

# Breaking Barriers:

## Static Hedging of Barrier Securities

Peter Carr  
Morgan Stanley  
1585 Broadway, 6th floor  
New York, NY 10036  
(212) 761-7340  
carrp@morgan.com

Andrew Chou  
MIT Computer Science  
545 Technology Square  
Cambridge, MA 02139  
(617) 253-3796  
achou@theory.lcs.mit.edu

Current Version: November 7, 1996

This version is extremely preliminary. Any errors are our own. We would like to thank Nick Firoozye, Galin Georgiev, Ciamac Moallemi, Andrew Smith, and Ravi Sundaram for fruitful discussions.

# Breaking Barriers: Static Hedging of Barrier Securities

## 1 Introduction

Exotic options are very sophisticated instruments, and the techniques used to hedge and value them are fairly complex. The most common type of exotic options are barrier options, which were introduced in American over-the-counter markets years before vanilla options were listed (see Snyder[19]). Barrier options are special cases of barrier *securities*, which may involve single or multiple barriers. Examples of the latter include double barrier options, rolldown calls, and even lookback options. In this paper, we focus on single barrier securities for simplicity, leaving multiple barrier options for future research. Single barrier securities allow for an arbitrary payoff at maturity provided that the barrier has been touched (in-barrier securities) or not touched (out-barrier securities). They are usually further classified into “down securities” (barrier below spot) and “up securities” (barrier above spot).

The standard methodology for hedging and valuing barrier securities applies dynamic replication strategies in the underlying assets. In this paper, we hope to add insight into these structures by looking at them in a non-traditional manner. In particular, we show how barrier securities can be broken up into more fundamental securities, which in turn can be created out of vanilla European options. This allows us to hedge path-dependent barrier securities with path-independent vanilla options, with trading in the latter options occurring only at the initiation and the expiration<sup>1</sup> of the hedge. Due to the relative infrequency of trading, such hedges are commonly termed static.

In common with dynamic hedging, static hedging provides valuation formulas for barrier securities. When both types of hedging strategies are cast in the same economic model, the formulas result in identical values. In this paper, we will also show how the valuation formulas based on dynamic replication can be used to uncover the static hedge. Since formulas for barrier options have been available since the seminal paper of Merton[13], these formulas are now widely available (see Nelken[15] and Zhang[21] for surveys on

---

<sup>1</sup>We define the expiration of the hedge as the earlier of maturity and the first hitting time of the barrier.

exotic options). This result can be used to determine static hedges for a host of exotic options beyond those explicitly presented in this paper. (eg. double and partial barrier options.)

We will illustrate our decomposition results with several commonly available types of barrier securities. We give explicit results for down barriers, and exhort the reader to use published formulas for up barrier options to uncover the static hedge. Special cases of down securities covered include down calls, down puts, and several types of binary options. By definition, a European one-touch binary put pays one dollar at maturity if the lower barrier has been hit, while an American one-touch binary pays a dollar at the first hitting time, if any. By contrast, a no-touch binary pays one dollar at maturity if the barrier is never reached. We indicate the static hedge for all three types of binaries.

While we will employ a model in which dynamic and static replication strategies both work perfectly well in theory, there are a couple of reasons to believe that static hedging may be the method of choice in practice. First, a literal interpretation of dynamic replication requires continuous trading, which would generate ruinous transactions costs if implemented in practice. The standard kludge for this problem is to trade periodically, which leads to acceptably low approximation error when the gamma of the security is low. However, barrier options often have regions of high gamma, which while catastrophic to these periodically rebalanced strategies, is of no consequence whatsoever to the static hedger, so long as the investor can trade at the first passage time to the barrier. Jumps across the barrier can induce sub or super-replication for both types of strategies, with the superior strategy usually being identifiable in advance. Second, dynamic replication requires estimation of the future carrying costs and volatility rate of the underlying asset. The error arising from using the wrong volatility rate in dynamic replication is directly proportional to the option's vega, which is again often high<sup>2</sup> for barrier securities. By contrast, static replication relies only on knowing the *implied* volatility of the vanilla options at the entry of the static hedge and when the underlying is at the barrier. The realized volatility during the life of the hedge is of no consequence to the static hedger, except

---

<sup>2</sup>Carr[5] shows that high gammas imply high vegas for any European-style path-independent claim. Since this paper shows that barrier securities may be viewed as such claims, the same results apply to them.

to the extent that it impacts implied volatility.

The above benefits of static hedging may well be embedded in the implied volatility, which is typically greater than historical volatility. However, if this premium is paid at the initiation of the static hedge, it should at least be partially returned if the hedge is liquidated at the barrier. Thus, the main disadvantage of static hedging over dynamic hedging in practice appears to be the relative illiquidity of the standard options market when compared to the market for the underlying asset. Perhaps, this paper will help mitigate this disadvantage. In any case, empirical work is needed to compare the relative viability of these approaches.

Like most work on derivatives, the literature on static hedging may be traced back to work by Ross (see [18]), who showed how the availability of vanilla options in all strikes may be used to statically replicate path-independent payoffs. Breeden and Litzenberger[4] showed that butterfly spreads can be used to observe the prices of fundamental derivatives called Arrow Debreu securities. Green and Jarrow[11], Nachman[14], and Carr and Madan[7] extend this line of research to multiple underlying assets. Ho[12] and this paper extend this line of research to path-dependent securities.

Bowie and Carr[3] introduce static hedging for single barrier options, using the Black[1] model which restricts the drift of the underlying to be zero. Carr, Ellis, and Gupta[6] extend these results to a symmetric volatility structure and more complex instruments, such as double and partial barrier options. Derman, Ergener, and Kani[9],[10] relax the drift restriction in the foregoing papers by introducing an algorithm for hedging single barrier options in a binomial model, using options with a single strike but multiple expiries. By contrast, this paper provides explicit formulas for static hedges in the standard Black Scholes[2] model using options with the same expiry but multiple strikes.

## 2 Static Replication with Barrier Options

### 2.1 Assumptions

As stated, we will eventually use the standard Black Scholes[2] model. However, we first develop a set of results in a more general setting. Thus, we initially assume only that markets are frictionless and arbitrage-free. To

simplify notation, we also assume that investors can borrow or lend at a constant riskless rate  $r$  and that the underlying asset is a stock, with a constant dividend yield,  $d$ . The results in this section easily extend to stochastic interest rates and dividend yields.

An implication of frictionless markets is that investors can continuously trade in all barrier securities. For the purposes of this section, hedges will only require a liquid market in knockin options with the same trigger as the barrier security, but with any positive strike. Just as continuous trading is accepted as a reasonable approximation to reality even though markets close daily, we treat the availability of a continuum of strikes as an approximation of the over-the-counter market in barrier options. While we fully recognize that markets for barrier securities have limited liquidity in practice, we note that path-independent payoffs arise as special cases<sup>3</sup> of the payoffs from barrier securities. Given the advent of flex options in the listed market and the emergence of a substantial over-the-counter market in vanilla options, our liquidity assumption is tenable for this important special case. Hopefully, the skeptical reader will find the results of this section on barrier securities to be of interest, even when restricted to the path-independent case.

## 2.2 Decomposition into Fundamental Securities

In this subsection, we first show how a butterfly spread of down-and-in options can be used to create a fundamental security called a “down-and-in Arrow”. These securities are fundamental in the sense that an arbitrary payoff from a down-and-in security may be easily decomposed into a portfolio of such securities. It follows that the arbitrary down-and-in security can be statically hedged and valued by a portfolio of down-and-in options. The next section shows how the down-and-in options can in turn be replicated with vanilla options. The net result is that a barrier security can be statically hedged with a portfolio of vanilla options.

Our decomposition of barrier securities into down-and-in options can be applied to the path-independent case by moving the barrier appropriately. In particular, we can decompose claims with an arbitrary path-independent payoff into a static portfolio consisting of listed instruments such as bonds,

---

<sup>3</sup>To achieve results for path-independent securities from the corresponding results for in-barrier securities, the barrier is moved to the spot. Conversely, to obtain results from out-barrier securities, the barrier is moved an infinite distance away from the spot.

forward contracts, and vanilla options. A special case of this formula leads to a new decomposition of the claim value into intrinsic and time value. Besides being of intrinsic interest, our results on the path-independent case can be combined with our results for in-securities to obtain the the corresponding static hedge and valuation formula for “out-securities”.

### 2.3 Static Replication with “Arrows”

A simple way to bet that the underlying will finish around  $K$  is to form a butterfly spread with vanilla calls, which costs:

$$BS(K) = C(K - \Delta K) - 2C(K) + C(K + \Delta K),$$

where  $C(K)$  is the current price of a call struck at  $K$ . If we wish to also bet that a lower barrier was hit, the butterfly spread should be formed from down-and-in calls with the same barrier:

$$DIBS(K, H) = DIC(K - \Delta K, H) - 2DIC(K, H) + DIC(K + \Delta K, H),$$

where  $DIC(K, H)$  is the current price of a down-and-in call struck at  $K$  with barrier  $H$ . So long as the barrier has been hit, the final payoff of this position is a triangle as shown in Figure 1.

Note that the area under the triangle is:

$$\frac{1}{2} \times 2\Delta K \times \Delta K = (\Delta K)^2.$$

Thus, this bet can be normalized so that the area under the triangle is one:

$$NDIBS(K, H) = \frac{DIC(K - \Delta K, H) - 2DIC(K, H) + DIC(K + \Delta K, H)}{(\Delta K)^2}.$$

As  $\Delta K$  approaches zero, the base of the triangular payoff gets smaller and the height gets taller, so that the area is maintained at one. The limiting payoff approaches what is known as a Dirac delta function. The security providing this payoff is called an *Arrow-Debreu security*, named after their Nobel prize winning founders, Kenneth Arrow and Gerard Debreu. Given their heritage and their payoff, we will refer to these securities as “down-and-in Arrows” (see Figure 2).

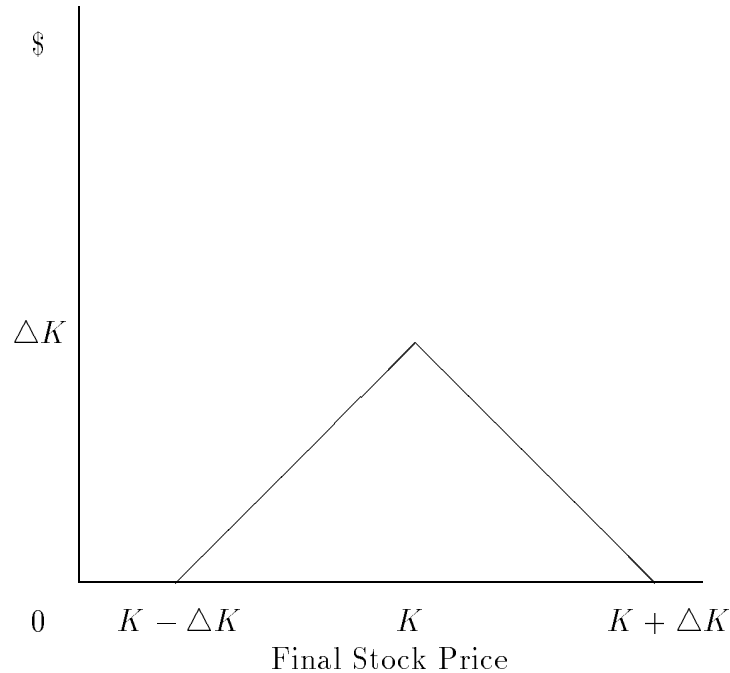


Figure 1: Payoff of a Down-and-In Butterfly Spread.

Since the payoff of a down-and-in Arrow is non-standard, it may be properly called a second generation derivative security. The name is appropriate in another sense since the value of a down-and-in Arrow is given by the second derivative of the down-and-in call value with respect to its strike:

$$DIA(K, H) = \frac{\partial^2 DIC(K, H)}{\partial K^2}, \quad (1)$$

where  $DIA(K, H)$  denotes the current value of a down-and-in Arrow struck at  $K$  with barrier  $H$ .

We next show how Arrows can alternatively be observed from put values. Let  $DIB(H)$  be the value of a down-and-in bond which pays one dollar at  $T$ , so long as the stock price has hit the barrier  $H$  previously. Similarly, let  $DIS(H)$  denote the value of a down-and-in stock, which pays the stock price at  $T$ , so long as the barrier has been hit beforehand. There is a simple

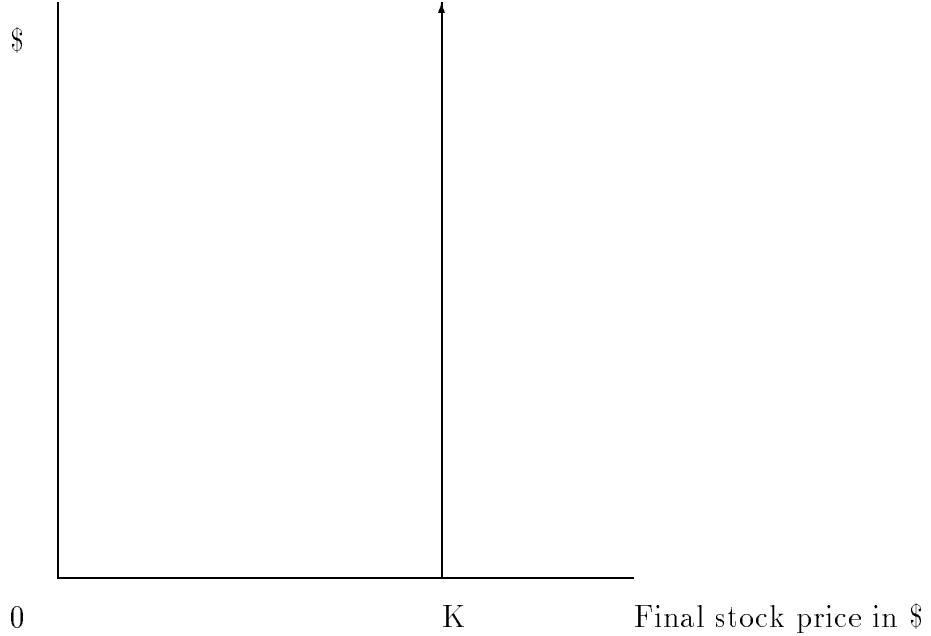


Figure 2: Payoff of a Down-and-In Arrow.

generalization of put-call-parity involving these contracts:

$$DIC(K, H) = DIS(H) - KDIB(H) + DIP(K, H). \quad (2)$$

Differentiating twice with respect to the strike  $K$  implies that the down-and-in Arrow's value can alternatively be derived from down-and-in put values:

$$DIA(K, H) = \frac{\partial^2 DIP(K, H)}{\partial K^2}. \quad (3)$$

Now, let  $f(S)$  denote an arbitrary final payoff received so long as the barrier has been hit. By buying and holding a portfolio of down-and-in Arrows of all strikes with the number of Arrows at strike  $K$  given by  $f(K)dK$ , an investor can synthesize the payoff  $f(S)$ . Absence of arbitrage thereby requires that the value of the down-and-in claim  $DIV(H)$  paying  $f(S)$  at maturity is simply:

$$DIV(H) = \int_0^\infty f(K)DIA(K, H)dK. \quad (4)$$



When viewed as functions of their strike, down-and-in puts have zero value and slope at  $K = 0$ , while down-and-in calls have zero value and slope at  $K = \infty$ . This observation motivates rewriting (4) as:

$$DIV(H) = \int_0^\kappa f(K) \frac{\partial^2 DIP(K, H)}{\partial K^2} dK + \int_\kappa^\infty f(K) \frac{\partial^2 DIC(K, H)}{\partial K^2} dK, \quad (5)$$

where  $\kappa$  is an arbitrary positive constant. Integrating by parts twice and using (2) yields the following decomposition of an arbitrary down-and-in claim into down-and-in versions of zeros, forward contracts<sup>4</sup>, and options<sup>5</sup> :

$$\begin{aligned} DIV(H) = & f(\kappa)DIB(H) + f'(\kappa)[DIS(H) - \kappa DIB(H)] \\ & + \int_0^\kappa f''(K)DIP(K, H)dK + \int_\kappa^\infty f''(K)DIC(K, H)dK. \end{aligned} \quad (6)$$

Thus, to synthesize the payoff  $f(S)$  received at  $T$  if the barrier has been hit, buy and hold a portfolio consisting of  $f(\kappa)$  down-and-in bonds,  $f'(\kappa)$  down-and-in forward contracts with delivery price  $\kappa$ ,  $f''(K)dK$  down-and-in puts of all strikes  $K \leq \kappa$ , and  $f''(K)dK$  down-and-in calls of all strikes  $K > \kappa$ .

Setting the barrier  $H$  to infinity in (6) yields the corresponding result for path-independent payoffs as a special case:

$$V = f(\kappa)e^{-rT} + f'(\kappa)[Se^{-dT} - \kappa e^{-rT}] + \int_0^\kappa f''(K)P(K)dK + \int_\kappa^\infty f''(K)C(K)dK. \quad (7)$$

Further setting  $\kappa$  to the forward price  $F \equiv Se^{(r-d)T}$  yields a new decomposition of a claim with an arbitrary path-independent payoff into its intrinsic value,  $f(F)e^{-rT}$ , and its time value:

$$V = f(F)e^{-rT} + \int_0^F f''(K)P(K)dK + \int_F^\infty f''(K)C(K)dK. \quad (8)$$

---

<sup>4</sup>Note that barrier forward contracts are easily synthesized by buying a barrier call and writing a barrier put.

<sup>5</sup>Technically, (6) holds only for payoffs  $f(K)$  which satisfy:

$$\begin{aligned} \lim_{K \downarrow 0} f(K) \frac{\partial DIP(K, H)}{\partial K} = 0 & \quad \lim_{K \downarrow 0} f'(K)DIP(K, H) = 0 \\ \lim_{K \uparrow \infty} f(K) \frac{\partial DIC(K, H)}{\partial K} = 0 & \quad \lim_{K \uparrow \infty} f'(K)DIC(K, H) = 0. \end{aligned}$$

It is difficult to imagine payoffs arising in practice not satisfying these restrictions.

The time value is expressed in terms of the prices of out-of-the-money-forward puts and calls. Since puts and calls with the same strike have the same time value, the time value of an arbitrary claim is simply a linear combination of the time values of an option, with coefficients given by the second derivative of the payoff. If the payoff is linear, then  $f''(K) = 0$  for all  $K$  and there is no time value. Conversely, if the payoff is globally convex ( $f''(K) \geq 0$  for all  $K$ ), then the time value is positive. Finally, note that if we restrict attention to Black's model, then as the underlying gets more volatile, the option values grow, and therefore so does the time value.

If we set  $\kappa$  to infinity or zero in (6), then certain types of contracts can be eliminated from the static hedge, provided  $f$  behaves reasonably at these extremes. For example, setting  $\kappa$  to zero and  $H$  to infinity in (6) eliminates puts from the static hedge, provided that  $f(0)$  and  $f'(0)$  are bounded:

$$V = f(0)B + f'(0)Se^{-dT} + \int_0^\infty f''(K)C(K)dK. \quad (9)$$

If  $f$  is not smooth, generalized functions such as Heaviside step functions and its derivatives may be needed. For example, to replicate a vanilla put's payoff of  $f(S) = \max(0, K_p - S)$ , (9) indicates that one should buy  $K$  zeros, sell  $e^{-dT}$  shares, and buy a call struck at  $K_p$ . Thus, (7) generates put-call-parity as a special case. Similarly, (7) can be used to generate the replication of digital options using vertical spreads.

To generate the corresponding results for down-and-out securities, subtract (6) from (7) and use in-out parity:

$$\begin{aligned} DOV(H) = & f(\kappa)DOB(H) + f'(\kappa)[DOS(H) - \kappa DOB(H)] \\ & + \int_0^\kappa f''(K)DOP(K, H)dK + \int_\kappa^\infty f''(K)DOC(K, H)dK. \end{aligned} \quad (10)$$

To generate results for up-securities, replace  $D$  by  $U$  in all the above results.

### 3 Static Replication with Vanilla Options

The last section showed how barrier securities can be statically replicated with barrier options. This section shows how the same securities can be statically replicated with vanilla options. Replicability of this kind is important since vanilla options are more liquid in practice and their prices are more

transparent. However, the replication is achieved at the theoretical cost of requiring the imposition of the rest of the Black Scholes assumptions. We therefore assume that the stock price obeys a lognormal process with a constant volatility rate  $\sigma$ . Importantly, the price process is continuous, so that the underlying cannot jump across the barrier<sup>6</sup>.

### 3.1 Main Intuition

In the appendix, we give a formal derivation of our static hedging technique. In this subsection, we provide the main intuition for our results. The actual technique is fairly direct, although its simplicity may be lost in the details.

Consider a down-and-in call option. If the barrier is never reached, it will expire worthless at maturity. Upon reaching the barrier, it becomes identical to a vanilla call. To replicate this exotic, we want a portfolio of European options to imitate this behavior. If the barrier is never reached, our portfolio should be worthless at maturity, and at the barrier, it should always be equivalent to a call.

Depending on its strike, a down-and-in call can have payoffs both above and below the barrier. For payoffs below the barrier, the requirement that the in-barrier be touched is superfluous, and so we can replicate with European options using the results of the last section. For payoffs above the barrier, we will reflect these payoffs below the barrier. The reflected payoffs will be constructed to have a value matching that of the original payoffs whenever the stock price is at the barrier. Thus, we can also replicate the reflected payoffs with European options to complete our static hedge.

### 3.2 Adjusted Payoff

More generally, suppose a European security has final payoff  $f(S_T)$ . The appendix shows that the down-and-in version of this security with barrier  $H$  has the same value as a portfolio of European options with a payoff of:

$$\hat{f}(S_T) \equiv \begin{cases} 0 & \text{if } S_T > H, \\ f(S_T) + \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H, \end{cases}$$

---

<sup>6</sup>If jumps were possible, one can forecast whether our static portfolio will overvalue or undervalue the exotic.

Barrier Security	Adjusted Payoff	
No-touch binary put	1	for $S_T > H$
	$-(S_T/H)^p$	for $S_T < H$
One-touch binary put (European)	0	for $S_T > H$
	$1 + (S_T/H)^p$	for $S_T < H$
Down-and-out call	$\max(S_T - K_c, 0)$	for $S_T > H$
	$-(S_T/H)^p \max((H^2/S_T) - K_c, 0)$	for $S_T < H$
Down-and-out put	$\max(K_p - S_T, 0)$	for $S_T > H$
	$-(S_T/H)^p \max(K_p - (H^2/S_T), 0)$	for $S_T < H$

Table 1: Adjusted Payoffs for Down Securities. ( $p = 1 - \frac{2(r-d)}{\sigma^2}$ )

where the power  $p \equiv 1 - \frac{2(r-d)}{\sigma^2}$ . We call  $\hat{f}(S_T)$  the *adjusted payoff* for the down-and-in security. For a down-and-out security, in-out parity implies that the adjusted payoff is:

$$\hat{f}(S_T) \equiv \begin{cases} f(S_T) & \text{if } S_T > H, \\ -\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H. \end{cases}$$

In Table 1 and Figure 3, we show the adjusted payoff for some common securities.

Upon inspection of the table, the adjusted payoffs are usually not piecewise linear. Thus, an exact replication using a finite number of European puts and calls is usually not possible. However, as Figure 3 makes clear, the payoffs are close to linear. Furthermore, a few special cases are worth mentioning. When  $r = d$ , then  $p = 1$  and all payoffs are linear. The resulting payoffs are identical to the results given in Bowie and Carr[3]. Also, for  $r - d = \frac{\sigma^2}{2}$ , then  $p = 0$  and the binary payoffs are linear. In particular, a one-touch binary can be exactly replicated by two digitals.

Given the adjusted payoff, the value of the replicating portfolio can be determined by risk-neutral valuation:

$$V(S, \tau) = \int_0^\infty \hat{f}(K) A(S, \tau; K) dK, \quad S > 0,$$

where the Appendix shows that the value of an Arrow in the Black Scholes

model is:

$$A(S, \tau; K) = e^{-r\tau} \frac{1}{K \sqrt{2\pi\sigma^2\tau}} \exp \left\{ -\frac{(\ln(K/S) - (r - d - \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau} \right\}.$$

The value of the down-and-in claim is obtained by restricting this value to stock prices above the barrier:

$$DIV(S, \tau; H) = V(S, \tau), \quad S > H.$$

### 3.3 Derivation from Pricing Formula

In this section, we derive the static hedge in another manner. We suppose that a pricing formula for a barrier security is known, either because it exists in the literature (see eg. [17]), or because it has been derived using dynamic replication arguments. We then show how this formula can be used to generate a static hedge using vanilla options. The advantage of approaching static hedging in this manner is that it is very simple and the approach can be used to generate static hedges for a wide set of securities.

For simplicity, we again work with down securities only. We essentially work backwards from the results of last section. Thus, we assume we know the formula  $D(S, \tau)$  for a down security as a function of the current stock price  $S$  and the time to maturity  $\tau$ . The first step is to find the value of the replicating option portfolio for any stock price by simply removing the restriction that stock prices are above the barrier:

$$V(S, \tau) = D(S, \tau), \quad S > 0. \quad (11)$$

The second step is to obtain the adjusted payoff which gave rise to this value. Since values converge to their payoff at maturity, simply take the limit of the value as the time to maturity approaches zero:

$$\hat{f}(S) = \lim_{\tau \downarrow 0} V(S, \tau), \quad S > 0. \quad (12)$$

The third step is to use (7) with  $\kappa = H$  to uncover the requisite static position in bonds, forward contracts, and vanilla options.

We illustrate this three step procedure with a down-and-in call struck at  $K_c > H$ . From Merton[13], the valuation formula is:

$$\begin{aligned}
DIC(S, \tau; H) &= S e^{-d\tau} \left(\frac{S}{H}\right)^{p-2} N\left(\frac{\ln\left(\frac{H^2}{SK_c}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) \\
&\quad - K_c e^{-r\tau} \left(\frac{S}{H}\right)^p N\left(\frac{\ln\left(\frac{H^2}{SK_c}\right) + \left(r - d - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right), S > H.
\end{aligned}$$

Removing the requirement that  $S > H$ , letting  $\tau \downarrow 0$ , and denoting the indicator function by  $1(\cdot)$  gives:

$$\begin{aligned}
\lim_{\tau \downarrow 0} DIC(S, \tau; H) &= S \left(\frac{S}{H}\right)^{p-2} 1\left(\frac{H^2}{S} > K_c\right) - K_c \left(\frac{S}{H}\right)^p 1\left(\frac{H^2}{S} > K_c\right) \\
&= \left(\frac{S}{H}\right)^p \left(\frac{H^2}{S} - K_c\right) 1\left(\frac{H^2}{S} > K_c\right) \\
&= \left(\frac{S}{H}\right)^p \max\left(0, \frac{H^2}{S} - K_c\right).
\end{aligned}$$

Thus, using in-out parity, the adjusted payoff for a down-and-out call agrees with Table 1 (recall  $K_c > H$ ). The third step is to statically replicate the down-and-in call's adjusted payoff of  $\hat{f}(S) = \left(\frac{S}{H}\right)^p \max\left(0, \frac{H^2}{S} - K_c\right)$  using listed instruments. Setting  $\kappa = H$  in (7) and replacing  $f$  by  $\hat{f}$  gives:

$$V_0 = \hat{f}(H)e^{-rT} + \hat{f}'(H)[Se^{-dT} - He^{-rT}] + \int_0^H \hat{f}''(K)P(K)dK + \int_H^\infty \hat{f}''(K)C(K)dK. \tag{13}$$

The reader can verify that:

$$\begin{aligned}
\hat{f}(H) &= 0 \\
\hat{f}'(H) &= 0 \\
\hat{f}''(K) &= \left(\frac{H}{K_c}\right)^{p-2} \delta\left(K - \frac{H^2}{K_c}\right) + \left(\frac{K}{H}\right)^{p-2} (p-1) \left(\frac{p-2}{K} - \frac{pK_c}{H^2}\right) 1\left(K < \frac{H^2}{K_c}\right)
\end{aligned}$$

Applying (13), our replicating portfolio becomes:

1.  $\left(\frac{H}{K_c}\right)^{p-2}$  puts at strike  $\frac{H^2}{K_c}$ .

2.  $\left(\frac{K}{H}\right)^{p-2} (p-1) \left(\frac{p-2}{K} - \frac{pK_\epsilon}{H^2}\right) dK$  puts at strike  $K$  for  $K < \frac{H^2}{K_\epsilon}$ .

To show how this approach can be used to generate adjusted payoffs for other securities, consider the valuation of an American binary put paying a dollar at the first passage time to  $H$ . From [17], the valuation formula is:

$$ABP(S, \tau; H) = \left(\frac{S}{H}\right)^{\gamma+\epsilon} N\left(\frac{\ln\left(\frac{H}{S}\right) - \epsilon\sigma^2\tau}{\sigma\sqrt{\tau}}\right) + \left(\frac{S}{H}\right)^{\gamma-\epsilon} N\left(\frac{\ln\left(\frac{H}{S}\right) + \epsilon\sigma^2\tau}{\sigma\sqrt{\tau}}\right),$$

for  $S > H$ , where  $\gamma \equiv \frac{1}{2} - \frac{r-d}{\sigma^2}$ ,  $\epsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$ . Removing the requirement that  $S > H$  and letting  $\tau \downarrow 0$  gives the adjusted payoff as (see Figure 4):

$$\lim_{\tau \downarrow 0} ABP(S, \tau; H) = \left[ \left(\frac{S}{H}\right)^{\gamma+\epsilon} + \left(\frac{S}{H}\right)^{\gamma-\epsilon} \right] 1(S < H).$$

Another way to interpret our results is to use an alternative framework. Financial mathematics is fortunate in that there are many equivalent interpretations of the same phenomena, and one of the methods to evaluate derivative securities is through differential equations (as exemplified by Wilmott, *et al* [20]). All pricing formulas must satisfy the same Black Scholes differential equation. Different options are created by imposing different initial value and boundary conditions. In this section, we are essentially transforming a Dirichlet problem (incomplete initial value problem with boundary conditions at the barrier) into a Cauchy problem (complete initial value problem). For single barrier options, both types of problems give rise to unique solutions. Given the final solution, it is straightforward (as illustrated above) to transform between the two types of problems.

## 4 Summary and Future Research

All down-and-in securities can be decomposed into a static portfolio of down-and-in Arrows. In the Black Scholes model, the value of a down-and-in Arrow struck at some level  $K$  above the barrier  $H$  matches the value of a suitably weighted path-independent Arrow struck at the geometric reflection of  $K$  in  $H$ . It follows that the value of a down-and-in security can be represented by a static portfolio of Arrows struck below the barrier. Since any such

portfolio can be created out of a static portfolio of European options, down-and-in securities can be statically hedged with vanilla options. In-out parity implies that the same result holds for out options. The valuation formulas derived via static replication match those obtained by dynamic replication. It follows that one can start from a formula obtained by dynamic replication and uncover the implicit static hedge.

In future analytical work in this area, we plan to explore the effects of imposing multiple barriers. It will also be interesting to examine the static replication error arising from hedging with vanilla options when one can trade in only a finite number of strikes. One possibility is to attempt super-replication at the least cost. Another is to minimize mean squared error of the replicating portfolio's payoff from the target. The latter approach is likely to permit lower offering prices and at least some of the risk can be diversified away. Finally, it should be interesting to conduct empirical tests comparing static and dynamic hedging under realistic market conditions.



## References

- [1] Black, F., 1976, “The Pricing of Commodity Contracts”, *Journal of Financial Economics*, **3**, 167–79.
- [2] Black, F. and M. Scholes, 1973, “The Pricing of Options and Corporate Liabilities”, *The Journal of Political Economy*, **81**, 637–659.
- [3] Bowie, J., and P. Carr, 1994, “Static Simplicity”, *Risk*, **7**, 45–49.
- [4] Breeden, D. and R. Litzenberger, 1978, “Prices of State Contingent Claims Implicit in Option Prices”, *Journal of Business*, **51**, 621–651.
- [5] Carr, P., 1993, “Deriving Derivatives of Derivative Securities”, Cornell University working paper.
- [6] Carr P., K. Ellis, and V. Gupta, 1996, “Static Hedging of Exotic Options”, forthcoming in *Journal of Finance*.
- [7] Carr, P., and D. Madan, 1996, “Optimal Positioning in Derivative Securities”, Morgan Stanley working paper.
- [8] Cox, J. and S. Ross, 1976, “The Valuation of Options for Alternative Stochastic Processes”, *Journal of Financial Economics*, **3**, 145–166.
- [9] Derman, E., D. Ergener, and I. Kani, 1994, “Forever Hedged,” *Risk*, **7**, 139–45.
- [10] Derman, E., D. Ergener, and I. Kani, 1995, “Static Options Replication,” *Journal of Derivatives*, Summer, **2**, 4, 78–95.
- [11] Green and Jarrow, 1987, “Spanning and Completeness in Markets with Contingent Claims”, *Journal of Economic Theory*, **41**, 1, 202–207.
- [12] Ho, T., 1993, “Primitive Securities: Portfolio Building Blocks”, *Journal of Derivatives*, winter, 6–22.
- [13] Merton, R., 1973, “Theory of Rational Option Pricing”, *Bell Journal of Economics and Management Science*, **4**, 141–183.

- [14] Nachman, D., 1988, "Spanning and Completeness with Options", *Review of Financial Studies*, **1**, 3, 311-328.
- [15] Nelken I., 1995, Handbook of Exotic Options, Probus, Chicago IL.
- [16] Reiner, E. and M. Rubenstein, 1991, "Unscrambling the Binary Code", *Risk*, **4**, 28-35.
- [17] Rubenstein M. and E. Reiner, 1991, "Breaking down the Barriers", *Risk*, **4**, 8.
- [18] Ross, S., 1976, "Options and Efficiency", *Quarterly Journal of Economics*, **90**, 75-89.
- [19] Snyder, G., 1969, "Alternative Forms of Options," *Financial Analysts Journal*, **26** Sept.-Oct., 93-99.
- [20] P. Wilmott, J. DeWynne, and S. Howison, 1993, Option Pricing: Mathematical Models and Computation, Oxford Financial Press.
- [21] Zhang, P., 1995, Exotic Options: A Guide to the Second Generation Options, forthcoming manuscript.

## Appendix

This appendix shows how we obtain the adjusted payoff for down-and-in securities. Recall that Arrows are fundamental securities in the sense that the value of an arbitrary path-independent claim is easily represented in terms of them:

$$V(S, \tau) = \int_0^\infty f(K)A(S, \tau; K)dK,$$

where the current spot price  $S$  and the time to maturity  $\tau \equiv T - t$  have been introduced as arguments of the Arrow's value. Dynamic replication arguments such as those given by Cox and Ross[8] imply that the value can also be calculated using risk-neutral valuation:

$$V(S, \tau) = \int_0^\infty f(K)e^{-r\tau}\ell(S, \tau; K)dK,$$

where  $\ell(S, \tau; K)$  is the risk-neutral probability that the stock price finishes at  $K$  at  $T$ , conditional on starting from  $S$  at  $t$ . Since the above equations hold for any payoff function  $f$ , it follows that an Arrow's price is simply the discounted risk-neutral probability:

$$A(S, \tau; K) = e^{-r\tau}\ell(S, \tau; K).$$

In the Black Scholes model, the risk-neutral density is lognormal and it follows that the Arrow's price is given by:

$$A(S, \tau; K) = e^{-r\tau} \frac{1}{K\sqrt{2\pi\sigma^2\tau}} \exp\left\{-\frac{(\ln(K/S) - (r - d - \frac{\sigma^2}{2})\tau)^2}{2\sigma^2\tau}\right\}. \quad (14)$$

Recall from (4) that a down-and-in Arrow is fundamental in valuing barrier securities since:

$$\begin{aligned} DIV(S, \tau; H) &= \int_0^\infty f(K)DIA(S, \tau; K, H)dK \\ &= \int_0^H f(K)DIA(S, \tau; K, H)dK + \int_H^\infty f(K)DIA(S, \tau; K, H)dK \\ &= \int_0^H f(K)A(S, \tau; K)dK + Z, \end{aligned} \quad (15)$$

where  $Z \equiv \int_H^\infty f(K)DIA(S, \tau; K, H)dK$  is the portion of the value arising from payoffs above the barrier. For payoffs below the barrier (i.e.  $S \leq H$ ), it follows that  $DIA(S, \tau; K, H) = A(S, \tau; K)$ .

To evaluate  $Z$  more explicitly, let  $\phi(S, t; H)$  denote the risk-neutral probability density function for the first passage time to the barrier  $H$  when starting from  $S$ . Then,  $Z$  is the discounted expected value of the Arrow securities whenever the barrier is reached. Thus,

$$Z = \int_0^\tau e^{-rt} \phi(S, t; H) I(t) dt, \quad (16)$$

where  $I(t) \equiv \int_H^\infty f(K) A(H, t; K) dK$ . Consider the change of variables  $S_T = \frac{H^2}{K}$  in  $I(t)$ :

$$\begin{aligned} I(t) &= \int_H^0 f\left(\frac{H^2}{S_T}\right) A(H, t; H^2/S_T) \left(-\frac{H^2}{S_T^2} dS_T\right) \\ &= \int_0^H f\left(\frac{H^2}{S_T}\right) \frac{H^2}{S_T^2} A(H, t; H^2/S_T) dS_T. \end{aligned} \quad (17)$$

Using (14), we have:

$$\begin{aligned} \frac{H^2}{S_T^2} A(H, t; H^2/S_T) &= \frac{H^2}{S_T^2} e^{-rt} \frac{1}{(H^2/S_T) \sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[\ln((\frac{H^2}{S_T})/H) - (r - d - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t}\right\} \\ &= e^{-rt} \frac{1}{S_T \sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[\ln(H/S_T) - (r - d - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t}\right\} \\ &= e^{-rt} \left(\frac{S_T}{H}\right)^p \frac{1}{S_T \sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[\ln(S_T/H) - (r - d - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t}\right\} \\ &= \left(\frac{S_T}{H}\right)^p A(H, t; S_T), \end{aligned} \quad (18)$$

where  $p = 1 - \frac{2(r-d)}{\sigma^2}$ . Substituting into (17) gives:

$$I(t) = \int_0^H f\left(\frac{H^2}{S_T}\right) \left(\frac{S_T}{H}\right)^p A(H, t; S_T) dS_T,$$

and substituting into (16) gives:

$$Z = \int_0^\tau e^{-rt} \phi(S, t; H) \left[ \int_0^H f\left(\frac{H^2}{S_T}\right) \left(\frac{S_T}{H}\right)^p A(H, t; S_T) dS_T \right] dt$$

$$\begin{aligned}
&= \int_0^H f\left(\frac{H^2}{S_T}\right) \left(\frac{S_T}{H}\right)^p \left[ \int_0^\tau e^{-rt} \phi(S, t; H) A(H, t; S_T) dt \right] dS_T \\
&= \int_0^H f\left(\frac{H^2}{S_T}\right) \left(\frac{S_T}{H}\right)^p A(S, \tau; S_T) dS_T.
\end{aligned}$$

The last line follows since  $S_T < H$  in the range of the integral. Note that although we have used  $\phi(\cdot)$  in our derivation, we never needed to know its functional form. Substituting into (15) gives:

$$\begin{aligned}
DIV(S, \tau; H) &= \int_0^H f(S_T) A(S, \tau; S_T) dS_T + \int_0^H A(S, \tau; S_T) \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) dS_T \\
&= \int_0^\infty \hat{f}(S_T) A(S, \tau; S_T) dS_T,
\end{aligned}$$

where  $\hat{f}(S_T) \equiv \left[ f(S_T) + \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) \right] 1(S_T < H)$  is defined as the *adjusted payoff* for a down-and-in security.

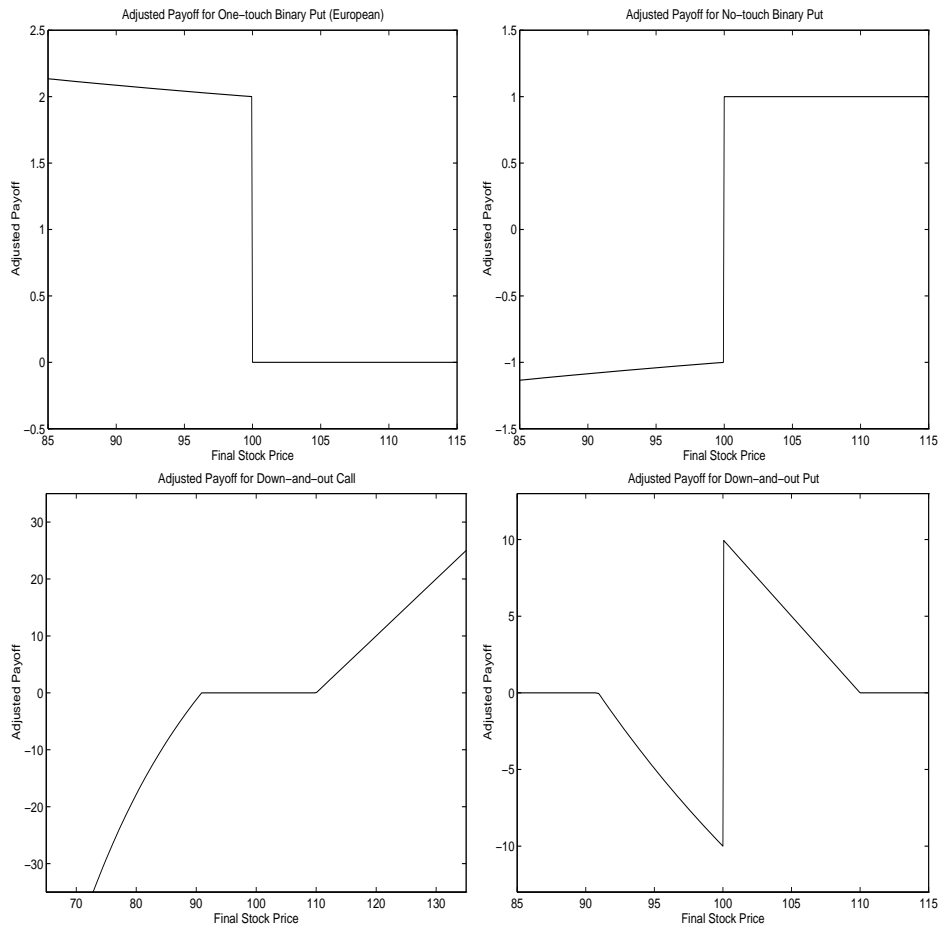


Figure 3: Adjusted payoffs for down securities ( $r = 0.05$ ,  $d = 0.03$ ,  $\sigma = .15$ ,  $K_c = K_p = 110$ ,  $H = 100$ ).

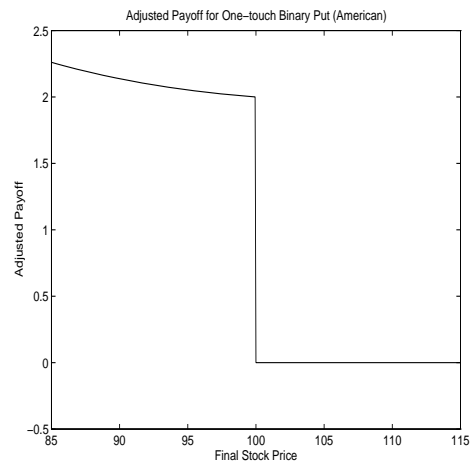


Figure 4: Adjusted payoff for American binary put ( $r = 0.05$ ,  $d = 0.03$ ,  $\sigma = .15$ ,  $H = 100$ ).