The Non-Commutative Gröbner Freaks

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0.1 Introduction

De occulto orbis terrarum situ interrogasti, et si tanta monstrorum essent genera credenda[1] : to the rhetorical question which open Adhelm's *Liber monstrorum de diversis generibus* [A] we are trying here to give a positive answer by studying Gröbner Fan and Universal Bases in the non-commutative case.

Our starting point was the question whether the well-known result that the Gröbner Fan is finite in the commutative case, was generalizable to the non-commutative case. The answer, of course, is negative and is presented in § 0.4. Building the example presented here, we were attracted by other fascinating examples. This suggested us the idea of preparing this teratological *Kunstkammer* where we are exhibiting:

- an example of a principal ideal whose reduced Gröbner basis is infinite (§ 0.3) – the example discussed in § 0.4 is already of this kind, but the one we present here is older and much nicer.
- an example of an ideal whose Gröbner Fan is non-enumerable (§ 0.5.1) – a similar example was recently presented for canonical bases in [D] and we generalized it in the non-commutative setting. Unfortunately to do so, we needed to rely to the dirty trick of adding initial and terminal variables to terms so to guarantee the inexistence of S-polynomials; this spoiled its artistic value.
- our best approximation to an example of an ideal whose universal basis is the enumerable set $G = \{g_0, \ldots, g_i, \ldots\}$ and the set of all possible Gröbner bases of I is $\{G_i : i \in \mathbb{N}\}$ where $G_i = \{g_0, \ldots, g_i\}$ (§ 0.5.2) – we failed to find a "pure" example of this kind.

Other, Gröbner Freaks are probably existing, but, quoting Adhelm, *de quibus iam nihil singulare et admiratione dignum reperi*[2] [A].

0.2 Gröbner Bases, Gröbner Fan, Universal Basis

Let K be any field, let $\mathbf{S} = \langle a_1, \ldots, a_n \rangle$ be the free monoid generated by the n symbols a_i and let $\mathcal{P} := K\langle a_1, \ldots, a_n \rangle$.

A *two-sided monomial ideal* $M \subset \mathbf{S}$ is a subset such that

$$\forall s, t, u \in \mathbf{S}, u \in M \Longrightarrow sut \in M.$$

A *monoid well-ordering* $<$ on $\mathbf{S}$ is a total ordering such that

[1] You asked of the occult places of the world and whether I believed that there were so many kinds of monsters

[2] of which I found nothing singular and worthwhile of interest

- $\forall s, t, u, v \in \mathbf{S}, u < v \Longrightarrow sut < svt$
- there is no infinite decreasing sequence

$$s_1 < s_2 < \ldots < s_i < s_{i+1} \ldots$$

Let us fix a monoid well-ordering $<$ on $\mathbf{S}$ and let $p \in \mathcal{P}$; p has a unique *ordered* representation as a linear combination of elements of $\mathbf{S}$:

$$p = \sum_{i=1}^{s} c_i t_i : c_i \in K \setminus \{0\}, t_i \in \mathbf{S}, t_1 > \cdots > t_s$$

so that we can associate to each non-zero element $p \in \mathcal{P}$ the monomial $in_<(p) := t_1$, and to any two-sided ideal $I \subset \mathcal{P}$ the two-sided monomial ideal $in_<(I) := \{in_<(f) : f \in I\}$.

If $<$ is a monoid well-ordering on $\mathbf{S}$ and $I \subset \mathcal{P}$ a two-sided ideal, a *Gröbner basis* of I with respect to $<$ is a subset $G \subset I$ s.t. $in_<(I)$ is generated as monomial ideal by $\{in_<(g) : g \in G\}$.

The unique *reduced Gröbner basis* ([M], Prop. 1.5) of I with respect to $<$ is the set

$$G_<(I) = \{g_1, \ldots, g_n, \ldots\} \subset I$$

such that denoting

$$g_j = \sum_{i=1}^{s_j} c_{ij} t_{ij} : c_{ij} \in K \setminus \{0\}, t_{ij} \in \mathbf{S}, t_{1j} > \cdots > t_{s_j j}$$

the following assertions hold

- $\{t_{11}, \ldots, t_{1n}, \ldots\}$ is an irredundant ideal basis of $in_<(I)$
- $c_{1j} = 1, \forall j$
- $t_{ij} \notin in_<(I), \forall i \neq 1, \forall j$.

Let us denote by $\mathcal{O}(\mathcal{P})$ the set of all the monoid well-orderings on $\mathbf{S}$. For a two-sided ideal $I \subset \mathcal{P}$, let us denote by $\mathcal{I}(I)$ the set $\{in_<(I) : < \in \mathcal{O}(\mathcal{P})\}$. For each two-sided monomial ideal $M \in \mathcal{I}(I)$, let

$$\mathcal{O}(M) := \{< \in \mathcal{O}(\mathcal{P}) : M = in_<(I)\}.$$

The *universal basis* [Sc, W] of I is the union of all the reduced Gröbner bases of I,

$$\mathcal{U}(I) := \bigcup_{< \, \in \mathcal{O}(\mathcal{P})} G_<(I).$$

The *Gröbner Fan* of I is the set

$$\mathcal{F}(I) = \{\mathcal{O}(M) : M \in \mathcal{I}(I)\}$$

which satisfies

- $\mathcal{O}(\mathcal{P}) = \bigcup_{\mathcal{O} \in \mathcal{F}(I)} \mathcal{O}$
- $\forall \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{F}(I), \mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset \Longrightarrow \mathcal{O}_1 = \mathcal{O}_2$.

In the rest of the paper, we need to test whether a countable set G is a Gröbner basis of the two-sided ideal generated by it. The tool to do that is the Buchberger Algorithm, which is surveyed in [M] and briefly recalled here.

Let $G = \{g_1, \ldots, g_i, \ldots\}$ and let us denote by $\tau_i = in_<(g_i), \forall i$. If $l, r, \lambda, \rho \in \mathbf{S}$ are s.t. $l\tau_j r = \lambda \tau_i \rho$ let us denote by

$$S(i,j;l,r;\lambda,\rho) := lg_j r - \lambda g_i \rho$$

and let us say that $S(i,j;l,r;\lambda,\rho)$ has a *weak Gröbner representation* (in terms of G) if

$$S(i,j;l,r;\lambda,\rho) = \sum_{k=0}^{t} c_k l_k g_{i_k} r_k$$

where

$$c_k \in K \setminus \{0\},\ l_k, r_k \in \mathbf{S},\ g_{i_k} \in G, \text{ and } l_k\ in_<(g_{i_k})\ r_k < l\ in_<(g_j)\ r\ \forall k.$$

For each pair i, j with $i \leq j$ let us denote by $M(i,j)$ to be the set of all tuples $(i,j;l,r;\lambda,\rho)$ such that:

- $l\tau_j r = \lambda \tau_i \rho$;
- at least two among l, r, λ, ρ are equal to 1;
- $\deg(l) + \deg(r) < \deg(\tau_i)$

where again $\tau_i = in_<(g_i), \forall i$.

Finally let $M(G) = \bigcup_{i \leq j} M(i,j)$.

Theorem 0.2.1 *The following conditions are equivalent:*

- *G is a Gröbner basis of the ideal generated by G;*
- *for each $(i,j;l,r;\lambda,\rho) \in M(G)$, $S(i,j;l,r;\lambda,\rho)$ has a weak Gröbner representation.*

Proof: See [M],§5. □

In our examples we will need only to compute with ideals generated by binomials; to simplify our argument, let us introduce the following notation. We will assume that we are given a monoid well-ordering $<$ and a basis $G = \{g_1, \ldots, g_i, \ldots\}$ where $g_i = l_i - r_i$ and $l_i > r_i$. Let us introduce on $\mathbf{S}$ the following relation (which implicitly depends on G and $<$): for $t_1, t_2 \in \mathbf{S}$, $t_1 \hookrightarrow t_2$ means there are $l, r \in \mathbf{S}$ and i such that $t_1 = l l_i r$, $t_2 = l r_i r$. Moreover $\rightarrow$ will denote the reflexive and transitive closure of $\hookrightarrow$.

Corollary 0.2.2 *Let $<$ be a monoid well-ordering and let $G = \{g_1, \ldots, g_i, \ldots\}$ where $g_i = l_i - r_i$ and $l_i > r_i$. Then the following conditions are equivalent:*

- *G is a Gröbner basis of the ideal generated by G;*
- *for each $(i, j; l, r; \lambda, \rho) \in M(G)$ there is $u \in \mathbf{S}$ s.t. $lr_j r \to u \leftarrow \lambda r_i \rho$.* □

Another way to prove that G is a Gröbner basis is to use Hilbert series. Let us recall that for a graded algebra $A = \oplus_0^\infty A_n$ the Hilbert series $H_A = H_A(t)$ is the generating function $\sum_0^\infty (\dim A_n) t^n$ and that it holds

Theorem 0.2.3 *Let $F = in_<(G), A =< X|G >, B =< X|F >$. Then the following conditions are equivalent:*

- *G is a Gröbner basis of the ideal generated by G;*
- *$H_A = H_B$.*

Proof: See [U], for example. □

0.3 The shortest principal ideal with an infinite reduced Gröbner basis

There are different examples of a principal ideal $(f) \subset \mathcal{P}$ whose Gröbner basis with respect to some monoid well-ordering is infinite; the most amazing is this one: let $f = xx - xy \in K\langle x, y\rangle$, $g_i = xy^i x - xy^{i+1}, i \geq 0$ and $<$ be any monoid well-ordering s.t. $x > y$.

Proposition 0.3.1 *$G := \{g_i : i \geq 0\}$ is the reduced Gröbner basis of (f) with respect to $<$.*

Proof: It is clear that $M(G) = \{(i, j; xy^i, 1; 1, y^j x), \forall i, j\}$. Moreover, for each i, j it holds:

$$S(i, j; xy^i, 1; 1, y^j x) = xy^i g_j - g_i y^j x = xy^{i+j+1}x - xy^i xy^{j+1} =$$

$$= -g_i y^{j+1} - xy^{i+j+2} + xy^{i+j+1}x = -g_i y^{j+1} + g_{i+j+1}.$$

As a consequence we conclude that

- G is a Gröbner basis of the ideal it generates, since $S(i, j; xy^i, 1; 1, y^j x)$ has a weak Gröbner representation $\forall i, j$.
- $G \subset (f)$ since $g_{i+1} = xy^i g_0 + g_i(y - x)$, $\forall i \geq 0$.

Therefore $in_<(I)$ is generated by $\{xy^ix : i \leq 0\}$ which is obviously an irredundant ideal basis of $in_<(I)$, so that the Gröbner basis G is the reduced one.

An alternative proof is to note that, thru the change of variables

$$x \mapsto y, y \mapsto x - y,$$

the algebra $< x, y|f >$ is isomorphic to $A =< x, y|xy >$ and

$$H_A = H_B = (1 - t)^{-2},$$

where $B =< x, y|xy^ix, i \geq 0 >$. □

Since $\{f\}$ is the reduced Gröbner basis for any monoid well-ordering s.t. $y > x$, G is also the universal basis, while the Gröbner Fan consists of two subsets.

Remark that, while ([Sq]) there are many two-sided ideals such that all their Gröbner bases are infinite (even if one change the generating set), (f) above is an elementary example of an ideal which has a finite Gröbner basis for a monoid well-ordering and an infinite one for a different monoid well-ordering. Examples of this kind are already well-known ([L]).

Remark 0.3.2 In our Kunstkammer, we are happy to quote the easiest (and perhaps the earliest) ideal satisfying Squier's property of having only infinite Gröbner bases, even if one change the generating set.

The ideal is generated by two polynomials only:

$$x^2, xy + zx.$$

In reality $A =< x, y, z, ...|x^2, xy + zx >$ is the only (up-to isomorphism) algebra with two quadratic relations with the property that the Poincaré series is not defined (and that is the reason why Gröbner basis can not be finite - see [B] for the details). □

Remark 0.3.3 In relation with the remark above, it would be nice to find a (homogeneous) principal two-sided ideal having an infinite Gröbner basis for every choice of variables and monoid well-orderings. Note nevertheless, that all algebras with one (homogeneous) relation only have (rational) Poincaré series.

On the other hand, it would be nice to prove that each homogeneous principal two-sided ideal is such that, for any monoid well-ordering $<$ and any change of coordinates, its reduced Gröbner basis G is *regular*, in the sense that $in_<(G)$ forms a regular language. □

Remark 0.3.4 Regarding principal ideals, generated by a non homogeneous (but non scalar) element, it would be nice to have a direct proof that their reduced Gröbner basis is different from 1 (in other words that the factor-algebra is not trivial).

First non easy case is $xxyx - xyxx - 1$ and the problem seems still to be open for the positive characteristic. □

0.4 A principal ideal whose Gröbner Fan is infinite

Let us preliminarily consider another principal two-sided ideal in $K\langle x, y\rangle$ which has an infinite Gröbner basis. As the monoid well-ordering $<$ we will preliminarily fix the degree-lexicographical ordering generated by $x < y$. To simplify the argument, we will write the leading term of elements in **bold**. Let $f = \mathbf{yxy} - xyx$ (so $yxy \to xyx$) and let I be the two sided ideal generated by f.

Lemma 0.4.1 *It holds*

1) For $m \geq 1$, $y^m xy \to xyx^m$;

2) For $m \geq 1$, $yxy^m \to x^m yx$.

Proof: Ad 1): It holds

$$y^m xy = y^{m-1} \;\; yxy \to y^{m-1} \;\; xyx = y^{m-1}xy \;\; x \to xyx^{m-1} \;\; x = xyx^m$$

where the last $\to$ follows by induction.

Ad 2): It holds

$$yxy^m = yxy \;\; y^{m-1} \to xyx \;\; y^{m-1} = x \;\; yxy^{m-1} \to x \;\; x^{m-1}yx = x^m yx$$

where the last $\to$ follows by induction — in fact 2) could be directly obtained by 1) because of symmetry. □

Let $p_0 := \mathbf{yxy} - xyx$, $p_i := \mathbf{yx^{i+1}yx} - xyxxy^i, \forall i \geq 1$, $G = \{p_i : i \geq 0\}$ and $l_i, r_i \in \mathbf{S}$ s.t. $p_i = l_i - r_i, \forall i$.

Proposition 0.4.2 *G is a Gröbner basis of I*

Proof: It holds

$$(xyx)xy \leftarrow (yxy)xy = yx(yxy) \to yx(xyx)$$

so that $p_1 \in (p_0) = I$ and, for all m,

$$yx^{m+1}yx = yx^m(xyx) \leftarrow yx^m(yxy) = (yx^m yx)y \to (xyxxy^{m-1})y = xyxxy^m$$

so that $p_m \in (p_0, p_1, \ldots, p_{m-1}) = I$. As a consequence G generates I.

Remarking that

$$\begin{aligned} M(G) \;\; = \;\; & \{(0,0;yx,1;1,xy)\} \bigcup \{(0,m;1,y;yx^{m+1},1) : m \geq 1\} \bigcup \\ & \bigcup \{(m,0;1,x^{m+1}yx;yx,1) : m \geq 1\} \bigcup \\ & \bigcup \{(m,n;yx^{m+1},1;1,x^n yx) : m,n \geq 1\} \end{aligned}$$

and

$$
\begin{array}{rcl}
yxr_0 & = & yxxyx \to xyxxy \leftarrow xyxxy = r_0xy; \\
r_my & = & xyxxy^{m+1} \to xyxxy^{m+1} \leftarrow yx^{m+2}yx = yx^{m+1}r_0; \\
r_0x^{m+1}yx & = & x(yx^{m+2}yx) \to \\
& \to & xxyxxy^{m+1} \leftarrow \\
& \leftarrow & x(yxxyx)y^m \leftarrow (yxxyx)xy^m = yxr_m; \\
yx^{m+1}r_n & = & (yx^{m+2}yx)xy^n \to xyxx(y^{m+1}xy)y^{n-1} \to \\
& \to & xyxx(xyx^{m+1})y^{n-1} = xyxx(xyx^{m-1})xxy^{n-1} \leftarrow \\
& \leftarrow & xyxx(y^{m-1}xy)xxy^{n-1} \leftarrow xyxxy^{m-1}(yx^nyx) = r_mx^nyx.
\end{array}
$$

we conclude that G is a Gröbner basis of I. □

Denoting

$$H_j := \{p_i : 0 \leq i \leq j\} \ \forall j \in \mathbb{N},$$

in order to show the freak whose existence we claim, our task is now to define monoid well-orderings $<_j$ for all $j \in \mathbb{N}, j \geq 1$ over the monoid $\langle x, y\rangle$ such that, if we denote by G_j the reduced Gröbner basis of I with respect to $<_j$, we have

$$H_j = \{f \in G_j : \deg(f) \leq j+4\},$$

$$in_{<_j}(p_i) = \begin{cases} xyxxy^i & i = j \\ yx^{i+1}yx & 0 < i < j \\ yxy & i = 0. \end{cases} \tag{0.1}$$

Corollary 0.4.3 *Let $j \in \mathbb{N}$ and assume that Eq. 0.1 is satisfied.*
Then $H_j = \{f \in G_j : \deg(f) \leq j+4\}$.

Proof: The computations included in the proof of Prop. 0.4.2 allow to derive the result, since each p_i is homogeneous and so it is sufficient to check that $S(i,h;l,r;\lambda,\rho)$ has a weak Gröbner representation in terms of H_j, for each $(i,h;l,r;\lambda,\rho) \in M(G)$ such that $\deg(lr_ir) \leq j+4$.

Note that the Hilbert series here being $(1-2t+t^3)^{-1}$, this could be used for an alternative proof. □

We have therefore to show the existence of $<_j$ satisfying Eq. 0.1 and to do so we use the class of monoid well-orderings proposed in [SK^3].

Their proposal is to define a weight $W^{\mathbf{s}}$ in $\langle x, y\rangle$ in terms of a real $\mathbf{s} > 0$ by $W^{\mathbf{s}}(t) = \sum \epsilon_i \mathbf{s}^{i-1}$ where

$$\epsilon_i = \begin{cases} 1 \\ 0 \end{cases} \quad \text{if the } i^{th} \text{ variable in } t \text{ from the left is } \begin{cases} y \\ x \end{cases}$$

(e.g., $W^{\mathbf{s}}(xyxxy^i) = \mathbf{s} + \sum_4^{i+3} \mathbf{s}^i$, $W^{\mathbf{s}}(yx^{i+1}yx) = 1 + \mathbf{s}^{i+2}$) and an order $<^{\mathbf{s}}$ on $\langle x, y\rangle$ by

$$t_1 <^{\mathbf{s}} t_2 \quad \text{if} \quad W^{\mathbf{s}}(t_1) < W^{\mathbf{s}}(t_2) \quad \text{or} \quad W^{\mathbf{s}}(t_1) = W^{\mathbf{s}}(t_2) \quad \text{and} \quad t_1 <_{dl} t_2$$

where $<_{dl}$ denotes the degree-lexicographical ordering with $y < x$.

For each binomial $p = t_1 - t_2$, we can therefore define a function $f^p : \mathbb{R}_+ \to \mathbb{R}$ by $f^p(\mathbf{s}) = W^{\mathbf{s}}(t_1) - W^{\mathbf{s}}(t_2)$. Denoting $in^{\mathbf{s}}(p) := in_{<^{\mathbf{s}}}(p)$, remark that

$$in^{\mathbf{s}}(p) = \begin{cases} t_1 \\ t_2 \end{cases} \iff f^p(\mathbf{s}) \begin{cases} > 0 \\ < 0 \end{cases} .$$

To simplify the notation we will write $f_i(\mathbf{s}) := f^{p_i}(\mathbf{s})$.

Example 0.4.4 We have

$$\begin{array}{ll} p_0 = yxy - xyx & f_0(\mathbf{s}) = 1 - \mathbf{s} + \mathbf{s}^2 \\ p_1 = yxxyx - xyxxy & f_1(\mathbf{s}) = 1 - \mathbf{s} + \mathbf{s}^3 - \mathbf{s}^4 \\ p_2 = yxxxyx - xyxxyy & f_2(\mathbf{s}) = 1 - \mathbf{s} - \mathbf{s}^5 \\ p_3 = yxxxxyx - xyxxyyy & f_3(\mathbf{s}) = 1 - \mathbf{s} - \mathbf{s}^4 - s^6. \end{array}$$

□

Theorem 0.4.5 *There is a sequence*

$$\mathbf{s}_1 = 1 > \mathbf{s}_2 > \mathbf{s}_3 > \ldots > 0$$

such that if we denote the monoid well-ordering $<^{\mathbf{s}_j}$ by $<_j$, then Eq. 0.1 holds.

Proof: Remark that obviously $in^{\mathbf{s}_1}(p_1) = xyxxy$ and

$$\mathbf{s} \in (0,1) \Longrightarrow f_1(\mathbf{s}) > 0 \iff in^{\mathbf{s}}(p_1) = yxxyx.$$

Now let us assume we have found a sequence $1 > \mathbf{s}_2 > \mathbf{s}_3 > \cdots > \mathbf{s}_i > 0$ such that for all $j \le i$ it holds $f_j(\mathbf{s}_j) = 0$ — and so $in^{\mathbf{s}_j}(p_j) = xyxxy^j$ — and

$$\mathbf{s} \in (0, \mathbf{s}_j) \Longrightarrow f_j(\mathbf{s}) \ge 0 \iff in^{\mathbf{s}}(p_j) = yx^{j+1}yx.$$

It is easy to verify that for each $i \ge 0$

$$f_{i+1}(\mathbf{s}) = f_i(\mathbf{s}) - \mathbf{s}^{i+2}(1 - \mathbf{s} + \mathbf{s}^2).$$

As a consequence, $f_{i+1}(\mathbf{s}_i) = -\mathbf{s}_i^{i+2}(1 - \mathbf{s}_i + \mathbf{s}_i^2) < 0$ and, since $f_{i+1}(0) = 1$, there is a minimal $\mathbf{s}_{i+1} \in (0, \mathbf{s}_i)$ such that $f_{i+1}(\mathbf{s}_{i+1}) = 0$ and so $in^{\mathbf{s}_{i+1}}(p_{i+1}) = xyxxy^{i+1}$ and

$$\mathbf{s} \in (0, \mathbf{s}_{i+1}) \Longrightarrow f_{i+1}(\mathbf{s}) > 0 \iff in^{\mathbf{s}}(p_{i+1}) = yx^{i+2}yx.$$

Therefore we proved that there exists an infinite sequence

$$1 > \mathbf{s}_2 > \mathbf{s}_3 > \ldots > 0$$

such that for all j, $in^{\mathbf{s}}(p_j) = xyxxy^j$ and

$$\mathbf{s} \in (0, \mathbf{s}_j) \Longrightarrow f_j(\mathbf{s}) > 0 \iff in^{\mathbf{s}}(p_j) = yx^{j+1}yx$$

from which we obtain

$$in_{<_j}(p_i) = \begin{cases} yx^{i+1}yx & i < j \\ xyxxy^i & i = j \end{cases}$$

and so Eq. 0.1 holds for each j. □

Corollary 0.4.6 *The two sided ideal* $I = (xyx - yxy) \in K\langle x, y\rangle$ *is such that* $\mathcal{I}(I)$ *and* $\mathcal{F}(I)$ *are infinite sets.* □

Remark 0.4.7 If we would choose $<$ to be the degree-antilex ordering, instead than the degree-lex one, i.e. the one where the tie between two monomials with the same number of symbols is broken comparing the symbols from right to left, obviously we could prove (just writing the proofs above from right to left) that, denoting $q_0 = p_0$, $q_i := \mathbf{xyx^iy} - y^{i-1}xxyx, \forall i \geq 1$, $G = \{q_i : i \geq 0\}$, $H_j := \{q_i : 0 \leq i \leq j\}$ $\forall j \in \mathbb{N}$, it holds:

- G is a Gröbner basis of I;

- $\forall j \in \mathbb{N}, j \geq 1$, there is an monoid well-ordering $<_j$ such that

  - $in_{<_j}(q_i) = \begin{cases} y^i xxyx & \text{i = j} \\ xyx^{i+1}y & 0 < \text{i} < \text{j} \\ yxy & \text{i} = 0 \end{cases}$;
  - denoting G_j the Gröbner basis of I, with respect to $<_j$, then $H_j = \{f \in G_j : \deg(f) \leq j + 4\}$.

□

0.5 The hunting of new freaks

0.5.1 Toward a two-sided ideal which has a non-enumerable Gröbner Fan

Let us consider the commutative polynomial ring $\mathbf{P} = K[X_1, \ldots, X_n]$ and a finitely generated subalgebra $R \subset \mathbf{P}$. To any term ordering $<$ on $\mathbf{P}$, let us denote by $in_<(R)$ the *initial algebra* of R, which is the K-vector space generated by $\{in_<(f) : f \in R\}$ (*cf.*[St]). It has been recently proved [D] that the set $\{in_<(R) : <$ is a term ordering on $\mathbf{P}\}$ is non-enumerable.

Mimicking his argument, let us consider the non-commutative polynomials

$$f_{\alpha\beta} := xy^\alpha z^\beta w - xt^\alpha u^\beta w \in K\langle x, y, z, w, t, u\rangle, \ \forall \alpha, \beta \in \mathbb{N}$$

and the two sided ideal $I = (f_{\alpha\beta} : \alpha, \beta \in \mathbb{N})$. For each $q \in \mathbb{R}$ let δ_q be the degree on $\mathbf{S} := \langle x, y, z, w, t, u\rangle$ such that

$$\delta_q(x) = \delta_q(z) = \delta_q(t) = \delta_q(w) = 0, \delta_q(y) = 1, \delta_q(u) = q$$

and $<_q$ be the monoid well-ordering on $\mathbf{S}$ defined by

$$m <_q n \iff \delta_q(m) < \delta_q(n) \quad \text{or} \quad \delta_q(m) = \delta_q(n) \quad \text{and} \quad m <_{dl} n$$

where $<_{dl}$ is the degree-lexicographical ordering with $x < y < z < w < t < u$. Then it is clear that

$$in_{<_q}(f_{\alpha\beta}) = \begin{cases} xy^\alpha z^\beta w & \text{if } \frac{\alpha}{\beta} < q \\ xt^\alpha u^\beta w & \text{if } \frac{\alpha}{\beta} \geq q. \end{cases}$$

Therefore

Lemma 0.5.1 *I is such that $\mathcal{F}(I)$ is non-enumerable.* □

Problem: is there a finitely generated or, better, principal two-sided ideal I such that $\mathcal{F}(I)$ is non-enumerable?

0.5.2 Toward a two-sided principal ideal such that its infinite universal basis coincides with its reduced Gröbner basis and which has an infinite Gröbner Fan

Let us go back to the example of § 0.3. If we could assume, that for each $i \in \mathbb{N}$ there is a monoid well-ordering $<_i$ s.t.

$$in_{<_i}(g_j) = \begin{cases} xy^j x & \text{if } j < i \\ xy^{j+1} & \text{if } j = i, \end{cases}$$

the obvious computation

$$g_j y^{i+1} + xy^j g_i = xy^j xy^i x - xy^{j+i+2} = g_j y^i x + g_i y^j (y - x) + xy^i g_j$$

would allow us to conclude that $\forall i \geq 0, \{g_0, \ldots, g_i\}$ is a reduced Gröbner basis of (f) with respect to $<_i$.

Unfortunately, if $xx > xy$ then $x > y$ and so $xy^i x > xy^{i+1} \forall i$, so the assumption is wrong. Nevertheless it can be realized as a term rewriting system[BO].

However it would be very nice to find out a principal ideal (g_0) s.t.

- for a fixed monoid well-ordering $\prec$ its reduced Gröbner basis is $G := \{g_i : i \in \mathbb{N}\}$;
- for any monoid well-ordering $<$ there is $n \in \mathbb{N} \cup \{\infty\}$ such that $G_n := \{g_i : i \leq n\}$ is its reduced Gröbner basis;
- for each $n \in \mathbb{N}$ there is a monoid well-ordering $<_n$ such that $G_n := \{g_i : i \leq n\}$ is its reduced Gröbner basis.

If such example exists, it would provide a two-sided principal ideal such that its infinite universal basis coincides with one of its Gröbner basis and which has an infinite Gröbner Fan.

What follows is our best approximation; it satisfies the assumption above except that it is not principal. Let $\mathcal{P} = K\langle v, x, y, u, w\rangle$, $g_i := vx^{i+1}w - uy^{i+2}$, $i \in \mathbb{N}$, $f := wy - xw$, $h = vwy - uy^2$ and $I \subset \mathcal{P}$ be the two sided ideal generated by (f, h) whose Gröbner bases are:

- $\{f, h\}$ if $wy < xw$.
- $\{f, g_0, \ldots, g_i\}$, $i \in \mathbb{N}$, if $wy > xw, vx^j w > uy^{j+1}$ for $j \leq i, uy^{i+2} > vx^{i+1}w$;
- $G := \{f, g_0, \ldots, g_i, \ldots\}$ if $wy > xw, vx^j w > uy^{j+1}$ $\forall j$.

As a consequence its universal basis is $G \cup \{h\}$ and the Gröbner Fan is infinite.

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