Keywords

Galois embeddings, partially-ordered spaces, isomorphism, order extension

Abstract

Partially-ordered spaces $(A, \leq_A)$ and $(B, \leq_B)$ that may be mutually embedded via Galois embeddings are not generally isomorphic — as was previously assumed. Even the additional assumption of the ascending and descending chain conditions (ACC and DCC) does not imply isomorphism. A counterexample proves this. However, we illustrate on the counterexample that $\leq_A$ and $\leq_B$ may be extended such that the resulting spaces are isomorphic. We call such extensible partially-ordered spaces *consistent*. We prove that two partially-ordered spaces with isotone functions in both directions are consistent if ACC and DCC hold: a theory of *zig-zags* enables the construction of partial isomorphisms on subsets of A and B giving rise to a general construction of extensions $\leq_A^*$ and $\leq_B^*$ of the relations $\leq_A$ and $\leq_B$ such that the combination of the partial isomorphisms proves $(A, \leq_A^*) \cong (B, \leq_B^*)$. Consistency in the presence of isotone functions implies consistency for Galois embeddings. In the remainder of the paper we illustrate that the result is best possible and discuss the impact of Galois embeddings on isomorphism.

Schlüsselworte

Galois-Einbettungen, partielle Ordnungen, Isomorphismus, Erweiterungen

Zusammenfassung

Partiell geordnete Mengen $(A, \leq_A)$ und $(B, \leq_B)$, die durch Galois-Einbettungen ineinander eingebettet werden können, sind — entgegen der bisherigen Auffassung — im allgemeinen nicht isomorph. Sogar die zusätzliche Annahme der aufsteigenden und absteigenden Kettenbedingungen (ACC und DCC) impliziert keinen Isomorphismus. Ein Gegenbeispiel beweist dies. Wir illustrieren aber an dem Gegenbeispiel, daß $\leq_A$ und $\leq_B$ erweitert werden können, so daß die resultierenden Ordnungen isomorph sind. Wir nennen solche erweiterbaren partiellen Ordnungen *konsistent*. Wir beweisen, daß zwei partiell geordnete Mengen mit isotonen Funktionen in beiden Richtungen konsistent sind, wenn ACC und DCC gelten: Eine Theorie von *zig-zags* ermöglicht die Konstruktion partieller Isomorphismen auf Teilmengen von A und B, was die allgemeine Konstruktion von Erweiterungen $\leq_A^*$ und $\leq_B^*$ der Relationen $\leq_A$ und $\leq_B$ erlaubt, so daß die Kombination der partiellen Isomorphismen $(A, \leq_A^*) \cong (B, \leq_B^*)$ beweist. Konsistenz in Gegenwart von isotonen Funktionen impliziert die Konsistenz für Galois-Einbettungen. Im Rest des Papiers illustrieren wir, daß das Resultat bestmöglich ist und diskutieren den Einfluß von Galois-Einbettungen auf Isomorphismus.

Contents

1 Introduction

Galois connections are important in relating models at different levels of abstraction. The concept is, by its simplicity, widely applicable; but as a consequence its theory is relatively restricted (compared, for instance, with that of adjunctions). In view of the goal-directed nature of system derivation the most common appearance of the symmetric notion of a Galois connection is in situations where one semantic domain is embedded in the other. We call the resulting asymmetric type of Galois connection a *Galois embedding* [7].

In [1] we showed that if there is a Galois embedding from each of two partial orders to the other then it is not neccessarily the case that the two partial orders are isomorphic. In the notation of [1] (introduced below) the implication

$$\left(\begin{array}{l} ge(\varepsilon_0, \pi_0; (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_1, \pi_1; (B, \leq_B), (A, \leq_A)) \end{array} \right) \Rrightarrow (A, \leq_A) \cong (B, \leq_B) \tag{1.1}$$

is false. In this paper, we develop the theory and present a positive result concerning isomorphism of Galois embeddings. First, we repeat the definitions and motivate the question of isomorphism of Galois embeddings in Section 1.1. In Section 2 we summarize the results of [1]. In Section 3, standard results from set theory and lattice theory are used to formalize a theory of substructures of partial orders with isotone maps, so-called *zig-zags*. This theory leads on to the main theorem of this paper presented in Section 4: in partial orders where there are only finite ascending and descending chains, the orders may be extended such that (1.1) holds in the original partially-ordered sets, but with the extended order relation. We will call such extensible orders *consistent*. In Section 5 we prove the result to be best possible by showing that consistency is not generally true, if the ascending and descending chain conditions do not hold.

Our Extension theorem might appear rather weak. However, although Galois embeddings on partially-ordered spaces in both directions are a strong assumptions it turns out that not much can be said about isomorphism. The proof that we present for this theorem is constructive. For Galois connections arising from the representations of a design at two different levels of abstraction, the isomorphism can be interpreted as a constructive data representation.

1.1 Definitions

We adopt the convention (from Z) that a conjunction $X \wedge Y$ is written

$$\begin{pmatrix} X \\ Y \end{pmatrix}.$$

We write the conditional 'p if r else q' as $p \lhd r \rhd q$; and we write $p \Rrightarrow q$ [$p \equiv q$] to mean that the implication $p \Rightarrow q$ [equivalence $p \Leftrightarrow q$] is a theorem.

Application of a function f to an argument x is written $f.x$ and the lifting of f which maps subsets of its domain to subsets of its range is written

$$f[E] \quad \equiv_{df} \quad \{f.e \mid e \in E\}.$$

The identity function on X is written $\imath_X$.

By a *partially-ordered space* $(A, \leq_A)$ we mean, as usual, that $\leq_A$ is a reflexive, antisymmetric and transitive relation on A. A *complete lattice* is a partially-ordered space $(A, \leq_A)$ where supremum and infimum exist for any $S \subseteq A$. Suppose that $(A, \leq_A)$ and $(B, \leq_B)$ are partially-ordered spaces. A function $f : A \to B$ is *monotone* [*isotone*] iff

$$\begin{array}{ll} x \leq_A y \quad \Rrightarrow \quad f.x \leq_B f.y \\ [\, x \leq_A y \quad \equiv \quad f.x \leq_B f.y \,]. \end{array}$$

Partially-ordered spaces are isomorphic, written $(A, \leq_A) \cong (B, \leq_B)$, iff there is an isotone bijection from A to B. They are *consistent*, iff there are extended orders $(\leq_A^*) \supseteq (\leq_A)$ and $(\leq_B^*) \supseteq (\leq_B)$ for which $(A, \leq_A^*) \cong (B, \leq_B^*)$. In that case, we call the extended partially-ordered spaces *conservative* extensions.

By a *Galois connection* between partially-ordered spaces $(A, \leq_A)$ and $(B, \leq_B)$ we mean a pair of monotone functions $\varepsilon : A \to B$ and $\pi : B \to A$ for which these (pointwise) inequalities hold:

$$\begin{pmatrix} \pi \circ \varepsilon & \geq_A & \imath_A \\ \varepsilon \circ \pi & \leq_B & \imath_B \end{pmatrix}.$$

In that case we write

$$gc(\varepsilon, \pi; (A, \leq_A), (B, \leq_B)).$$

By a *Galois embedding* we mean a Galois connection for which $\pi \circ \varepsilon \; = \; \imath_A$, in which case we write

$$ge(\varepsilon, \pi; (A, \leq_A), (B, \leq_B)).$$

Standard theory shows that a Galois connection is a Galois embedding iff either ε is injective or π is surjective.

2 Antisymmetry of Galois Embeddings

In [1] the authors show that implication (1.1) fails for arbitrary Galois embeddings. For comparison ([2, section 7] claims to present a proof.

This section first summarizes the major observation and results of [1] in order to give sufficient motivation and background for the theory presented in the remainder of this paper. We repeat the counterexample from [1] in full detail. But here we use the counterexample to illustrate how the order relation may be extended such that it becomes possible to construct an isomorphism; the main theorem proved in the remainder of this article generalizes this procedure.

2.1 Construction of Counterexample

In this section we present from [1] two partially-ordered spaces satisfying the ACC and DCC for which implication (1.1) fails.

The Partial Orders

We write $(\mathbb{N}, \leq)$ for the partially-ordered space of natural numbers and suppose that the subset $\{0, 1, 2\}$ inherits its order from $\mathbb{N}$. Define a partially-ordered space:

$$X \equiv_{df} (\mathbb{N}\times\{1,2\}) \cup \{(0,0)\}$$
$$(a,b) \leq_X (c,d) \quad \equiv_{df} \quad \begin{pmatrix} a = c \\ b \leq d \end{pmatrix} \vee \begin{pmatrix} a = 0 \\ b = 0 \end{pmatrix} .$$

We are interested in two subsets of X with their inherited orders:

$$A \equiv_{df} X$$
$$B \equiv_{df} X \setminus \{(0,2)\}$$

(see Figure 1). Evidently each is a partial order.

The First Galois Embedding

The first Galois embedding is defined (cf. Figure 2):

$$\varepsilon_1 : A \to B$$
$$\pi_1 : B \to A$$
$$\varepsilon_1.(a,b) \quad \equiv_{df} \quad (0,0) \lhd b = 0 \rhd (a{+}1, b)$$
$$\pi_1.(c,d) \quad \equiv_{df} \quad (0,0) \lhd c = 0 \rhd (c{-}1, d) .$$

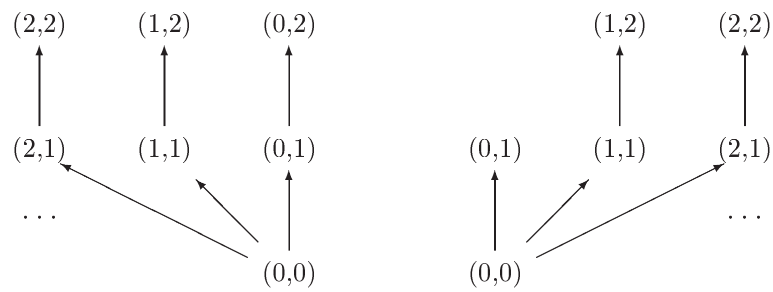

Figure 1: Partial orders $(A, \leq_A)$ on the left and $(B, \leq_B)$ on the right

Proposition 2.1 $ge(\varepsilon_1, \pi_1; (A, \leq_A), (B, \leq_B))$.

Proof: Firstly $\pi_1 \circ \varepsilon_1 = \imath_A$, for

$$
\begin{array}{lll}
\pi_1 \circ \varepsilon_1.(a, b) & = & \text{(definitions of } \varepsilon_1 \text{ and} \circ \text{)} \\
\pi_1.((0,0) \lhd \; b = 0 \; \rhd (a{+}1, b)) & = & \text{(application distributes } \lhd \; \rhd \text{)} \\
\pi_1.(0,0) \lhd \; b = 0 \; \rhd \pi_1.(a{+}1, b) & = & \text{(definition of } \pi_1 \text{)} \\
(0,0) \lhd \; b = 0 \; \rhd (a, b) & = & \text{(definitions of } \lhd \; \rhd \text{ and } A \text{)} \\
(a, b) \, .
\end{array}
$$

Secondly $\varepsilon_1 \circ \pi_1 \leq \imath_B$, for

$$
\begin{array}{lll}
\varepsilon_1 \circ \pi_1.(c, d) & = & \text{(definitions of } \pi_1 \text{ and} \circ \text{)} \\
\varepsilon_1.((0,0) \lhd \; c = 0 \; \rhd (c{-}1, d)) & = & \text{(application distributes } \lhd \; \rhd \text{)} \\
\varepsilon_1.(0,0) \lhd \; c = 0 \; \rhd \varepsilon_1.(c{-}1, d) & = & \text{(definition of } \varepsilon_1 \text{)} \\
(0,0) \lhd \; c = 0 \; \rhd (c, d) & \leq_B & \text{(definitions of } \lhd \; \rhd \text{ and } (B, \leq_B)\text{)}
\end{array}
$$

(c,d) .

In particular equality fails at (just) $(0,1)$. $\square$

The Second Galois Embedding

The second Galois embedding is defined as

$$
\begin{array}{lll}
\varepsilon_2 : B \to A \\
\pi_2 : A \to B \\
\varepsilon_2.(c,d) & \equiv_{df} & ((0,0) \lhd\ d=0\ \rhd (0,2)) \lhd\ c=0\ \rhd (c,d) \\
\pi_2.(a,b) & \equiv_{df} & ((0,0) \lhd\ b \le 1\ \rhd (0,b{-}1)) \lhd\ a=0\ \rhd (a,b)\,.
\end{array}
$$

Proposition 2.2 $\quad ge(\varepsilon_2, \pi_2; (B, \le_B), (A, \le_A))$.

Proof: Firstly $\pi_2 \circ \varepsilon_2 = \imath_B$, for

$$
\begin{array}{lll}
\pi_2 \circ \varepsilon_2.(c,d) & = & \text{(arguing as above)} \\
\\
\pi_2.(0,0) \lhd\ d=0\ \rhd \pi_2.(0,2) & & \\
\quad \lhd\ c=0\ \rhd & = & \text{(definition of } \pi_2) \\
\quad \pi_2.(c,d) & & \\
\\
(0,0) \lhd\ d=0\ \rhd (0,1) & & \\
\lhd\ c=0\ \rhd & = & \text{(definitions of } \lhd\ \rhd \text{ and } B) \\
(c,d) & & \\
\\
(c,d)\,. & &
\end{array}
$$

Secondly $\varepsilon_2 \circ \pi_2 \le \imath_A$, for

$$
\begin{array}{lll}
\varepsilon_2 \circ \pi_2.(a,b) & = & \text{(def } \pi_2\text{, application distributes } \lhd\ \rhd) \\
\\
\varepsilon_2.(0,0) \lhd\ b \le 1\ \rhd \varepsilon_2.(0,b{-}1) & & \\
\lhd\ a=0\ \rhd & = & \text{(definition of } \varepsilon_2) \\
\varepsilon_2.(a,b) & &
\end{array}
$$

$$(0,0) \lhd\ b \le 1\ \rhd (0,2)$$
$$\lhd\ a = 0\ \rhd \qquad \le \qquad \text{(definitions of } \lhd\ \rhd \text{ and } A\text{)}$$
$$(a,b)$$

$$(a,b)$$

with equality failing again at just $(0,1)$. □

Figure 2 illustrates these Galois embeddings.

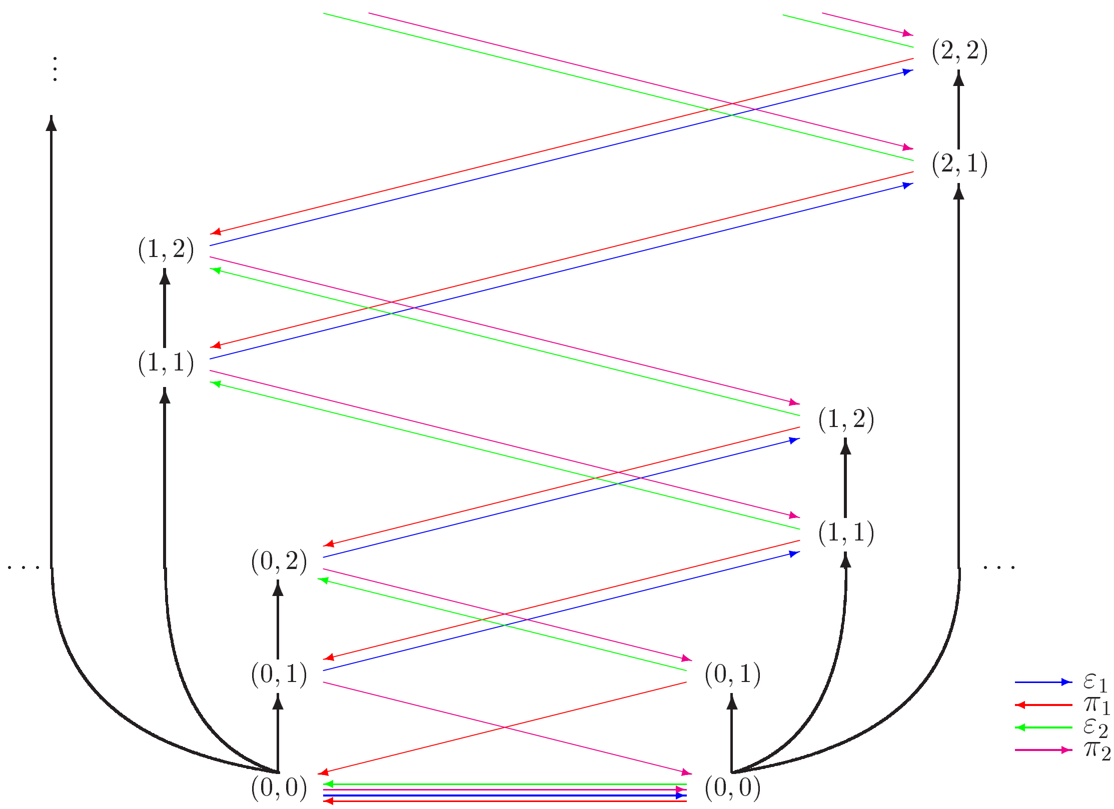

Figure 2: The partial orders with the Galois embeddings (A left, B right)

Now that we have established those Galois embeddings, we show that the structures are not isomorphic.

Proposition 2.3 $(A, \leq_A) \not\cong (B, \leq_B))$.

Proof: If $f : A \to B$ is an isomorphism of partially-ordered spaces then $a <_A b \Rrightarrow f.a <_B f.b$. For

$$
\begin{array}{lll}
a <_A b & \equiv & \text{(definition of } <_A) \\
\\
\left(\begin{array}{l} a \leq_A b \\ a \neq b \end{array}\right) & \Rrightarrow & (f \text{ monotone and injective}) \\
\\
\left(\begin{array}{l} f.a \leq_B f.b \\ f.a \neq f.b \end{array}\right) & \Rrightarrow & \text{(definition of } <_B) \\
\\
f.a <_B f.b\,. & &
\end{array}
$$

For each $n \in \mathbb{N}$, in space A

$$(0, 0) <_A (n, 1) <_A (n, 2)\,,$$

hence in space B

$$f.(0, 0) <_B f.(n, 1) <_B f.(n, 2)\,.$$

Since the only possibility is

$$f.(0, 0) = (0, 0)$$

we infer, for each $n \in \mathbb{N}$,

$$(0, 0) <_B f.(n, 1) <_B f.(n, 2)\,.$$

But that means the range of f cannot contain the maximal chain

$$(0, 0) <_B (0, 1)$$

contradicting surjectivity of f . □

Thereby, (1.1) is proved to be false. Since Galois embeddings are a special case of isotone functions naturally (1.1) neither holds for just isotone functions.

Moreover, the counterexample shows that conjecture (1.1) does not hold even restricted to A and B satisfying both the ascending chain condition and descending chain condition (i.e. for which every strictly increasing or decreasing chain is finite).

After a formal argument that shows that the implication (1.1) is at least as hard as the Cantor-Schröder-Bernstein theorem (CSB) [1, Section 3.2], the paper presents a proof of the restriction of (1.1) to finite partially-ordered sets. We present this theorem in Section 3.3 and prove it using the technique of zig-zags introduced here. The construction used in the proof of CSB for the bijection is not monotone let alone isotone.

2.2 Extending the Order of the Counterexample

Although we have seen that it is not possible to define an isomorphism from A to B, we are now going to illustrate why the orders are consistent. In Section 3 we will show that it is generally possible to define conservative extensions if the ascending or descending chain condition holds.

Assuming the same structures as previously defined, Figure 3 shows how the order may be extended and the isomorphism ρ can be defined. The pink

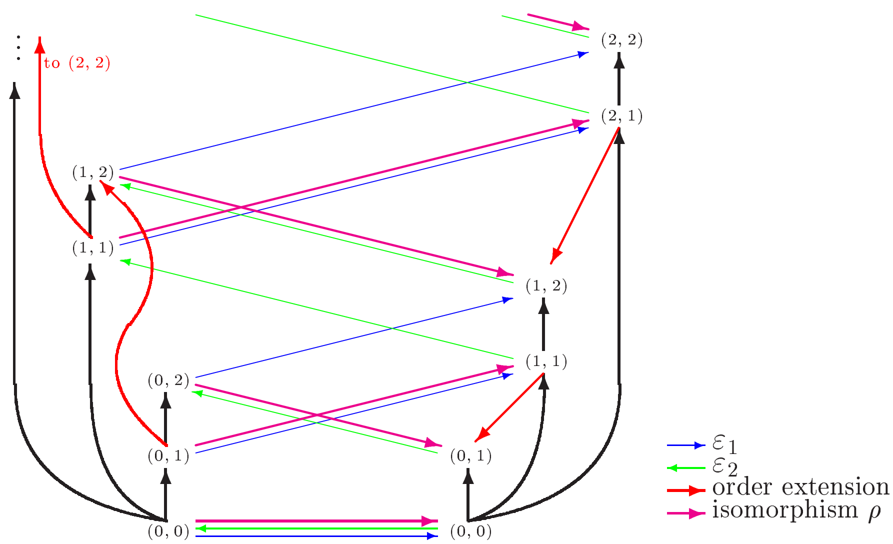

Figure 3: The extended partial orders with an isomorphism

fat arrows represent the isomorphism ρ, the red arrows the extension of the order relation. Without attempting to prove now that ρ is an isomorphism, we give an intuitive idea how the construction of ρ and the extension is motivated.

Ordered Part

There is a subset of A that is constant under ε_1 and ε_2, i.e. for elements of this set $\varepsilon_2 \circ \varepsilon_1 = \imath_A$. In Figure 4 this subset and its image in B are encircled by red lines. We will refer to these subsets as the *ordered part*, and call them A_{ord} and B_{ord}, respectively, when introducing them formally in Section 3.

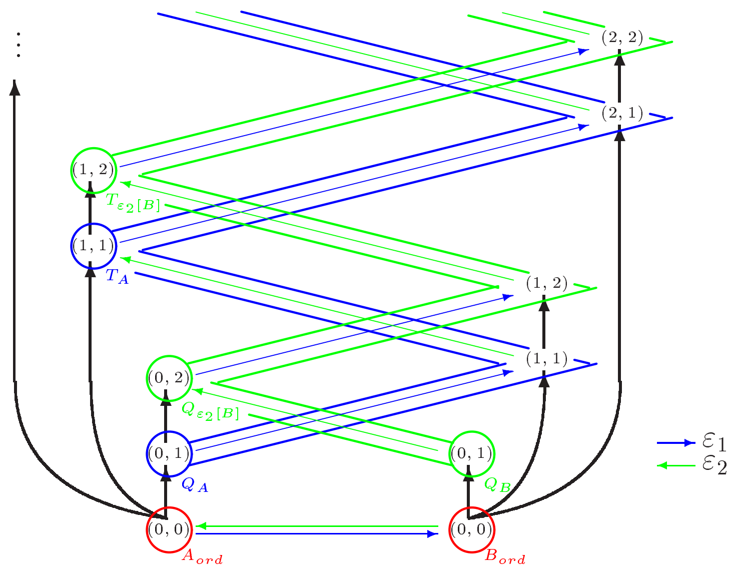

Figure 4: Subsets of $(A, \leq_A)$ and $(B, \leq_B)$ forming zig-zags

Complements of Ranges

The main phenomenon that creates the zig-zag-like structure of the function arrows are the complements of the ranges of the injections ε_1 and ε_2. We will call them Q_A and Q_B from now on. In Figure 4, $Q_A = \{(0, 1)\}$, the complement of the range of ε_2, is enclosed by a blue circle and $Q_B = \{(0, 1)\}$, the complement of the range of ε_1, by a green circle. In the example each set is singular but in general they need not be. Other elements enclosed in blue or green circles belong to zig-zags that are described below. It is clear by the definition of complement that, if the complements would be empty, ε_1 and ε_2 were surjective and thereby isomorphisms, as they are already isotone by assumption.

Zig-Zags

The image in A of Q_B under ε_2 we call $Q_{\varepsilon_2[B]}$. As may be seen in Figure 4, the images of Q_A and $Q_{\varepsilon_2[B]}$ under $\varepsilon \equiv_{df} \varepsilon_2 \circ \varepsilon_1$ are encircled in the same color as their origins. Highlighting the function arrows that connect origins and images we see that $\varepsilon^n, 0 \leq n$ has zig-zagging structure. We call the subset of elements of A created by an iterated application of ε to a single element of Q_A a *zig-zag*. Similarly, the subset of A containing the images of a single element of $Q_{\varepsilon_2[B]}$ under iterated application of ε is a *zag-zig*. By the plural, i.e. zig-zags and zag-zigs, we refer to the set of all zig-zags or zag-zigs that are started by an element of Q_A or $Q_{\varepsilon_2[B]}$. More formally, the set T_A, the zig-zags, may be defined as $\bigcup_{0 \leq n} \varepsilon^n[Q_A]$ and the set $T_{\varepsilon_2[B]}$ of zag-zigs as $\bigcup_{0 \leq n} \varepsilon^n[Q_{\varepsilon_2[B]}]$[1].

After a thorough definition and analysis of these structures in Section 3.4 we will see that there is a third kind of zig-zagging structures. These are both side infinite iterations of ε applications. We will call the elements contained in the image of such ε-applications *zigars*. The partial order considered in the present section does not contain zigars.

For reasons of simplicity we overload the term zig-zags and call the sets zig-zags, zag-zigs, and zigars collectively zig-zags.

Construction of the Isomorphism

Considering the zig-zags, i.e. the set T_A (which includes Q_A), Figure 4 illustrates that ε_1 is an isomorphism with respect to this subset and its image. With respect to A_{ord}, again ε_1 is also an isomorphism between A_{ord} and B_{ord}. However, for $T_{\varepsilon_2[B]}$ this equivalence does not hold: if we apply ε_1, we miss out Q_B. Consequently, to build a bijection for the whole of A, we use ε_2^{-1} for $T_{\varepsilon_2[B]}$. As mentioned earlier, a similar construction is also used in the proof

[1]In Section 3.4 we will use a different definition to gain fixpoint properties, but will then show that it is equivalent to the one presented here.

of the Cantor-Schröder-Bernstein theorem. Unfortunately, the bijection

$$\begin{array}{lll} \rho : A \to B & & \\ \rho.(a,b) & \equiv_{df} & (0,0) \\ & & \lhd (a,b) = (0,0) \rhd \\ & & (a{+}1,b) \lhd b = 1 \rhd ((0,1) \lhd a = 0 \rhd (a,b)) \end{array}$$

built from a combination of ε_1 and ε_2^{-1} (cf. Figure 3) is not isotone. For example, $(0,2) \geq_A (0,1)$, but $(0,1) \not\geq_B (1,1)$. Therefore, we have to extend the order to turn it into an isomorphism. The extension of the order just compensates for gaps in the order of the image of the bijection ρ. As an intuitive appeal that the extended partial orders in the example are actually isomorphic with respect to ρ, Figure 5 shows a rearranged illustration of the extended example. The new relation is transitive (the corresponding edges in Figure 5 are implicitly assumed).

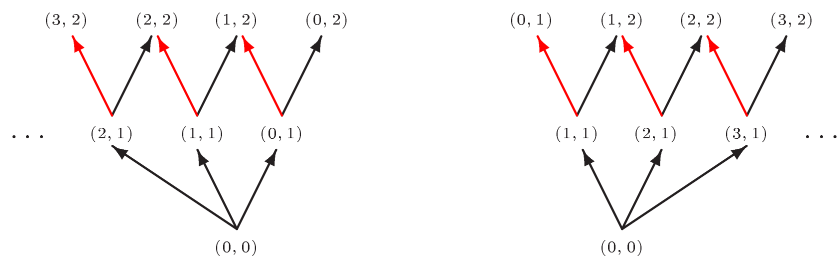

Figure 5: Redrawn extended partial orders $(A, \leq_A)$ and $(B, \leq_B)$

3 Zig-Zags, Zag-Zigs, and Zigars

The counterexample presented in the previous section shows that the ascending chain condition (ACC) and descending chain condition (DCC) together still do not suffice to ensure (1.1). However, using a more intricate argument relying on the isotonicity of ε_1 and ε_2 and the chain conditions, we will show in this and the following section that in that case the orders $\leq_A$ and $\leq_B$ are consistent with each other (see Section 4, Theorem 1). This generalizes the construction illustrated in the example of the previous section.

Before we present the proof of this theorem we analyze the structures of A and B by dividing them into substructures. In the following we define zig-zags formally and prove lemmas about them that will then be used in the final proof.

3.1 Preliminaries

Sets and Functions

Before we start the development of the theory of zig-zags, we want to present some basic facts about sets and functions that we are going to use. To ensure correctness, we have checked (3.1)-(3.11) with the theorem prover Isabelle [12] (the corresponding proof scripts are in Appendix A.2). Note that the symmetric set difference $\setminus$ associates to the left.

$$
\begin{aligned}
(\forall\ i \in I.\ E_i \subseteq F) &\equiv \bigcup_{i\in I} E_i \subseteq F && (3.1)\\
E \setminus F \setminus G &= E \setminus G \setminus F && (3.2)\\
F \subseteq G \Rrightarrow E \setminus F \setminus (G \setminus F) &= E \setminus G && (3.3)\\
\bigcup_{i\in I} f[E_i] &= f[\bigcup_{i\in I} E_i] && (3.4)\\
F \subseteq E \Rrightarrow (E \setminus F) \cup F &= E\,. && (3.5)
\end{aligned}
$$

We will need some more facts about injective functions. Let $f : A \to B$ be an injective function, and $X, Y \subseteq A$ or B. Then $f^{-1} : f[A] \to A$ is a well defined function and

$$
\begin{aligned}
\forall\ x \in f[A].\ f \circ f^{-1}.x &= x && (3.6)\\
\forall\ x \in A.\ f^{-1} \circ f.x &= x && (3.7)\\
f^{-1}[f[X]] &= X && (3.8)
\end{aligned}
$$

$$Y \subseteq f[A] \Rrightarrow f[f^{-1}[Y]] \quad = \quad Y \tag{3.9}$$
$$f[X \setminus Y] \quad = \quad f[X] \setminus f[Y] \tag{3.10}$$
$$(f[X] = f[Y]) \quad \equiv \quad (X = Y)\,. \tag{3.11}$$

A basic fact about isomorphism and image is

Lemma 3.1 $\qquad \left(\begin{array}{l} f : A \to B \\ f \text{ isotone} \end{array}\right) \Rrightarrow A \cong f[A]\,.$

Proof: By definition of image of a function f is surjective on $f[A]$. Since it is isotone, it is an isomorphism. □

Furthermore, we note that

$$f \text{ isotone} \quad \equiv \quad f^{-1} \text{ isotone} \tag{3.12}$$
$$f, g \text{ isotone} \quad \Rrightarrow \quad f \circ g \text{ isotone}\,. \tag{3.13}$$

Next, we present a 'disjoint-from' calculus also presented briefly in [1] that is very useful for proofs related to Galois embeddings.

The 'Disjoint-From' Calculus

We use the following simple laws of the 'disjoint-from' calculus. We write $E \diamond F$ to mean that sets E and F are disjoint, i.e. $E \cap F = \varnothing$.

$$(F \setminus E) \diamond E \tag{3.14}$$
$$\left(\begin{array}{c} E \diamond F \\ E' \subseteq E \end{array}\right) \quad \Rrightarrow \quad E' \diamond F \tag{3.15}$$
$$\left(\begin{array}{c} E \diamond F \\ E \neq \varnothing \end{array}\right) \quad \Rrightarrow \quad F \subset (E \cup F) \tag{3.16}$$
$$E \diamond (\bigcup_{i \in I} F_i) \quad \equiv \quad \forall\, i \in I.\; E \diamond F_i \tag{3.17}$$
$$\left(\begin{array}{l} E \diamond F \\ f \text{ injective} \end{array}\right) \quad \Rrightarrow \quad f[E] \diamond f[F] \tag{3.18}$$
$$\left(\begin{array}{c} E \diamond F \\ x \in E \\ y \in F \end{array}\right) \quad \Rrightarrow \quad x \neq y \tag{3.19}$$
$$\left(\begin{array}{c} E \subseteq F \\ E \diamond G \end{array}\right) \quad \Rrightarrow \quad E \subseteq (F \setminus G) \tag{3.20}$$

$$\begin{pmatrix} E \diamond F \\ e : E \to E' \\ f : F \to F' \\ e, f \text{ injective} \end{pmatrix} \Rrightarrow (e \cup f) : E \cup F \to E' \cup F' \text{ injective} \tag{3.21}$$

$$\begin{pmatrix} E \diamond F \\ E \cup F = G \end{pmatrix} \Rrightarrow E = G \setminus F\,. \tag{3.22}$$

Definitions and Local Assumptions

We assume for this section and the following the premises of the theorem we are going to prove finally in Section 4, i.e. for partially-ordered spaces $(A, \leq_A)$, $(B, \leq_B)$ let

$$\varepsilon_1 : A \to B,\ \varepsilon_1 \text{ isotone} \tag{3.23}$$

$$\varepsilon_2 : B \to A,\ \varepsilon_2 \text{ isotone} \tag{3.24}$$

$$ACC(A) \wedge DCC(A)\,. \tag{3.25}$$

We further define

$$Q_A \quad \equiv_{df} \quad A \setminus \varepsilon_2[B] \tag{3.26}$$

$$Q_B \quad \equiv_{df} \quad B \setminus \varepsilon_1[A] \tag{3.27}$$

and assume that

$$Q_A \neq \varnothing \tag{3.28}$$

$$Q_B \neq \varnothing \tag{3.29}$$

because if either of (3.28) or (3.29) were false, either ε_2 or ε_1 would be surjective. In that case, since both ε_1 and ε_2 are isotone, i.e. in particular injective, we would have an isotone bijection, that is $(A, \leq_A) \cong (B, \leq_B)$; thus the main theorem, Theorem 1, would be trivially true.

If it is clear from context that a set X is a subset of A or B, we write $\overline{X}$ to mean its *complement* with respect to its superset, e.g. for $X \subseteq A$, $\overline{X}$ means $A \setminus X$. We define the compositions

$$\varepsilon \quad \equiv_{df} \quad \varepsilon_2 \circ \varepsilon_1 \tag{3.30}$$

$$\hat{\varepsilon} \quad \equiv_{df} \quad \varepsilon_1 \circ \varepsilon_2 \tag{3.31}$$

and note that by (3.13) ε and $\hat{\varepsilon}$ are isotone. We write $x \gtrless y$ to annotate that x and y are not comparable, i.e. $x \not\leq y$ and $y \not\leq x$. We define

$$< \quad \equiv_{df} \quad (\leq) \cap (\neq) \tag{3.32}$$

assume the dual definition for $>$, and note that (cf. proof of Proposition 2.3)

$$< \text{ is transitive.} \tag{3.33}$$

$$f \text{ isotone } \Rrightarrow x < y \equiv f.x < f.y\,. \tag{3.34}$$

Where convenient we omit subscripts on orders.

Ascending and Descending Chain Conditions *ACC* and *DCC*

The ascending and descending chain condition (ACC and DCC) is the major assumption that enables the construction that leads to an isomorphism between extensions of the partial orders $(A, \leq_A)$ and $(B, \leq_B)$. There are a few facts concerned with these chain conditions that we want to establish before we characterize zig-zags.

Lemma 3.2 $\quad ACC(A) \equiv ACC(B)\,.$

Proof: Assume for contradiction there is an infinite strictly increasing chain $(x_i)_{i\in\mathbb{N}}$ in B. Then for any $i \in \mathbb{N}$, we have by (3.34)

$$x_i < x_{i+1} \equiv \varepsilon_2.x_i < \varepsilon_2.x_{i+1}\,.$$

Thereby, $(\varepsilon_2.x_i)_{i\in\mathbb{N}}$ is an infinite ascending chain in A contradicting $ACC(A)$.

The other direction works similarly using ε_1. □

The dual argument yields

Corollary 3.3 $\quad DCC(A) \equiv DCC(B)\,.$

Because of these equivalences of the ascending and descending chain conditions, we will in the following omit the reference to the partially-ordered spaces, i.e we will just write ACC and DCC.

We will now see that ACC and DCC are strong assumptions; the image of the composite function ε cannot be strictly less or greater than its origin.

Lemma 3.4 $\quad (\exists\, x \in A.\ \varepsilon.x > x) \Rrightarrow \neg ACC.$

Proof: Since ε is isotone, we can derive by induction

$$\varepsilon^{n+1}.x > \varepsilon^n.x\,.$$

Since $<$ is transitive (3.33), we can construct the infinitely *ascending* chain

$$x < \varepsilon.x < \varepsilon^2.x < \varepsilon^3.x < \ldots\,.$$ □

It is tempting to assume that the condition $\varepsilon.x > x$ is sufficient to exclude DCC as well. We will show now why one might have this idea and why it does not apply. Since ε is isotone,

$$\varepsilon.x > x \equiv \varepsilon^{-1} \circ \varepsilon.x > \varepsilon^{-1}.x$$

if $\varepsilon^{-1}.x$ is defined. Since ε is injective the latter is, by (3.7), equivalent to

$$x > \varepsilon^{-1}.x\,.$$

Since ε^{-1} is isotone, if ε is (3.12), we get by induction

$$\varepsilon^{-n}.x > \varepsilon^{-(n+1)}.x\,,$$

assuming $\varepsilon^{-(n+1)}.x$ is defined for each n. This enables, by transitivity, the construction of an infinitely *descending* chain

$$x > \varepsilon^{-1}.x > \varepsilon^{-2}.x > \varepsilon^{-3}.x > \ldots\,.$$

However, the previous argument assumes the definedness of ε^{-1} in each step. The simple counterexample of $(\mathbb{N}, \leq)$ and the successor function s as ε shows that this is not always the case: s is isotone and we have generally $s.x > x$, but s^{-1} is equivalent to the predecessor function p which is not defined for 0: since $s^{-1} : s[\mathbb{N}] \to \mathbb{N}$, we have that $p : \mathbb{N} \setminus \{0\} \to \mathbb{N}$.

However, since $\varepsilon : A \to A$, we can at least deduce with the dual argument of Lemma 3.4

Corollary 3.5 *$(\exists\, x \in A.\ \varepsilon.x < x) \Rrightarrow \neg DCC$.*

3.2 The Ordered Parts A_{ord}/B_{ord}

Define

$$A_{ord} \equiv_{df} \{x \in A \mid \varepsilon.x \geq x \ \vee\ \varepsilon.x \leq x\} \tag{3.35}$$

$$A_{\gtrless} \equiv_{df} \{x \in A \mid \varepsilon.x \gtrless x\} \tag{3.36}$$

and similarly for B,

$$B_{ord} \equiv_{df} \{x \in B \mid \varepsilon_1 \circ \varepsilon_2.x \geq x \ \vee\ \varepsilon_1 \circ \varepsilon_2.x \leq x\} \tag{3.37}$$

$$B_{\gtrless} \equiv_{df} \{x \in B \mid \varepsilon_1 \circ \varepsilon_2.x \gtrless x\}\,. \tag{3.38}$$

By definition, (similarly for B_{ord})

$$A_{ord} \cup A_{\gtrless} = A \tag{3.39}$$

$$A_{ord} = \overline{A_{\gtrless}}\,. \tag{3.40}$$

Our first observation is

Proposition 3.6 $\qquad A_{ord} = \{x \in A \mid \varepsilon.x = x\}\,.$

Proof: We have globally assumed $ACC \wedge DCC$. By contraposition with Lemma 3.4 and Corollary 3.5 we infer

$$\left(\begin{array}{l} \nexists\, x \in A.\ \varepsilon.x > x \\ \nexists\, x \in A.\ \varepsilon.x < x \end{array} \right).$$

Since we have

$$A_{ord} = \{x \in A \mid \varepsilon.x \geq x \vee \varepsilon.x \leq x\}$$

we must have for $x \in A_{ord}$ that $\varepsilon.x = x$. $\square$

The ascending and descending chain conditions also hold for B (Lemma 3.2 or Corollary 3.3, respectively). Hence we also have that for the subset B_{ord}

Corollary 3.7 $\qquad B_{ord} \equiv \{y \in B \mid (\varepsilon_1 \circ \varepsilon_2).y = y\}\,.$

Next we observe that for the subsets A_{ord} and B_{ord} the restriction $\varepsilon_1 \mid_{A_{ord}}$ is already an isomorphism such that $A_{ord} \cong B_{ord}$.

Proposition 3.8 $\qquad A_{ord} \cong \varepsilon_1[A_{ord}] = B_{ord}\,.$

Proof: By Lemma 3.1, we immediately get $A_{ord} \cong \varepsilon_1[A_{ord}]$. So, it remains to show $\varepsilon_1[A_{ord}] = B_{ord}$. We proceed in two directions.
For $\subseteq$:

$$
\begin{array}{lll}
y \in \varepsilon_1[A_{ord}] & \Rrightarrow & \text{(image property, Prop. 3.6)} \\[2ex]
\exists\, x \in A_{ord}. \left(\begin{array}{l} \varepsilon.x = x \\ \varepsilon_1.x = y \end{array} \right) & \Rrightarrow & \text{(substitution, def. } \varepsilon\text{)} \\[2ex]
\varepsilon_2.y = \varepsilon_2 \circ \varepsilon_1.x = x & \Rrightarrow & (\varepsilon_1 \text{ function}) \\[2ex]
\varepsilon_1 \circ \varepsilon_2.y = \varepsilon_1.x = y & & \text{i.e. } y \in B_{ord}\,.
\end{array}
$$

Conversely $\supseteq$:

$y \in B_{ord}$ $\Rrightarrow$ (definition of B_{ord})

$\varepsilon.(\varepsilon_2.y) = \varepsilon_2 \circ \varepsilon_1.(\varepsilon_2.y) =$
$\varepsilon_2.(\varepsilon_1 \circ \varepsilon_2.y) = \varepsilon_2.y$ $\Rrightarrow$ (definition of A_{ord})

$\varepsilon_2.y \in A_{ord}$. □

In the following, we concentrate on the remainder of A and B, i.e. $A_{\gtrless}$ and $B_{\gtrless}$, respectively.

3.3 $A_{\gtrless}/B_{\gtrless}$

In this section we formally define the subsets of $A_{\gtrless}$ and $B_{\gtrless}$ called zig-zags. This will enable us to analyze the structure of $(A, \leq_A)$ and $(B, \leq_B)$ and provide the techniques to extend the orders in the subsequent section.

Getting Started

Before we develop the theory of zig-zags, we define their starting sets. In Section 2.2, we described intuitively how elements that are not in the range of ε_1 or ε_2 start off zig-zags. E.g., each element of Q_A is the start of a zig-zag. Hence Q_A is the starting set of all zig-zags. Similarly, $Q_{\varepsilon_2[B]}$ starts off zag-zigs. Now, we are going to define these starting sets, i.e. Q_A and $Q_{\varepsilon_2[B]}$, formally and present some basic facts about them.

The set Q_A defined in Section 3.1 is the complement of the range of ε_2. We identify another subset in A

$$Q_{\varepsilon_2[B]} \equiv_{df} \varepsilon_2[B] \setminus \varepsilon[A] .$$

By definition $Q_{\varepsilon_2[B]} = \varepsilon_2[B] \setminus \varepsilon_2 \circ \varepsilon_1[A]$. Since ε_2 is injective, we can apply (3.10) and obtain

$$Q_{\varepsilon_2[B]} = \varepsilon_2[B \setminus \varepsilon_1[A]] .$$

That is, $Q_{\varepsilon_2[B]}$ is the image under ε_2 of the complement of $\varepsilon_1[A]$. It is the subset of A that is the starting set for the zag-zigs. Although zag-zigs are started in B, the elements of A that are 'contained' in zag-zigs are those that are in $Q_{\varepsilon_2[B]}$ or ε-applications and its iterates to elements of $Q_{\varepsilon_2[B]}$.

We observe that

Proposition 3.9 $(i) \quad \varepsilon[Q_A] \diamond Q_A$
$(ii) \quad \varepsilon[Q_{\varepsilon_2[B]}] \diamond Q_{\varepsilon_2[B]}$.

Proof: The reason for (i) is

$$\varepsilon[Q_A] = \varepsilon_2 \circ \varepsilon_1[Q_A] \subseteq \varepsilon_2[B] \diamond A \setminus \varepsilon_2[B] = Q_A ,$$

whereby with the disjoint-from law (3.15) we arrive at $\varepsilon[Q_A] \diamond Q_A$. Similarly, (ii) is justified by

$$\varepsilon[Q_{\varepsilon_2[B]}] \subseteq \varepsilon[A] \subseteq \varepsilon[A] \cup \overline{\varepsilon_2[B]} = \overline{Q_{\varepsilon_2[B]}} \diamond Q_{\varepsilon_2[B]} ,$$

and again by (3.15) we conclude $\varepsilon[Q_{\varepsilon_2[B]}] \diamond Q_{\varepsilon_2[B]}$. □

That is, application of ε to the sets Q_A and $Q_{\varepsilon_2[B]}$ produces disjoint sets. Iteration of this process continues to map to elements outside Q_A and $Q_{\varepsilon_2[B]}$, i.e. Proposition 3.9 can be extended for arbitrary $n \in \mathbb{N}$.

Corollary 3.10 $(i) \quad \varepsilon^{n+1}[Q_A] \diamond Q_A$
$(ii) \quad \varepsilon^{n+1}[Q_{\varepsilon_2[B]}] \diamond Q_{\varepsilon_2[B]}$

Furthermore, the disjointness of the image sets carries on when iterating the application of ε and it can be shown that the iterations are pairwise disjoint.

Proposition 3.11 $(i) \quad n \neq m \;\Rrightarrow\; \varepsilon^n[Q_A] \diamond \varepsilon^m[Q_A]$
$(ii) \quad n \neq m \;\Rrightarrow\; \varepsilon^n[Q_{\varepsilon_2[B]}] \diamond \varepsilon^m[Q_{\varepsilon_2[B]}]$

Proof: For (i):
Assume the contrary, i.e. for some $n \neq m$ let

$$\varepsilon^n[Q_A] \cap \varepsilon^m[Q_A] \neq \varnothing .$$

That is,

$$\exists\, x \in A.\; x \in \varepsilon^n[Q_A] \wedge x \in \varepsilon^m[Q_A] .$$

Let, without loss of generality, $n < m$ (otherwise just switch sides). Then, $x = \varepsilon^n.y$ for some $y \in Q_A$ and $x = \varepsilon^m.y'$ for some $y' \in Q_A$, by definition of image. Hence,

$$\begin{array}{lll} \varepsilon^n.y = \varepsilon^m.y' & \equiv & (n < m,\ \varepsilon \text{ injective}) \\ y = \varepsilon^{m-n}.y' & \Rrightarrow & (m{-}n > 0,\ y' \in A,\ \varepsilon^{m-n} : A \to A) \\ y \in \varepsilon[A] & \Rrightarrow & (\varepsilon[A] = \varepsilon_2 \circ \varepsilon_1[A] \subseteq \varepsilon_2[B]) \\ y \in \varepsilon_2[B] & & \end{array}$$

contradicting $y \in Q_A = A \setminus \varepsilon_2[B] \diamond \varepsilon_2[B]$. For (ii), a similar argument applies. We get $y \in \varepsilon[A]$ (corresponding to the penultimate step of the previous proof) for some $y \in Q_{\varepsilon_2[B]}$ contradicting $Q_{\varepsilon_2[B]} = \varepsilon_2[B] \setminus \varepsilon[A]$. □

What we are going to show in the subsequent section is that Q_A and $Q_{\varepsilon_2[B]}$ start 'zig-zags' and 'zag-zigs' respectively. Those will then be collected in T_A and $T_{\varepsilon_2[B]}$. Furthermore, in the following section, we will identify the remainder of $A_{\gtrless}$. This remainder, called T_A^∞, turns out to be the set that can be described as 'both side infinite zig-zags', which we call here *zigars*. Note, that zigars may be circular, i.e. we can have repetitions of elements when applying ε. However, this will not affect our considerations, because they are sufficiently general to cover those cases as well.

3.4 Definition and Properties of Zig-Zags

As described in [1], the proof of isomorphism must be at least that of the Cantor-Schröder-Bernstein theorem (CSB). The (wrong) conjecture (1.1) looks at first sight quite plausible, and one might be tempted to try to find an alternative proof of CSB that respects order in the construction of an isomorphism. Moschovakis [11, Chapter 4] gives a sketch of an alternative proof of CSB in his exercises. Since (1.1) is false, obviously the proof cannot work for isomorphism of Galois embedding. However, it inspired the following constructions.

Define *zig-zags* and *zag-zigs* as

$$\begin{array}{lll} T_A & \equiv_{df} & \bigcap\{X \mid Q_A \cup \varepsilon[X] \subseteq X\} \\ T_{\varepsilon_2[B]} & \equiv_{df} & \bigcap\{X \mid Q_{\varepsilon_2[B]} \cup \varepsilon[X] \subseteq X\}. \end{array}$$

These are the constructions used by Moschovakis. We observe that there is a correspondence with Tarski's fixpoint theorem [14]. We identify the functions

$$\begin{array}{lll} F_{Q_A} & \equiv_{df} & \lambda X.\ Q_A \cup \varepsilon[X] \\ F_{Q_{\varepsilon_2[B]}} & \equiv_{df} & \lambda X.\ Q_{\varepsilon_2[B]} \cup \varepsilon[X]. \end{array}$$

Since these functions are monotone on the complete lattice $(\mathbb{P}(A), \subseteq)$ we can apply Tarski's fixpoint theorem [3, Theorem 4.11] and obtain that T_A and $T_{\varepsilon_2[B]}$ are their least fixpoints[2]. That is, we have that[3]

$$T_A = Q_A \cup \varepsilon[T_A] \tag{3.41}$$

$$T_{\varepsilon_2[B]} = Q_{\varepsilon_2[B]} \cup \varepsilon[T_{\varepsilon_2[B]}]\,. \tag{3.42}$$

Tarski's theorem is an elegant way of achieving the closure properties (3.41) and (3.42) of zig-zags and zag-zigs, but there is a more constructive one. We observe that F_{Q_A} and $F_{Q_{\varepsilon_2[B]}}$ are even continuous.

Lemma 3.12 *The functions F_{Q_A} and $F_{Q_{\varepsilon_2[B]}}$ are continuous functions on the complete lattice $(\mathbb{P}(A), \subseteq)$.*

Proof: The least upper bound in the complete lattice is given by the union. We have that for any chain of subsets $(X_i)_{i\in I}$ in $(\mathbb{P}(A), \subseteq)$

$$\begin{array}{lll} \bigcup_{i\in I} F_{Q_A}[X_i] & = & (\text{definition of } F_{Q_A}) \\[2ex] \bigcup_{i\in I} Q_A \cup \varepsilon[X_i] & = & (\lambda X.\ Q_A \cup X \text{ idempotent}) \\[2ex] Q_A \cup \bigcup_{i\in I} \varepsilon[X_i] & = & (3.4) \\[2ex] Q_A \cup \varepsilon[\bigcup_{i\in I} X_i] & = & (\text{definition of } F_{Q_A}) \\[2ex] F_{Q_A}[\bigcup_{i\in I} X_i]\,. & & \end{array}$$

[2] Tarski's theorem used in the definitions of T_A and $T_{\varepsilon_2[B]}$ constructs the fixpoint as the greatest lower bound of the pre-fixpoints of F_{Q_A} and $F_{Q_{\varepsilon_2[B]}}$ — hence it is smallest.

[3] These properties are needed for Moschovakis proof of CSB.

That is, F_{Q_A} is continuous. A similar argument works for $F_{Q_{\varepsilon_2[B]}}$. □

Now, there is an alternative formulation of the structures T_A and $T_{\varepsilon_2[B]}$ (cf. Section 2.2) that is better suited for our constructive proofs. Since F_{Q_A} and $F_{Q_{\varepsilon_2[B]}}$ are not only monotone but continuous we can apply a stronger theorem from the theory of fixpoints [3, Theorem 4.5]. It enables the construction of the fixpoints T_A and $T_{\varepsilon_2[B]}$ by building the least upper bound over iterated applications of F_{Q_A} and $F_{Q_{\varepsilon_2[B]}}$ to the bottom element of the complete lattice. Since in $(\mathbb{P}(A), \subseteq)$ the least upper bound is given by the union and the bottom element by the empty set $\varnothing$ this stronger fixpoint theorem directly implies that

$$T_A = \bigcup_{0 \le n} F^n_{Q_A}[\varnothing] \tag{3.43}$$

$$T_{\varepsilon_2[B]} = \bigcup_{0 \le n} F^n_{Q_{\varepsilon_2[B]}}[\varnothing]\,. \tag{3.44}$$

Using these characterizations we can derive that T_A and $T_{\varepsilon_2[B]}$ are actually the zig-zags and zag-zigs as we intuitively described them in Section 2.2.

Lemma 3.13

$$\begin{aligned} T_A &= \bigcup_{0 \le n} \varepsilon^n[Q_A] \\ T_{\varepsilon_2[B]} &= \bigcup_{0 \le n} \varepsilon^n[Q_{\varepsilon_2[B]}] \end{aligned}$$

Proof: For T_A, we can perform the following algebraic transformations. First we derive by induction

$$\forall\ n \in \mathbb{N}.\ F^{n+1}_{Q_A}[\varnothing] = \bigcup_{0 \le i \le n} \varepsilon^i[Q_A]\,. \tag{3.45}$$

For the induction base we have

$$\begin{aligned}
& F^1_{Q_A}[\varnothing] & = \quad & (\text{definition of } F_{Q_A}) \\
& (\lambda X.\ Q_A \cup \varepsilon[X])[\varnothing] & = \quad & (\beta\text{-reduction}) \\
& Q_A \cup \varepsilon[\varnothing] & = \quad & (\forall\ f.\ f[\varnothing] = \varnothing,\ Q_A = \varepsilon^0[Q_A]) \\
& \varepsilon^0[Q_A] & = \quad &
\end{aligned}$$

$$\bigcup_{0\le i\le 0} \varepsilon^i[Q_A]\,.$$

For the step $n \to n{+}1$ consider

$$
\begin{aligned}
&F^{n+2}_{Q_A}[\varnothing] && = && \text{(definition of } F_{Q_A}\text{)}\\
&(\lambda X.Q_A \cup \varepsilon[X])[F^{n+1}_{Q_A}[\varnothing]] && = && (\beta\text{-reduction})\\
&Q_A \cup \varepsilon[F^{n+1}_{Q_A}[\varnothing]] && = && \text{(induction hypothesis)}\\
&Q_A \cup \varepsilon[\bigcup_{0\le i\le n} \varepsilon^i[Q_A]] && = && (3.4)\\
&Q_A \cup \bigcup_{0\le i\le n} \varepsilon^{i+1}[Q_A] && = && \text{(rewrite indices)}\\
&Q_A \cup \bigcup_{1\le i\le n+1} \varepsilon^{i}[Q_A] && = && (Q_A = \varepsilon^0[Q_A])\\
&\bigcup_{0\le i\le n+1} \varepsilon^{i}[Q_A]\,.
\end{aligned}
$$

Since $F^0_{Q_A}[\varnothing] = \varnothing$, we derive with some rewriting of indices

$$\bigcup_{0\le n} F^n_{Q_A}[\varnothing] = F^0_{Q_A}[\varnothing] \cup \bigcup_{1\le n} F^n_{Q_A}[\varnothing] = \bigcup_{0\le n} F^{n+1}_{Q_A}[\varnothing]\,. \tag{3.46}$$

Putting everything together, we have

$$
\begin{aligned}
&T_A && = && (3.43)\\
&\bigcup_{0\le n} F^n_{Q_A}[\varnothing] && = && (3.46)\\
&\bigcup_{0\le n} F^{n+1}_{Q_A}[\varnothing] && = && (3.45)
\end{aligned}
$$

$$\bigcup_{0\le n}\bigcup_{0\le i\le n}\varepsilon^i[Q_A] \quad = \quad \text{(definition of union)}$$

$$\bigcup_{0\le n}\varepsilon^n[Q_A]\,.$$

A similar argument applies to $T_{\varepsilon_2[B]}$. □

Now we have achieved two things. First, we have defined zig-zags as least fixpoints providing us with the closure properties (3.41) and (3.42). Second, we have proved in the previous lemma a constructive description of zig-zags. The latter constructive characterization immediately gives us

Corollary 3.14 *(i)* $\forall\, x \in T_A.\ \exists\, n \in \mathbb{N}.\quad x \in \varepsilon^n[Q_A]$
(ii) $\forall\, x \in T_{\varepsilon_2[B]}.\ \exists\, n \in \mathbb{N}.\quad x \in \varepsilon^n[Q_{\varepsilon_2[B]}]\,.$

In [1] the proof of (1.1) for the finite case contains as a main argument[4]:

Proposition 3.15 $\forall\, n \in \mathbb{N}.\ \varepsilon^{n+1}[Q_A] \diamond \bigcup_{0\le i\le n}\varepsilon^i[Q_A]\,.$

Proof: We apply disjoint-law (3.17) backwards to reduce the conjecture to

$$\forall_{0\le i\le n}.\ \varepsilon^{n+1}[Q_A] \diamond \varepsilon^i[Q_A]$$

which is contained in Proposition 3.11*(i)*. □

The corresponding result for $Q_{\varepsilon_2[B]}$ immediately follows with Proposition 3.11*(ii)*.

Corollary 3.16 $\forall n \in \mathbb{N}.\ \varepsilon^{n+1}[Q_{\varepsilon_2[B]}] \diamond \bigcup_{0\le i\le n}\varepsilon^i[Q_{\varepsilon_2[B]}]\,.$

As already presented in [1], conjecture 1.1 holds if restricted to finite partial orders.

Proposition 3.17 *If A is finite, then* $\varepsilon_2 : (B, \le_B) \cong (A, \le_A)\,.$

[4]The proof differs from [1] since we can use more general results.

Proof: Assume for contradiction that $Q_A \neq \varnothing$. Then, we can apply Proposition (3.15) and deduce that the sequence $(\bigcup_{0 \le i \le n} .\varepsilon^i[Q_A])_{n \in \mathbb{N}}$ is an infinite strictly-increasing sequence of subsets of A, contradicting A finite.

Hence, $Q_A = A \setminus \varepsilon_2[B] = \varnothing$ and thereby ε_2 is surjective. Since ε_2 is isotone by assumption, it is an isomorphism. □

We can strengthen Corollary 3.14 to unique existence.

Proposition 3.18 (*i*) $\forall\ x \in T_A.\ \exists^1\ n \in \mathbb{N}.\ x \in \varepsilon^n[Q_A]$
(*ii*) $\forall\ x \in T_{\varepsilon_2[B]}.\ \exists^1\ n \in \mathbb{N}.\ x \in \varepsilon^n[Q_{\varepsilon_2[B]}]$

Proof: (For (*i*), (*ii*) similar)

Let $x \in T_A$. From Corollary 3.14 we get that there is an n such that $x \in \varepsilon^n[Q_A]$. Since, by Proposition 3.11(*i*), $\varepsilon^n[Q_A] \diamond \varepsilon^m[Q_A]$, for $n \neq m$, $\varepsilon^n[Q_A]$ is the only one containing x . □

After the construction of T_A and $T_{\varepsilon_2[B]}$ it is very important to note that they are subsets of $A_{\gtrless}$.

Lemma 3.19 (*i*) $T_A \subseteq A_{\gtrless}$
(*ii*) $T_{\varepsilon_2[B]} \subseteq A_{\gtrless}$

Proof:

(*i*). Because of Proposition 3.6 and (3.40) it follows

$$A_{\gtrless} = \{x \in A \mid \varepsilon.x \neq x\} . \tag{3.47}$$

So, it suffices to show for $x \in T_A$ that $\varepsilon.x \neq x$, then $x \in A_{\gtrless}$.

Now, $x \in T_A$ implies by Proposition 3.18(*i*) that $x \in \varepsilon^n[Q_A]$ for a unique $n \in \mathbb{N}$. Hence, $\varepsilon.x \in \varepsilon^{n+1}[Q_A]$. Since by Proposition 3.11(*i*)

$$\varepsilon^{n+1}[Q_A] \diamond \varepsilon^n[Q_A]$$

we have by disjoint-from law (3.19) $\varepsilon.x \neq x$.

(*ii*). With Proposition 3.11 (*ii*) the proof is similar. □

Disjointness of Zig-Zags and Zag-Zigs

For further arguments, we need that the set of zig-zags and zag-zigs are disjoint, i.e. $T_A \diamond T_{\varepsilon_2[B]}$. We will develop this result step by step, starting with the disjointness of the complements of the ranges.

First we observe that

Lemma 3.20 $\quad Q_A \diamond Q_{\varepsilon_2[B]}$.

Proof:

$$\begin{pmatrix} \varepsilon_2[B] \diamond A \setminus \varepsilon_2[B] \\ \varepsilon_2[B] \setminus \varepsilon[A] \subseteq \varepsilon_2[B] \end{pmatrix} \qquad \Rrightarrow \quad \text{(disjoint-from law (3.15))}$$

$$\varepsilon_2[B] \setminus \varepsilon[A] \diamond A \setminus \varepsilon_2[B] \qquad \equiv \quad \text{(definition of } Q_A \text{ and } Q_{\varepsilon_2[B]}\text{)}$$

$$Q_{\varepsilon_2[B]} \diamond Q_A$$

□

Similar to Corollary 3.10 we can derive

Lemma 3.21 $\quad (i) \quad \forall\, n \in \mathbb{N}.\ \varepsilon^{n+1}[Q_A] \diamond Q_{\varepsilon_2[B]}$
$(ii) \quad \forall\, n \in \mathbb{N}.\ \varepsilon^{n+1}[Q_{\varepsilon_2[B]}] \diamond Q_A$

Proof: Since, for (i), $Q_A \subseteq A$ we have $\varepsilon^n[Q_A] \subseteq A$ and thereby

$$\varepsilon^{n+1}[Q_A] \subseteq \varepsilon[A].$$

However,

$$Q_{\varepsilon_2[B]} = \varepsilon_2[B] \setminus \varepsilon[A]$$

so by disjoint-from laws (3.14) and (3.15) we get $\varepsilon^{n+1}[Q_A] \diamond Q_{\varepsilon_2[B]}$.

For (ii), since $Q_{\varepsilon_2[B]} \subseteq A$ we obtain $\varepsilon_1[\varepsilon^n[Q_{\varepsilon_2[B]}]] \subseteq B$ and consequently

$$\varepsilon^{n+1}[Q_{\varepsilon_2[B]}] = \varepsilon_2[\varepsilon_1[\varepsilon^n[Q_{\varepsilon_2[B]}]]] \subseteq \varepsilon_2[B].$$

Since, by definition,

$$Q_A = A \setminus \varepsilon_2[B]$$

laws (3.14) and (3.15) again apply and we reach $\varepsilon^{n+1}[Q_{\varepsilon_2[B]}] \diamond Q_A$. □

The previous lemma extends to

Proposition 3.22 $\forall\, n,m \in \mathbb{N}.\ \varepsilon^n[Q_A] \diamond \varepsilon^m[Q_{\varepsilon_2[B]}]\,.$

Proof: A first application of natural number induction reduces to the induction base and step

$$\begin{array}{c} \forall\, m \in \mathbb{N}.\ Q_A \diamond \varepsilon^m[Q_{\varepsilon_2[B]}] \\ \forall\, m \in \mathbb{N}.\ \varepsilon^n[Q_A] \diamond \varepsilon^m[Q_{\varepsilon_2[B]}] \ \Rrightarrow\ \forall\, m \in \mathbb{N}.\ \varepsilon^{n+1}[Q_A] \diamond \varepsilon^m[Q_{\varepsilon_2[B]}]\,. \end{array}$$

The base is solved by Lemma 3.20 for $m = 0$ and by Lemma 3.21(ii) for $m \geq 1$. To solve the step we apply again natural number induction to the step conclusion — this time over m. This leaves the premise

$$\forall\, m \in \mathbb{N}.\ \varepsilon^n[Q_A] \diamond \varepsilon^m[Q_{\varepsilon_2[B]}] \tag{3.48}$$

untouched and reduces to the base and step case

$$\begin{array}{c} \varepsilon^{n+1}[Q_A] \diamond Q_{\varepsilon_2[B]} \\ \varepsilon^{n+1}[Q_A] \diamond \varepsilon^{m'}[Q_{\varepsilon_2[B]}] \ \Rrightarrow\ \varepsilon^{n+1}[Q_A] \diamond \varepsilon^{m'+1}[Q_{\varepsilon_2[B]}]\,. \end{array}$$

The base case is now solved by Lemma 3.21(i). For the step case we discard its premise and reduce the conclusion by disjoint-from law (3.18) to

$$\varepsilon^n[Q_A] \diamond \varepsilon^{m'}[Q_{\varepsilon_2[B]}]\,.$$

However, this remaining subgoal is an instance of the first induction hypothesis (3.48). □

These preparations enable the proof of

Lemma 3.23 $T_A \diamond T_{\varepsilon_2[B]}\,.$

Proof: We merely apply law (3.17) twice and the unions become disjoint as well. Starting from Proposition 3.22,

$$\begin{array}{ll} \forall\, n,m \in \mathbb{N}.\ \varepsilon^n[Q_A] \diamond \varepsilon^m[Q_{\varepsilon_2[B]}] \Rrightarrow & (3.17) \\ \forall\, n \in \mathbb{N}.\ \varepsilon^n[Q_A] \diamond \bigcup_{0 \leq m} \varepsilon^m[Q_{\varepsilon_2[B]}] \Rrightarrow & (3.17) \\ \bigcup_{0 \leq n} \varepsilon^n[Q_A] \diamond \bigcup_{0 \leq m} \varepsilon^m[Q_{\varepsilon_2[B]}] \quad \equiv & \text{(Lemma 3.13)} \\ T_A \diamond T_{\varepsilon_2[B]}\,. & \end{array}$$

□

Definition and Properties of Zigars

Define both side infinite zig-zags, so-called *zigars*, as

$$T_A^\infty \quad \equiv_{df} \quad A_{\gtrless} \setminus T_A \setminus T_{\varepsilon_2[B]} \,.$$

By construction

$$T_A^\infty \subseteq A_{\gtrless} \tag{3.49}$$

$$T_{\varepsilon_2[B]} \diamond T_A^\infty \diamond T_A \,. \tag{3.50}$$

Furthermore, with Lemma 3.19 we immediately get

$$A_{\gtrless} = T_A^\infty \cup T_A \cup T_{\varepsilon_2[B]} \,. \tag{3.51}$$

The subset properties from Lemma 3.19 and (3.49) also imply with (3.40)

$$T_A \diamond A_{ord} \tag{3.52}$$

$$T_{\varepsilon_2[B]} \diamond A_{ord} \tag{3.53}$$

$$T_A^\infty \diamond A_{ord} \,. \tag{3.54}$$

We already know by construction, that $T_A \diamond T_A^\infty$ and $T_A^\infty \diamond T_{\varepsilon_2[B]}$ (see 3.50). With Lemma 3.23 we get that all three are mutually disjoint.

The set T_A^∞ is actually the set of what we described intuitively as 'both side infinite zig-zags'.

Lemma 3.24 $\quad T_A^\infty \quad = \quad T_A^\infty \cup \varepsilon[T_A^\infty] \cup \varepsilon^{-1}[T_A^\infty]$

As this property is not needed for the following derivations and since its proof is a rather technical case analysis we do not prove it here. The interested reader may find the proof in Appendix A.1. Zigars are not present in the various examples presented in this work. However, one may easily extend any of those examples consistently by zigars by adding an infinite number of elements to A and B connecting those elements accordingly with ε_1 and ε_2. Alternatively, an extension by finite sets of elements is possible. In the latter case ε_1 and ε_2 have to be constructed circularly, i.e. iterated interleaving application of these isotone function must build a repeating enumeration of the elements of a zigar. Although zigars seem to be pathological cases we do have to consider them for the general case.

After we have defined zig-zags, zag-zigs, and zigars and clarified their basic features, we are going to use them to show that they are isomorphic to the zig-zags in B.

3.5 Partial Isomorphisms of Zig-Zags

The partially-ordered set B has the following corresponding substructures.

$$\begin{array}{lll} Q_{\varepsilon_1[A]} & \equiv_{df} & \varepsilon_1[A] \setminus \hat{\varepsilon}[B] \\ T_{\varepsilon_1[A]} & \equiv_{df} & \bigcap\{X \mid Q_{\varepsilon_1[A]} \cup \hat{\varepsilon}[X] \subseteq X\} \\ T_B & \equiv_{df} & \bigcap\{X \mid Q_B \cup \hat{\varepsilon}[X] \subseteq X\} \\ T_B^\infty & \equiv_{df} & B_{\gtrless} \setminus T_B \setminus T_{\varepsilon_1[A]} \end{array}$$

The equivalences, closure, and disjointness properties we established for A in the current section obviously hold for their counterparts in B. In particular we have

$$B_{\gtrless} = T_B^\infty \cup T_B \cup T_{\varepsilon_1[A]} \,. \tag{3.55}$$

These preparations enable the proof of partial isomorphisms between the substructures of $A_{\gtrless}$ and their counterparts in $B_{\gtrless}$. The partial isomorphisms are

$$\begin{array}{lll} T_A & \cong & \varepsilon_1[T_A] = T_{\varepsilon_1[A]} \\ T_{\varepsilon_2[B]} & \cong & \varepsilon_2^{-1}[T_{\varepsilon_2[B]}] = T_B \\ T_A^\infty & \cong & \varepsilon_1[T_A^\infty] = T_B^\infty \,. \end{array}$$

Recall (cf. Section 2.2), that we can use ε_1 as isomorphism for all subsets (including A_{ord}, see Proposition 3.6) except $T_{\varepsilon_2[B]}$. This is because the 'starting points' of $T_{\varepsilon_2[B]}$ in B are the elements of Q_B, the complement of the range of ε_1, hence cannot be reached by it.

We are going to prove these equivalences between zig-zags in A and B now.

Proposition 3.25 $\qquad T_A \cong \varepsilon_1[T_A] = T_{\varepsilon_1[A]}$.

Proof: Clearly, with Lemma 3.1, $\varepsilon_1[T_A] \cong T_A$.

For Q_A, we have (using (3.10) to distribute ε_1, and the definition of $Q_{\varepsilon_1[A]}$ on page 36)

$$\varepsilon_1[Q_A] = \varepsilon_1[A \setminus \varepsilon_2[B]] = \varepsilon_1[A] \setminus \varepsilon_1 \circ \varepsilon_2[B] = Q_{\varepsilon_1[A]} \,.$$

Similarly for T_A,

$$T_{\varepsilon_1[A]} \quad = \quad \text{(correspondence of Lemma 3.13)}$$

$$\bigcup_{0\le n} (\varepsilon_1 \circ \varepsilon_2)^n [Q_{\varepsilon_1[A]}] \quad = \quad \text{(previous argument)}$$

$$\bigcup_{0\le n} \varepsilon_1[(\varepsilon_2 \circ \varepsilon_1)^n [Q_A]] \quad = \quad \text{((3.4), definition of } \varepsilon\text{)}$$

$$\varepsilon_1[\bigcup_{0\le n} \varepsilon^n [Q_A]] \quad = \quad \text{(Lemma 3.13)}$$

$$\varepsilon_1[T_A]\,. \qquad \square$$

In order to prove the corresponding propositions for $T_{\varepsilon_2[B]}$ and T_A^∞, we need two lemmata.

Lemma 3.26 $\quad T_{\varepsilon_2[B]} \subseteq \varepsilon_2[B]$.

Proof: By definition $Q_{\varepsilon_2[B]}$ is a subset of $\varepsilon_2[B]$:

$$Q_{\varepsilon_2[B]} = \varepsilon_2[B] \setminus \varepsilon[A] \subseteq \varepsilon_2[B]\,.$$

Since furthermore by (3.42) we have that

$$T_{\varepsilon_2[B]} = Q_{\varepsilon_2[B]} \cup \varepsilon[T_{\varepsilon_2[B]}]$$

and $\varepsilon[T_{\varepsilon_2[B]}] = \varepsilon_2[\varepsilon_1[T_{\varepsilon_2[B]}]]$ is obviously a subset of $\varepsilon_2[B]$ as well, we attain $T_{\varepsilon_2[B]} \subseteq \varepsilon_2[B]$ because it is the union of two subsets of $\varepsilon_2[B]$. $\square$

Lemma 3.27 $\quad (\varepsilon_2^{-1}[T_{\varepsilon_2[B]}] = T_B) \equiv (T_{\varepsilon_2[B]} = \varepsilon_2[T_B])$

Proof:

$$\varepsilon_2^{-1}[T_{\varepsilon_2[B]}] = T_B \quad \equiv \quad \text{(}\varepsilon_2\text{ injective, (3.11))}$$

$$\varepsilon_2[\varepsilon_2^{-1}[T_{\varepsilon_2[B]}]] = \varepsilon_2[T_B] \quad \equiv \quad \text{(Lemma 3.26, (3.9))}$$

$$T_{\varepsilon_2[B]} = \varepsilon_2[T_B]\,. \qquad \square$$

Proposition 3.28 $T_{\varepsilon_2[B]} \cong \varepsilon_2^{-1}[T_{\varepsilon_2[B]}] = T_B$.

Proof: With ε_2, ε_2^{-1} is isotone, so as before, with Lemma 3.1, $T_{\varepsilon_2[B]} \cong \varepsilon_2^{-1}[T_{\varepsilon_2[B]}]$.

Similar to the proof of Proposition 3.25, we have for Q_B (this time using (3.10) to distribute ε_2, and the definitions of Q_B and $Q_{\varepsilon_2[B]}$ on page 21 and 25)

$$\varepsilon_2[Q_B] = \varepsilon_2[B \setminus \varepsilon_1[A]] = \varepsilon_2[B] \setminus \varepsilon[A] = Q_{\varepsilon_2[B]} .$$

Using Lemma 3.27 we prove the equivalent $T_{\varepsilon_2[B]} = \varepsilon_2[T_B]$ here.

$$\begin{aligned}
& T_{\varepsilon_2[B]} && = \quad \text{(Lemma 3.13)} \\
& \bigcup_{0 \le n} \varepsilon^n[Q_{\varepsilon_2[B]}] && = \quad \text{(previous argument, def. } \varepsilon) \\
& \bigcup_{0 \le n} (\varepsilon_2 \circ \varepsilon_1)^n[\varepsilon_2[Q_B]] && = \quad \text{(rearrange)} \\
& \bigcup_{0 \le n} \varepsilon_2[(\varepsilon_1 \circ \varepsilon_2)^n[Q_B]] && = \quad \text{((3.4), definition of } \hat{\varepsilon}) \\
& \varepsilon_2[\bigcup_{0 \le n} \hat{\varepsilon}^n[Q_B]] && = \quad \text{(correspondence of Lemma 3.13)} \\
& \varepsilon_2[T_B] && \qquad \square
\end{aligned}$$

For the third partial isomorphism, we need two more lemmata.

Lemma 3.29 $\varepsilon_1[T_{\varepsilon_2[B]}] = T_B \setminus Q_B$

Proof: By the Tarski property for T_B (corresponding to (3.41)) we get

$$T_B = Q_B \cup \hat{\varepsilon}[T_B] .$$

Starting from that, expanding the definition of $\hat{\varepsilon}$, we derive

$$\varepsilon_1[\varepsilon_2[T_B]] \cup Q_B = T_B \quad \Rrightarrow \quad ((3.22),\ Q_B \diamond \varepsilon_1[A],\ (3.15))$$

$$\varepsilon_1[\varepsilon_2[T_B]] = T_B \setminus Q_B \quad \Rrightarrow \quad \text{(Lemma 3.27, Proposition 3.28)}$$

$$\varepsilon_1[T_{\varepsilon_2[B]}] = T_B \setminus Q_B\,. \qquad \square$$

Lemma 3.30 $\varepsilon_1[A_{\gtrless}] = B_{\gtrless} \setminus Q_B$

Proof: By definition and disjoint-from law (3.14), we have $Q_B \diamond \varepsilon_1[A]$. Since $\varepsilon_1[A_{\gtrless}] \subseteq \varepsilon_1[A]$, we have with disjoint-from law (3.15) that $Q_B \diamond \varepsilon_1[A_{\gtrless}]$. So, using disjoint-from law (3.22) we can reduce the goal to

$$\varepsilon_1[A_{\gtrless}] \cup Q_B = B_{\gtrless}\,.$$

First, we observe

$$Q_B \cup \varepsilon_1[A_{\gtrless}] \cup \varepsilon_1[A_{ord}] = B \tag{3.56}$$

because

$$Q_B \cup \varepsilon_1[A_{\gtrless}] \cup \varepsilon_1[A_{ord}] \quad = \quad (3.4)$$

$$Q_B \cup \varepsilon_1[A_{\gtrless} \cup A_{ord}] \quad = \quad (3.39)$$

$$Q_B \cup \varepsilon_1[A] \quad = \quad ((3.27),\ (3.5),\ \varepsilon_1[A] \subseteq B)$$

$$B\,.$$

As before starting from $Q_B \diamond \varepsilon_1[A]$ we derive $\varepsilon_1[A_{\gtrless}] \cup Q_B = B_{\gtrless}$.

$$Q_B \diamond \varepsilon_1[A] \quad \Rrightarrow \quad ((3.39),(3.4))$$

$$Q_B \diamond \varepsilon_1[A_{ord}] \cup \varepsilon_1[A_{\gtrless}] \quad \Rrightarrow \quad ((3.17),(3.40),(3.14),(3.18))$$

$$\begin{pmatrix} Q_B \diamond \varepsilon_1[A_{ord}] \\ \varepsilon_1[A_{\gtrless}] \diamond \varepsilon_1[A_{ord}] \end{pmatrix} \quad \Rrightarrow \quad \text{(disjoint-from law (3.17))}$$

$$\varepsilon_1[A_{\gtrless}] \cup Q_B \diamond \varepsilon_1[A_{ord}] \qquad \Rrightarrow \quad \text{(disjoint-from law (3.22), (3.56))}$$

$$\varepsilon_1[A_{\gtrless}] \cup Q_B = \overline{\varepsilon_1[A_{ord}]} \qquad \Rrightarrow \quad \text{(Prop. 3.8, corr. of (3.40))}$$

$$\varepsilon_1[A_{\gtrless}] \cup Q_B = \overline{B_{ord}} = B_{\gtrless} \,. \qquad \square$$

Proposition 3.31 $\quad T_A^\infty \cong \varepsilon_1[T_A^\infty] = T_B^\infty$.

Proof: Since ε_1 is isotone we get with Lemma 3.1, $T_A^\infty \cong \varepsilon_1[T_A^\infty]$.
The previously derived lemmata prove $\varepsilon_1[T_A^\infty] = T_B^\infty$, because

$$\begin{aligned}
&\varepsilon_1[T_A^\infty] && = \quad \text{(definition of } T_A^\infty\text{)} \\
&\varepsilon_1[A_{\gtrless} \setminus T_A \setminus T_{\varepsilon_2[B]}] && = \quad (\varepsilon_1 \text{ injective, (3.10))} \\
&\varepsilon_1[A_{\gtrless}] \setminus \varepsilon_1[T_A] \setminus \varepsilon_1[T_{\varepsilon_2[B]}] && = \quad \text{(Lemmata 3.29, 3.30, Prop. 3.25)} \\
&B_{\gtrless} \setminus Q_B \setminus T_{\varepsilon_1[A]} \setminus (T_B \setminus Q_B) && = \quad \text{(Lemma 3.2)} \\
&B_{\gtrless} \setminus Q_B \setminus (T_B \setminus Q_B) \setminus T_{\varepsilon_1[A]} && = \quad \text{((3.3), } Q_B \subseteq T_B\text{)} \\
&B_{\gtrless} \setminus T_B \setminus T_{\varepsilon_1[A]} && = \quad \text{(definition of } T_B^\infty\text{, page 36)} \\
&T_B^\infty \,. && \qquad \square
\end{aligned}$$

We have identified the sets of substructures intuitively described as zig-zags and zag-zigs in $A_{\gtrless}$. Moreover we have seen that these sets are individually isomorphic to the corresponding sets in B . However, as we have seen in the counterexample, this still does not suffice to ensure that A and B are isomorphic. Nevertheless, these insights will help us to define a bijection based on the partial isomorphisms. We then show that this bijection is an isomorphism on A and B with respect to an extended order. The critical bit is the mapping of $T_{\varepsilon_2[B]}$. It is necessary to map it with ε_2^{-1} to obtain a bijection, but this violates the isotonicity of the combined mapping.

4 Order Extension Theorem

In this section we show that if the ascending chain condition (ACC) and the descending chain condition (DCC) hold, the partial orders are consistent. More formally, we are going to prove the following theorem:

$$\begin{pmatrix} \varepsilon_1 : A \to B,\ \varepsilon_1 \text{ isotone} \\ \varepsilon_2 : B \to A,\ \varepsilon_2 \text{ isotone} \\ ACC \wedge DCC \end{pmatrix} \Rrightarrow \begin{pmatrix} \exists \quad \leq^*_A,\ \leq^*_B . \\ (\leq_A) \subseteq (\leq^*_A) \ \wedge\ (\leq_B) \subseteq (\leq^*_B) \\ (A, \leq^*_A) \cong (B, \leq^*_B) \end{pmatrix} .$$

To that end, we first define a function ρ based on ε_1 and ε_2 and prove that ρ is a bijection. It is partly isotone, i.e. on each of the subsets A_{ord}, $T_A \cup T_A^\infty$, and $T_{\varepsilon_2[B]}$ individually it is an isomorphism. However, there may be elements that are in relation $\leq_A$ or $\leq_B$ that are in different of these subsets. We are going to consider in detail the various different cases, exhibiting the critical ones that make an extension necessary. Before we define the extensions $\leq^*_A$ and $\leq^*_B$ of the order relations $\leq_A$ and $\leq_B$, we have to develop a certain amount of theory concerning finite cycles and transitivity. Finally, we will show that these extensions are partial orders and that the bijection ρ is an isomorphism from $(A, \leq^*_A)$ to $(B, \leq^*_B)$.

Bijection ρ

After the preparations of the previous section, we are able to define a function ρ which proves to be a bijection.

$$\rho.x \equiv_{df} \begin{cases} \varepsilon_1.x & \text{if} \quad x \in A_{ord} \cup T_A \cup T_A^\infty, \\ \varepsilon_2^{-1}.x & \text{if} \quad x \in T_{\varepsilon_2[B]}. \end{cases}$$

Now, we have

$$\begin{aligned}
&\rho[A] && = \quad (3.39) \\
&\rho[A_{ord} \cup A_{\gtrless}] && = \quad (3.51) \\
&\rho[A_{ord} \cup T_{\varepsilon_2[B]} \cup T_A \cup T_A^\infty] && = \quad (3.4) \\
&\rho[A_{ord}] \cup \rho[T_{\varepsilon_2[B]}] \cup \\
&\rho[T_A] \cup \rho[T_A^\infty] && = \quad (\text{definition of } \rho)
\end{aligned}$$

$$\begin{aligned}
&\varepsilon_1[A_{ord}] \cup \varepsilon_2^{-1}[T_{\varepsilon_2[B]}] \cup \\
&\varepsilon_1[T_A] \cup \varepsilon_1[T_A^\infty] && = \quad \text{(Propositions 3.8, 3.28, 3.25, 3.31)} \\
&B_{ord} \cup T_B \cup T_{\varepsilon_1[A]} \cup T_B^\infty && = \quad \text{(3.55)} \\
&B_{ord} \cup B_{\gtrless} && = \quad \text{(correspondence of (3.39))} \\
&B
\end{aligned}$$

whereby ρ is a surjection.

Furthermore, since the sets A_{ord}, T_A, $T_{\varepsilon_2[B]}$, and T_A^∞ are pairwise disjoint ((3.50), (3.52)–(3.54), and Lemma 3.23), and ε_1 as well as ε_2^{-1} are injective, on each of the substructures individually, we can apply disjoint-law (3.21) repeatedly and obtain that the composite function ρ is injective. Hence, ρ is a bijection.

To investigate ρ with respect to conflicts with the isomorphism property, we consider the relationship between $T_{\varepsilon_2[B]}$ and each of A_{ord}, T_A, and T_A^∞. This suffices because on the latter three substructures ρ corresponds to ε_1 which is isotone by assumption. Hence, ρ is a partial isomorphism on $A \setminus T_{\varepsilon_2[B]}$, i.e. $\rho : A \setminus T_{\varepsilon_2[B]} \cong B \setminus T_B$.

However, we will exhibit two cases in which the isotonicity property will be violated — this will then be where we extend the order.

Case A_{ord} vs. $T_{\varepsilon_2[B]}$

For pairs of elements $(x, y) \in ((A_{ord} \times T_{\varepsilon_2[B]} \cup T_{\varepsilon_2[B]} \times A_{ord}) \cap (\leq_A))$ we can use the strong property of A_{ord} expressed in Proposition 3.6 to show that there is no conflict there, i.e. for (x, y) the bijection ρ is isotone. Let $y \in A_{ord}$, $x \in T_{\varepsilon_2[B]}$, then[5]

$$\begin{aligned}
&x \leq_A y && \equiv \quad (\varepsilon_2 \text{ injective, Proposition 3.6}) \\
&(\varepsilon_2 \circ \varepsilon_2^{-1}).x \leq_A (\varepsilon_2 \circ \varepsilon_1).y && \equiv \quad (\varepsilon_2 \text{ isotone})
\end{aligned}$$

[5]We only show the case $x \leq_A y$, the other direction is similar.

$$\varepsilon_2^{-1}.x \leq_B \varepsilon_1.y \quad \equiv \quad \text{(definition of } \rho\text{)}$$

$$\rho.x \leq_B \rho.y\,.$$

This argument shows that the isotonicity of ρ between $T_{\varepsilon_2[B]}$ and A_{ord} is ensured because of the defining property of A_{ord} .

Case $T_A \cup T_A^{\infty}$ vs $T_{\varepsilon_2[B]}$

We deal with T_A and T_A^{∞} simultaneously because they are mapped by ρ in the same way, i.e by ε_1 . However, in contrast to A_{ord} we get conflicts with isotonicity. For $x \in T_{\varepsilon_2[B]}$ and $y \in T_A \cup T_A^{\infty}$ we can rule out the case contradicting isotonicity of ρ, i.e. we have

$$x <_A y \Rrightarrow \neg(\rho.x \geq_B \rho.y) \tag{4.1}$$

because

$$\rho.x \geq \rho.y \quad \equiv \quad (x \in T_{\varepsilon_2[B]},\, y \in T_A \cup T_A^{\infty},\ \text{def. } \rho)$$

$$\varepsilon_2^{-1}.x \geq_B \varepsilon_1.y \quad \equiv \quad (\varepsilon_2 \text{ isotone})$$

$$(\varepsilon_2 \circ \varepsilon_2^{-1}).x \geq_A (\varepsilon_2 \circ \varepsilon_1).y \quad \equiv \quad (\varepsilon_2 \text{ injective, definition of } \varepsilon)$$

$$x \geq_A \varepsilon.y \quad \Rrightarrow \quad (x <_A y,\ \text{transitivity})$$

$$y >_A \varepsilon.y \quad \equiv \quad \text{(Corollary 3.5)}$$

$$\neg DCC \quad \text{contradicting assumptions}\,.$$

Similarly, we get $x >_A y \Rrightarrow \neg(\rho.x \leq_B \rho.y)$: with the dual argument we arrive with Proposition 3.4 at $\neg ACC$, contradicting the global assumptions.

We can exclude the case $x = y$, because by assumption $x \in T_{\varepsilon_2[B]}$ and $y \in T_A \cup T_A^{\infty}$ and these are disjoint ((3.50) and Lemma 3.23), so by law (3.19) of the disjoint-from calculus $x \neq y$. Similarly, we can exclude $\rho.x = \rho.y$

because they are in T_B and $T_{\varepsilon_1[A]} \cup T_B^\infty$ and these are again disjoint. Hence, the remaining cases are

$$\begin{pmatrix} x <_A y \\ \rho.x <_B \rho.y \end{pmatrix} \vee \begin{pmatrix} x >_A y \\ \rho.x >_B \rho.y \end{pmatrix} \tag{4.2}$$

$$\begin{pmatrix} x \gtrless y \\ \rho.x <_B \rho.y \vee \rho.x >_B \rho.y \end{pmatrix} \tag{4.3}$$

$$\begin{pmatrix} x <_A y \vee x >_A y \\ \rho.x \gtrless \rho.y \end{pmatrix} \tag{4.4}$$

Now, case (4.2) conforms to isomorphism. Cases (4.3) and (4.4) are possible and are responsible for $(A, \leq_A)$ and $(B, \leq_B)$ not generally being isomorphic. These facts are used in the construction of the counterexample in Section 2, e.g. $(1,1) \in T_{\varepsilon_1[A]}$, $(1,2) \in T_B$, and $(1,1) \leq_B (1,2)$, whereas $(1,1) \in T_A$, $(0,2) \in T_{\varepsilon_2[B]}$, and $(1,1) \gtrless_A (0,2)$. However, we use the two cases here as the basis for the extension of the orders. In principle, we add (x, y) to $\leq_A$ whenever $(\rho.x, \rho.y)$ has been in $\leq_B$ and vice versa. The previous case analysis enables to restrict the extension to edges between $T_A \cup T_A^\infty$ and $T_{\varepsilon_2[B]}$ (or $T_{\varepsilon_1[A]} \cup T_B^\infty$ and T_B, respectively). However, since the resulting structures are again partial orders, we need to build the transitive closure over the extended relations[6]. We introduce some more theory, to be able to properly define and prove the desired properties.

Graph of a Relation and Finite Cycles

In the following, we consider transitivity and antisymmetry of relations. To that end, we talk about the *graph of a relation*. For any relational structure $(X, \sim)$ the graph of the relation $\sim$ is the directed graph $G(\sim)$ with the set X of nodes and for arbitrary $x, y \in X$ the graph contains an edge (x, y) with source x and target y iff $x \sim y$. A *path* of length n in $\sim$ is a sequence of edges $e_0, \ldots, e_n$ in $G(\sim)$ such that $\mathrm{target}(e_i) = \mathrm{source}(e_{i+1})$, $i \in 0, \ldots, n-1$. A *finite cycle* of length n in $\sim$ is a path such that $\mathrm{target}(e_n) = \mathrm{source}(e_1)$ for $n \geq 2$, i.e. cycles are at least of length 2. Circular permutations of cycles are considered equal, e.g.

$$(x_1, x_2)(x_2, x_3)(x_3, x_1) = (x_3, x_1)(x_1, x_2)(x_2, x_3)\,.$$

[6] For an example that shows that the basic extension is not generally transitive see Section 5.2.

For paths p and edges e, we write $e \in_e p$ meaning that edge e is part of path p. Similarly, we write $x \in_n p$ if $x \in X$ and there is an $e \in_e p$ such that $\text{source}(e) = x$ or $\text{target}(e) = x$. For a path p, let

$$\begin{array}{lll} \text{Nodes}(p) & \equiv_{df} & \{x \mid x \in_n p\} \\ \text{Edges}(p) & \equiv_{df} & \{e \mid e \in_e p\}\,. \end{array}$$

Finally, we define an order $\leq_\circ$ on finite cycles c_1, c_2 of a relation $\sim$ as

$$c_1 \leq_\circ c_2 \equiv_{df} \text{Nodes}(c_1) \subseteq \text{Nodes}(c_2)\,. \tag{4.5}$$

The relation $\leq_\circ$ is a partial order on finite cycles. This is clear, as it is defined by the subset relation. Furthermore, the relation is *well founded*: every nonempty subset has a minimum [4].

This is less trivial since the subset relation on subsets of a set is not generally well founded[7]. However, the subset relation on *finite* subsets of a set is well-founded because every nonempty set of finite subsets has a minimal element. Similarly, the relation $\leq_\circ$, defined by finite subsets, is well founded.

That is, any nonempty set of finite cycles of a graph has a minimal element. In particular, the set C_G of *all* finite cycles of a graph G has a minimal element.

This immediately implies the following proposition.

Corollary 4.1 *If the set C_G of finite cycles of a graph G contains no minimal cycles, then G contains no finite cycles.*

Transitive Closure and Antisymmetry

The transitive closure $\sim^*$ of a relation $\sim \subseteq X \times X$ is (cf. [3])

$$a \sim^* b \equiv_{df} \exists\ n \in \mathbb{N}.\ \exists\ z_0, \ldots, z_n \in X.\ a = z_0 \sim z_1 \ldots z_{n-1} \sim z_n = b\,. \tag{4.6}$$

Clearly, the transitive closure of a relation is transitive. We observe that transitivity is related to finite cycles.

Proposition 4.2 *Let $\sim$ be a relation on a set X. If $\sim$ has no finite cycles, its transitive closure $\sim^*$ is antisymmetric.*

[7]Consider, the infinitely decreasing chain of subsets $([n, \infty))_{n \in \mathbb{N}}$ in $\mathbb{N}$ for example.

Proof: We prove that if $\sim^*$ is *not* antisymmetric, then there is a finite cycle.

Let $x, y \in X$, such that

$$x \sim^* y \;\wedge\; y \sim^* x \;\wedge\; x \neq y$$

then, by definition,

$$\begin{aligned} &\exists\, n, m \in \mathbb{N}.\ \exists x_0, \ldots, x_n, y_0, \ldots, y_m \in X. \\ &x = x_0 \sim x_1 \ldots x_{n-1} \sim x_n = y \wedge \\ &y = y_0 \sim y_1 \ldots y_{m-1} \sim y_m = x\,. \end{aligned}$$

Then, we have the finite cycle

$$(x, x_1)(x_1, x_2) \ldots (x_{n-1}, y)(y, y_1) \ldots (y_{m-1}, x)$$

of length $n{+}m$. □

We define isotone for arbitrary relations. Let $\sim_X$ and $\sim_Y$ be arbitrary relations over X and Y. Then we call a function $f : X \to Y$ *pseudo-isotone* iff

$$x \sim_X y \equiv_{df} f.x \sim_Y f.y\,. \tag{4.7}$$

Clearly, if $\sim$ is pseudo-isotone on a partially-ordered space, it is isotone. Note, that pseudo-isotone does not imply injective. Furthermore, pseudo-isotone bijections are persistent under transitive closure.

Proposition 4.3 *Let $f : X \to Y$, f be bijective, and f be pseudo-isotone on $(X, \sim_X)$ and $(Y, \sim_Y)$. Then f is pseudo-isotone on $(X, \sim_X^*)$ and $(Y, \sim_Y^*)$.*

Proof: Let, for one direction, $x, y \in X$ such that $x \sim_X^* y$. By definition, we have

$$\exists\, n \in \mathbb{N}.\ \exists\, x_0, \ldots, x_n \in X.\ x = x_0 \sim_X x_1 \ldots x_{n-1} \sim_X x_n = y\,.$$

For these $x_1, \ldots, x_n$, we have since f is pseudo-isotone on $\sim$

$$f.x = f.x_0 \sim_Y f.x_1 \ldots f.x_{n-1} \sim_Y f.x_n = f.y\,.$$

Since all these f-images are in Y, we have according to the definition of transitive closure

$$f.x \sim_Y^* f.y\,.$$

Similarly, for the other direction, we get from $f.x \sim_Y^* f.y$ by the definition of transitive closure

$$\exists\, n \in \mathbb{N}.\ \exists\, y_0, \ldots, y_n \in Y.\ f.x = y_0 \sim_X y_1 \ldots y_{n-1} \sim_X y_n = f.y\,.$$

However, now we need that f is surjective to conclude that

$$\exists\, x_1, \ldots, x_n \in X.\ f.x_0 = y_0 \wedge \ldots \wedge f.x_n = y_n\,.$$

Finally, we can combine these facts to obtain

$$\begin{aligned}&\exists\, n \in \mathbb{N}.\ \exists\, x_1, \ldots, x_n \in X.\\ &\qquad f.x = f.x_0 \sim_X f.x_1 \ldots f.x_{n-1} \sim_X f.x_n = f.y\,.\end{aligned}$$

Now, since f is pseudo-isotone and injective, this is equivalent to

$$\exists\, n \in \mathbb{N}.\ \exists\, x_1, \ldots, x_n \in X.\ x = x_0 \sim_X x_1 \ldots x_{n-1} \sim_X x_n = y\,,$$

which in turn is by definition equivalent to $x \sim_X^* y$. □

Extension of Orders

We define now the extension of the order relations of $(A, \leq_A)$ and $(B, \leq_B)$. Let, for A,

$$\begin{aligned}
E_{A\uparrow} &\equiv_{df} \{(x,y) \mid x \in T_{\varepsilon_2[B]} \wedge y \in T_A \cup T_A^\infty \wedge \rho.x <_B \rho.y\}\\
E_{A\downarrow} &\equiv_{df} \{(y,x) \mid x \in T_{\varepsilon_2[B]} \wedge y \in T_A \cup T_A^\infty \wedge \rho.y <_B \rho.x\}\\
\widetilde{\leq}_A &\equiv_{df} E_{A\uparrow} \cup E_{A\downarrow} \cup \leq_A
\end{aligned}$$

and similarly for B

$$\begin{aligned}
E_{B\uparrow} &\equiv_{df} \{(\rho.x,\rho.y) \mid x \in T_{\varepsilon_2[B]} \wedge y \in T_A \cup T_A^\infty \wedge x <_A y\}\\
E_{B\downarrow} &\equiv_{df} \{(\rho.y,\rho.x) \mid x \in T_{\varepsilon_2[B]} \wedge y \in T_A \cup T_A^\infty \wedge y <_A x\}\\
\widetilde{\leq}_B &\equiv_{df} E_{B\uparrow} \cup E_{B\downarrow} \cup \leq_B\ .
\end{aligned}$$

Note, that by construction ρ is pseudo-isotone on the (in general not partially-ordered) spaces $(A, \widetilde{\leq}_A)$ and $(B, \widetilde{\leq}_B)$. This is clear, since the only contradicting cases are (4.3) and (4.4) and the extensions are defined to compensate those cases.

To obtain partially-ordered spaces, we define the extended orders as the transitive closure of the old orders unified with the latter extensions.

$$\begin{aligned} \leq_A^* &\equiv_{df} (\widetilde{\leq}_A)^* \\ \leq_B^* &\equiv_{df} (\widetilde{\leq}_B)^* \end{aligned}$$

Now, we have to show that $\leq_A^*$ and $\leq_B^*$ are reflexive, transitive and antisymmetric, i.e. that the extended structures are partial orders. Furthermore, we have to show that ρ is an isomorphism from $(A, \leq_A^*)$ to $(B, \leq_B^*)$.

Since $\leq_A$ and $\leq_B$ are reflexive, clearly also the extended relations are reflexive. As they are built by transitive closure, $\leq_A^*$ and $\leq_B^*$ are also transitive.

To show antisymmetry, we concentrate on the existence of finite cycles in the graphs of the extended relations. We define a special class of finite cycles in order to prove that there are no finite cycles at all.

Crossing Cycles

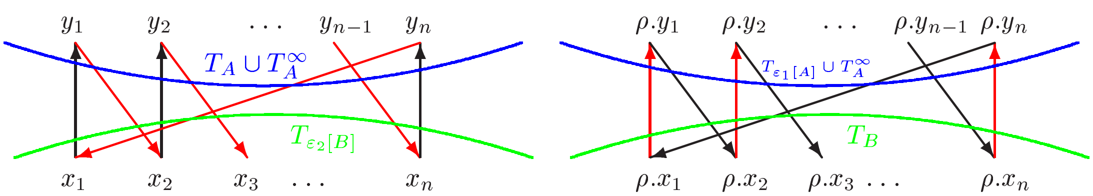

Figure 6: Crossing cycles with $(x_j, y_j) \in (\leq_A) \wedge (y_i, x_{i+1}) \in E_{A\downarrow}$

First, we define a *crossing edge* in the graphs of the relations $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$. For $x, y \in A$,

$$\begin{aligned} (x,y) \text{ is a crossing edge } \equiv_{df}\ & (x,y) \in (E_{A\uparrow} \cup E_{A\downarrow}) \vee \\ & (x,y) \in (\leq_A) \wedge \\ & \quad ((x \in T_A \cup T_A^\infty \wedge y \in T_{\varepsilon_2[B]}) \vee \\ & \quad (y \in T_A \cup T_A^\infty \wedge x \in T_{\varepsilon_2[B]}))\,. \end{aligned}$$

The corresponding definition for B is assumed. A *crossing cycle* is a finite cycle of elements $x_j \in T_{\varepsilon_2[B]} \wedge y_j \in T_A \cup T_A^\infty$ with $j \in \{1, \ldots, n\}$, $n \geq 1$

$$(x_1, y_1) \ldots (y_n, x_1)$$

such that either

$$(x_j, y_j) \in (\leq_A) \wedge (y_i, x_{i+1}) \in E_{A\downarrow}\,,$$

or

$$(x_j, y_j) \in E_{A_\uparrow} \wedge (y_i, x_{i+1}) \in (\leq_A)$$

for $i \in \{1, \ldots, n-1\}$. The corresponding crossing cycles will be defined in B. Crossing cycles of the first kind are illustrated in Figure 6, of the second kind in Figure 7. We call crossing cycles *alternating*, because, walking along

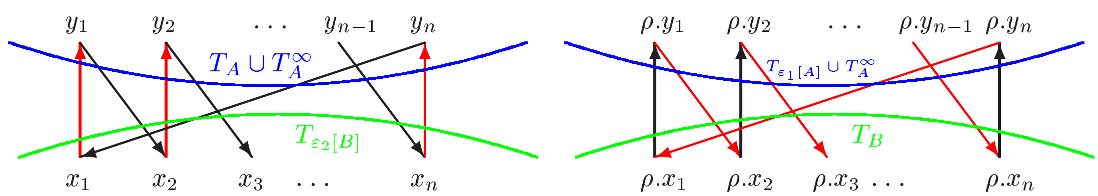

Figure 7: Crossing cycles for $(x_j, y_j) \in E_{A_\uparrow} \wedge (y_i, x_{i+1}) \in (\leq_A)$

the cycle, the edges alternate between the original relations $\leq_A$ or $\leq_B$ and the extensions. Alternation is a defining property of crossing cycles, i.e. a cycle having crossing edges that is non-alternating is not a crossing cycle.

Proposition 4.4 *All minimal cycles wrt.* $<_\circ$ *in the set of finite cycles of the relations* $\widetilde{\leq}_A$ *and* $\widetilde{\leq}_B$ *are crossing cycles.*

Proof: We have to show that any minimal cycle is a crossing cycle[8]. We do so by assuming a minimal cycle m that is *not* a crossing cycle and derive a contradiction that m is not minimal, i.e. $\exists\, m'.m' <_\circ m$. We conduct the proof for A: that for B is similar.

Assume now for contradiction there is a finite cycle m in A that is minimal and *not* a crossing cycle.

The cycle m cannot be entirely contained in $A_{ord} \cup T_A \cup T_A^\infty$, because there we did not extend the relation and $\leq_A$ is transitive and antisymmetric, so there are no finite cycles.

[8]The converse is not generally true: crossing cycles may have other crossing cycles as proper subcycles.

So, m must contain at least one crossing edge. Since it is a cycle it must lead back, so it has to have an even number of crossing edges. To be different from a crossing cycle it must either contain at least one non-crossing edge, or the sequence of the edges in m must be non-alternating. We show that both cases lead to a cycle m' with $m' <_\circ m$, whereby m is not minimal.

For the first case (see Figure 8), let m contain a consecutive segment

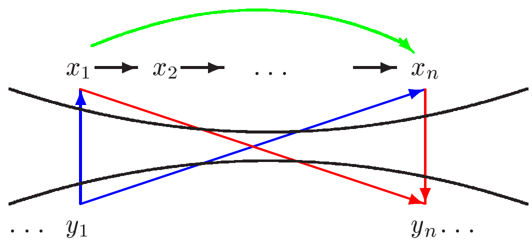

Figure 8: Cycle contains non-crossing edges and one connecting edge in $\leq_A$

$(x_1, x_2) \ldots (x_{n-1}, x_n)$ of non-crossing edges between two crossing edges. Since they are non-crossing, they are in $\leq_A$ and not in the extension. Since $\leq_A$ is transitive, m must contain the edge (x_1, x_n) to be minimal. Now, let the connecting crossing edges be (y_1, x_1) and (x_n, y_n). If either of them were already in $\leq_A$, by transitivity, a shorter cycle not containing (x_1, x_n) would be possible: for $(y_1, x_1) \in (\leq_A)$ we would have $(y_1, x_n) \in (\leq_A)$, for $(x_n, y_n) \in (\leq_A)$ we would have $(x_1, y_n) \in (\leq_A)$.

The only remaining possibility is that both crossing edges (y_1, x_1) and (x_n, y_n) are in the extensions (see Figure 9). However, that means that there

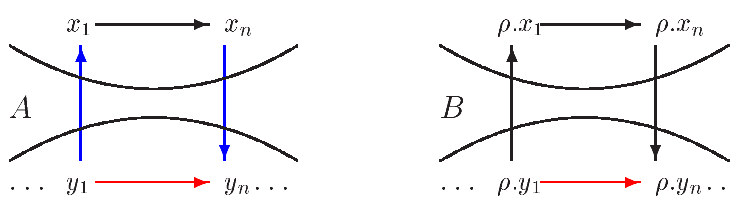

Figure 9: Non-crossing edge (x_1, x_n), both connecting edges ($\rightarrow$) in extension

are crossing edges $(\rho.y_1, \rho.x_1)$ and $(\rho.x_n, \rho.y_n)$ in $\leq_B$. Since the non-crossing edge $(x_1, x_n) \in (\leq_A)$ we also have $(\rho.x_1, \rho.x_n) \in (\leq_B)$. Since $\leq_B$ is transitive, we have an edge $(\rho.y_1, \rho.y_n) \in (\leq_B)$. This is not a crossing edge, so we must have also $(y_1, y_n) \in (\leq_A)$. For the cycle m' containing (y_1, y_n) instead of the segment $(y_1, x_1)(x_1, x_n)(x_n, y_n)$ we have by definition $m' <_\circ m$, whereby m is not minimal. Consequently, m cannot contain any non-crossing edges.

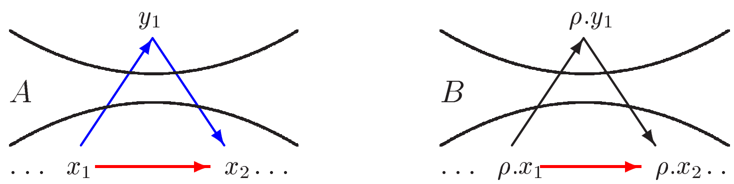

Figure 10: Two consecutive crossing edges ($\rightarrow$) in the extension

For the second case consider that m contains only crossing edges, but a non-alternating sequence of crossing edges, i.e. there are two consecutive crossing edges (x_1, y_1) and (y_1, x_2) in m that are either both in $\leq_A$ or both in the extensions. If they are both in $\leq_A$ we also have $(x_1, x_2) \in (\leq_A)$ whereby the cycle m' containing (x_1, x_2) instead of the segment $(x_1, y_1)(y_1, x_2)$ becomes strictly less than m. Otherwise, if the two consecutive crossing edges are both in the extensions, their ρ images must have both been in $\leq_B$ (see Figure 10). Thereby we get $(\rho.x_1, \rho.x_2) \in (\leq_B)$. This is again a non-crossing edge, hence not in the extension. So, we have again $(x_1, x_2) \in (\leq_A)$ and again we have as before that for the cycle m' containing (x_1, x_2) instead of $(x_1, y_1)(y_1, x_2)$ that $m' <_\circ m$. $\square$

Proposition 4.5 *There are no crossing cycles.*

Proof: For crossing cycles of length 2, we have already disproved their existence in (4.1). For all other crossing cycles, with even length > 2 we will show now that their existence contradicts ACC or DCC, respectively.

First, we demonstrate the nonexistence of crossing cycles of length $n = 4$ and finally show how the proof scales up for arbitrary n.

We consider only one of the possible crossing cycles of length 4; the proof for the other one is similar (contradicting ACC rather than DCC though). We illustrate the following consideration in Figure 11. Let $x_1, x_2 \in T_{\varepsilon_2[B]}$, $y_1, y_2 \in T_A \cup T_A^\infty$ and the cycle be $(x_1, y_1)(y_1, x_2)(x_2, y_2)(y_2, x_1)$ such that $(x_1, y_1), (x_2, y_2) \in \leq_A$ and (y_1, x_2), $(y_2, x_1) \in E_{A\downarrow}$.

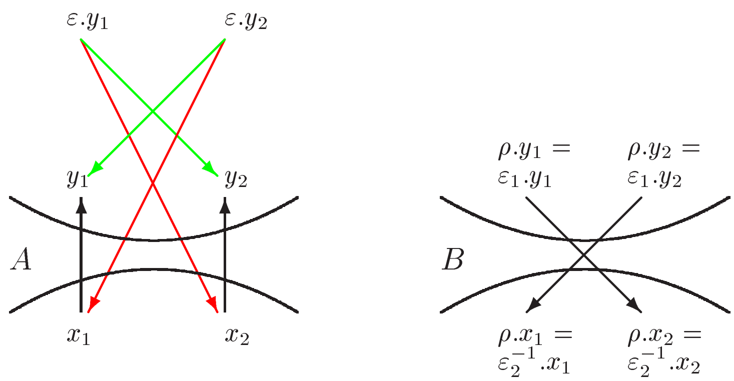

Figure 11: $\rightarrow$ by isotonicity, $\rightarrow$ by transitivity

Then $\rho.y_1 = \varepsilon_1.y_1$, $\rho.y_2 = \varepsilon_1.y_2$, $\rho.x_1 = \varepsilon_2^{-1}.x_1$, and $\rho.x_1 = \varepsilon_2^{-1}.x_1$. Since $(y_1, x_2) \in E_{A\downarrow}$ we have $\rho.y_1 <_B \rho.x_2$, and since $(y_2, x_1) \in E_{A\downarrow}$, we have $\rho.y_2 <_B \rho.x_1$. This is equivalent to $\varepsilon_1.y_2 <_B \varepsilon_2^{-1}.x_1$ and implies by isotonicity of ε_2 that

$$\varepsilon.y_2 = \varepsilon_2 \circ \varepsilon_1.y_2 <_A \varepsilon_2 \circ \varepsilon_2^{-1}.x_1 = x_1 .$$

From $\varepsilon.y_2 <_A x_1$ we get by transitivity of $<_A$ with $x_1 <_A y_1$ finally

$$\varepsilon.y_2 <_A y_1 . \tag{4.8}$$

Similarly, we get from $\varepsilon_1.y_1 <_B \varepsilon_2^{-1}.x_2$ that $\varepsilon.y_1 <_A x_2$, and then with the premise $x_2 <_A y_2$ we arrive at

$$\varepsilon.y_1 <_A y_2 . \tag{4.9}$$

From (4.8) we derive

$$\varepsilon^n.y_1 >_A \varepsilon^{n+1}.y_2$$

by applying isotonicity and induction. Similarly, we get from (4.9)

$$\varepsilon^n.y_2 >_A \varepsilon^{n+1}.y_1 .$$

From those we can construct by interleaving

$$y_1 >_A \varepsilon.y_2 >_A \varepsilon^2.y_1 >_A \varepsilon^3.y_2 >_A \ldots$$

an infinitely decreasing chain, contradicting DCC.

For crossing cycles of length $2n$ with $n \geq 2$, we can apply the corresponding argument to obtain

$$\begin{array}{lll} y_1 & >_A & \varepsilon.y_n \\ y_n & >_A & \varepsilon.y_{n-1} \\ \vdots & \vdots & \vdots \\ y_3 & >_A & \varepsilon.y_2 \\ y_2 & >_A & \varepsilon.y_1 \end{array}$$

which enables the construction of an infinitely decreasing chain

$$y_1 >_A \varepsilon.y_n >_A \varepsilon^2.y_{n-1} >_A \ldots >_A \varepsilon^{n-1}.y_2 >_A \varepsilon^n.y_1 >_A \varepsilon^{n+1}.y_n >_A \ldots$$

contradicting again DCC. □

With Corollary 4.1 the previous proposition immediately implies:

Corollary 4.6 *The relations $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$ contain no finite cycles.*

Finally, we have all the ingredients at hand to prove the main theorem.

Theorem 1

$$\left(\begin{array}{l} \varepsilon_1 : A \rightarrow B,\ \varepsilon_1 \text{ isotone} \\ \varepsilon_2 : B \rightarrow A,\ \varepsilon_2 \text{ isotone} \\ ACC \wedge DCC \end{array}\right) \Rrightarrow \left(\begin{array}{l} \exists \quad \leq_A^*,\ \leq_B^* . \\ (\leq_A) \subseteq (\leq_A^*) \ \wedge \ (\leq_B) \subseteq (\leq_B^*) \\ (A, \leq_A^*) \cong (B, \leq_B^*) \end{array}\right).$$

Proof: As we have seen before, $\leq_A^*$ and $\leq_B^*$ are reflexive and transitive.

We have shown in Corollary 4.6 that $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$ contain no finite cycles. From that, with Proposition 4.2, we can immediately conclude that $\leq_A^*$ and $\leq_B^*$ are antisymmetric.

The bijection ρ is pseudo-isotone by construction on $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$ (see page 47). Since it is also bijective we conclude by Proposition 4.3 that ρ is isotone on the transitive closures $\leq_A^*$ and $\leq_B^*$.

Summarizing, ρ is an isomorphism between the partial orders $(A, \leq_A^*)$ and $(B, \leq_B^*)$. □

The actual theorem we are interested in is implied by the previous result.

Corollary 4.7

$$\left(\begin{array}{l} ge(\varepsilon_1, \pi_1, (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_2, \pi_2, (B, \leq_B), (A, \leq_A)) \\ ACC \wedge DCC \end{array}\right) \Rrightarrow \left(\begin{array}{l} \exists \quad \leq_A^*,\ \leq_B^* . \\ (\leq_A) \subseteq (\leq_A^*) \ \wedge \ (\leq_B) \subseteq (\leq_B^*) \\ (A, \leq_A^*) \cong (B, \leq_B^*) \end{array}\right).$$

Similarly, we can infer the corresponding conclusion for Lagois connections [9] as these are also special cases of isotone functions — so Theorem 1 applies. Other possible application of our main theorem are in graph theory because a partial order may as well be interpreted as a directed acyclic graph, and an isotone function as a graph embedding.

5 Strengthening the Result

In this section we want to show that the result obtained in the previous section is meaningful. We found out so far that the condition $ACC(A) \wedge DCC(A)$ is sufficient to make an extension. However, is it necessary at all? We will see that it is not, but will try to illustrate that our result is best possible.

We are going to establish that if neither ACC nor DCC hold then the orders cannot generally be extended to create isomorphic order structures. To that end, we will define a counterexample, two partial orders with Galois embeddings between them. We will then show that they are not isomorphic and that there is no way to extend the orders such that they may be made isomorphic.

Finally, in Section 5.2, we address the Galois embeddings property again. Since we only used the isotonicity of the two functions for the main result of this article, the question arises what is the benefit of having the stronger assumption.

5.1 Counterexample

The Partial Orders

Define a partially ordered space X:

$$X \equiv_{df} (\mathbb{N} \times \mathbb{Z})$$
$$(a,b) \leq_X (c,d) \quad \equiv_{df} \quad a <_{\mathbb{N}} c \vee \begin{pmatrix} a = c \\ b \leq_{\mathbb{Z}} d \end{pmatrix} .$$

We define substructures A and B by selecting certain subsets from X .

$$A \equiv_{df} \{(0,0)\} \cup (\bigcup_{n\in\mathbb{N}} \{(3n+1,0),(3n+2,0)\}) \cup \bigcup_{n\in\mathbb{N}} \bigcup_{z\in\mathbb{Z}} \{(3(n+1),z)\}$$
$$B \equiv_{df} (\bigcup_{n\in\mathbb{N}} \{(3n,0),(3n+1,0)\}) \cup \bigcup_{n\in\mathbb{N}} \bigcup_{z\in\mathbb{Z}} \{(3n+2,z)\}$$

Each is a total order[9], since their superset $(X, \leq_X)$ is one, and any subset of a totally ordered set is clearly a totally ordered set. We note that in A and B

$$(a,b) \neq (0,0) \Rrightarrow a > 0 . \qquad (5.1)$$

[9] That is $\forall\, x,y \in X . x \leq_X y \vee y \leq_X x$.

The First Galois Embedding

The first Galois embedding is defined:

$$\begin{array}{lll}
\varepsilon_1 : A \to B \\
\pi_1 : B \to A \\
\varepsilon_1.(a,b) & \equiv_{df} & (0,0) \lhd (a,b) = (0,0) \rhd (a{+}2,b) \\
\pi_1.(c,d) & \equiv_{df} & (0,0) \lhd c \le 2 \rhd (c{-}2,d)\,.
\end{array}$$

Proposition 5.1 $ge(\varepsilon_1, \pi_1; (A, \le_A), (B, \le_B))$.

Proof: Firstly $\pi_1 \circ \varepsilon_1 = \imath_A$, for

$$\begin{array}{lll}
\pi_1 \circ \varepsilon_1.(a,b) & = & \text{(definitions of } \varepsilon_1 \text{ and} \circ) \\[2ex]
\begin{array}{l}\pi_1.(0,0) \\ \quad \lhd (a,b) = (0,0) \rhd \\ \quad (a{+}2,b)\end{array} & = & \text{(application distributes } \lhd\ \rhd) \\[2ex]
\begin{array}{l}\pi_1.(0,0) \\ \lhd (a,b) = (0,0) \rhd \\ \pi_1.(a{+}2,b)\end{array} & = & \text{(definition of } \pi_1\text{, (5.1)} \Rrightarrow a{+}2 > 2) \\[2ex]
(0,0) \lhd (a,b) = (0,0) \rhd (a,b) & = & \text{(definitions of } \lhd\ \rhd \text{ and } A) \\[2ex]
(a,b)\,.
\end{array}$$

Secondly $\varepsilon_1 \circ \pi_1 \le \imath_B$, for

$$\begin{array}{lll}
\varepsilon_1 \circ \pi_1.(c,d) & = & \text{(definitions of } \pi_1 \text{ and} \circ) \\[2ex]
\varepsilon_1.((0,0) \lhd\ c \le 2\ \rhd (c{-}2,d)) & = & \text{(application distributes } \lhd\ \rhd) \\[2ex]
\varepsilon_1.(0,0) \lhd\ c \le 2\ \rhd \varepsilon_1.(c{-}2,d) & = & \text{(definition of } \varepsilon_1) \\[2ex]
(0,0) \lhd\ c \le 2\ \rhd (c,d) & \le_B & \text{(definitions of } \lhd\ \rhd \text{ and } B)
\end{array}$$

(c, d) .

The Second Galois Embedding

The second Galois embedding is defined as

$$\begin{array}{lll} \varepsilon_2 : B \to A & & \\ \pi_2 : A \to B & & \\ \varepsilon_2.(c,d) & \equiv_{df} & (0,0) \lhd (c,d) = (0,0) \rhd (c{+}1, d) \\ \pi_2.(a,b) & \equiv_{df} & (0,0) \lhd (a,b) = (0,0) \rhd (a{-}1, b) \,. \end{array}$$

Proposition 5.2 $\quad ge(\varepsilon_2, \pi_2; (B, \leq_B), (A, \leq_A))$.

Proof: Firstly $\pi_2 \circ \varepsilon_2 = 1_B$, for

$\pi_2 \circ \varepsilon_2.(c, d)$ $\quad$ = $\quad$ (arguing as above)

$\pi_2.(0, 0)$
$\lhd\ (c, d) = (0, 0)\ \rhd$ $\quad$ = $\quad$ (definition of π_2, $c{+}1 > 0$)
$\pi_2.(c{+}1, d)$

$(0, 0)$
$\lhd\ (c, d) = (0, 0)\ \rhd$ $\quad$ = $\quad$ definitions of $\lhd\ \rhd$ and B
(c, d)

(c, d) .

Secondly $\varepsilon_2 \circ \pi_2 \leq 1_A$, for

$\varepsilon_2 \circ \pi_2.(a, b)$ $\quad$ = $\quad$ (application distributes $\lhd\ \rhd$)

$\varepsilon_2.(0, 0)$
$\lhd\ (a, b) = (0, 0)\ \rhd$ $\quad$ = $\quad$ (definition of ε_2, (5.1))
$\varepsilon_2.(a{-}1, b)$

$$(0,0)$$
$$\lhd\ (a,b) = (0,0)\ \rhd \qquad \leq \qquad \text{(definitions of } \lhd\ \rhd \text{ and } A)$$
$$(a,b)$$

$$(a,b)\,. \qquad \square$$

Figure 12 illustrates these Galois embeddings.

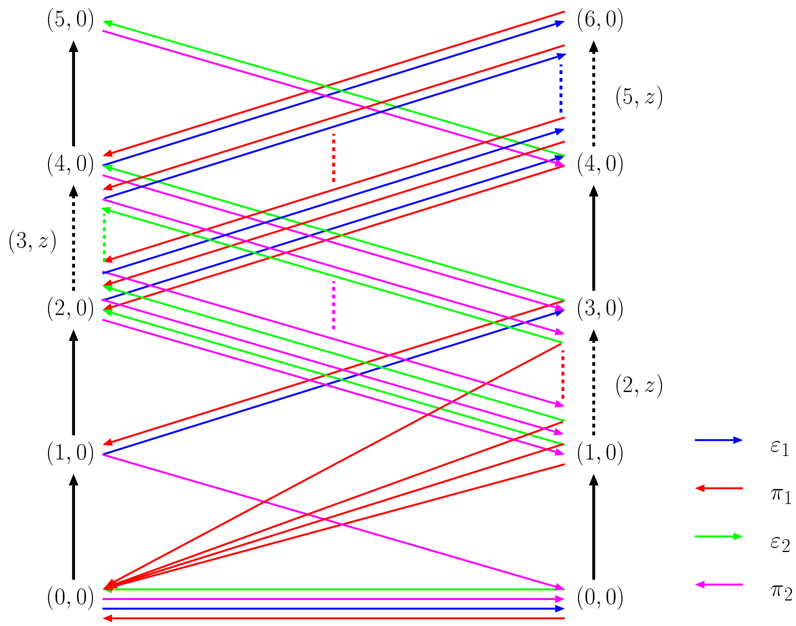

Figure 12: The partial orders with the Galois embeddings

Disproving Isomorphism

Let $(X, \leq_X)$ be a partial order and $x \in X$, We define the set of direct successors $s(x)$ as

$$s(x) \quad \equiv_{df} \quad \{y \in X \mid x <_X y \ \wedge \nexists\ z \in X.\ x <_X z <_X y\}$$

Lemma 5.3 *Let* $(X, \leq_X)$ *and* $(Y, \leq_Y)$ *be partial orders and* $f : X \to Y$ *be isotone and* $x \in X$ *such that* $s(x) \neq \varnothing$ *. Let* $y \in s(x)$*, then*

$$f.y \in s(f.x)\,.$$

Proof: Since f is isotone, we have for $x, y \in X$

$$x <_X y \equiv f.x <_Y f.y\,.$$

Since $x <_X y$, we can derive $f.x <_X f.y$. Assume now that there is a $z \in Y$ such that

$$f.x <_Y z <_Y f.y\,. \tag{5.2}$$

Since f is an isomorphism, in particular surjective, there must be a $q \in X$ with $f.q = z$. However, since f is isotone, this implies

$$(f.x <_Y f.q = z <_Y f.y) \equiv (x <_X q <_X y)$$

contradicting y is a direct successor of x . Consequently, there is no element $z \in Y$ with property (5.2), whereby, by definition of direct successor, $f.y \in s(f.x)$. □

Proposition 5.4 $(A, \leq_A) \ncong (B, \leq_B)\,.$

Proof: Assume there was an isomorphism $f : A \to B$. As we have seen before (see proof of Proposition 2.3), we must have

$$f.(0,0) = (0,0)\,.$$

In A, $s((0,0)) = \{(1,0)\}$. According to the previous lemma, this direct successor must be mapped to a direct successor of $(0,0)$ in B . The only direct successor of $(0,0)$ in B is $(1,0)$. Consequently, we must have $f.(1,0) = (1,0)$. Now, $(2,0) \in s((1,0))$ in A . Hence, by Lemma 5.3

$$f.(2,0) \in s(f.(1,0)) = s((1,0))\,.$$

However $s((1,0)) = \varnothing$, because for any element $(2,z) \in B$, for $z \in \mathbb{Z}$ we have

$$(2,z) >_B (1,0)\,,$$

but there is always an element

$$(2, z-1)$$

with

$$(2, z) >_B (2, z-1) >_B (1, 0)\,.$$

Consequently, there cannot be an isomorphism $f : A \to B$. □

Consequences for Theorem 1

Finally, we come to the main point of the counterexample: the proof of the nonexistence of extensions.

Proposition 5.5 *Partially-ordered spaces* $(A, \leq_A)$ *and* $(B, \leq_B)$ *are not consistent.*

Proof: Since $(A, \leq_A)$ and $(B, \leq_B)$ are totally ordered sets, they cannot be conservatively extended. That is, we cannot introduce new edges into the graph of the relations $\leq_A$ and $\leq_B$ without destroying antisymmetry. □

The counterexample proves that there are partially ordered structures with Galois embeddings in both directions that are not isomorphic and cannot be extended to be isomorphic. The example we have found is one where our condition $(ACC \wedge DCC)$ was violated.

This shows that apart from being sufficient, as mentioned before, our ascending or descending chain condition is meaningful.

However, it is not a necessary condition, because there are examples, where ACC or DCC do *not* hold and still the partial orders may be extended to an isomorphism. For example, take $\mathbb{N}$, the identity in both directions as trivial Galois embeddings and the empty set as trivial extension of the order relation. The ascending chain condition ACC does not hold in $\mathbb{N}$ and still we can (trivially) extend the orders to get isomorphic spaces.

An indication that our result is best possible is illustrated by looking at partially-ordered spaces with weaker assumptions than ACC and DCC: every partial order for which ACC holds is a complete partially-ordered set[10]

[10] A CPO is a partially-ordered space $(A, \leq_A)$ where each chain (directed subset) has a least upper bound in A.

(CPO) [5, exercise 4.11]. Hence, one might be tempted to believe that our notion of consistency might already be valid for CPOs. However, reconsidering the counterexample of this section we easily see that this is not the case. Extending A and B with top elements T_A and T_B, i.e. $\forall\ x \in A.\ x \leq T_A$ and $\forall\ y \in B.\ y \leq T_B$, the resulting structures $(A_T, \leq_{A_T})$ and $(B, \leq_{B_T})$ form CPOs: for example, the chain $((2, z))_{z\in\mathbb{N}}$ has least upper bound $(3, 0)$ in B. Furthermore, the entire sets A_T and B_T are chains; their top elements are the least upper bounds.

However, as we have seen before, $(A, \leq_A)$ and $(B, \leq_B)$ are not consistent and with the same argument we find that the CPOs $(A_T, \leq_{A_T})$ and $(B_T, \leq_{B_T})$ are neither consistent.

5.2 What about Galois Embeddings?

It seems rather surprising that in the context of the stronger assumption of Galois embedding not more is true than is anyway under the assumption of isotone maps in both directions. We do have the strong feeling that there must be something to Galois embeddings. However, in the present context, i.e assuming general partially-ordered spaces and ACC as well as DCC, isotonicity is already sufficient to ensure the extension property entailed in Theorem 1.

In the present section we want to do two things: first, we want to show one proposition that distinguishes the Galois embeddings property from mere isotonicity, second we will present a counterexample. It illustrates that the transitive closure in the construction of the extension described in Section 4 is generally necessary. Furthermore it shows, as it is an example with two Galois embeddings, that even the Galois embedding property does not generally guarantee transitivity of the basic order extension, as one might think when looking at the first counterexample in Section 2.

Galois Embedding Property

The methods developed to characterize the structure of two spaces $(A, \leq_A)$ and $(B, \leq_B)$ also yields the following property of Galois embeddings.

Proposition 5.6

$$\left(\begin{array}{l} ge(\varepsilon_1, \pi_1, (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_2, \pi_2, (B, \leq_B), (A, \leq_A)) \end{array} \right) \Rrightarrow \left(\begin{array}{lccc} (i) & x \in Q_A & \equiv & \varepsilon_2 \circ \pi_2 . x < x \\ (ii) & y \in Q_B & \equiv & \varepsilon_1 \circ \pi_1 . x < x \end{array} \right).$$

Proof: We show (i), (ii) is similar.

By the Galois embedding property (see Section 1.1) we have already that $\varepsilon_2 \circ \pi_2.x \leq x$. Hence, we just have to show $\varepsilon_2 \circ \pi_2.x \neq x$. Assume for contradiction $\varepsilon_2 \circ \pi_2.x = x$. This implies that $x \in \varepsilon_2[B] \diamond Q_A$ contradicting $x \in Q_A$.

For the other direction, i.e. $\varepsilon_2 \circ \pi_2.x < x \Rrightarrow x \in Q_A$, assume for contradiction $\varepsilon_2 \circ \pi_2.x < x$ and $x \notin Q_A$. Since $x \notin Q_A = \overline{\varepsilon_2[B]}$, we have $x \in \varepsilon_2[B]$ which is equivalent to $\exists\ y \in B.\ \varepsilon_2.y = x$. For this y, we substitute into $\varepsilon_2 \circ \pi_2.x < x$ and get $\varepsilon_2 \circ \pi_2.x < \varepsilon_2.y$. Since ε_2 is isotone this is equivalent to $\pi_2.x < y$. However, we get from that

$$\pi_2 \circ \varepsilon_2.y = \pi_2.x < y\,.$$

In particular, we have $\pi_2 \circ \varepsilon_2.y \neq y$ contradicting the Galois embedding property $\pi \circ \varepsilon = \imath$. □

The other Galois embedding property, i.e. $\varepsilon \circ \pi \leq x$ yields

$$(\varepsilon \circ \pi.x \not< x) \equiv (\varepsilon \circ \pi.x = x)\,.$$

and thereby we can immediately infer

Corollary 5.7

$$\left(\begin{array}{l} ge(\varepsilon_1, \pi_1, (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_2, \pi_2, (B, \leq_B), (A, \leq_A)) \end{array}\right) \Rrightarrow \left(\begin{array}{llll} (i) & x \notin Q_A & \equiv & \varepsilon_2 \circ \pi_2.x = x \\ (ii) & y \notin Q_B & \equiv & \varepsilon_1 \circ \pi_1.x = x \end{array}\right).$$

These propositions seem rather strong and one would expect to infer some stronger result from them.

Yet Another Counterexample

A proposition that one might be tempted to believe, is the following: the extension of the order defined on page 47 could be simplified if we use the stronger assumption of Galois embeddings, instead just ε_1, ε_2 isotone. Motivated by the counterexample of Section 2, where the basic extensions $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$ are already transitive, the conjecture is that in the case of Galois embeddings we do not need to build a transitive closure. That is, one might be tempted to believe that

$$\left(\begin{array}{l} ge(\varepsilon_1, \pi_1, (A, \leq_A), (B, \leq_B)) \\ ge(\varepsilon_2, \pi_2, (B, \leq_B), (A, \leq_A)) \end{array}\right) \Rrightarrow \left(\begin{array}{l} \widetilde{\leq}_A \text{ transitive} \\ \widetilde{\leq}_B \text{ transitive} \end{array}\right)$$

provided that ACC and DCC hold. Unfortunately, this is not the case.

We consider now two partially-ordered spaces $(A, \leq_A)$ and $(B, \leq_B)$ and two Galois embeddings between them such that the basic order extension described in Theorem 1 results in nontransitive relations $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$. The two spaces, ε_1, ε_2, and the basic order extension are illustrated in Figure 13, the formal definition follows.

In a similar fashion to the example in Section 2 we define a partially-ordered space:

$$X \equiv_{df} (\mathbb{N} \times \{1, 2, 3\}) \cup \{(0, 0)\}$$
$$(a, b) \leq_X (c, d) \quad \equiv_{df} \quad \begin{pmatrix} a = c \\ b \leq d \end{pmatrix} \vee \begin{pmatrix} a = 0 \\ b = 0 \end{pmatrix}.$$

Again, we are interested in two subsets of X with their inherited orders:

$$A \equiv_{df} X$$
$$B \equiv_{df} X \setminus \{(0, 2), (0, 3)\}$$

Clearly each is a partial order.

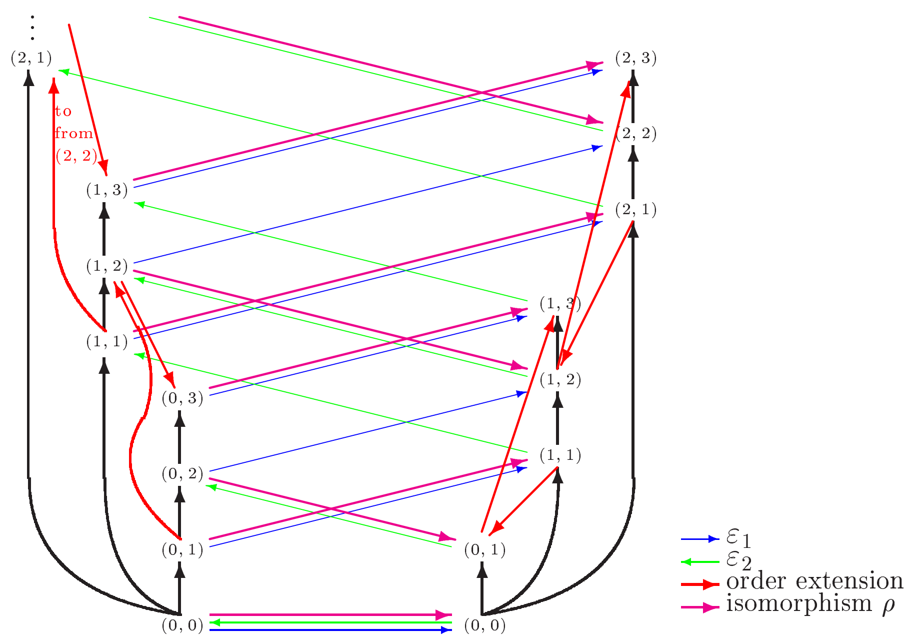

Figure 13: Partial orders with Galois embeddings and basic extensions

The first Galois embedding is defined as in the former counterexample:

$$\begin{aligned}
&\varepsilon_1 : A \to B\\
&\pi_1 : B \to A\\
&\varepsilon_1.(a,b) \quad \equiv_{df} \quad (0,0) \lhd b = 0 \rhd (a{+}1, b)\\
&\pi_1.(c,d) \quad \equiv_{df} \quad (0,0) \lhd c = 0 \rhd (c{-}1, d)\,.
\end{aligned}$$

The second Galois embedding is slightly different to the first example:

$$\begin{aligned}
&\varepsilon_2 : B \to A\\
&\pi_2 : A \to B\\
&\varepsilon_2.(c,d) \quad \equiv_{df} \quad ((0,0) \lhd\ d = 0\ \rhd (0,2)) \lhd\ c = 0\ \rhd (c,d)\\
&\pi_2.(a,b) \quad \equiv_{df} \quad (0, b \bmod 2) \lhd\ a = 0\ \rhd (a,b)\,.
\end{aligned}$$

The proof that these functions are Galois embeddings and that the two spaces are not isomorphic is similar to before.

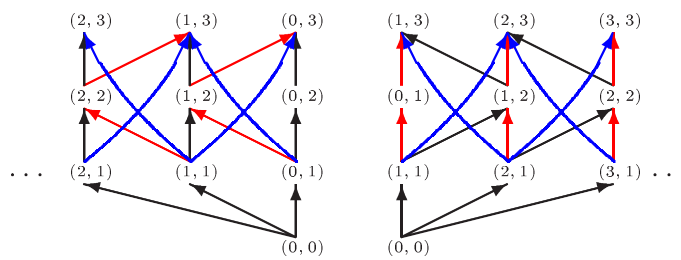

Figure 14: Rearranged extended partial orders, → only by transitive closure

The basic extension of the order is nontransitive: for example, in A, $(0,1)\widetilde{\leq}_A(1,2)$ and $(1,2)\widetilde{\leq}_A(1,3)$, but *not* $(0,1)\widetilde{\leq}_A(1,3)$. Similarly, in B, we have $(2,1)\widetilde{\leq}_B(1,2)$ and $(1,2)\widetilde{\leq}_B(1,3)$, but *not* $(2,1)\widetilde{\leq}_B(1,3)$. Figure 14 shows the result of applying Theorem 1: the rearranged isomorphic spaces $(A, \leq_A^*)$ and $(B, \leq_B^*)$ with blue arrows pointing out the edges that only come in through the transitive closure. Thus, this example proves that the closure is also necessary in case of Galois embeddings in both directions and not just isotone functions.

6 Summary and Future Work

This report is concerned with the question of isomorphism of partially-ordered spaces $(A, \leq_A)$ and $(B, \leq_B)$ in the presence of Galois embeddings in both directions. First, we recollect the facts already presented in [1] that the general isomorphism property (1.1) — contrary to a previously published result — is false. Moreover, the counterexample presented in Section 2 is chosen in such a way that it also proves that even the strong ascending and descending chain conditions ACC and DCC do not suffice to ensure isomorphism. However, we demonstrate, again on the counterexample, that we can extend the order relations such that the resulting partially-ordered spaces are isomorphic. Generalizing this extension of the order has been the major subject of the paper. Section 3 presented a formal definition of *zig-zags*, subsets of the partially-ordered sets A and B that are closed under repetitive application of $\varepsilon = \varepsilon_1 \circ \varepsilon_2$. They are the basis for the general construction of an order extension. We showed various results about zig-zags that give rise to partial isomorphism properties of the subsets (Propositions 3.25, 3.28, and 3.31). In Section 4 we combined these 'partial' isomorphism obtaining a bijection ρ. We pointed out where precisely the isotonicity property is violated by ρ. This analysis gives rise to the basic order extensions $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$. However, prior to the definition of the extension, we presented some theory on finite cycles in the graph of relations. This theory shows that we can safely build the transitive closure $\sim^*$ over a relation $\sim$ while preserving antisymmetry provided that there are no finite cycles in $\sim$. We used this property to show that the basic extensions $\widetilde{\leq}_A$ and $\widetilde{\leq}_B$ of $\leq_A$ and $\leq_B$ can be made reflexive, transitive, and antisymmetric by building their transitive closure $\leq_A^*$ and $\leq_A^*$. We concluded Section 4 with the main result Theorem 1 stating $(A, \leq_A^*) \cong (B, \leq_B^*)$. Finally, in Section 5, we showed with another counterexample that the conditions ACC and DCC — already proved to be sufficient to build an isomorphic extension — are also meaningful. In general, an extension is not possible as is shown by two partially-ordered spaces where neither ACC nor DCC hold and there is no way to extend the orders of these structures to achieve isomorphic partially-ordered spaces. We finished the paper by showing a couple of facts that distinguish Galois embeddings from mere isotone functions. A final counterexample illustrates the necessity of the transitive closure in the extension of the order — necessary even for Galois embeddings.

Open Questions

As discussed in Section 5.2 we did not use the Galois embedding properties in most of our considerations and in particular not for Theorem 1. It is sufficient to assume isotone functions ε_1 and ε_2. Hence, despite the insights developed in the present paper, we consider it an open question to find a precise characterization for the antisymmetry of Galois embeddings.

In the light of the counterexamples from [1] (see also Section 2) and Section 5.2, we observe that the element $(0, 0)$ has an infinite number of edges in the graph of the relations. If in addition to the assumptions of Theorem 1, i.e. (3.23), (3.24), and (3.25), we further assume that in A there are no infinitely branching nodes in the graph of the partial orders, then the chances for A and B being isomorphic are improved.

Note that ACC and DCC together with no infinitely branching nodes does *not* imply that A is finite (whereby the problem would be solvable by Proposition 3.17). Just consider a partially ordered infinite set with no edges in the graph of the relation. However, if we assume not just partially-ordered spaces but complete partial orders (i.e. a bottom element) then the restriction to finite branching yields finiteness even without ACC and DCC. In that case we can infer isomorphism by Proposition 3.17. The key is the connectedness of the graph of the relation. Since bottom has an edge to each element, it would have infinite branching degree, if the set was infinite. However, two isotone functions in both directions are sufficient to achieve isomorphism, i.e. this result is again independent of Galois embeddings.

Possibly, the assumption of completeness of the order in addition to finite branching degree can be weakened to the more general condition that the graph of the relation is connected[11]. For general partially-ordered spaces with finite branching degree the connectedness does not imply that the spaces are finite (an example proving this is left as an exercise). However, the connectedness in addition to the finite branching condition should be sufficient to exclude pathological cases.

[11] Note that the counterexample in Section 5.1 has no finite branching degree.

A Appendix

A.1 Closure of Zigars

For reasons of completeness we present here the proof of Lemma 3.24 already quoted in Section 3.4.

Lemma 3.24 $\qquad T_A^\infty \;=\; T_A^\infty \cup \varepsilon[T_A^\infty] \cup \varepsilon^{-1}[T_A^\infty]$

Proof: The inclusion $T_A^\infty \subseteq T_A^\infty \cup \varepsilon[T_A^\infty] \cup \varepsilon^{-1}[T_A^\infty]$ is obviously true. So we only have to show $\supseteq$. That is, for $x \in T_A^\infty \cup \varepsilon[T_A^\infty] \cup \varepsilon^{-1}[T_A^\infty]$ we need $x \in T_A^\infty$. We only need to prove that if $x \in \varepsilon[T_A^\infty] \cup \varepsilon^{-1}[T_A^\infty]$ then $x \in T_A^\infty$.

We proceed by case analysis:

(i). Let $x \in \varepsilon[T_A^\infty]$. We need to show that $x \in T_A^\infty$. We do so by showing

- (a) $x \in A$
- (b) $x \notin A_{ord}$
- (c) $x \notin T_A$
- (d) $x \notin T_{\varepsilon_2[B]}$.

Then from $(i)a$ and $(i)b$

$$\begin{aligned}
&\begin{pmatrix} x \in A \\ x \notin A_{ord} \end{pmatrix} && \Rrightarrow \quad (3.40) \\
&x \in A_{\gtrless} && \equiv \quad (3.51) \\
&x \in T_A^\infty \cup T_A \cup T_{\varepsilon_2[B]} && \equiv \quad ((i)c,(i)d) \\
&x \in T_A^\infty .
\end{aligned}$$

(ii). Let $x \in \varepsilon^{-1}[T_A^\infty]$. As in (i) we show $x \in T_A^\infty$ by proving

- (a) $x \in A$
- (b) $x \notin A_{ord}$

(c) $x \notin T_A$

(d) $x \notin T_{\varepsilon_2[B]}$.

For the same reasons as before it follows that $x \in T_A^\infty$.

Case analysis:

(i). $x \in \varepsilon[T_A^\infty]$.

Let $\varepsilon.y = x$, $y \in T_A^\infty$, then

(a) $y \in A$, because $T_A^\infty \subseteq A$ by definition. Since $\varepsilon : A \to A$, we have $\varepsilon.y = x \in A$.

(b) Assume for contradiction $\varepsilon.y \in A_{ord}$. Then by Proposition 3.6

$$\begin{array}{lll} x = \varepsilon.y = y & \Rrightarrow & \text{(assumptions)} \\ \left(\begin{array}{l} x = y \in T_A^\infty \\ x = y \in A_{ord} \end{array} \right) & \Rrightarrow & \\ x \in T_A^\infty \cap A_{ord} & & \text{contradicting (3.54)} . \end{array}$$

(c) Assume for contradiction $\varepsilon.y \in T_A$. This implies by Corollary 3.14

$$\exists\ n \in \mathbb{N}, q \in Q_A.\ \varepsilon^n.q = \varepsilon.y\,.$$

Now, for $n = 0$, we have $q = \varepsilon.y \in \varepsilon_2[B]$ contradicting $q \in Q_A = A \setminus \varepsilon_2[B]$ and law (3.14). For $n \geq 1$, the injectivity of ε implies

$$y = \varepsilon^{n-1}.q \in T_A$$

by Lemma 3.13. However, $y \in T_A$ contradicts $y \in T_A^\infty \diamond T_A$ (3.50).

(d) Assume again for contradiction $\varepsilon.y \in T_{\varepsilon_2[B]}$. Then

$$\exists\ n \in \mathbb{N}, q \in Q_{\varepsilon_2[B]}.\ \varepsilon^n.q = \varepsilon.y$$

For $n = 0$, $q \in \varepsilon[A]$ contradicting $q \in Q_{\varepsilon_2[B]} = \varepsilon_2[B] \setminus \varepsilon[A]$ by definition of $Q_{\varepsilon_2[B]}$. Otherwise, if $n \geq 1$, because ε is injective,

$$y = \varepsilon^{n-1}.q \in T_{\varepsilon_2[B]}$$

by the closedness of $T_{\varepsilon_2[B]}$ (3.42). This contradicts $y \in T_A^\infty \diamond T_{\varepsilon_2[B]}$ (3.50).

(ii). $x \in \varepsilon^{-1}[T_A^\infty]$.

Let $\varepsilon^{-1}.y = x$, $y \in T_A^\infty$, then

(a) $y \in A$, because $T_A^\infty \subseteq A$ by definition. Since $\varepsilon : A \to A$, we have $\varepsilon^{-1} : A \to A$ and consequently $\varepsilon^{-1}.y = x \in A$.

(b) Assume for contradiction $\varepsilon^{-1}.y \in A_{ord}$. Then

$$\begin{aligned}
& \varepsilon^{-1}.y = x \\
\equiv \quad & (\varepsilon \text{ function}) \\
& \varepsilon.(\varepsilon^{-1}.y) = \varepsilon.x \\
\equiv \quad & (\varepsilon \text{ injective}) \\
& y = \varepsilon.x \\
\equiv \quad & (x \in A_{ord}) \\
& y = x \in A_{ord} \qquad \text{contradicting } x \in T_A^\infty \text{ (3.54).}
\end{aligned}$$

(c) Assume for contradiction $\varepsilon.y \in T_A$. This implies

$$\exists\, n \in \mathbb{N}, q \in Q_A.\ \varepsilon^n.q = \varepsilon^{-1}.y$$

which implies for this n and q that

$$y = \varepsilon^{n+1}.q \in T_A$$

contradicting $y \in T_A^\infty$ (3.50).

(d) Finally, we assume for contradiction $\varepsilon^{-1}.y \in T_{\varepsilon_2[B]}$. Then

$$\exists\, n \in \mathbb{N}, q \in Q_{\varepsilon_2[B]}.\ \varepsilon^n.q = \varepsilon^{-1}.y\,.$$

As in the previous case this implies for these n and q

$$\varepsilon^{n+1}.q = y \in T_{\varepsilon_2[B]}$$

contradicting the assumption and $T_A^\infty \diamond T_{\varepsilon_2[B]}$ (3.50). □

A.2 Proof scripts for Isabelle

The propositions (3.1)-(3.11) have been checked in Isabelle/HOL version 99. The proof scripts are contained here.

```
(*3.1*)
Goal "(∀ e ∈ E. e ⊆ F) = ((Union E) ⊆ F)";
auto();
val prop3_1 = result();

(*3.2*)
Goal "((A :: 'a set) - B) - C = A - C - B";
auto();
val prop3_2 = result();

(*3.3*)
Goal "B ⊆ (C :: 'a set) ==> (A - B - (C - B)) = (A - C)";
auto();
val prop3_3 = result();

(*3.4*)
Goal "Union {f `` ai  | ai. ai ∈ A} = f `` (Union A)";
auto();
val prop3_4 = result();

(*3.5*)
Goal "A ⊆ B ==> (A - B) ∪ B = B";
auto();
val prop3_5 = result();

(*3.6*)
Goal "∀ x ∈ f `` A. f (inv f x) = x";
auto();
br f_inv_f 1;
br rangeI 1;
val prop3_6 = result();

(*3.7*)
Goal "inj f ==> ∀ x ∈ A. inv f(f x) = x";
auto();
val prop3_7 = result();
```

```
Goal "[| inj f; x ∈ A |] ==> x ∈ inv f `` f `` A";
by (res_inst_tac [("t","x")] (inv_f_f RS subst) 1);
ba 1;
br imageI 1;
br imageI 1;
ba 1;
val lemma1 = result();

(*3.8*)
Goal "inj f ==> (inv f) `` (f `` A) = A";
auto();
be lemma1 1;
ba 1;
val prop3_8 = result();

(*3.9*)
Goal "[| inj f; X ⊆ f `` A |] ==> f ``(inv f `` X) = X";
auto();
by (res_inst_tac [("t","x"),("f1","f")] (f_inv_f RS subst) 1);
br imageI 2;
br subsetD 1;
ba 2;
auto();
val prop3_9 = result();

(*3.10*)
Goal "f `` A - f `` B ⊆ f `` (A - B)";
auto();
val image_diff1 = result();

Goal "inj f ==> f `` (A - B) ⊆ f `` A - f `` B";
auto();
be notE 1;
be (injD RS ssubst) 1;
ba 1;
ba 1;
val image_diff2 = result();

Goal "inj f ==> f `` (X - Y) = (f `` X) - (f `` Y)";
```

```
by (asm_full_simp_tac (simpset()
    addsimps [equalityI,image_diff1,image_diff2]) 1);
val prop3_10 = result();

(*3.11*)
Goal "[| inj f; f x: f `` A|] ==> x: A";
auto();
be (injD RS ssubst) 1;
ba 1;
ba 1;
val lemma2 = result();

Goal "inj f ==> (f `` A = f `` B) = (A = B)";
auto();
br lemma2 1;
ba 1;
be subst 1;
be imageI 1;
br lemma2 1;
ba 1;
be ssubst 1;
be imageI 1;
val prop3_11 = result();
```

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