *axioms* of T. A theory serves as a formal mathematical model or as a specification of a family of mathematical models.

A formula A of L is *valid* in T, written $T \models A$, if A is a logical consequence of Γ.

Example 6.1 (Orders) Let $K = (\mathcal{A}, \mathcal{B})$ be a type language of BESTT such $\mathcal{B} = \{\mathbf{E}\} \cup \mathcal{B}_0$, and let $L = (K, \mathcal{V}, \mathcal{C}, \tau)$ be a language of BESTT such that $\mathcal{C} = \{\leq\} \cup \mathcal{C}_0$ and $\tau(\leq) = ((\mathbf{E} \times \mathbf{E}) \to *)$. Then let $T_1 = (L, \{A_1, A_2\})$, $T_2 = (L, \{A_1, A_2, A_3\})$, and $T_3 = (L, \{A_1, A_2, A_3, A_4\})$ where:

(1) A_1 is $\forall x : \mathbf{E} \,.\, x \leq x$ (Reflexivity).

(2) A_2 is $\forall x, y, z : \mathbf{E} \,.\, (x \leq y \wedge y \leq z) \supset x \leq z$ (Transitivity).

(3) A_3 is $\forall x, y : \mathbf{E} \,.\, (x \leq y \wedge y \leq x) \supset x = y$ (Antisymmetry).

(4) A_4 is $\forall x, y : \mathbf{E} \,.\, x \leq y \vee y \leq x$ (Comparability).

T_1, T_2, and T_3 are theories of a preorder, partial order, and total order, respectively. □

Example 6.2 (Peano arithmetic) Let $K = (\mathcal{A}, \mathcal{B})$ be a type language of BESTT such $\mathcal{B} = \{\mathbf{N}\} \cup \mathcal{B}_0$, and let $L = (K, \mathcal{V}, \mathcal{C}, \tau)$ be a language of BESTT such that $\mathcal{C} = \{0, S\} \cup \mathcal{C}_0$, $\tau(0) = \mathbf{N}$, and $\tau(S) = (\mathbf{N} \to \mathbf{N})$. Let $T = (L, \{A_1, A_2, A_3\})$ where:

(1) A_1 is $\forall x : \mathbf{N} \,.\, 0 \neq S(x)$ (0 has no predecessor).

(2) A_2 is $\forall x, y : \mathbf{N} \,.\, S(x) = S(y) \supset x = y$ (S is injective).

(3) A_3 is

$$\forall P : (\mathbf{N} \to *) \,.\, (P(0) \wedge \forall x : \mathbf{N} \,.\, P(x) \supset P(S(x))) \supset \forall x : \mathbf{N} \,.\, P(x)$$

(Induction principle).

T is (second-order) Peano arithmetic, a theory that formalizes natural number arithmetic. Addition and multiplication on the natural numbers can be defined in this theory by primitive recursion. □

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