# De Nugis Groebnerialium 1: Eagon, Northcott, Gröbner

*Ferdinando Mora* [1]

> En el nombre del Padre que fizo toda cosa.
> Et de Don Jhesucristo, fijo de la Gloriosa.
> Et del Spiritu Sancto, que egual dellos prosa,
> De un confesor sancto quiero fer una prosa.
>
> Quiero fer una prosa en roman paladino,
> En qual suele el pueblo fablar a su vecino,
> Ga non se tan letrado por fer otro latino,
> Bien valdra, como credo, un vaso de bon vino.
>
> Gonzalo de Berceo

## Remembrance

It was Autumn 1983, when the researchers on Gröbner could have been counted on the fingers of two hands. Michael and me were completing our algorithm to compute resolutions (Mora, Möller 1986a, 1986b) and I was invited in Naples to give an introductory tutorial on Gröbner bases.

I had plenty of free time and, since somebody had just quoted me the Eagon-Northcott formula expressing the resolution of the ideals generated by the majors of a matrix whose entries are independent variables (Eagon, Northcott 1962)), I decided to try to see whether our tools allowed me to tackle the $5 \times 3$ case.

I was really surprised when not only I got the resolution but I realized that it was sufficient to give a look to the solution to devise the complete formula (Th. 1.1) and that proving it required only to generalize the computation I did[2]: it was the first time that I realized the amazing power of Buchberger's tool.

My notes ended in a pile of other computations, and probably would have died there... until I thought it could have been curious to present here this "archaeological" result to show a piece of research in those times when the researchers on Gröbner could have been counted on the fingers of two hands ...

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[2]Honestly I must confess that I needed to do a few computations over the $7 \times 4$ case to fix a bug in my guess and complete the proof.

Up to a small polishing, removing useless remarks, adding a pair of footnotes and a chapter (§ 2) aimed to summarize what I knew at that time, the note here is nothing more than my original one, including the hand-computed example[3].

**Acknowledgements**

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Mainly I thanks Michael for the wonderful research together.

# 1 Notation

Let $m, n \in \mathbb{N}$ s.t. $m < n$.

For each $t$, $0 \leq t \leq n - m$, we denote

$$\mathcal{C}_t := \{(\gamma_1, \ldots, \gamma_{m+t}) : 1 \leq \gamma_1 < \cdots < \gamma_{m+t} \leq n\}.$$

If $\mathsf{c} := (\gamma_1, \ldots, \gamma_{m+t}) \in \mathcal{C}_t$, $t > 0$, we denote for each $i, 1 \leq i \leq m + t$,

$$\mathsf{c}(i) := (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{m+t}) \in \mathcal{C}_{t-1},$$

and, if $t > 1$, we denote also for each $i, l, 1 \leq i < l \leq m + t$,

$$\mathsf{c}(i, l) := (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_{l-1}, \gamma_{l+1}, \ldots, \gamma_{m+t}) \in \mathcal{C}_{t-2}.$$

For each $t$, $0 \leq t \leq n - m$, we denote

$$\mathcal{R}_t := \{(\rho_1, \ldots, \rho_m) : \sum_{j=1}^{m} \rho_j = m + t, 1 \leq \rho_j, \forall j\}.$$

If $\mathsf{r} := (\rho_1, \ldots, \rho_m) \in \mathcal{R}_t$, $t > 0$, we denote $I_\mathsf{r} := \{j : \rho_j > 1\}$ and, $\forall j \in I_\mathsf{r}$,

$$\mathsf{r}(j) := (\rho_1, \ldots, \rho_{j-1}, \rho_j - 1, \rho_{j+1}, \ldots, \rho_m) \in \mathcal{R}_{t-1}.$$

For $t > 1$ we denote

$$I_\mathsf{r}^{(2)} := \{(j, k) : j \neq k, \rho_j > 1, \rho_k > 1\} \cup \{(k, k) : \rho_k > 2\}$$

and, remarking that

[3] Within the proof of the Eagon-Northcott formula, I remarked (Lemma 3.5.1.) that the set of the majors of a generic matrix are a Gröbner basis.

In (Narasimhan 1986), (Caniglia et. al. 1990) and (Sturmfels 1990) it is proved that the minors of any given order of a generic matrix are a Gröbner basis with respect to a diagonal term order . More strong statements of this kind can be found in (Conca 1994), (Conca 1995), (Bruns, Conca 1996).

- $(j,k) \in I_{\mathsf{r}}^{(2)} \iff (k,j) \in I_{\mathsf{r}}^{(2)}$,
- $I_{\mathsf{r}}^{(2)} = \bigcup_{j \in I_{\mathsf{r}}} \{(j,k) : k \in I_{\mathsf{r}(j)}\}$,

we denote

$$\begin{aligned} \mathsf{r}(j,k) &:= \mathsf{r}(j)(k) = \mathsf{r}(k)(j) = \\ &= \begin{cases} (\ldots, \rho_{j-1}, \rho_j - 1, \rho_{j+1}, \ldots, \rho_{k-1}, \rho_k - 1, \rho_{k+1}, \ldots) & j < k \\ (\ldots, \rho_{j-1}, \rho_j - 2, \rho_{j+1}, \ldots) & j = k \\ (\ldots, \rho_{k-1}, \rho_k - 1, \rho_{k+1}, \ldots, \rho_{j-1}, \rho_j - 1, \rho_{j+1}, \ldots) & j > k. \end{cases} \end{aligned}$$

For each $\mathsf{r} := (\rho_1, \ldots, \rho_m) \in \mathcal{R}_t$, let us define $d_i, 1 \leq i \leq m+t$, as $d_i := j$ where $j$ is the unique integer s.t. $\sum_{k<j} \rho_k < i \leq \sum_{k \leq j}$ and we will set $d(\mathsf{r}) := (d_1, \ldots, d_{m+t})$ [4].

For each $t$, $0 \leq t \leq n-m$, we denote

$$\mathcal{S}_t := \mathcal{R}_t \times \mathcal{C}_t.$$

Let us now consider the polynomial ring

$$\mathfrak{P} := \mathbb{Q}[X_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$$

and, for each $t$, the free module $\mathfrak{P}^{\mathcal{S}_t}$ generated by the canonical basis

$$\{E_{\mathsf{s}} : \mathsf{s} \in \mathcal{S}_t\}.$$

Remark that, since $\mathrm{card}(\mathcal{R}_0) = 1$, it holds $\mathcal{S}_0 \cong \mathcal{C}_0$.

Let $\mathcal{M}$ be the $m \times n$ matrix $\mathcal{M} := (X_{ij})_{ij}$ and for each

$$\mathsf{c} = (\gamma_1, \ldots, \gamma_m) \in \mathcal{C}_0 \cong \mathcal{S}_0$$

let $M_{\mathsf{c}}$ be the (determinant of the) major[5] of $\mathcal{M}$ on the $m$ $\gamma_j^{th}$ columns and let

$$\mathfrak{I} := (M_{\mathsf{c}} : \mathsf{c} \in \mathcal{C}_0) \subset \mathfrak{P}.$$

Finally denote

$$\delta_0 : \mathfrak{P}^{\mathcal{S}_0} \mapsto \mathfrak{P}$$

to be the map s.t.

$$\delta_0(E_{\mathsf{c}}) = M_{\mathsf{c}}$$

and, for every $t > 0$,

$$\delta_t : \mathfrak{P}^{\mathcal{S}_t} \mapsto \mathfrak{P}^{\mathcal{S}_{t-1}}$$

to be the map s.t., for each $\mathsf{s} := (\mathsf{r}, \mathsf{c}) \in \mathcal{S}_t$ with $\mathsf{c} = (\gamma_1, \ldots, \gamma_{m+t})$, it holds

$$\delta_t(E_{\mathsf{s}}) := \sum_{i=1}^{m+t} \sum_{j \in I_{\mathsf{r}}} (-1)^i X_{j\gamma_i} E_{(\mathsf{r}(j), \mathsf{c}(i))}.$$

**Theorem 1.1** *The minimal free resolution of* $\mathfrak{P}/\mathfrak{I}$ *is*

$$0 \to \mathfrak{P}^{\mathcal{S}_{n-m}} \xrightarrow{\delta_{n-m}} \mathfrak{P}^{\mathcal{S}_{n-m-1}} \xrightarrow{\delta_{n-m-1}} \cdots \mathfrak{P}^{\mathcal{S}_t} \xrightarrow{\delta_t} \mathfrak{P}^{\mathcal{S}_{t-1}} \cdots \xrightarrow{\delta_1} \mathfrak{P}^{\mathcal{S}_0} \xrightarrow{\delta_0} \mathfrak{P}/\mathfrak{I} \to 0.$$

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[4] in other words, the vector $d$ consists of $\rho_1$ 1's, $\rho_2$ 2's, $\rho_3$ 3's,....

[5] maximal minor.

## 2 Recall

This section is essentially a fast resume of the results in Mora, Möller (1986)[6] which will be applied to prove the claim above.

Once we are given a well-ordering $<$ on the set $\mathsf{T}$ of the terms in $\mathfrak{P}$, we can use $<$ to impose an ordering $<_t$ on the set

$$\mathsf{T}_t := \{mE_{\mathsf{s}} : m \in \mathsf{T}, \mathsf{s} \in \mathcal{S}_t\}$$

of the terms in $\mathfrak{P}^{\mathcal{S}_t}$ which is "compatible" with $<$ in the sense that

$$\forall m_1, m_2 \in \mathsf{T}, \forall \mu_1, \mu_2 \in \mathsf{T}_t, \mu_1 \leq_t \mu_2, m_1 \leq m_2 \Longrightarrow m_1\mu_1 \leq_t m_2\mu_2.$$

The order $<_t$ is defined by fixing

- monomials $m_{\mathsf{s}}$ for each $\mathsf{s} \in \mathcal{S}_t$ and
- an ordering $\prec$ on $\mathcal{S}_t$

and setting

$$mE_{\mathsf{s}} <_t m'E_{\mathsf{s}'} \iff \begin{array}{l} \mu := mm_{\mathsf{s}} < m'm'_{\mathsf{s}} =: \mu' \\ \text{or } \mu = \mu', \mathsf{s} \prec \mathsf{s}'. \end{array}$$

For each term $\mu := mE_{\mathsf{s}} \in \mathsf{T}_t$ we denote

$$Tdeg(\mu) := mm_{\mathsf{s}};$$

for each module element $f := \sum_{\mu \in \mathsf{T}_t} c_\mu \mu \in \mathfrak{P}^{\mathcal{S}_t}$ we denote

$$Tdeg(f) := \max_{<}\{Tdeg(\mu) : c_\mu \neq 0\},$$

$$Hterm(f) := \max_{<_t}\{\mu : c_\mu \neq 0\};$$

for each submodulo $V \subset \mathfrak{P}^{\mathcal{S}_t}$ we consider a finite subset $G = \{g_1, \ldots g_h\} \subset V$ generating $V$; $G$ is a *Gröbner basis* of $V$ if

$$\forall f \in V, \exists g \in G : Hterm(g) \text{ divides } Hterm(f).$$

If $G$ is a Gröbner basis of $V$, for each $f \in V$ there is a *G-representation*

$$f = \sum_i h_i g_i : Hterm(h_i g_i) \leq Hterm(f), \forall i.$$

[6] Of course I must also quote Bayer (1982) whose algorithm I used to compute the resolution.

Let $G := \{g_1, \dots g_h\} \subset V$ be a basis of $V$ and for each pair $g', g'' \in G$, with $Hterm(g') =: m'E_{\mathsf{s}'}$, $Hterm(g'') =: m''E_{\mathsf{s}''}$, let us define

$$\operatorname{lcm}(Hterm(g'), Hterm(g'')) :=$$
$$:= \begin{cases} \operatorname{lcm}(Hterm(m'), Hterm(m''))E_{\mathsf{s}'} & \text{if } \mathsf{s}' = \mathsf{s}'' \\ 0 & \text{otherwise.} \end{cases}$$

If $\operatorname{lcm}(Hterm(g'), Hterm(g'')) \neq 0$, then there are $t', t'' \in \mathsf{T}$ s.t.

$$t'Hterm(g') = \operatorname{lcm}(Hterm(g'), Hterm(g'')) = t''Hterm(g''),$$

in which case we define

$$S(g', g'') := t'g' - t''g''$$

and we remark that

$$Hterm(S(g', g'')) < \operatorname{lcm}(Hterm(g'), Hterm(g'')).$$

Denote

$$\mathfrak{S}(G) := \{(g_i, g_j) \in G \times G : i < j, \operatorname{lcm}(Hterm(g_i), Hterm(g_j)) \neq 0\}$$

and let $\mathfrak{S}_u(G) \subseteq \mathfrak{S}(G)$ be s.t. for all $(g', g'') \in \mathfrak{S}(G) \setminus \mathfrak{S}_u(G)$, there is $g''' \in G$ s.t. $\operatorname{lcm}(Hterm(g'), Hterm(g'''))$ and $\operatorname{lcm}(Hterm(g''), Hterm(g'''))$ divide properly $\operatorname{lcm}(Hterm(g'), Hterm(g''))$; then the following conditions are equivalent:

- $G$ is a Gröbner basis of $V$;
- each $(g', g'') \in \mathfrak{S}_u(G)$ has a $G$-representation.

In case the conditions above are satisfied, consider the module $\mathfrak{P}^G$ generated by the canonical basis $\{E_g : g \in G\}$ and the map $\Delta : \mathfrak{P}^G \mapsto \mathfrak{P}$ defined by

$$\Delta(E_g) := g, \forall g \in G.$$

For each $(g', g'') \in \mathfrak{S}_u(G)$, let $\sum_{g \in G} h_g g$ be a $G$-representation of $S(g', g'')$, so that

$$\sum_{g \in G} h_g g = S(g', g'') = t'g' - t''g'';$$

let us define

$$\Sigma(g', g'') := \sum_{g \in G} h_g E_g - t'E_{g'} + t''E_{g''}.$$

With this notation it holds

- $Im(\Delta) = V$;
- $\{\Sigma(g', g'') : (g', g'') \in \mathfrak{S}_u(G)\}$ generates $Ker(\Delta)$.

# 3 (Re)solution

**Lemma 3.1** $\delta_0\delta_1 = 0$.

**Proof:** For any $(\mathsf{r},\mathsf{c}) \in \mathcal{S}_1$ with $\mathsf{c} = \{(\gamma_1, \ldots, \gamma_{m+1})$, denoting

$$\mathfrak{M}_j := \begin{pmatrix} X_{j\gamma_1} & \ldots & X_{j\gamma_i} & \ldots & X_{j\gamma_{m+1}} \\ X_{1\gamma_1} & \ldots & X_{1\gamma_i} & \ldots & X_{1\gamma_{m+1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{j\gamma_1} & \ldots & X_{j\gamma_i} & \ldots & X_{j\gamma_{m+1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{m\gamma_1} & \ldots & X_{m\gamma_i} & \ldots & X_{m\gamma_{m+1}} \end{pmatrix},$$

it holds

$$\begin{aligned} \delta_0\delta_1(E_{(\mathsf{r},\mathsf{c})}) &= \sum_{i=1}^{m+1}\sum_{j\in I_\mathsf{r}}(-1)^i X_{j\gamma_i} M_{\mathsf{c}(i)} = \\ &= \sum_{j\in I_\mathsf{r}}\sum_{i=1}^{m+1}(-1)^i X_{j\gamma_i} M_{\mathsf{c}(i)} = \\ &= \sum_{j\in I_\mathsf{r}} -\det(\mathfrak{M}_j) \\ &= 0. \quad \square \end{aligned}$$

**Lemma 3.2** *If* $t > 1$, $\delta_{t-1}\delta_t = 0$.

**Proof:** For any $(\mathsf{r},\mathsf{c}) \in \mathcal{S}_t$ with $\mathsf{c} = \{(\gamma_1, \ldots, \gamma_{m+t})$, it holds

$$\begin{aligned} &\delta_{t-1}\delta_t(E_{(\mathsf{r},\mathsf{c})}) = \\ &= \sum_{i=1}^{m+t}\sum_{j\in I_\mathsf{r}}(-1)^i X_{j\gamma_i}\delta_{t-1}\left(E_{(\mathsf{r}(j),\mathsf{c}(i))}\right) = \\ &= \sum_{i=1}^{m+t}\sum_{j\in I_\mathsf{r}}(-1)^i X_{j\gamma_i}\sum_{l=1}^{i-1}\sum_{k\in I_{\mathsf{r}(j)}}(-1)^l X_{k\gamma_l} E_{(\mathsf{r}(k,j),\mathsf{c}(l,i))} + \\ &\quad + \sum_{i=1}^{m+t}\sum_{j\in I_\mathsf{r}}(-1)^i X_{j\gamma_i}\sum_{l=i+1}^{m+t}\sum_{k\in I_{\mathsf{r}(j)}}(-1)^{l-1} X_{k\gamma_l} E_{(\mathsf{r}(k,j),\mathsf{c}(i,l))} = \\ &= \sum_{i=1}^{m+t}\sum_{l=1}^{i-1}(-1)^{i+l}\sum_{j\in I_\mathsf{r}}\sum_{k\in I_{\mathsf{r}(j)}} X_{j\gamma_i}X_{k\gamma_l}E_{(\mathsf{r}(k,j),\mathsf{c}(l,i))} + \\ &\quad + \sum_{i=1}^{m+t}\sum_{l=i+1}^{m+t}(-1)^{i+l-1}\sum_{j\in I_\mathsf{r}}\sum_{k\in I_{\mathsf{r}(j)}} X_{j\gamma_i}X_{k\gamma_l}E_{(\mathsf{r}(k,j),\mathsf{c}(i,l))} = \\ &= \sum_{i=1}^{m+t}\sum_{l=1}^{i-1}(-1)^{i+l}\sum_{(k,j)\in I_\mathsf{r}^{(2)}} X_{j\gamma_i}X_{k\gamma_l}E_{(\mathsf{r}(k,j),\mathsf{c}(l,i))} + \end{aligned}$$

$$+\sum_{i=1}^{m+t}\sum_{l=i+1}^{m+t}(-1)^{i+l-1}\sum_{(k,j)\in I_{\mathsf{r}}^{(2)}} X_{j\gamma_i}X_{k\gamma_l}E_{(\mathsf{r}(k,j),\mathsf{c}(i,l))} =$$

$$= \sum_{(k,j)\in I_{\mathsf{r}}^{(2)}}\sum_{i=1}^{m+t}\sum_{l=1}^{i-1}\left((-1)^{i+l}-(-1)^{i+l}\right)X_{j\gamma_i}X_{k\gamma_l}E_{(\mathsf{r}(k,j),\mathsf{c}(l,i))} =$$

$$= 0. \quad \square$$

Let us now fix the well-ordering $<$ on the set $\mathsf{T}$ by choosing the total degree ordering [7] induceded by

$$X_{1n} < \cdots X_{mn} < X_{1n-1} < \cdots < X_{m2} < \cdots < X_{11} < \cdots X_{m1}.$$

With this choice it holds

**Lemma 3.3** *For each* $\mathsf{c} := (\gamma_1, \ldots, \gamma_m) \in \mathcal{S}_0$, *we have*

$$Hterm(M_{\mathsf{c}}) = \prod_{i=1}^{m} X_{i\gamma_i}.^{8}$$

**Proof:** $M_{\mathsf{c}}$ is just the combination of all the possible terms $\prod_{i=1}^{m} X_{i\sigma_i}$ where $(\sigma_1, \ldots, \sigma_m)$ runs among the permutations of $(\gamma_1, \ldots, \gamma_m)$. Therefore each term of $M_{\mathsf{c}}$ is divisible by exactly one among the variables $X_{i\gamma_m}$ and, by definition, the greatest ones are those divisible by $X_{m\gamma_m}$. The thesis then follows by induction. $\square$

We now define the monomials $m_{\mathsf{s}}$ for each $\mathsf{s} \in \mathcal{S}_t$ and for each $t$ as follows: let $\mathsf{s} := (\mathsf{c}, \mathsf{r})$, with $\mathsf{c} := (\gamma_1, \ldots, \gamma_{m+t})$, and $d(\mathsf{r}) =: (d_1, \ldots, d_{m+t})$; then

$$m_{\mathsf{s}} = Tdeg(E_{\mathsf{s}}) := \prod_{i=1}^{m+t} X_{d_i\gamma_i}{}^{9}.$$

In particular we have

**Corollary 3.4** $\forall \mathsf{c} := (\gamma_1, \ldots, \gamma_m) \in \mathcal{S}_0, m_{\mathsf{c}} = Hterm(M_{\mathsf{c}}) = \prod_{i=1}^{m} X_{i\gamma_i}$.

[7] The total degree ordering $<$ induced by $X_1 < X_2 < \ldots X_n$ is the one defined by $X_1^{a_1}\cdots X_n^{a_n} =: m_1 < m_2 := X_1^{b_1}\cdots X_n^{b_n}$ iff

$$\deg(m_1) < \deg(m_2) \text{ or } \deg(m_1) = \deg(m_2) \text{ and } \exists j : a_j > b_j, a_i = b_i \forall i < j.$$

[8] In other words the maximal term $Hterm(M_{\mathsf{c}})$ of the determinant of $M_{\mathsf{c}}$ is its diagonal. This choice induces the following generalized definition of $Hterm(m_{\mathsf{s}})$ and justify the introduction of the vector $d(\mathsf{r})$.

[9] You can interpret this formula in this way: build an $(m+n)$ square matrix by writing $\rho_1$ copies of the $1^{st}$ row of $\mathcal{M}$, $\rho_2$ copies of the $2^{nd}$ row of $\mathcal{M}$,..., and then cancelling the columns which are not indexed by the elements $\gamma_i$ of $\mathsf{c}$; then $Tdeg(E_{\mathsf{s}})$ is the diagonal of this matrix.

**Proof:** In fact we have $\mathcal{R} = \{(1,1,\ldots,1)\}$ so that $d_i = i, \forall i$. $\square$

**Lemma 3.5** *It holds*

1. $\{d_0(E_{\mathsf{c}}), \mathsf{c} \in \mathcal{S}_0\}$ *is the Gröbner basis of* $Im(d_0)$;
2. $\{d_1(E_{\mathsf{s}}), \mathsf{s} \in \mathcal{S}_1\}$ *is a basis of* $Ker(d_0)$;
3. $Tdeg(d_1(E_{\mathsf{s}})) = Tdeg(E_{\mathsf{s}}), \forall \mathsf{s} \in \mathcal{S}_1$;
4. $\forall \mathsf{s} := (\mathsf{c}, \mathsf{r}) \in \mathcal{S}_1$, *with* $\mathsf{c} := (\gamma_1, \ldots, \gamma_{m+1})$ *and* $\mathsf{r} = (\rho_1, \ldots, \rho_m)$, *it holds*
$$Hterm(d_1(E_{\mathsf{s}}) = X_{\nu\gamma_\nu} E_{\mathsf{c}(\nu)},$$
*where* $\nu$ *is the single index s.t.* $\rho_\nu = 2$.

**Proof:** The argument follows by considering all possible S-pairs between the elements in $G := \{d_0(E_{\mathsf{c}}), \mathsf{c} \in \mathcal{S}_0\}$; so let

$$\mathsf{c}^{(1)} = (\gamma_1^{(1)}, \ldots, \gamma_m^{(1)}) \text{ and } \mathsf{c}^{(2)} = (\gamma_1^{(2)}, \ldots, \gamma_m^{(2)}).$$

There are two cases to be considered:

- $\mathsf{c}^{(1)}$ and $\mathsf{c}^{(2)}$ differ in more than one position; in this case, let $\nu$ be the higher index s.t. $\gamma_\nu^{(1)} \neq \gamma_\nu^{(2)}$ and w.l.o.g. $\gamma_\nu^{(1)} < \gamma_\nu^{(2)}$. Define then
$$\mathsf{c}^{(3)} = (\gamma_1^{(1)}, \ldots, \gamma_{\nu-1}^{(1)}, \gamma_\nu^{(2)}, \ldots, \gamma_m^{(2)}).$$
Since there is another index $\mu < \nu$ s.t. $\gamma_\mu^{(1)} \neq \gamma_\mu^{(2)}$, we know that both $\mathrm{lcm}(Tdeg(E_{\mathsf{c}^{(1)}}), Tdeg(E_{\mathsf{c}^{(3)}}))$ and $\mathrm{lcm}(Tdeg(E_{\mathsf{c}^{(2)}}), Tdeg(E_{\mathsf{c}^{(3)}}))$ divide and have less degree than $\mathrm{lcm}(Tdeg(E_{\mathsf{c}^{(1)}}), Tdeg(E_{\mathsf{c}^{(2)}}))$. This guarantees that $S(E_{\mathsf{c}^{(1)}}, E_{\mathsf{c}^{(2)}})$ is an element in $\mathfrak{S}(G) \setminus \mathfrak{S}_u(G)$.
- If $\mathsf{c}^{(1)}$ and $\mathsf{c}^{(2)}$ differ exactly in a single position, say the $\nu^{th}$ one, supposing w.l.o.g. $\gamma_\nu^{(1)} < \gamma_\nu^{(2)}$, we define $\mathsf{c} := (\gamma_1, \ldots, \gamma_{m+1})$ by
$$\gamma_i := \begin{cases} \gamma_i^{(1)} = \gamma_i^{(2)} & i < \nu \\ \gamma_\nu^{(1)} & i = \nu \\ \gamma_\nu^{(2)} & i = \nu + 1 \\ \gamma_{i-1}^{(1)} = \gamma_{i-1}^{(2)} & i > \nu + 1. \end{cases}$$
We define then $\mathsf{r} := (\rho_1, \ldots, \rho_m)$ by $\rho_i = \begin{cases} 1 & i \neq \nu \\ 2 & i = \nu \end{cases}$ and $\mathsf{s} := (\mathsf{c}, \mathsf{r})$.
By Lemma 3.1 we know that
$$0 = \delta_0 \delta_1 = \sum_{i=1}^{m+1} (-1)^i X_{\nu\gamma_i} M_{\mathsf{c}(i)};$$

by Lemma 3.3 that

$$Tdeg(X_{\nu\gamma_i}M_{\mathsf{c}(i)}) = \left(\prod_{j=1}^{i-1} X_{j\gamma_j}\right) X_{\nu\gamma_i} \left(\prod_{j=i+1}^{m+1} X_{j-1\ \gamma_j}\right),$$

so that

$$Tdeg(X_{\nu\gamma_i}M_{\mathsf{c}(i)}) \le Tdeg(E_{\mathsf{s}}) = \left(\prod_{j=1}^{\nu} X_{j\gamma_j}\right) \left(\prod_{j=\nu+1}^{m+1} X_{j-1\ \gamma_j}\right)$$

and the equality holds if $i \in \{\nu, \nu+1\}$. As a consequence

$$\begin{aligned} \pm S(E_{\mathsf{c}^{(1)}}, E_{\mathsf{c}^{(2)}}) &= -(-1)^{\nu} X_{\nu\gamma_\nu} M_{\mathsf{c}(\nu)} - (-1)^{\nu+1} X_{\nu\gamma_\nu} M_{\mathsf{c}(\nu+1)} = \\ &= \sum_{i \notin \{\nu,\nu+1\}} (-1)^i X_{\nu\gamma_i} M_{\mathsf{c}(i)} \end{aligned}$$

is a G-representation.

Therefore the existence of a G-representation for any element in $\mathfrak{S}_u(G)$ proves that $G$ is a Gröbner basis (proving 1.) and that the elements

$$\delta_1(E_{\mathsf{s}}) = \sum_{i=1}^{m+1} (-1)^i X_{\nu\gamma_i} E_{\mathsf{c}}(i)$$

form a basis of $Ker(\delta_0)$ (proving 2.). 3. and 4. follow from the computation above. □

On the basis of Lemma 3.5 we introduce the following properties for $t \ge 1$:

**P1**(t) $\{d_t(E_{\mathsf{s}}), \mathsf{s} \in \mathcal{S}_t\}$ is a basis of $Ker(d_{t-1})$;

**P2**(t) $\{d_t(E_{\mathsf{s}}), \mathsf{s} \in \mathcal{S}_t\}$ is the Gröbner basis of $Im(d_t)$;

**P3**(t) $Tdeg(d_t(E_{\mathsf{s}})) = Tdeg(E_{\mathsf{s}}), \forall \mathsf{s} \in \mathcal{S}_t$;

**P4**(t) for each $\mathsf{s} := (\mathsf{c}, \mathsf{r}) \in \mathcal{S}_1$, with $\mathsf{c} := (\gamma_1, \dots, \gamma_{m+1})$, let $j := \min(I_{\mathsf{r}})$; then $Tdeg(d_t(E_{\mathsf{s}})) = X_{j\gamma_j} E_{(\mathsf{c}(j),\mathsf{r}(j))}$.

**Lemma 3.6** *If* $t \ge 1$,

$$\mathbf{P1}(t), \mathbf{P3}(t), \mathbf{P4}(t) \Longrightarrow \mathbf{P1}(t+1), \mathbf{P2}(t), \mathbf{P3}(t+1), \mathbf{P4}(t+1).$$

**Proof:** As we did in Lemma 3.5, we have to consider all the S-pairs of the elements in $\{d_t(E_{\mathsf{s}}), \mathsf{s} \in \mathcal{S}_t\}$: let $\mathsf{s}^{(1)}, \mathsf{s}^{(2)} \in \mathcal{S}_t$ be s.t.

$$\operatorname{lcm}(Hterm(\delta_t(E_{\mathsf{s}^{(1)}})), Hterm(\delta_t(E_{\mathsf{s}^{(2)}}))) = mE_{\mathsf{s}^{(3)}}, \mathsf{s}^{(3)} \in \mathcal{S}_{t-1}.$$

For $i = 1..3$ we denote

$$\mathsf{s}^{(i)} := (\mathsf{c}^{(i)}, \mathsf{r}^{(i)}), \mathsf{c}^{(i)} = (\gamma_1^{(i)}, \gamma_2^{(i)}, \ldots), \mathsf{r}^{(i)} = (\rho_1^{(i)}, \ldots \rho_m^{(i)}),$$

and we set $j := min(I_{\mathsf{r}^{(1)}})$ and $k := min(I_{\mathsf{r}^{(2)}})$ so that

$$\mathsf{r}^{(3)} = \mathsf{r}^{(1)}(j) = \mathsf{r}^{(2)}(k).$$

Assuming w.l.o.g. $j \leq k$ we have then

$$\rho_i^{(3)} = \rho_i^{(1)} = \rho_i^{(2)} = 1, \forall i < j, \tag{3.1}$$

and either $j = k$ or

- $j < k$,
- $\rho_j^{(3)} = \rho_j^{(2)} = 1, \rho_j^{(1)} = 2;$
- $\rho_i^{(1)} = \rho_i^{(3)} = \rho_i^{(2)} = 1, \forall j < i < k;$
- $\rho_k^{(1)} = \rho_k^{(3)} = 1, \rho_k^{(2)} = 2;$
- $\rho_i^{(1)} = \rho_i^{(3)} = \rho_i^{(2)}, \forall i > k.$

We then define $\mathsf{r} := \mathsf{r}^{(4)} = (\rho_1^{(4)}, \ldots \rho_m^{(4)})$ in the case $j = k$ as

$$\rho_i^{(4)} := \begin{cases} \rho_i^{(3)} = \rho_i^{(2)} = \rho_i^{(1)} & i \neq j = k \\ \rho_i^{(3)} + 2 = \rho_i^{(2)} + 1 = \rho_i^{(1)} + 1 & i = j = k, \end{cases}$$

while in case $j < k$ we define it as

$$\rho_i^{(4)} := \begin{cases} \rho_i^{(3)} = \rho_i^{(2)} = \rho_i^{(1)} & j \neq i \neq k \\ 2 & i = j \\ \rho_i^{(2)} = \rho_i^{(1)} + 1 = \rho_i^{(3)} + 1 & i = k, \end{cases}$$

so that, in both cases

$$\mathsf{r}^{(1)} = \mathsf{r}^{(4)}(k), \mathsf{r}^{(2)} = \mathsf{r}^{(4)}(j).$$

Since $\mathsf{c}^{(3)} = \mathsf{c}^{(1)}(j) = \mathsf{c}^{(2)}(k)$, we then define $\mathsf{c} := \mathsf{c}^{(4)} = (\gamma_1^{(4)}, \ldots \gamma_{m+t+1}^{(4)})$ as follows:

- if $j = k$, supposing w.l.o.g. $\gamma_j^{(1)} < \gamma_j^{(2)}$, we set

$$\gamma_i^{(4)} := \begin{cases} \gamma_i^{(3)} = \gamma_i^{(2)} = \gamma_i^{(1)} & i < j \\ \gamma_i^{(1)} & i = j \\ \gamma_{i-1}^{(2)} & i = j+1 \\ \gamma_{i-2}^{(3)} = \gamma_{i-1}^{(2)} = \gamma_{i-1}^{(1)} & i > j+1; \end{cases}$$

- if $j < k$ we set

$$\gamma_i^{(4)} := \begin{cases} \gamma_i^{(3)} = \gamma_i^{(2)} = \gamma_i^{(1)} & i < j \\ \gamma_i^{(1)} & i = j \\ \gamma_{i-1}^{(3)} = \gamma_{i-1}^{(2)} = \gamma_i^{(1)} & j < i \leq k \\ \gamma_{i-1}^{(2)} & i = k+1 \\ \gamma_{i-2}^{(3)} = \gamma_{i-1}^{(2)} = \gamma_{i-1}^{(1)} & i > k+1. \end{cases}$$

In both cases

$$\mathsf{c}^{(1)} = \mathsf{c}^{(4)}(k+1), \mathsf{c}^{(2)} = \mathsf{c}^{(4)}(j).$$

Then we can define $\mathsf{s}^{(4)} := \mathsf{s} := (\mathsf{r}, \mathsf{c}) \in \mathcal{S}_{t+1}$, so that

$$d_{t+1}(E_\mathsf{s}) = \sum_{\mu=1}^{m+t+1} \sum_{\nu \in I_\mathsf{r}} (-1)^\mu X_{\nu\mu} E_{(\mathsf{r}(\nu), \mathsf{c}(\mu))}.$$

We have then to verify that it holds

$$Tdeg(X_{\nu\mu} E_{(\mathsf{r}(\nu),\mathsf{c}(\mu))}) \leq Tdeg(d_{t+1}(E_\mathsf{s})),$$

and that the equality is satisfied only in the cases $\nu = j = \mu$, $\nu = k = \mu - 1$, so that

$$Tdeg(d_{t+1}(E_\mathsf{s})) = Tdeg(X_{j\gamma_j} E_{(\mathsf{r}(j),\mathsf{c}(j))}) = Tdeg(E_\mathsf{s}). \tag{3.2}$$

Since this analysis holds for all possible pairs in $\mathfrak{S}_u$, we can conclude **P1**(t+1) and **P2**(t); **P3**(t+1) follows from Eq. 3.2 and **P4**(t+1) from Eq. 3.1[10]. □

Obviously this series of lemmata, implies a proof of Theorem.

# 4 Example

We can present here the computation I did with $m = 3, n = 5$ in which I denote $E_{(\mathsf{r},\mathsf{c})}$, with $\mathsf{c} := (\gamma_1, \ldots, \gamma_\mu)$ and $\mathsf{r} := (\rho_1, \ldots, \rho_\nu)$ as $E_{\rho_1,\ldots,\rho_\nu;\gamma_1,\ldots,\gamma_\mu}$.

$$\begin{aligned} E_{111;123} := \; & X_{11}X_{22}X_{33} - X_{11}X_{32}X_{23} - X_{21}X_{12}X_{33} + X_{21}X_{32}X_{13} + \\ & + X_{31}X_{12}X_{23} - X_{31}X_{22}X_{13} \end{aligned}$$

[10] To be more precise we must remark that

$$Tdeg(d_t(E_\mathsf{s})) = X_{j\gamma_h} Tdeg(E_{(\mathsf{c}(h),\mathsf{r}(j))}), j \leq h \leq j + \rho_j - 1.$$

To guarantee the claimed result, one must apply the ordering $<_t$ on $\mathsf{T}_t$; such ordering depends on an ordering $\prec$ on $\mathcal{S}_t$.

My note suggested to use the ordering $(\mathsf{r}, \mathsf{c}) \prec (\mathsf{r}', \mathsf{c}')$ iff $\mathsf{c}$ "is lexicographically less than" $\mathsf{c}'$ but, unfortunately it is not clear to me of which ordering I was thinking.

$$
\begin{aligned}
E_{111;124} &:= X_{11}X_{22}X_{43} - X_{11}X_{42}X_{23} - X_{21}X_{12}X_{43} + X_{21}X_{42}X_{13} + \\
&\quad +X_{41}X_{12}X_{23} - X_{41}X_{22}X_{13} \\
E_{111;125} &:= X_{11}X_{22}X_{53} - X_{11}X_{52}X_{23} - X_{21}X_{12}X_{53} + X_{21}X_{52}X_{13} + \\
&\quad +X_{51}X_{12}X_{23} - X_{51}X_{22}X_{13} \\
E_{111;134} &:= X_{11}X_{32}X_{43} - X_{11}X_{42}X_{33} - X_{31}X_{12}X_{43} + X_{31}X_{42}X_{13} + \\
&\quad +X_{41}X_{12}X_{33} - X_{41}X_{32}X_{13} \\
E_{111;135} &:= X_{11}X_{32}X_{53} - X_{11}X_{52}X_{33} - X_{31}X_{12}X_{53} + X_{31}X_{52}X_{13} + \\
&\quad +X_{51}X_{12}X_{33} - X_{51}X_{32}X_{13} \\
E_{111;145} &:= X_{11}X_{42}X_{53} - X_{11}X_{52}X_{43} - X_{41}X_{12}X_{53} + X_{41}X_{52}X_{13} + \\
&\quad +X_{51}X_{12}X_{43} - X_{51}X_{42}X_{13} \\
E_{111;234} &:= X_{21}X_{32}X_{43} - X_{21}X_{42}X_{33} - X_{31}X_{22}X_{43} + X_{31}X_{42}X_{23} + \\
&\quad +X_{41}X_{22}X_{33} - X_{41}X_{32}X_{23} \\
E_{111;235} &:= X_{21}X_{32}X_{53} - X_{21}X_{52}X_{33} - X_{31}X_{22}X_{53} + X_{31}X_{52}X_{23} + \\
&\quad +X_{51}X_{22}X_{33} - X_{51}X_{32}X_{23} \\
E_{111;245} &:= X_{21}X_{42}X_{53} - X_{21}X_{52}X_{43} - X_{41}X_{22}X_{53} + X_{41}X_{52}X_{23} + \\
&\quad +X_{51}X_{22}X_{43} - X_{51}X_{42}X_{23} \\
E_{111;345} &:= X_{31}X_{42}X_{53} - X_{31}X_{52}X_{43} - X_{41}X_{32}X_{53} + X_{41}X_{52}X_{33} + \\
&\quad +X_{51}X_{32}X_{43} - X_{51}X_{42}X_{33}
\end{aligned}
$$

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$$
\begin{aligned}
E_{211;2345} &:= -X_{15}E_{111;234} + X_{14}E_{111;235} - X_{13}E_{111;245} + X_{12}E_{111;345} \\
E_{211;1345} &:= -X_{15}E_{111;134} + X_{14}E_{111;135} - X_{13}E_{111;145} + X_{11}E_{111;345} \\
E_{211;1245} &:= -X_{15}E_{111;124} + X_{14}E_{111;125} - X_{12}E_{111;145} + X_{11}E_{111;245} \\
E_{211;1235} &:= -X_{15}E_{111;123} + X_{13}E_{111;125} - X_{12}E_{111;135} + X_{11}E_{111;235} \\
E_{211;1234} &:= -X_{14}E_{111;123} + X_{13}E_{111;124} - X_{12}E_{111;134} + X_{11}E_{111;234} \\
E_{121;2345} &:= -X_{25}E_{111;234} + X_{24}E_{111;235} - X_{23}E_{111;245} + X_{22}E_{111;345} \\
E_{121;1345} &:= -X_{25}E_{111;134} + X_{24}E_{111;135} - X_{23}E_{111;145} + X_{21}E_{111;345} \\
E_{121;1245} &:= -X_{25}E_{111;124} + X_{24}E_{111;125} - X_{22}E_{111;145} + X_{21}E_{111;245} \\
E_{121;1235} &:= -X_{25}E_{111;123} + X_{23}E_{111;125} - X_{22}E_{111;135} + X_{21}E_{111;235} \\
E_{121;1234} &:= -X_{14}E_{111;123} + X_{23}E_{111;124} - X_{22}E_{111;245} + X_{21}E_{111;234} \\
E_{112;2345} &:= -X_{35}E_{111;234} + X_{34}E_{111;235} - X_{33}E_{111;245} + X_{32}E_{111;345} \\
E_{112;1345} &:= -X_{35}E_{111;134} + X_{34}E_{111;135} - X_{33}E_{111;145} + X_{31}E_{111;345} \\
E_{112;1245} &:= -X_{35}E_{111;124} + X_{34}E_{111;125} - X_{32}E_{111;145} + X_{21}E_{111;245} \\
E_{112;1235} &:= -X_{35}E_{111;123} + X_{33}E_{111;125} - X_{32}E_{111;135} + X_{31}E_{111;235} \\
E_{112;1234} &:= -X_{34}E_{111;123} + X_{33}E_{111;124} - X_{32}E_{111;245} + X_{31}E_{111;234}
\end{aligned}
$$

---

$$
\begin{aligned}
E_{113;12345} &:= X_{31}E_{112;2345} - X_{32}E_{112;1345} + X_{33}E_{112;1245} - \\
&\quad -X_{34}E_{112;1235} + X_{35}E_{112;1234} \\
E_{131;12345} &:= X_{21}E_{121;2345} - X_{22}E_{121;1345} + X_{23}E_{121;1245} - \\
&\quad -X_{24}E_{121;1235} + X_{25}E_{121;1234} \\
E_{311;12345} &:= X_{11}E_{211;2345} - X_{12}E_{211;1345} + X_{13}E_{211;1245} - \\
&\quad -X_{14}E_{211;1235} + X_{15}E_{211;1234} \\
E_{122;12345} &:= X_{31}E_{121;2345} + X_{21}E_{112;2345} - X_{32}E_{121;1345} - \\
&\quad -X_{22}E_{112;1345} + X_{33}E_{121;1245} + X_{23}E_{112;1245} - \\
&\quad -X_{34}E_{121;1235} - X_{24}E_{112;1235} + X_{35}E_{121;1234} + \\
&\quad +X_{25}E_{112;1234} \\
E_{212;12345} &:= X_{31}E_{211;2345} + X_{11}E_{112;2345} - X_{32}E_{211;1345} - \\
&\quad -X_{12}E_{112;1345} + X_{33}E_{211;1245} + X_{13}E_{112;1245} - \\
&\quad -X_{34}E_{211;1235} - X_{14}E_{112;1235} + X_{35}E_{211;1234} + \\
&\quad +X_{15}E_{112;1234} \\
E_{221;12345} &:= X_{21}E_{211;2345} + X_{11}E_{121;2345} - X_{22}E_{211;1345} - \\
&\quad -X_{12}E_{121;1345} + X_{23}E_{211;1245} + X_{13}E_{121;1245} - \\
&\quad -X_{24}E_{211;1235} - X_{14}E_{121;1235} + X_{25}E_{211;1234} + \\
&\quad +X_{15}E_{121;1234}
\end{aligned}
$$

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