Essentials of Fractional Calculus

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Abstract

The aim of these introductory lectures is to provide the reader with the essentials of the fractional calculus according to different approaches that can be useful for our applications in the theory of probability and stochastic processes. We discuss the linear operators of fractional integration and fractional differentiation, which were introduced in pioneering works by Abel, Liouville, Riemann, Weyl, Marchaud, M. Riesz, Feller and Caputo. Particular attention is devoted to the techniques of Fourier and Laplace transforms for treating these operators in a way accessible to applied scientists, avoiding unproductive generalities and excessive mathematical rigor. Furthermore, we discuss the approach based on limit of difference quotients, formerly introduced by Grünwald and Letnikov, which provides a discrete view-point to the fractional calculus. Such approach is very useful for actual numerical computation and is complementary to the previous integral approaches, which provide the continuous view-point. Finally, we give some information on the transcendental functions of the Mittag-Leffler and Wright type which, together with the most common Eulerian functions, turn out to play a fundamental role in the theory and applications of the fractional calculus.

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A. Historical Notes on Fractional Calculus

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term *fractional* is a misnomer, but it is retained following the prevailing use.

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The fractional calculus may be considered an *old* and yet *novel* topic. It is an *old* topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to nowadays. In fact the idea of generalizing the notion of derivative to non integer order, in particular to the order 1/2, is contained in the correspondence of Leibniz with Bernoulli, L'Hôpital and Wallis. Euler took the first step by observing that the result of the evaluation of the derivative of the power function has a a meaning for non-integer order thanks to his Gamma function.

A list of mathematicians, who have provided important contributions up to the middle of the 20-th century, includes P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1837), B. Riemann (1847), H. Holmgren (1865-67), A.K. Grünwald (1867-1872), A.V. Letnikov (1868-1872), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892-1912), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924-1936), A. Zygmund (1935-1945), E.R. Love (1938-1996), A. Erdélyi (1939-1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949), W. Feller (1952).

However, it may be considered a *novel* topic as well, since only from less than thirty years it has been object of specialized conferences and treatises. For the first conference the merit is ascribed to B. Ross who organized the *First Conference on Fractional Calculus and its Applications* at the University of New Haven in June 1974, and edited the proceedings [79]. For the first monograph the merit is ascribed to K.B. Oldham and J. Spanier [71], who, after a joint collaboration started in 1968, published a book devoted to fractional calculus in 1974.

Nowadays, to our knowledge, the list of texts in book form devoted to fractional calculus includes less than ten titles, namely Oldham & Spanier (1974) [71] McBride (1979) [63], Samko, Kilbas & Marichev (1987-1993) [83], Nishimoto (1991) [70], Miller & Ross (1993) [64], Kiryakova (1994) [46], Rubin (1996) [80], Podlubny (1999) [74], among which the encyclopaedic treatise by Samko, Kilbas & Marichev is the most prominent. Furthermore, we recall the attention to the treatises by Davis (1936) [17], Erdélyi (1953-1954) [22], Gel'fand & Shilov (1959-1964) [26], Dzherbashian [19,20], Caputo [13], Babenko [4], Gorenflo & Vessella [37], which contain a detailed analysis of some mathematical aspects and/or physical applications of fractional calculus, although without explicit mention in their titles.

For details on the historical development of the fractional calculus we refer the interested reader to Ross' bibliography in [71] and to the historical notes generally available in the above quoted texts.

In recent years considerable interest in fractional calculus has been stimulated by the applications that it finds in different fields of science, including numerical analysis, physics, engineering, biology, economics and finance. In this respect we quote the collection of articles on the topic of *Fractional Differencing and Long Memory Processes*, edited by Baillie & King (1996), appeared as a special issue in the Journal of Econometrics [5], the book edited by Carpinteri and Mainardi (1997), containing lecture notes of a CISM Course devoted *Fractals and Fractional Calculus in Continuum Mechanics* [15], and finally the forthcoming book edited by R. Hilfer (2000), containing invited contributions in some areas of physics [41].

B. Approaches to Fractional Calculus

There are different approaches to the fractional calculus which, not being all equivalent, have lead to a certain degree of confusion and several misunderstandings in the literature. Probably for this the fractional calculus is in some way the "black sheep" of the analysis. In spite of the numerous eminent mathematicians who have worked on it, still now the fractional calculus is object of so many prejudices.

In these review lectures we essentially consider and develop two different view-points to the fractional calculus: the *continuous* view-point based on the Riemann-Liouville fractional integral and the *discrete* view-point based on the Grünwald-Letnikov fractional derivative. Both approaches turn out to be useful in treating our generalized diffusion processes in the theory of probability and stochastic processes.

We use the standard notation $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ to denote the sets of natural, integer, real and complex numbers, respectively; furthermore, $\mathbf{R}^+$ and $\mathbf{R}_0^+$ denote the sets of positive real numbers and of non-negative real numbers, respectively. Let us remark that, wanting our lectures to be accessible to various kinds of people working in applications (*e.g.* physicists, chemists, theoretical biologists, economists, engineers) we have deliberately and consciously as far as possible avoided the language of functional analysis. We have used vague phrases like "for a sufficiently well behaved function" instead of constructing a stage of precisely defined spaces of admissible functions. We have devoted particular attention to the techniques of Fourier and Laplace transforms: correspondingly our functions are required to belong to the space $L_1(\mathbf{R})$ (summable functions in all of $\mathbf{R}$) or $L_{loc}(\mathbf{R}_0^+)$ (summable function in any finite inerval of $\mathbf{R}_0^+$) . We have extended the Fourier and Laplace transforms to the Dirac "delta function" in the typical way suitable for applications in physics and engineering, without adopting the language of distributions. We kindly ask specialists of these fields of pure mathematics to forgive us. Our notes are written in a way that makes it easy to fill in details of precision which in their opinion might be lacking.

The starting point of the so called Riemann-Liouville fractional calculus is the integral formula (attributed to Cauchy) that reduces the calculation of the $n-$fold primitive of a (sufficiently well behaved) function $\phi(x)$ ($x \in [a,b] \subset \mathbf{R}$, $-\infty \le a < b \le +\infty$) to a single integral of convolution type. Indeed, for any $n \in \mathbf{N}$, the repeated integral

$$I^n_{a+}\,\phi(x) := \int_a^x \int_a^{x_{n-1}} \ldots \int_a^{x_1} \phi(x_0)\,dx_0\,\ldots\,dx_{n-1}\,, \quad a \le x < b\,,$$

which provides the $n-$fold primitive $\phi_n(x)$, vanishing at $x = a$ with its derivatives of order $1, 2, \ldots, n-1$, can be written because of the Cauchy formula as

$$I^n_{a+}\,\phi(x) = \phi_n(x) = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1}\,\phi(\xi)\,d\xi\,, \quad a \le x < b\,. \qquad (B.1)$$

Then, in a natural way, one is led to extend the above formula from positive integer values of the index n to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary *positive* real number α, one defines the *fractional integral of order* $\alpha > 0$ as

$$I^\alpha_{a+}\,\phi(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \quad a < x < b \quad \alpha > 0\,. \qquad (B.2)$$

A dual form of the above integral is

$$I^\alpha_{b-}\,\phi(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (\xi-x)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \quad a < x < b \quad \alpha > 0\,. \qquad (B.3)$$

We refer to the fractional integrals I^α_{a+} and I^α_{b-} as *progressive* and *regressive*, respectively. For complementation we define $I^0_{a+} = I^0_{b-} := \mathbf{I}$ (Identity operator), *i.e.* we mean $I^0_{a+}\,\phi(x) = I^0_{b-}\,\phi(x) = \phi(x)$. Furthermore, by $I^\alpha_{a+}\phi(a^+)$, $I^\alpha_{b-}\phi(b^-)$, we mean the limits (if they exists) of $I^\alpha_{a+}\phi(x)$ for $x \to a^+$ and $I^\alpha_{b-}\phi(x)$ for $x \to b^-$, respectively; these limits may be infinite.

The most common choices concerning the interval $[a,b]$ are the whole set of real numbers $\mathbf{R}$ (*i.e.* $a = -\infty$, $b = +\infty$), considered by Liouville [50–52], and the set of non negative real numbers $\mathbf{R}^+_0$ (*i.e.* $a = 0$, $b = +\infty$), considered by Riemann [77]. We note that the fractional integrals over infinite intervals, especially the regressive one, were named in many later papers as Weyl integrals. Weyl [89] arrived at these indirectly by defining fractional integrals suitable for periodic functions. We thus agree to refer to these integrals as *Liouville-Weyl fractional integrals.* In these lectures we shall use the simplified notation I^α_+ and I^α_- when $a = -\infty$ and $b = +\infty$, respectively.

We also note that, before Riemann, fractional integrals with $a = 0$ have been considered by Abel [1,2] when he introduced his integral equation, named after him, to treat the problem of the tautochrone. It was Abel who gave the first application of fractional calculus to mechanics in solving his problem by inverting the fractional integral, see *e.g.* Gorenflo & Vessella (1991) [37]. In these lectures we agree to refer to the fractional integral I^{α}_{0+} as to Abel-Riemann fractional integral; for them we use the special and simplified notation J^{α} in agreement with the notation introduced by Gorenflo & Vessella (1991) [37] and then followed in any paper of ours.

Before introducing the fractional derivative let us point out the fundamental property of the fractional integrals, namely the *semi-group property* according to which

$$I^{\alpha}_{a+}\, I^{\beta}_{a+} = I^{\alpha+\beta}_{a+}\,, \quad I^{\alpha}_{b-}\, I^{\beta}_{b-} = I^{\alpha+\beta}_{b-}\,, \quad \alpha\,,\, \beta \ge 0\,. \qquad (B.4)$$

The fractional derivative of order α can be introduced as the *left inverse* of the corresponding fractional integral, so extending the similar property of the common derivative of integer order. In fact it is straightforward to recognize that the derivative of any integer order $n = 0, 1, 2, \dots$

$$D^n\, \phi(x) = \frac{d^n}{dx^n}\, \phi(x) = \phi^{(n)}(x)\,, \quad a < x < b$$

satisfies the following composition rules with respect to the repeated integrals of the same order n, $I^n_{a+}\, \phi(x)$ and $I^n_{b-}\, \phi(x)$

$$\begin{cases} D^n\, I^n_{a+}\, \phi(x) = \phi(x)\,, \\ I^n_{a+}\, D^n\, \phi(x) = \phi(x) - \displaystyle\sum_{k=0}^{n-1} \frac{\phi^{(k)}(a^+)}{k!}\, (x-a)^k\,, \end{cases} \quad a < x < b\,, \qquad (B.5a)$$

$$\begin{cases} D^n\, I^n_{b-}\, \phi(x) = (-1)^n \phi(x)\,, \\ I^n_{b-}\, D^n\, \phi(x) = (-1)^n \left\{ \phi(x) - \displaystyle\sum_{k=0}^{n-1} \frac{\phi^{(k)}(b^-)}{k!}\, (b-x)^k \right\}\,, \end{cases} \quad a < x < b\,. \quad (B.5b)$$

As a consequence of (B.3-5) the left-inverse of the fractional integrals I^{α}_{a+}, I^{α}_{b-} may be defined by introducing the positive integer m such that $m-1 < \alpha \le m$. Then one defines the *fractional derivative of order* $\alpha > 0$ as

$$\begin{cases} D^{\alpha}_{a+}\, \phi(x) := D^m\, I^{m-\alpha}_{a+}\, \phi(x)\,, \\ D^{\alpha}_{b-}\, \phi(x) := (-1)^m\, D^m\, I^{m-\alpha}_{b-}\, \phi(x)\,, \end{cases} \quad a < x < b\,, \quad m-1 < \alpha \le m\,. \quad (B.6)$$

In fact, taking for example the progressive operators, we get

$$D^{\alpha}_{a+}\, I^{\alpha}_{a+} = D^m\, I^{m-\alpha}_{a+}\, I^{\alpha}_{a+} = D^m\, I^m_{a+} = \mathbf{I}\,.$$

The fractional derivatives, like the fractional integrals, turn out to be continuous with respect to the order, reducing to the to the standard repeated derivatives when the order is an integer. However, when the order is not integer, the fractional derivatives (namely the "proper" fractional derivatives) do not follow necessarily the "semi-group" property of the fractional integrals: in this respect the starting point $a \neq -\infty$ (or the ending point $b \neq +\infty$) plays a "disturbing" role. Furthermore we stress the fact that "proper" fractional derivatives are *non-local operators* being expressed by ordinary derivatives of convolution integrals with a weakly singular kernel. We note that the fractional integrals contain a weakly singular kernel only when the order is less than one.

The discrete view-point to fractional calculus

In all the above approaches in which the fractional derivatives are defined as the left inverse of the corresponding fractional integrals, the fractional differentiation is seen as a sort of integration of order $-\alpha$. A totally different approach arises from the desire to properly generalize the fact that ordinary derivatives are limits of difference quotients. Let T^h denote the translation by a step of length $h > 0$

$$T^h \, \phi(x) = \phi(x-h) \, . \tag{B.7}$$

The backward finite difference of order α is defined as

$$\Delta_h^\alpha \, \phi(x) := (\mathbf{I} - T^h)^\alpha \, \phi(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \phi(x - kh) \, , \tag{B.8}$$

where $\mathbf{I} = T^0$, the identity, and

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\,\dots\,(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)} \, . \tag{B.9}$$

For $\alpha \in \mathbf{N}$ it reduces to the familiar backward finite difference of integer order. For $\alpha \notin \mathbf{N}$ we note the asymptotic behabviour

$$\left|\binom{\alpha}{k}\right| = \Gamma(\alpha+1) \, \frac{|\sin(\pi\alpha)|}{\pi} \, \frac{|\Gamma(k-\alpha)|}{\Gamma(k+1)} \sim \Gamma(\alpha+1) \, \frac{|\sin(\pi\alpha)|}{\pi} \, k^{-(\alpha+1)} \text{ as } k \to \infty \, .$$

Grünwald (1867) [39] and Letnikov (1868) [47] developed an approach to fractional differentiation based on the definition

$$D_+^\alpha \, \phi(x) = \lim_{h \to 0^+} \frac{\Delta_h^\alpha \, \phi(x)}{h^\alpha} \, . \tag{B.10}$$

While the arguments of the first author were rather formal, the latter gave a rigorous construction of the theory of fractional integro-differentiation on the basis of such a definition. Letnikov had in particular shown that thus defined

$D_+^{-\alpha}\,\phi(x)$ coincides with our Liouville-Weyl fractional integral I_+^α and with our Abel-Riemann fractional integral J^α under the appropriate interpretation of the fractional difference $\Delta_h^\alpha\,\phi(x)$. He proved the semigroup property within the framework of definition (B.10).

Hilfer (1997) [40] has pointed out that there are several possibilities to define the limit of the fractional finite difference quotient, *e.g.* point-wise, almost everywhere, or in the norm of a Banach space. The choice depends upon the question at hand. Furthermore he notes that eq. (B.8) suggests it is also possible to define fractional derivatives as fractional powers of the differentiation operator following the approach started by Balakrishnan (1958, 1959, 1960) [6–8] and Westphal (1974) [88]. More generally one may consider fractional powers of the infinitesimal generators of strongly continuous semi-groups.

Finally, a mixed approach to the fractional calculus, namely containing integrals and finite differences, is to define the fractional derivatives trying to replace α with $-\alpha$ directly in the above Riemann-Liouville fractional integrals. However the resulting integrals are divergent (with *hyper-singular* kernels) and need to be regularized by using the techniques of Hadamard's finite part. This approach was successfully pursued by Marchaud (1927) [62]. It is interesting to note that both Liouville and Riemann dealt with the so-called "complementary" functions which arise when one attempts to treat fractional differentiation of order α as fractional integration of order $-\alpha$, see Samko, Kilbas & Marichev (1993) ([83], p. xxix and Historical Notes in §4.1, §9.1).

The plan for the following sections

Let us now explain the contents of the following sections. Sections C and D will be devoted to the continuous view-point to fractional calculus. In section C we start from the Liouville-Weyl fractional integrals to arrive at the fractional derivatives in the sense of Riesz and Feller. These derivatives are suitable to generalize the standard diffusion equation by replacing the second-order space derivative. In view of this we shall consider functions of the space variable x, denoted by $\phi(x)$, and apply the Fourier transform. In section D we start from the Abel-Riemann fractional integrals to arrive at the fractional derivatives in the sense of Caputo. These derivatives are suitable to generalize the standard diffusion equation by replacing the first-order time derivative. In view of this we shall consider functions of the time variable t, denoted by $\psi(t)$, and apply the Laplace transform. In section E we shall provide some details on the discrete view-point to fractional calculus based on the Grünwald-Letnikov difference scheme and its variants. These schemes turn out to be useful in the interpretation of the space or time fractional diffusion processes by random-walk models. Finally sections F and G are devoted respectively to the special functions of the Mittag-leffler and Wright type, which play a fundamental role in our applications of the fractional calculus.

C. Fractional Calculus according to Riesz and Feller

In this Section the functions under consideration are assumed sufficiently well behaved in $L_1(\mathbf{R})$ to ensure the existence of the Fourier transform or its inverse, as required. In our notation the Fourier transform and its inverse read

$$\begin{cases} \widehat{\phi}(\kappa) = \mathrm{F}\,[\phi(x)] = \displaystyle\int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x}\,\phi(x)\,dx\,, \\ \phi(x) = \mathrm{F}^{-1}\left[\widehat{\phi}(\kappa)\right] = \displaystyle\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-i\kappa x}\,\widehat{\phi}(\kappa)\,d\kappa\,, \end{cases} \qquad (C.1)$$

where $\kappa \in \mathbf{R}$, denotes the Fourier parameter. In this framework we also consider the class of pseudo-differential operators of which the ordinary repeated integrals and derivatives are special cases. A pseudo-differential operator A, acting with respect to the variable $x \in \mathbf{R}$, is defined through its Fourier representation, namely

$$\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\,A\,\phi(x)\,dx = \widehat{A}(\kappa)\,\widehat{\phi}(\kappa)\,, \qquad (C.2)$$

where $\widehat{A}(\kappa)$ is referred to as symbol of A. An often applicable practical rule is

$$\widehat{A}(\kappa) = \left(A\,\mathrm{e}^{-i\kappa x}\right)\,\mathrm{e}^{+i\kappa x}\,, \quad \kappa \in \mathbf{R}\,. \qquad (C.3)$$

If B is another pseudo-differential operator, then we have $\widehat{A\,B}(\kappa) = \widehat{A}(\kappa)\,\widehat{B}(\kappa)$.

For the sake of convenience we shall adopt the notation $\div$ to denote the juxtaposition of a function with its Fourier transform and that of a pseudo-differential operator with its symbol, namely

$$\phi(x) \div \widehat{\phi}(\kappa)\,, \quad A \div \widehat{A}\,.$$

The Liouville-Weyl fractional integrals and derivatives

We now consider the pseudo-differential operators represented by the Liouville-Weyl fractional integrals and derivatives. The *Liouville-Weyl fractional integrals* (of order $\alpha > 0$ for a well-behaved function $\phi(x)$ with $x \in \mathbf{R}$) are defined

$$\begin{cases} I_+^\alpha\,\phi(x) := \displaystyle\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \quad \alpha > 0\,, \\ I_-^\alpha\,\phi(x) := \displaystyle\frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty}(\xi-x)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \quad \alpha > 0\,. \end{cases} \qquad (C.4)$$

For complementation we put $I_{\pm}^{0} := \mathbf{I}$ (Identity operator), as it can be justified by passing to the limit $\alpha \to 0$.

The Liouville-Weyl integrals possess the semigroup property, *i.e.*

$$I_{+}^{\alpha}\, I_{+}^{\beta} = I_{+}^{\alpha+\beta}\,, \quad I_{-}^{\alpha}\, I_{-}^{\beta} = I_{-}^{\alpha+\beta}\,, \quad \text{for all } \alpha\,,\, \beta \geq 0\,. \tag{C.5}$$

The *Liouville-Weyl fractional derivatives* (of order $\alpha > 0$) are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (of order $\alpha > 0$), *i.e.*

$$D_{+}^{\alpha}\, I_{+}^{\alpha} = D_{-}^{\alpha}\, I_{-}^{\alpha} = \mathbf{I}\,. \tag{C.6}$$

Therefore, introducing the positive integer m such that $m-1 < \alpha \leq m$ and recalling that $D^m\, I_{+}^{m} = \mathbf{I}$ and $D^m\, I_{-}^{m} = (-1)^m\, \mathbf{I}$, we have

$$D_{\pm}^{\alpha}\, \phi(x) = \begin{cases} \pm(D^m\, I_{\pm}^{m-\alpha})\, \phi(x)\,, & \text{if} \quad m-1 < \alpha \leq m\,, \quad m \ \text{odd;} \\ (D^m\, I_{\pm}^{m-\alpha})\, \phi(x)\,, & \text{if} \quad m-1 < \alpha \leq m\,, \quad m \ \text{even.} \end{cases} \tag{C.7}$$

For complementation we put $D_{\pm}^{0} := \mathbf{I}$ (Identity operator).

We note that a sufficient condition that the integrals entering $I_{\pm}^{\alpha}$ in (C.4) converge is that

$$\phi\,(x) = O\left(|x|^{-\alpha-\epsilon}\right)\,, \quad \epsilon > 0\,, \quad x \to \mp\infty\,.$$

Integrable functions satisfying these properties are sometimes referred to as functions of Liouville and Weyl class, respectively, see Miller & Ross (1993) [64]. For example power functions $|x|^{-\delta}$ with $\delta > \alpha > 0$ and $x < 0$ and e^{cx} with $c > 0$ are of Liouville class. For these functions we obtain

$$\begin{cases} I_{+}^{\alpha}\, |x|^{-\delta} = \dfrac{\Gamma(\delta-\alpha)}{\Gamma(\delta)}\, |x|^{-\delta+\alpha}\,, \\ D_{+}^{\alpha}\, |x|^{-\delta} = \dfrac{\Gamma(\delta+\alpha)}{\Gamma(\delta)}\, |x|^{-\delta-\alpha}\,, \end{cases} \quad \delta > \alpha > 0\,, \quad x < 0\,, \tag{C.8}$$

and

$$\begin{cases} I_{+}^{\alpha}\, \mathrm{e}^{\,cx} = c^{-\alpha}\, \mathrm{e}^{\,cx}\,, \\ D_{+}^{\alpha}\, \mathrm{e}^{\,cx} = c^{\alpha}\, \mathrm{e}^{\,cx}\,, \end{cases} \quad c > 0\,, \quad x \in \mathbf{R}\,. \tag{C.9}$$

The symbols of the fractional Liouville-Weyl integrals and derivatives can be easily derived according to

$$\begin{cases} I_{\pm}^{\alpha} \div (\mp i\kappa)^{-\alpha} = |\kappa|^{-\alpha}\, \mathrm{e}^{\pm i\,(\mathrm{sign}\,\kappa)\,\alpha\pi/2}\,, \\ D_{\pm}^{\alpha} \div (\mp i\kappa)^{+\alpha} = |\kappa|^{+\alpha}\, \mathrm{e}^{\mp i\,(\mathrm{sign}\,\kappa)\,\alpha\pi/2}\,. \end{cases} \tag{C.10}$$

The Liouville-Weyl fractional integrals can be combined to give rise to the *Riesz fractional integral* (usually called *Riesz potential*) of order α, defined as

$$I_0^\alpha \, \phi(x) = \frac{I_+^\alpha \phi(x) + I_-^\alpha \phi(x)}{2\, \cos(\alpha\pi/2)} = \frac{1}{2\,\Gamma(\alpha)\, \cos(\alpha\pi/2)} \int\limits_{-\infty}^{+\infty} |x-\xi|^{\alpha-1}\, \phi(\xi)\, d\xi \,, \quad (C.11)$$

for any positive α with the exclusion of odd integer numbers for which $\cos(\alpha\pi/2)$ vanishes. Using (C.10-11) the symbol of the Riesz potential turns out to be

$$I_0^\alpha \div |\kappa|^{-\alpha} \,, \qquad \alpha > 0 \,, \quad \alpha \neq 1\,, 3\,, 5\, \dots \,. \qquad (C.12)$$

In fact

$$I_+^\alpha + I_-^\alpha \div \left[\frac{1}{(-i\kappa)^\alpha} + \frac{1}{(+i\kappa)^\alpha} \right] = \frac{(+i)^\alpha + (-i)^\alpha}{|\kappa|^\alpha} = \frac{2\, \cos(\alpha\pi/2)}{|\kappa|^\alpha} \,.$$

We note that, at variance with the Liouville-Weyl fractional integral, the Riesz potential has the semigroup property only in restricted ranges, *e.g.*

$$I_0^\alpha I_0^\beta = I_0^{\alpha+\beta} \quad \text{for} \quad 0 < \alpha < 1 \,, \quad 0 < \beta < 1 \,, \quad \alpha + \beta < 1 \,. \qquad (C.13)$$

From the Riesz potential we can define by analytical continuation the *Riesz fractional derivative* D_0^α, including also the singular case $\alpha = 1$, by formally setting $D_0^\alpha := -I_0^{-\alpha} \div -|\kappa|^\alpha$, where the minus sign has been put in order to recover for $\alpha = 2$ the standard second derivative. Restricting our attention to the range $0 < \alpha \le 2$ the explicit correct definition turns out to be

$$D_0^\alpha \, \phi(x) := -I_0^{-\alpha} \, \phi(x) := \begin{cases} -\dfrac{D_+^\alpha \, \phi(x) + D_-^\alpha \, \phi(x)}{2\, \cos(\alpha\pi/2)} & \text{if } \alpha \neq 1 \,, \\ -D\, H\, \phi(x) \,, & \text{if } \alpha = 1 \,, \end{cases} \qquad (C.14)$$

where H denotes the Hilbert transform operator defined by

$$H\, \phi(x) := \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\phi(\xi)}{x-\xi}\, d\xi = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\phi(x-\xi)}{\xi}\, d\xi \,, \qquad (C.15)$$

the integral understood in the Cauchy principal value sense. Incidentally, we note that $H^{-1} = -H$. By using the rule (C.3) we can derive the symbol of H, namely

$$H \div \widehat{H} = i\, \mathrm{sign}\, \kappa \,. \qquad (C.16)$$

The expressions in (C.14) can be easily verified by manipulating with symbols of "good" operators as below

$$D_0^\alpha := -I_0^{-\alpha} \div -|\kappa|^\alpha = \begin{cases} -\dfrac{(-i\kappa)^\alpha + (+i\kappa)^\alpha}{2\,\cos(\alpha\pi/2)} = -|\kappa|^\alpha\,, & \text{if } \alpha \neq 1\,, \\ +i\kappa \cdot i\mathrm{sign}\,\kappa = -\kappa\,\mathrm{sign}\,\kappa = -|\kappa|\,, & \text{if } \alpha = 1\,. \end{cases}$$

In particular, from (C.14) we recognize that

$$D_0^2 = \frac{1}{2}\left(D_+^2 + D_-^2\right) = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{d^2}{dx^2}\right) = \frac{d^2}{dx^2}\,, \quad \text{but} \quad D_0^1 \neq \frac{d}{dx}\,.$$

We also recognize that the symbol of D_0^α is just the logarithm of the characteristic function of a *symmetric* Lévy stable *pdf*.

In general, the Riesz fractional derivative D_0^α turns out to be related to the $\alpha/2$-power of the positive definite operator $-D^2 = -\frac{d^2}{dx^2}$ since, as noted by Feller (1952) [23],

$$-|\kappa|^\alpha = -\left(\kappa^2\right)^{\alpha/2} \div D_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\,, \qquad (C.17)$$

whereas the two Liouville-Weyl fractional derivatives are related to the α-power of the first order differential operator $D = \frac{d}{dx}$. We note that it was Bochner (1949) [11] who first introduced the fractional powers of the Laplacian to generalize the diffusion equation.

We would like to mention the "illuminating" notation introduced by Zaslavsky, see *e.g.* Saichev & Zaslavsky (1997) [81], to denote our Liouville-Weyl and Riesz fractional derivatives

$$D_\pm^\alpha = \frac{d^\alpha}{d(\pm x)^\alpha}\,, \quad D_0^\alpha = \frac{d^\alpha}{d|x|^\alpha}\,, \quad 0 < \alpha \le 2\,. \qquad (C.18)$$

We now point out that other expressions for $D_\pm^\alpha$ and henceforth D_0^α are obtained by "regularizing" the *hyper-singular* integrals $I_\pm^{-\alpha}$ for $0 < \alpha \le 2$. This "regularization", based on a former idea by Marchaud, see *e.g.* Marchaud (1927) [62], Samko, Kilbas & Marichev (1993) [83], Hilfer (1997) [40], is noteworthy since it leads to a discretization of the operators of fractional derivative, alternative to that based on the Grünwal-Letnikov method.

Let us first consider from (C.7) the operator

$$D_+^\alpha := I_+^{-\alpha} := \frac{d}{dx}\,I_+^{1-\alpha}\,, \quad 0 < \alpha < 1\,. \qquad (C.19)$$

We have, see Hilfer (1997) [40],

$$
\begin{aligned}
D_+^\alpha\,\phi(x) := I_+^{-\alpha}\,\phi(x) &= \frac{d}{dx}\,I_+^{1-\alpha}\,\phi(x)\\
&= \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int\limits_{-\infty}^{x}(x-\xi)^{-\alpha}\,\phi(\xi)\,d\xi\\
&= \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int\limits_{0}^{\infty}\xi^{-\alpha}\,\phi(x-\xi)\,d\xi\\
&= \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{0}^{\infty}\phi'(x-\xi)\int\limits_{\xi}^{\infty}\frac{1}{\eta^{1+\alpha}}\,d\eta\,d\xi\,,
\end{aligned}
$$

so that, interchanging the order of integration,

$$
D_+^\alpha\,\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{0}^{\infty}\frac{\phi(x)-\phi(x-\xi)}{\xi^{1+\alpha}}\,d\xi\,,\quad 0<\alpha<1\,. \tag{C.20}
$$

Here ϕ' denotes the first derivative of ϕ with respect to its argument. The coefficient in front to the integral in (C.20) can be re-written, using known formulas for the Gamma function, as

$$
\frac{\alpha}{\Gamma(1-\alpha)} = -\frac{1}{\Gamma(-\alpha)} = \Gamma(1+\alpha)\,\frac{\sin\,\alpha\pi}{\pi}\,. \tag{C.21}
$$

Similarly we get for

$$
D_-^\alpha := I_-^{-\alpha} = -\frac{d}{dx}\,I_-^{1-\alpha}\,,\quad 0<\alpha<1\,, \tag{C.22}
$$

$$
D_-^\alpha\,\phi(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{0}^{\infty}\frac{\phi(x+\xi)-\phi(x)}{\xi^{1+\alpha}}\,d\xi\,,\quad 0<\alpha<1\,. \tag{C.23}
$$

Similar results can be given for $1<\alpha<2$.

Recalling from (C.14) the fractional derivative in Riesz's sense

$$
D_0^\alpha\,\phi(x) := -\frac{D_+^\alpha\,\phi(x)+D_-^\alpha\,\phi(x)}{2\,\cos(\alpha\pi/2)}\,,\quad 0<\alpha<1\,,\quad 1<\alpha<2\,,
$$

and using (C.20), (C.23), we get for it the following regularized representation, valid also in $\alpha=1$,

$$
D_0^\alpha\,\phi(x) = \Gamma(1+\alpha)\,\frac{\sin\,(\alpha\pi/2)}{\pi}\int\limits_{0}^{\infty}\frac{\phi(x+\xi)-2\phi(x)+\phi(x-\xi)}{\xi^{1+\alpha}}\,d\xi\,,\quad 0<\alpha<2\,. \tag{C.24}
$$

A generalization of the Riesz fractional integral and derivative has been proposed by Feller (1952) [23] in a pioneering paper, recalled by Samko, Kilbas & Marichev (1993) [83], but only recently revised and used by Gorenflo & Mainardi (1998) [34]. The purpose of Feller was indeed to generalize the second order space derivative entering the standard diffusion equation with a pseudo-differential operator whose symbol is the logarithm of the characteristic function of a general Lévy stable *pdf* according to his his parameterization, which is closer to ours.

Let us now show how obtain the Feller derivative by inversion of a properly generalized Riesz potential, later called *Feller potential* by Samko, Kilbas & Marichev (1993) [83]. Using our notation we define the Feller potential I_θ^α by its symbol obtained by the Riesz potential by a suitable "rotation" of an angle $\theta\,\pi/2$ where θ is a real number properly restricted. We have

$$I_\theta^\alpha \div |\kappa|^{-\alpha}\,\mathrm{e}^{-i\,(\mathrm{sign}\,\kappa)\,\theta\pi/2}\,, \quad \begin{cases} |\theta| \le \alpha & \text{if } 0<\alpha<1\,, \\ |\theta| \le 2-\alpha & \text{if } 1<\alpha\le 2\,. \end{cases} \qquad (C.25)$$

As in the Riesz potential the case $\alpha = 1$ is omitted.

The integral representation of the pseudo-differential operator I_θ^α turns out to be

$$I_\theta^\alpha\,\phi(x) = c_-(\alpha,\theta)\,I_+^\alpha\,\phi(x) + c_+(\alpha,\theta)\,I_-^\alpha\,\phi(x)\,, \qquad (C.26)$$

where, if $0<\alpha<2\,,\ \alpha\neq 1\,,$

$$c_+(\alpha,\theta) = \frac{\sin\,[(\alpha-\theta)\,\pi/2]}{\sin\,(\alpha\pi)}\,, \qquad c_-(\alpha,\theta) = \frac{\sin\,[(\alpha+\theta)\,\pi/2]}{\sin(\alpha\pi)}\,, \qquad (C.27)$$

and, by passing to the limit (with $\theta = 0$)

$$c_+(2,0) = c_-(2,0) = -1/2\,. \qquad (C.28)$$

In the particular case $\theta = 0$ we get

$$c_+(\alpha,0) = c_-(\alpha,0) = \frac{1}{2\,\cos\,(\alpha\pi/2)}\,, \qquad (C.29)$$

and thus, from (C.26) and (C.29) we recover the Riesz potential (C.11). Like the Riesz potential also the Feller potential has the (range-restricted) semi-group property, *e.g.*

$$I_\theta^\alpha I_\theta^\beta = I_\theta^{\alpha+\beta} \quad \text{for} \quad 0<\alpha<1\,, \quad 0<\beta<1\,, \quad \alpha+\beta<1\,. \qquad (C.30)$$

From the Feller potential we can define by analytical continuation the *Feller fractional derivative* D_θ^α, including also the singular case $\alpha = 1$, by setting

$$D_\theta^\alpha := -I_\theta^{-\alpha} \div -|\kappa|^{\alpha}\,\mathrm{e}^{+i\,(\mathrm{sign}\,\kappa)\,\theta\pi/2}\,, \quad 0<\alpha\le 2\,, \qquad (C.31)$$

with θ restricted as in (C.25). We have

$$D_\theta^\alpha \, \phi(x) := \begin{cases} - \left[c_+(\alpha,\theta) \, D_+^\alpha + c_-(\alpha,\theta) \, D_-^\alpha \right] \, \phi(x) \, , & \text{if } \ \alpha \neq 1 \, , \\ \\ \left[\cos(\theta\pi/2) \, D_0^1 + \, \sin(\theta\pi/2) \, D \right] \, \phi(x) \, , & \text{if } \ \alpha = 1 \, . \end{cases} \qquad (C.32)$$

For $\alpha \neq 1$ it is sufficient to note that $c_\mp(-\alpha,\theta) = c_\pm(\alpha,\theta)$. For $\alpha = 1$ we need to recall the symbols of the operators D and $D_0^1 = -DH$, namely $\hat{D} = (-i\kappa)$ and $\widehat{D_0^1} = -|\kappa|$, and note that

$$\begin{aligned} \widehat{D_\theta^1} &= -|\kappa| \, \mathrm{e}^{+i \, (\text{sign } \kappa) \, \theta\pi/2} = -|\kappa| \, \cos(\theta\pi/2) - (i\kappa) \, \sin(\theta\pi/2) \\ &= \cos(\theta\pi/2) \, \widehat{D_0^1} + \sin(\theta\pi/2) \, \widehat{D} \, . \end{aligned}$$

For later use we find it convenient to return to the "weight" coefficients $c_\pm(\alpha,\theta)$ in order to outline some properties along with some particular expressions, which can be easily obtained from (C.27) with the restrictions on θ given in (C.25). We obtain

$$c_\pm \begin{cases} \geq 0 \, , & \text{if } \ 0 < \alpha < 1 \, , \\ \leq 0 \, , & \text{if } \ 1 < \alpha \leq 2 \, , \end{cases} \qquad (C.33)$$

and

$$c_+ + c_- = \frac{\cos\,(\theta\pi/2)}{\cos\,(\alpha\pi/2)} \begin{cases} > 0 \, , & \text{if } \ 0 < \alpha < 1 \, , \\ < 0 \, , & \text{if } \ 1 < \alpha \leq 2 \, . \end{cases} \qquad (A.34)$$

In the *extremal* cases we find

$$0 < \alpha < 1 \, , \begin{cases} c_+ = 1 \, , \ c_- = 0 \, , & \text{if } \ \theta = -\alpha \, , \\ c_+ = 0 \, , \ c_- = 1 \, , & \text{if } \ \theta = +\alpha \, , \end{cases} \qquad (C.35)$$

$$1 < \alpha < 2 \, , \begin{cases} c_+ = 0 \, , \ c_- = -1 \, , & \text{if } \ \theta = -(2-\alpha) \, , \\ c_+ = -1 \, , \ c_- = 0 \, , & \text{if } \ \theta = +(2-\alpha) \, . \end{cases} \qquad (C.36)$$

We also note that in the extremal cases of $\alpha = 1$ we get

$$D_{\pm 1}^1 = \pm D = \pm \frac{d}{dx} \, . \qquad (C.37)$$

In view of the relation of the Feller operators in the framework of *stable* probability density functions, we agree to refer to θ as to the *skewness* parameter.

We must note that in his original paper Feller (1952) used a skewness parameter δ different from our θ; the potential introduced by Feller is

$$I_\delta^\alpha \div \left(|\kappa| \, \mathrm{e}^{-i \, (\text{sign } \kappa) \, \delta} \right)^{-\alpha} \, , \quad \text{so} \quad \delta = -\frac{\pi}{2} \frac{\theta}{\alpha} \, , \quad \theta = -\frac{2}{\pi} \, \alpha\delta \, . \qquad (C.38)$$

Then Samko, Kilbas & Marichev (1993) named I_δ^α as the *Feller potential* operator. We note that Feller considered the inversion of his potential for $\alpha = 1$ but limiting himself to the symmetric case ($\delta = \theta = 0$) for which he provided the representation as the first derivative of the Hilbert transform. Samko, Kilbas & Marichev (1993) apparently ignored the singular case $\alpha = 1$.

D. Fractional Calculus according to Abel-Riemann and Caputo

In this Section the functions under consideration are assumed sufficiently well behaved in $L_{loc}(\mathbf{R}_0^+)$ to ensure the existence of the Laplace transform or its inverse, as required. In our notation the Laplace transform and its inverse read

$$\begin{cases} \widetilde{\psi}(s) = \mathrm{L}\left[\psi(t)\right] = \displaystyle\int_0^\infty \mathrm{e}^{-st}\, \psi(t)\, dt\,, \\ \psi(t) = \mathrm{L}^{-1}\left[\widetilde{\psi}(s)\right] = \displaystyle\frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\, \widetilde{\psi}(s)\, ds\,, \end{cases} \tag{D.1}$$

where Br denotes a Bromwich path and $s \in \mathbf{C}$ is the Laplace parameter. For the sake of convenience we shall adopt the notation $\div$ to denote the juxtaposition of a function with its Laplace transform, namely $\psi(t) \div \widetilde{\psi}(s)$.

The Abel-Riemann fractional integral and derivatives

We first define the Abel-Riemann (A-R) fractional integral and derivative of any order $\alpha > 0$ for a generic (well-behaved) function $\psi(t)$ with $t \in \mathbf{R}^+$.

For the A-R *fractional integral* (of order α) we have

$$J^\alpha\, \psi(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\, \psi(\tau)\, d\tau\,, \quad t > 0 \quad \alpha > 0\,. \tag{D.2}$$

For complementation we put $J^0 := \mathbf{I}$ (Identity operator), as it can be justified by passing to the limit $\alpha \to 0$.

The A-R integrals possess the semigroup property, *i.e.*

$$J^\alpha\, J^\beta = J^{\alpha+\beta}\,, \text{ for all } \alpha\,,\, \beta \geq 0\,. \tag{D.3}$$

The A-R *fractional derivative* (of order $\alpha > 0$) is defined as the left-inverse operator of the corresponding A-R fractional integral (of order $\alpha > 0$), *i.e.*

$$D^\alpha\, J^\alpha = \mathbf{I}\,. \tag{D.4}$$

Therefore, introducing the positive integer m such that $m-1 < \alpha \leq m$ and noting that $(D^m\, J^{m-\alpha})\, J^\alpha = D^m\, (J^{m-\alpha}\, J^\alpha) = D^m\, J^m = \mathbf{I}$, we define $D^\alpha := D^m\, J^{m-\alpha}$ namely

$$D^\alpha \psi(t) := \begin{cases} \displaystyle\frac{1}{\Gamma(m-\alpha)}\, \frac{d^m}{dt^m} \int_0^t \frac{\psi(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\,, & m-1 < \alpha < m\,, \\ \displaystyle\frac{d^m}{dt^m}\, \psi(t)\,, & \alpha = m\,. \end{cases} \tag{D.5}$$

For complementation we put $D^0 := \mathbf{I}$. For $\alpha \to m^-$ we thus recover the standard derivative of order m but the integral formula loses its meaning for $\alpha = m$.

By using the properties of the Eulerian beta and gamma functions it is easy to show the effect of our operators J^α and D^α on the power functions: we have

$$\begin{cases} J^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha}\,, \\ D^\alpha\, t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,, \end{cases} \qquad t > 0\,, \quad \alpha \geq 0\,, \quad \gamma > -1\,. \qquad (D.6)$$

These properties are of course a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha\, \psi(t)$ is not zero for the constant function $\psi(t) \equiv 1$ if $\alpha \notin \mathbf{N}$. In fact, the second formula in (D.6) with $\gamma = 0$ teaches us that

$$D^\alpha\, 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad \alpha \geq 0\,, \quad t > 0\,. \qquad (D.7)$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbf{N}$, due to the poles of the gamma function in the points $0, -1, -2, \ldots$.

The Caputo fractional derivative

We now observe that an alternative definition of fractional derivative, originally introduced by Caputo (1967) (1969) in the late sixties and adopted by Caputo & Mainardi (1971) in the framework of the theory of linear viscoelasticity, is $D_*^\alpha = J^{m-\alpha}\, D^m$ with $m-1 < \alpha \leq m$, namely

$$D_*^\alpha\, \psi(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \frac{\psi^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\,, & m-1 < \alpha < m\,, \\ \dfrac{d^m}{dt^m}\psi(t)\,, & \alpha = m\,. \end{cases} \qquad (D.8)$$

The definition (D.8) is of course more restrictive than (D.5), in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha\, \psi(t) := D^m\, J^{m-\alpha}\, \psi(t) \neq J^{m-\alpha}\, D^m\, \psi(t) := D_*^\alpha\, \psi(t)\,, \qquad (D.9)$$

unless the function $\psi(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, one recognizes that, for $m-1<\alpha<m$ and $t>0$,

$$D^{\alpha}\,\psi(t)=D_*^{\alpha}\,\psi(t)+\sum_{k=0}^{m-1}\frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\,\psi^{(k)}(0^+)\,, \tag{D.10}$$

and therefore, recalling the fractional derivative of the power functions, see (D.6),

$$D^{\alpha}\left(\psi(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}\,\psi^{(k)}(0^+)\right)=D_*^{\alpha}\,\psi(t)\,. \tag{D.11}$$

The subtraction of the Taylor polynomial of degree $m-1$ at $t=0^+$ from $\psi(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero can be easily recognized,

$$D_*^{\alpha}1\equiv 0\,,\quad \alpha>0\,. \tag{D.12}$$

We now explore the most relevant differences between the two fractional derivatives (D.5) and (D.8). We agree to denote (D.8) as the *Caputo fractional derivative* to distinguish it from the standard A-R fractional derivative (D.5). We observe, again by looking at (D.6), that $D^{\alpha}t^{\alpha-1}\equiv 0\,,\ \alpha>0\,,\ t>0\,.$ We thus recognize the following statements about functions which for $t>0$ admit the same fractional derivative of order α, with $m-1<\alpha\le m\,,\ m\in\mathbf{N}$,

$$D^{\alpha}\,\psi(t)=D^{\alpha}\,\phi(t)\iff \psi(t)=\phi(t)+\sum_{j=1}^{m}c_j\,t^{\alpha-j}\,, \tag{D.13}$$

$$D_*^{\alpha}\,\psi(t)=D_*^{\alpha}\,\phi(t)\iff \psi(t)=\phi(t)+\sum_{j=1}^{m}c_j\,t^{m-j}\,, \tag{D.14}$$

where the coefficients c_j are arbitrary constants.

For the two definitions we also note a difference with respect to the *formal* limit as $\alpha\to(m-1)^+$; from (D.5) and (D.8) we obtain respectively,

$$D^{\alpha}\,\psi(t)\to D^m\,J\,\psi(t)=D^{m-1}\,\psi(t)\,; \tag{D.15}$$

$$D_*^{\alpha}\,\psi(t)\to J\,D^m\,\psi(t)=D^{m-1}\,\psi(t)-\psi^{(m-1)}(0^+)\,. \tag{D.16}$$

We now consider the *Laplace transform* of the two fractional derivatives. For the A-R fractional derivative D^{α} the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k=1,2,\dots,m-1$. The corresponding rule reads, in our notation,

$$D^{\alpha}\,\psi(t)\div s^{\alpha}\,\widetilde{f}(s)-\sum_{k=0}^{m-1}D^k\,J^{(m-\alpha)}\,\psi(0^+)\,s^{m-1-k}\,,\quad m-1<\alpha\le m\,. \tag{D.17}$$

For the *Caputo* fractional derivative the Laplace transform technique requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \ldots, m-1$, in analogy with the case when $\alpha = m$. In fact, noting that $J^\alpha D_*^\alpha = J^\alpha J^{m-\alpha} D^m = J^m D^m$, we have

$$J^\alpha D_*^\alpha \psi(t) = \psi(t) - \sum_{k=0}^{m-1} \psi^{(k)}(0^+) \frac{t^k}{k!}, \qquad (D.18)$$

so we easily prove the following rule for the Laplace transform,

$$D_*^\alpha \psi(t) \div s^\alpha \widetilde{\psi}(s) - \sum_{k=0}^{m-1} \psi^{(k)}(0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \le m. \qquad (D.19)$$

Indeed the result (D.19), first stated by Caputo (1969) [13], appears as the "natural" generalization of the corresponding well known result for $\alpha = m$.

Gorenflo & Mainardi (1997) [33] have pointed out the major utility of the Caputo fractional derivative in the treatment of differential equations of fractional order for *physical applications*. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, despite the fact that the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.

E. Fractional Calculus according to Grünwald-Letnikov

According to the familiar definition based on the "backward difference limit", the first-order derivative of a function $\phi(x)$ is obtained as

$$\frac{d}{dx}\phi(x) = \phi^{(1)}(x) = \lim_{h\to 0} \frac{\phi(x) - \phi(x-h)}{h}.$$

Applying this rule twice gives the second-order derivative

$$\begin{aligned}\frac{d^2}{dx^2}\phi(x) = \phi^{(2)}(x) &= \lim_{h\to 0} \frac{\phi^{(1)}(x) - \phi^{(1)}(x-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left\{\frac{\phi(x) - \phi(x-h)}{h} - \frac{\phi(x-h) - \phi(x-2h)}{h}\right\} \\ &= \lim_{h\to 0} \frac{\phi(x) - 2\phi(x-h) + \phi(x-2h)}{h^2}.\end{aligned}$$

We then obtain

$$\frac{d^3}{dx^3}\phi(x) = \phi^{(3)}(x) = \lim_{h\to 0} \frac{\phi(x) - 3\phi(x-h) + 3\phi(x-2h) - \phi(x-3h)}{h^3}.$$

and, by induction,

$$D^n \phi(x) = \frac{d^n}{dx^n} \phi(x) = \phi^{(n)}(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \phi(x-kh) \,. \qquad (E.1)$$

We note that the n-th derivative is seen to be a linear combination of the function's values at the $(n+1)$ values $x, x-h, x-2h, \dots x-nh$ of the independent variable. The coefficients are binomial coefficients and alternate in sign. There is a divisor of h^n .

Starting from the so-called Riemann sum definition of an integral and iterating we get, see *e.g.* Oldham & Spanier (1974) [71], Podlubny (1999) [74]

$$I^n_{a+} \phi(x) = \lim_{h\to 0} h^n \sum_{k=0}^{(x-a)/h} (-1)^k \binom{-n}{k} \phi(x-kh) \,, \qquad (E.2)$$

with

$$(-1)^k \binom{-n}{k} = \frac{n(n+1)\dots(n+k-1)}{k!} \quad \text{for} \quad k = 1, 2, \dots \qquad (E.3)$$

We have implicitly assumed that $(x-a)/h$ is an integer. We then note that the n-th repeated integral is again a weighted sum of the functions's values, but that the length of the sum tends to *infinity* as $h \to 0$. The weights are again defined by binomial coefficients, but they are all positive. In this framework we also recognize the formal identities $I^n_{a+} = D^{-n}$, or $D^n = I^{-n}_{a+}$.

The definition of the Grünwald-Letnikov fractional derivative is an extension of (E.1) to any real positive order α, namely

$${}_{GL}D^\alpha_{a+} \phi(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[(x-a)/h]} (-1)^k \binom{\alpha}{k} \phi(x-kh) \,, \quad a < x < b \,. \qquad (E.4a)$$

Similarly, by taking the "forward difference limit", we define the dual Grünwald-Letnikov fractional derivative of order α as

$${}_{GL}D^\alpha_{b-} \phi(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[(b-x)/h]} (-1)^k \binom{\alpha}{k} \phi(x+kh) \,, \quad a < x < b \,. \qquad (E.4b)$$

In eqs (E.4) the $[\cdot]$ denotes the integer part. In analogy with the Riemann-Liouville fractional calculus we may refer to (E.4a) and (E.4b) as the *progressive* and *regressive* Grünwald-Letnikov fractional derivative, respectively.

For our future purposes it is convenient to introduce a general notation to denote the (finite or infinite) series of the backward/forward difference quotients entering the limit in the above definitions, namely

$$ {}_hD^{\alpha}_{\pm} := \frac{1}{h^{\alpha}} \sum_{k=0} (-1)^k \binom{\alpha}{k} \phi(x \mp kh)\,, \qquad (E.5) $$

where the upper limit of the sum is the appropriate one.

Under certain conditions one can prove the following connections with the Riemann-Liouville fractional integrals and derivatives, (B.2-3) and (B.6), for $a < x < b$,

$$ {}_{GL}D^{-\alpha}_{a+}\,\phi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi = I^{\alpha}_{a+}\,\phi(x)\,, \quad \alpha > 0\,, \qquad (E.6a) $$

$$ {}_{GL}D^{-\alpha}_{b-}\,\phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1}\,\phi(\xi)\,d\xi = I^{\alpha}_{b-}\,\phi(x)\,, \quad \alpha > 0\,, \qquad (E.6b) $$

and

$$ {}_{GL}D^{\alpha}_{a+}\,\phi(x) = D^m\,I^{m-\alpha}_{a+}\,\phi(x) = D^{\alpha}_{a+}\,\phi(x)\,, \quad m-1 < m \le m\,, \qquad (E.7a) $$

$$ {}_{GL}D^{\alpha}_{b-}\,\phi(x) = (-1)^m\,D^m\,I^{m-\alpha}_{b-}\,\phi(x) = D^{\alpha}_{b-}\,\phi(x)\,, \quad m-1 < \alpha \le m\,. \qquad (E.7b) $$

We point out the relation with the Caputo fractional derivative (D.8), for $t > 0$ and $m-1 < \alpha < m$,

$$ {}_{GL}D^{\alpha}_{0+}\,\psi(t) = \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\psi^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\,d\tau\,. \qquad (E.8) $$

The Grünwald-Letnikov scheme may be useful in numerical computations of fractional integrals and derivatives. However, as recently reviewed by Gorenflo (1997), see [28] and references therein, it has mostly a conceptual value since it yields only a first-order approximation in the step-length h. Higher-order methods have been introduced for getting better approximations; so, in [87] second-order accuracy has been obtained by shifting the scheme by a non-integer multiple of h.

Let us conclude this section by noting that the formal identity between the fractional derivative and the fractional integral (with the obvious change of sign in the order) stated in in (E.6a) has induced several authors to unify the notions of fractional derivatives and integrals with the joint name of *differintegration*. Giving priority to the derivative they use the notation $D^{-\alpha}$ for the fractional integral, attributing to it a negative order of differentiation. The present authors oppose to this notation since it is misleading and generally not correct, even if it is used in distinguished treatises as Davis (1936) [17], Oldham & Spanier (1974) [71], Miller & Ross (1993) [64] and Podlubny (1999) [74]. It is well known that derivation and integration operators are not inverse to each other, even if their order is integer, and therefore such unification of symbols appears not justified.

F. The Mittag-Leffler type functions

The Mittag-Leffler function is so named from the great Swedish mathematician who introduced it at the beginning of the XX-th century in a sequence of five notes [65–69]. In this Section we shall consider the Mittag-Leffler function and some of the related functions which are relevant for their connection with fractional evolution processes. It is our purpose to provide a short reference-historical background and a review of the main properties of these functions, based on our papers, see [32], [33], [57], [58], [59].

Reference-historical background

We note that the Mittag-Leffler type functions, being ignored in the common books on special functions, are unknown to the majority of scientists. Even in the 1991 Mathematics Subject Classification these functions cannot be found. However, in the new AMS classification foreseen for the year 2000, a place for them has been reserved: 33E12 ("Mittag-Leffler functions").

A description of the most important properties of these functions with relevant references can be found in the third volume of the Bateman Project [21], in the chapter $XVIII$ of the *miscellaneous functions*. The treatises where great attention is devoted to them are those by Dzherbashian [19], [20]. We also recommend the classical treatise on complex functions by Sansone & Gerretsen [84] and the recent book on fractional calculus by Podlubny [74].

Since the times of Mittag-Leffler several scientists have recognized the importance of the Mittag-Leffler type functions, providing interesting mathematical and physical applications, which unfortunately are not much known.

As pioneering works of mathematical nature in the field of fractional integral and differential equations, we like to quote those by Hille & Tamarkin [42] and Barret [9]. In 1930 Hille & Tamarkin have provided the solution of the Abel integral equation of the second kind in terms of a Mittag-Leffler function, whereas in 1954 Barret has expressed the general solution of the linear fractional differential equation with constant coefficients in terms of Mittag-Leffler functions.

As former applications in physics we like to quote the contributions by Cole in 1933 [16] in connection with nerve conduction, see also [17], and by Gross [38] in 1947 in connection with mechanical relaxation. Subsequently, in 1971, Caputo & Mainardi [14] have shown that Mittag-Leffler functions are present whenever derivatives of fractional order are introduced in the constitutive equations of a linear viscoelastic body. Since then, several other authors have pointed out the relevance of these functions for fractional viscoelastic models, see *e.g.* Mainardi [57].

The Mittag-Leffler functions $E_\alpha(z)$, $E_{\alpha,\alpha}(z)$

The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane,

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}\,, \quad \alpha > 0\,, \quad z \in \mathbf{C}\,. \qquad (F.1)$$

It turns out that $E_\alpha(z)$ is an *entire function* of order $\rho = 1/\alpha$ and type 1. This property is still valid but with $\rho = 1/\mathrm{Re}\{\alpha\}$, if $\alpha \in \mathbf{C}$ with *positive real part*, as formerly noted by Mittag-Leffler himself in [68].

The Mittag-Leffler function provides a simple generalization of the exponential function because of the substitution of $n! = \Gamma(n+1)$ with $(\alpha n)! = \Gamma(\alpha n+1)$. Particular cases of (F.1), from which elementary functions are recovered, are

$$E_2\left(+z^2\right) = \cosh\, z\,, \qquad E_2\left(-z^2\right) = \cos\, z\,, \qquad z \in \mathbf{C}\,, \qquad (F.2)$$

and

$$E_{1/2}(\pm z^{1/2}) = \mathrm{e}^z\,\left[1 + \mathrm{erf}\,(\pm z^{1/2})\right] = \mathrm{e}^z\,\mathrm{erfc}\,(\mp z^{1/2})\,, \quad z \in \mathbf{C}\,, \qquad (F.3)$$

where erf (erfc) denotes the (complementary) error function defined as

$$\mathrm{erf}\,(z) := \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-u^2}\,du\,, \quad \mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z)\,, \quad z \in \mathbf{C}\,. \qquad (F.4)$$

In (F.4) by $z^{1/2}$ we mean the principal value of the square root of z in the complex plane cut along the negative real semi-axis. With this choice $\pm z^{1/2}$ turns out to be positive/negative for $z \in \mathbf{R}^+$. A straightforward generalization of the Mittag-Leffler function, originally due to Agarwal in 1953 based on a note by Humbert, see [3], [44], [45], is obtained by replacing the additive constant 1 in the argument of the Gamma function in (F.1) by an arbitrary complex parameter β. For the generalized Mittag-Leffler function we agree to use the notation

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}\,, \quad \alpha > 0\,, \;\; \beta \in \mathbf{C}\,, \quad z \in \mathbf{C}\,. \qquad (F.5)$$

Particular simple cases are

$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}\,, \quad E_{2,2}(z) = \frac{\sinh\,(z^{1/2})}{z^{1/2}}\,. \qquad (F.6)$$

We note that $E_{\alpha,\beta}(z)$ is still an entire function of order $\rho = 1/\alpha$ and type 1. For lack of space we prefer to continue with the Mittag-Leffler functions in one parameter: since here we shall limit ourselves to consider evolution processes characterized by a single fractional order, the more general functions with two parameters turn out to be redundant. However, we find it convenient to introduce other functions depending on a single parameter which turn out to be related by simple relations to the original Mittag-Leffler functions, and to consider them as belonging to the class of Mittag-Leffler type functions.

The Mittag-Leffler integral representation and asymptotic expansions

Many of the most important properties of $E_\alpha(z)$ follow from Mittag-Leffler's *integral representation*

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\alpha-1}\, \mathrm{e}^{\zeta}}{\zeta^\alpha - z}\, d\zeta\,, \quad \alpha > 0\,, \quad z \in \mathbf{C}\,, \tag{F.7}$$

where the path of integration Ha (the *Hankel path*) is a loop which starts and ends at $-\infty$ and encircles the circular disk $|\zeta| \le |z|^{1/\alpha}$ in the positive sense: $-\pi \le \arg\zeta \le \pi$ on Ha. To prove (F.7), expand the integrand in powers of ζ, integrate term-by-term, and use Hankel's integral for the reciprocal of the Gamma function.

The integrand in (F.7) has a branch-point at $\zeta = 0$. The complex ζ-plane is cut along the negative real semi-axis, and in the cut plane the integrand is single-valued: the principal branch of ζ^α is taken in the cut plane. The integrand has poles at the points $\zeta_m = z^{1/\alpha}\, \mathrm{e}^{2\pi\, i\, m/\alpha}$, m integer, but only those of the poles lie in the cut plane for which $-\alpha\,\pi < \arg z + 2\pi\, m < \alpha\,\pi$. Thus, the number of the poles inside Ha is either $[\alpha]$ or $[\alpha+1]$, according to the value of $\arg z$.

The most interesting properties of the Mittag-Leffler function are associated with its asymptotic developments as $z \to \infty$ in various sectors of the complex plane. These properties can be summarized as follows. For the case $0 < \alpha < 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \quad |z| \to \infty\,, \quad |\arg z| < \alpha\pi/2\,, \tag{F.8}$$

$$E_\alpha(z) \sim - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \quad |z| \to \infty\,, \quad \alpha\pi/2 < \arg z < 2\pi - \alpha\pi/2\,. \tag{F.9}$$

For the case $\alpha \ge 2$ we have

$$E_\alpha(z) \sim \frac{1}{\alpha} \sum_m \exp\left(z^{1/\alpha} \mathrm{e}^{2\pi\, i\, m/\alpha}\right) - \sum_{k=1}^{\infty} \frac{z^{-k}}{\Gamma(1-\alpha k)}\,, \quad |z| \to \infty\,, \tag{F.10}$$

where m takes all integer values such that $-\alpha\pi/2 < \arg z + 2\pi\, m < \alpha\pi/2$, and $\arg z$ can assume any value between $-\pi$ and $+\pi$ inclusive. From the asymptotic properties (F.8-10) and the definition of the order of an entire function, we infer that the Mittag-Leffler function is an *entire function of order* $1/\alpha$ for $\alpha > 0$;

in a certain sense each $E_\alpha(z)$ is the simplest entire function of its order. The Mittag-Leffler function also furnishes examples and counter-examples for the growth and other properties of entire functions of finite order.

The Laplace transform pairs related to the Mittag-Leffler functions

The Mittag-Leffler functions are connected to the Laplace integral through the equation

$$\int_0^\infty \mathrm{e}^{-u}\, E_\alpha\left(u^\alpha\, z\right)\, du = \frac{1}{1-z}\,, \quad \alpha > 0\,. \tag{F.11}$$

The integral at the L.H.S. was evaluated by Mittag-Leffler who showed that the region of its convergence contains the unit circle and is bounded by the line $\mathrm{Re}\, z^{1/\alpha} = 1$. Putting in (F.11) $u = st$ and $u^\alpha\, z = -\lambda\, t^\alpha$ with $t \ge 0$ and $\lambda \in \mathbf{C}$, and using the sign $\div$ for the juxtaposition of a function depending on t with its Laplace transform depending on s, we get the following Laplace transform pairs

$$E_\alpha\left(-\lambda\, t^\alpha\right) \div \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,, \quad \mathrm{Re}\, s > |\lambda|^{1/\alpha}\,. \tag{F.12}$$

In our CISM Lecture notes [33] we have shown the key role of the Mittag-Leffler type functions $E_\alpha\left(-\lambda\, t^\alpha\right)$ in treating Abel integral equations of the second kind and fractional differential equations, so improving the former results by Hille & Tamarkin (1930) [42] and Barret (1954) [9], respectively. In particular, assuming $\lambda > 0$, we have discussed the peculiar characters of these functions (*power-law decay*) for $0 < \alpha < 1$ and for $1 < \alpha < 2$ related to *fractional relaxation* and *fractional oscillation* processes, respectively, see also [56], [32].

Other formulas: summation and integration

For completeness hereafter we exhibit some formulas related to summation and integration of ordinary Mittag-Leffler functions (in one parameter α), referring the interested reader to [19], [74] for their proof and for their generalizations to two parameters. Concerning summation we outline

$$E_\alpha(z) = \frac{1}{p} \sum_{h=0}^{p-1} E_{\alpha/p}\left(z^{1/p} e^{i2\pi h/p}\right)\,, \quad p \in \mathbf{N}\,, \tag{F.13}$$

from which we derive the *duplication formula*

$$E_\alpha(z) = \frac{1}{2}\left[E_{\alpha/2}(+z^{1/2}) + E_{\alpha/2}(-z^{1/2})\right]\,. \tag{F.14}$$

Concerning integration we outline another interesting *duplication formula*

$$\int_0^\infty \mathrm{e}^{-x^2/(4t)}\, E_\alpha(x^\alpha)\, dx = \sqrt{\pi\, t}\, E_{\alpha/2}(t^{\alpha/2})\,, \quad t > 0\,. \tag{F.15}$$

G. The M-Wright type functions

Let us first recall the more general *Wright* function $W_{\lambda,\mu}(z)\,, z \in \mathbf{C}$, with $\lambda > -1$ and $\mu > 0$. This function, so named from the British mathematician who introduced it between 1933 and 1940 [92–95] is defined by the following series and integral representation, valid in the whole complex plane,

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n+\mu)} := \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}}\,. \qquad (G.1)$$

Here Ha denotes an arbitrary Hankel path, namely a contour consisting of pieces of the two rays $\arg\sigma = \pm\phi$ extending to infinity, and of the circular arc $\sigma = \epsilon\,\mathrm{e}^{i\theta}\,,\ |\theta| \le \phi$, with $\phi \in (\pi/2, \pi)$, and $\epsilon > 0$, arbitrary.

It is possible to prove that the *Wright* function is entire of order $1/(1+\lambda)$, hence of exponential type if $\lambda \ge 0$. The case $\lambda = 0$ is trivial since $W_{0,\mu}(z) = \mathrm{e}^{z}/\Gamma(\mu)$. The case $\lambda = -\nu\,,\ \mu = 1-\nu$ with $0 < \nu < 1$ provides the function M function of the Wright type, $M_\nu(z)$, that is of special interest for us, see *e.g.* Mainardi (1995, 1996a, 1996b, 1997) [54–57] and Gorenflo, Luchko & Mainardi (1999, 2000) [30,31]. Specifically, we have

$$M_\nu(z) := W_{-\nu,1-\nu}(-z) = \frac{1}{\nu\, z}\, W_{-\nu,0}(-z)\,, \quad 0 < \nu < 1\,, \qquad (G.2)$$

and therefore from (G.1-2)

$$M_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (1-\nu)]} := \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\sigma - z\sigma^{\nu}} \frac{d\sigma}{\sigma^{1-\nu}}\,. \qquad (G.3)$$

It turns out that $M_\nu(z)$ is an entire function of order $\rho = 1/(1-\nu)$, which provides a generalization of the Gaussian and of the Airy function. In fact we obtain

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\,z^2/4\right)\,, \qquad (G.4)$$

$$M_{1/3}(z) = 3^{2/3}\,\mathrm{Ai}\left(z/3^{1/3}\right)\,, \qquad (G.5)$$

where Ai denotes the *Airy function*. Furthermore it can be proved that $M_{1/q}(z)$ satisfies the differential equation of order $q-1$

$$\frac{d^{q-1}}{dz^{q-1}} M_{1/q}(z) + \frac{(-1)^q}{q}\, z\, M_{1/q}(z) = 0\,, \qquad (G.6)$$

subjected to the $q-1$ initial conditions at $z = 0$, derived from the series expansion in (G.1),

$$M_{1/q}^{(h)}(0) = \frac{(-1)^h}{\pi}\, \Gamma[(h+1)/q]\, \sin[\pi\,(h+1)/q]\,, \quad h = 0,\,1,\,\ldots\,q-2\,. \quad (G.7)$$

We note that, for $q \geq 4$, Eq. (G.6) is akin to the *hyper-Airy* differential equation of order $q-1$, see *e.g.* Bender & Orszag (1987)[10]. Consequently, the function $M_\nu(z)$ is a generalization of the hyper-Airyfunction. In the limiting case $\nu = 1$ we get $M_1(z) = \delta(z-1)$.

From now on let us consider only the M function for positive (real) argument, $M_\nu(r)$, which will be the relevant function for our purposes. The asymptotic representation of $M_\nu(r)$, as $r \to \infty$ can be obtained by using the ordinary saddle-point method. Choosing as a variable r/ν rather than r the computation is easier and yields, see Mainardi & Tomirotti (1995) [61],

$$M_\nu(r/\nu) \sim a(\nu)\, r^{\frac{\nu-1/2}{1-\nu}} \exp\left[-b(\nu)\, r^{\frac{1}{1-\nu}}\right], \quad r \to +\infty, \qquad (G.8)$$

where

$$a(\nu) = \frac{1}{\sqrt{2\pi\,(1-\nu)}} > 0, \quad b(\nu) = \frac{1-\nu}{\nu} > 0. \qquad (G.9)$$

Because of the above exponential decay, any moment of order $\delta > -1$ for $M_\nu(r)$ is finite and results

$$\int_0^\infty r^{\,\delta}\, M_\nu(r)\, dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}, \quad \delta > -1. \qquad (G.9)$$

The following Laplace transform pairs can be proved, see Mainardi (1997) [57],

$$M_\nu(r) \div E_\nu(-s), \quad 0 < \nu < 1. \qquad (G.10)$$

and

$$\frac{\nu}{t^{\nu+1}} M_\nu\left(1/t^\nu\right) \div \exp\left(-s^\nu\right), \quad 0 < \nu < 1, \qquad (G.11)$$

$$\frac{1}{t^\nu} M_\nu\left(1/t^\nu\right) \div s^{1-\nu} \exp\left(-s^\nu\right), \quad 0 < \nu < 1, \qquad (G.12)$$

As a particular case of (G.10) we recover the well-known Laplace transform pair, see *e.g.* Doetsch (1974),

$$M_{1/2}(r) := \frac{1}{\sqrt{\pi}} \exp\left(-\,r^2/4\right) \div E_{1/2}(-s) := \exp(s^2)\,\mathrm{erfc}\,(s). \qquad (G.13)$$

We also note that, transforming term-by-term the Taylor series of $M_\nu(r)$ (not being of exponential order) yields a series of negative powers of s, which represents the asymptotic expansion of $E_\nu(-s)$ as $s \to \infty$ in a sector around the positive real axis. As particular cases of (G.11-12) we recover the well-known Laplace transform pairs, see *e.g.* Doetsch (1974) [18] ,

$$\frac{1}{2t^{3/2}} M_{1/2}(1/t^{1/2})) := \frac{1}{2\sqrt{\pi}}\, t^{-3/2} \exp\left(-1/(4t^2)\right) \div \exp\left(-s^{1/2}\right), \qquad (G.14)$$

$$\frac{1}{t^{1/2}} M_{1/2}(1/t^{1/2})) := \frac{1}{\sqrt{\pi}}\, t^{-1/2} \exp\left(-1/(4t^2)\right) \div s^{-1/2} \exp\left(-s^{1/2}\right). \qquad (G.15)$$

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