Exact Yangian Symmetry in the classical Euler-Calogero-Moser Model

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Abstract

We compute the r-matrix for the elliptic Euler-Calogero-Moser model. In the trigonometric limit we show that the model possesses an exact Yangian symmetry.

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1 Introduction

The Euler-Calogero-Moser model was defined in [1, 2]. In [3] we considered the rational case and we derived the r-matrix. In this paper we are interested in its trigonometric and elliptic generalizations. In the elliptic case we compute the r-matrix and show that the usual elliptic Calogero-Moser model and its r-matrix are obtained by Hamiltonian reduction. In the trigonometric case we show that the current algebra symmetry exhibited by Gibbons and Hermsen [1] in the rational case, is deformed into a Yangian symmetry algebra.

We consider a system of N particles on a line with pairwise interactions. The degrees of freedom consist of the positions and momenta $(p_i, q_i)_{i=1\cdots N}$ and of antisymmetric additional variables $(f_{ij} = -f_{ji})_{i,j=1\cdots N}$, with the Poisson brackets

$$\{p_i, q_j\} = \delta_{ij} \tag{1}$$

$$\{f_{ij}, f_{kl}\} = \frac{1}{2}\,(\delta_{il}\; f_{jk} + \delta_{ki}\; f_{lj} + \delta_{jk}\; f_{il} + \delta_{lj}\; f_{ki}). \tag{2}$$

The Poisson brackets of the f_{ij} just reproduce the $O(N)$ Lie algebra. The Hamiltonian will be taken of the form

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} f_{ij}\; f_{ji}\; V(q_{ij}), \qquad q_{ij} = q_i - q_j \tag{3}$$

with an even potential $V(-x) = V(x)$.

The equations of motion are easily derived:

$$\begin{aligned}
\dot{q}_i &= p_i \\
\dot{p}_i &= \sum_{\substack{j=1\\ j\neq i}}^{N} f_{ij}\; f_{ji}\; V'(q_{ij}) \\
\dot{f}_{ij} &= \sum_{\substack{k=1\\ k\neq i,j}}^{N} f_{ik}\; f_{jk}\; \left[V(q_{ik}) - V(q_{jk})\right].
\end{aligned}$$

Such a system admits a Lax representation only for specific potentials. Indeed writing the following ansatz for the Lax pair

$$L(\lambda) = \sum_{i=1}^{N} p_i\; e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} l(q_{ij}, \lambda)\; f_{ij}\; e_{ij} \tag{4}$$

$$M(\lambda) = \sum_{\substack{i,j=1\\ i\neq j}}^{N} m(q_{ij}, \lambda)\; f_{ij}\; e_{ij} \tag{5}$$

where e_{ij} is the $N\times N$ matrix $(e_{ij})_{kl} = \delta_{ik}\; \delta_{jl}$ and $\lambda \in \mathbb{C}$ is the spectral parameter, we find that the equations of motion can be written in the Lax form

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)] \tag{6}$$

if and only if the following equalities are satisfied:

$$m(x,\lambda) = -\frac{\partial}{\partial x} l(x,\lambda) = -l'(x,\lambda) \tag{7}$$

$$l'(x,\lambda)\; l(y,\lambda) - l'(y,\lambda)\; l(x,\lambda) = l(x+y,\lambda)\; [V(x) - V(y)] \tag{8}$$

$$l(x) \sim -\frac{1}{x} \;\text{ when }\; x \to 0. \tag{9}$$

Eq.(8) is the famous functional equation of Calogero. Its general solution is [4, 5]

$$l(x,\lambda) = -\frac{\sigma(x+\lambda)}{\sigma(x)\ \sigma(\lambda)}, \qquad V(x) = \wp(x) \tag{10}$$

where σ and $\wp$ are Weierstrass elliptic functions, the relevant properties of which are recalled in the appendix. The elliptic $O(N)$ Euler-Calogero-Moser model is precisely defined by eq.(3) with $V(x) = \wp(x)$ together with the Poisson brackets (1,2).

2 The r-matrix

From eq.(6) it follows that $trL^n(\lambda)$ is a set of conserved quantities. In particular

$$trL(\lambda) = \sum_{i=1}^{N} p_i, \qquad trL^2(\lambda) = 2H + \wp(\lambda).$$

The involution property of these quantities $trL^n(\lambda)$ will follow from the existence of an r-matrix which we now calculate [6, 7]. Introducing the notations $L_1(\lambda) = L(\lambda)\otimes 1$ and $L_2(\lambda) = 1\otimes L(\lambda)$ we show that the Poisson brackets of the Lax matrix elements can be recast as

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda,\mu), L_1(\lambda)] - [r_{21}(\mu,\lambda), L_2(\mu)]. \tag{11}$$

Following [3] we assume that r is of the form

$$r_{12}(\lambda,\mu) = a(\lambda,\mu)\sum_{i=1}^{N} e_{ii}\otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} b_{ij}(\lambda,\mu) e_{ij}\otimes e_{ji} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} c_{ij}(\lambda,\mu) e_{ij}\otimes e_{ij}.$$

Requiring that $r_{12}(\lambda,\mu)$ be independent of the p_i variables we obtain

$$b_{ij}(\lambda,\mu) = -b_{ji}(\mu,\lambda) \tag{12}$$
$$c_{ij}(\lambda,\mu) = c_{ij}(\mu,\lambda). \tag{13}$$

Moreover assuming that $r_{12}(\lambda,\mu)$ is independent of the f_{ij} variables yields the following system:

$$a(\lambda,\mu)\ l(q_{ij},\lambda) - b_{ij}(\lambda,\mu)\ l(q_{ij},\mu) + c_{ij}(\lambda,\mu)\ l(q_{ji},\mu) = -l'(q_{ij},\lambda) \tag{14}$$
$$b_{ij}(\lambda,\mu)\ l(q_{jk},\lambda) - b_{ik}(\lambda,\mu)\ l(q_{jk},\mu) = -\frac{1}{2}\ l(q_{ik},\lambda)\ l(q_{ji},\mu) \tag{15}$$
$$c_{ij}(\lambda,\mu)\ l(q_{jk},\lambda) + c_{ik}(\lambda,\mu)\ l(q_{kj},\mu) = \frac{1}{2}\ l(q_{ik},\lambda)\ l(q_{ij},\mu) \tag{16}$$
$$c_{ij}(\lambda,\mu)\ l(q_{ki},\lambda) + c_{kj}(\lambda,\mu)\ l(q_{ik},\mu) = \frac{1}{2}\ l(q_{kj},\lambda)\ l(q_{ij},\mu). \tag{17}$$

A solution to these equations is

$$a(\lambda,\mu) = -\frac{1}{2}\left[\zeta(\lambda+\mu)+\zeta(\lambda-\mu)\right]$$
$$b_{ij}(\lambda,\mu) = \frac{1}{2}\ l(q_{ij},\lambda-\mu)$$
$$c_{ij}(\lambda,\mu) = \frac{1}{2}\ l(q_{ij},\lambda+\mu).$$

Indeed substituting the preceding expressions in eq.(15,16,17) leads to the same relation:

$$l(q_{ij},\lambda-\mu)\ l(q_{jk},\lambda) + l(q_{ki},\mu-\lambda)\ l(q_{jk},\mu) + l(q_{ik},\lambda)\ l(q_{ji},\mu) = 0$$

which upon setting $x = \frac{1}{2}\ (\lambda + q_{ij}),\ y = \frac{1}{2}\ (2\mu - \lambda + q_{ji}),\ z = \frac{1}{2}\ (\lambda + q_{ji})$ and $t = \frac{1}{2}\ (-\lambda - q_{ki} + q_{kj})$ is a direct consequence of relation (53). The expression for $a(\lambda, \mu)$ is then given by eq.(14), and is simplified using eq.(51) and (55). Finally the r-matrix reads

$$\begin{aligned} r_{12}(\lambda,\mu) &= \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij}, \lambda-\mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij}, \lambda+\mu) e_{ij} \otimes e_{ij} \\ &- \frac{1}{2} \left[\zeta(\lambda+\mu) + \zeta(\lambda-\mu)\right] \sum_{i=1}^{N} e_{ii} \otimes e_{ii}. \end{aligned} \tag{18}$$

3 The $sl(N)$ model

The above $O(N)$ model can be obtained from the more general $sl(N)$ model by a mean procedure [8, 9, 10]. The $sl(N)$ elliptic Euler-Calogero Moser model is defined by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} f_{ij}\ f_{ji}\ \wp(q_{ij}) \tag{19}$$

and the Poisson brackets

$$\{p_i, q_j\} = \delta_{ij} \tag{20}$$

$$\{f_{ij}, f_{kl}\} = \delta_{jk}\ f_{il} - \delta_{li}\ f_{kj}. \tag{21}$$

For this model we define a Lax matrix as

$$L(\lambda) = \sum_{i=1}^{N} (p_i - \zeta(\lambda) f_{ii})\ e_{ii} + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij}, \lambda)\ f_{ij}\ e_{ij}. \tag{22}$$

The Hamiltonian is given by $H = \frac{1}{2} \int \frac{d\lambda}{2i\pi\lambda} trL^2(\lambda)$. A direct calculation gives

$$\begin{aligned} \{L_1(\lambda), L_2(\mu)\} &= [r_{12}(\lambda,\mu), L_1(\lambda)] - [r_{21}(\mu,\lambda), L_2(\mu)] \\ &- \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l'(q_{ij}, \lambda-\mu)\ (f_{ii} - f_{jj})\ e_{ij} \otimes e_{ji} \end{aligned} \tag{23}$$

with the beautifully simple r-matrix

$$r_{12}(\lambda,\mu) = -\zeta(\lambda-\mu) \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij}, \lambda-\mu)\ e_{ij} \otimes e_{ji}. \tag{24}$$

At this point let us make two remarks:

- Because of the third term in the right member of eq.(23) the integrals of motion $trL^n(\lambda)$ are not in involution. However we can restrict ourselves to the manifolds $(f_{ii} = \text{constant})_{i=1\cdots N}$ since $trL^n(\lambda)$ Poisson-commute with f_{ii}. On these manifolds $trL^n(\lambda)$ are in involution.
- The r-matrix for the $O(N)$ model eq.(24) is immediately seen to be of the form

  $$r_{12}^{O(N)} = \frac{1}{2}\ (1 + \sigma \otimes 1) r_{12}^{sl(N)}$$

  where σ is the involutive automorphism

  $$\sigma : \lambda^n e_{ij} \longmapsto -(-\lambda)^n e_{ij}.$$

  This is typical of a mean construction.

In the following we will restrict the f_{ij} to a symplectic leaf of the Poisson manifold (21). Introducing vectors

$$
\begin{array}{ll}
(\xi_i)_{i=1\cdots N} & \text{with} \quad \xi_i = (\xi_i^a)_{a=1\cdots r} \\
(\eta_i)_{i=1\cdots N} & \text{with} \quad \eta_i = (\eta_i^a)_{a=1\cdots r}
\end{array}
$$

with the Poisson brackets

$$
\{\xi_i^a, \xi_j^b\} = 0, \qquad \{\eta_i^a, \eta_j^b\} = 0, \qquad \{\xi_i^a, \eta_j^b\} = -\delta_{ij}\ \delta_{ab}, \tag{25}
$$

we parametrize the f_{ij} as follows:

$$
f_{ij} = \langle \xi_i | \eta_j \rangle = \sum_{a=1}^{r} \xi_i^a \eta_j^a. \tag{26}
$$

The phase space now becomes a true symplectic manifold.

4 The r-matrix of the elliptic Calogero model

We show here that the r-matrix for the elliptic Calogero model [6, 12] can be obtained from eq.(23) by a Hamiltonian reduction procedure [8, 9, 10].

We choose $r = 1$ in eq.(26). On the manifold $f_{ij} = \xi_i\ \eta_j$ acts an Abelian Lie group

$$
\xi_i \longrightarrow \lambda_i\ \xi_i, \quad \eta_i \longrightarrow \lambda_i^{-1} \eta_i. \tag{27}
$$

Remark that the group acts on $L(\lambda)$ as conjugation by the matrix $\mathrm{diag}(\lambda_i)_{i=1\cdots N}$ and therefore all the Hamiltonians $trL^n(\lambda)$ are invariant. Thus one can apply the method of Hamiltonian reduction. The infinitesimal generator of this action is

$$
H_\epsilon = \sum_{i=1}^{N} \epsilon_i\ f_{ii}, \qquad \lambda_i = 1 + \epsilon_i.
$$

We fix the momentum by choosing

$$
f_{ii} = \alpha.
$$

To compute the reduced Poisson brackets of the Lax matrix, we remark that the matrix

$$
\begin{aligned}
L^{Cal}(\lambda) &= g^{-1} L(\lambda)\ g \qquad \text{with } g = \mathrm{diag}(\xi_i)_{i=1\cdots N} \\
&= \sum_{i=1}^{N} [p_i - \alpha\zeta(\lambda)]\ e_{ii} + \alpha \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij,\lambda})\ e_{ij}
\end{aligned} \tag{28}
$$

is invariant under the isotropy group G_α of α (which is the whole group itself since it is Abelian) and we can compute the Poisson brackets of its matrix elements directly. We find

$$
\{L_1^{Cal}(\lambda), L_2^{Cal}(\mu)\} = [r_{12}^{Cal}(\lambda,\mu), L_1^{Cal}(\lambda)] - [r_{21}^{Cal}(\mu,\lambda), L_2^{Cal}(\mu)] \tag{29}
$$

with

$$
r_{12}^{Cal}(\lambda,\mu) = g_1^{-1} g_2^{-1} \left[r_{12}(\lambda,\mu) - \{g_1, L_2(\mu)\} g_1^{-1} + \frac{1}{2}[u_{12}, L_2(\mu)] \right] g_1 g_2
$$

where $u_{12} = \{g_1, g_2\} g_1^{-1} g_2^{-1}$ is here equal to zero. Redefining

$$
r_{12}^{Cal}(\lambda,\mu) \longrightarrow r_{12}^{Cal}(\lambda,\mu) + \left[\frac{1}{2\alpha} \sum_{i=1}^{N} e_{ii} \otimes e_{ii}, L_2(\mu) \right]
$$

does not change eq.(29) and yields exactly the r-matrix found in [11, 12]

$$
\begin{aligned}
r_{12}^{Cal}(\lambda,\mu) &= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij}, \lambda-\mu)\ e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} l(q_{ij}, \mu)\ (e_{ii} + e_{jj}) \otimes e_{ij} \\
&\quad - [\zeta(\lambda-\mu) + \zeta(\mu)] \sum_{i=1}^{N} e_{ii} \otimes e_{ii}.
\end{aligned} \tag{30}
$$

5 Yangian symmetry in the trigonometric case

The parametrization (26) of f_{ij} introduces a $sl(r)$ symmetry into the theory. The transformation

$$\begin{aligned}\eta_i^a &\longrightarrow \sum_{b=1}^{r} u^{ab}\eta_i^b \\ \xi_i^a &\longrightarrow \sum_{b=1}^{r}(u^{-1})^{ab}\xi_i^b\end{aligned}$$

leaves the f_{ij} invariant and therefore also the Hamiltonians. This symmetry is generated by a set of conserved currents

$$J_0^{ab} = \sum_{i=1}^{N} \xi_i^b \eta_i^a. \tag{31}$$

It is remarkable that this current was shown, in the rational case [1], to be the first of a hierarchy building a current algebra commuting with the Hamiltonian — and more generally with a subset of the commuting Hamiltonians.

We now extend this result to the trigonometric case, and we will show that the hierarchy of currents form a Yangian symmetry in this case. Taking the trigonometric limit ($\omega_1 = \infty$ and $\omega_2 = i\frac{\pi}{2}$) in the above formulas, we see that the Lax matrix can be taken of the form

$$L(\lambda) = L_0 - \coth(\lambda)F \tag{32}$$

with

$$L_0 = \sum_{i=1}^{N} p_i e_{ii} - \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \coth(q_{ij})\ f_{ij}\ e_{ij}, \qquad F = \sum_{i,j=1}^{N} f_{ij}\ e_{ij}. \tag{33}$$

By a straightforward calculation, or taking the limit of the elliptic case, we find

$$\begin{aligned}\{L_1(\lambda), L_2(\mu)\} &= [r_{12}^0, L_1(\lambda)] - [r_{21}^0, L_2(\mu)] \\ &\quad -\frac{1}{2}\left(1-\coth(\lambda)\coth(\mu)\right)\left([\mathcal{C}, F_1] - [\mathcal{C}, F_2]\right) \\ &\quad -\sum_{\substack{i,j=1 \\ i\neq j}}^{N} (f_{ii} - f_{jj})\frac{1}{\sinh^2(q_{ij})} e_{ij}\otimes e_{ji}\end{aligned} \tag{34}$$

where

$$r_{12}^0 = -\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \coth(q_{ij})\ e_{ij}\otimes e_{ji} \tag{35}$$

and $\mathcal{C}$ is the Casimir element of $sl(N)$

$$\mathcal{C} = \sum_{i,j=1}^{N} e_{ij}\otimes e_{ji}. \tag{36}$$

In spite of the unusual second term in eq.(34) the quantities $trL^n(\lambda)$ are still in involution on the manifolds $\Sigma_\alpha : (f_{ii} = \alpha)_{i=1\cdots N}$. Indeed,

$$\begin{aligned}\{trL^n(\lambda), trL^m(\mu)\} &= n\ m \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{f_{ii} - f_{jj}}{\sinh^2(q_{ij})}\ [L^{n-1}(\lambda)]_{ij}\ [L^{m-1}(\mu)]_{ji} \\ &\quad -\frac{n\ m}{2}(1-\coth(\lambda)\coth(\mu))\ tr_{12}\left(L_1^{n-1}(\lambda)L_2^{m-1}(\mu)[\mathcal{C}, F_1 - F_2]\right)\end{aligned}$$

and since $tr_2\ ((1\otimes A)\ \mathcal{C}) = A$, we obtain

$$\begin{aligned}\{trL^n(\lambda), trL^m(\mu)\} &= n\ m \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{f_{ii}-f_{jj}}{\sinh(q_{ij})^2}\ [L^{n-1}(\lambda)]_{ij}\ [L^{m-1}(\mu)]_{ji} \\ &\quad -\frac{n\ m}{2}(1-\coth(\lambda)\coth(\mu))\ tr\{L^{n-1}(\lambda)[L^{m-1}(\mu),F] - L^{m-1}(\mu)[L^{n-1}(\lambda),F]\}.\end{aligned}$$

If we notice that

$$F = -\frac{L(\lambda)-L(\mu)}{\coth(\lambda)-\coth(\mu)}$$

we immediately get the involution property.

We consider now the subset $tr(L^n) = tr(L_0+F)^n$ of commuting Hamiltonians; notice that H belongs to this subset, since $H = \frac{1}{2}trL^2 - \alpha\ trL + \frac{1}{2}N\alpha^2$.

We introduce the following quantities:

$$J_n^{ab} = tr(L^n F^{ab}), \qquad a,b = 1,\cdots,r \qquad n = 0,1,\cdots,\infty \tag{37}$$

where F^{ab} is the $N\times N$ matrix of elements

$$(F^{ab})_{ij} = f_{ij}^{ab} = \xi_i^b \eta_j^a. \tag{38}$$

We define the generating functional of the currents J_n^{ab}. It is the $r\times r$ matrix $T(z)$ of elements

$$T^{ab}(z) = -\frac{1}{2}\delta_{ab} - \sum_{n\in\mathbb{N}} \frac{1}{z^{n+1}} J_n^{ab} = -\frac{1}{2}\delta_{ab} + tr\left(\frac{1}{L-z}F^{ab}\right). \tag{39}$$

Proposition. *On the manifolds Σ_α we have the following two properties:*

1. *The currents J_n^{ab} Poisson commute with all the quantities of the form $tr(L^n)$.*
2. *The generating functional $T(z)$ satisfies the defining relation of a (classical) Yangian algebra:*

$$\{T(y) \overset{\otimes}{,} T(z)\} = [R(y,z), T(y)\otimes T(z)] \tag{40}$$

with

$$R(y,z) = -2\frac{\Pi}{y-z}\ , \qquad \Pi = \sum_{a,b=1}^{r} e_{ab}\otimes e_{ba}. \tag{41}$$

Proof. To prove this proposition we need the Poisson brackets

$$\{L_1, L_2\} = [r_{12}^0, L_1] - [r_{21}^0, L_2] + \sum_{\substack{i,j=1\\ i\neq j}}^{N} (f_{ii}-f_{jj})\frac{1}{\sinh^2(q_{ij})} e_{ij}\otimes e_{ji} \tag{42}$$

$$\{L_1, F_2^{ab}\} = [-r_{21}^0 + \mathcal{C}, F_2^{ab}] \tag{43}$$

$$\{F_1^{ab}, F_2^{cd}\} = (\delta_{ad}F_1^{cb} - \delta_{cb}F_2^{ad})\ \mathcal{C}. \tag{44}$$

Remark that the currents J_n^{ab} and the Hamiltonians $tr(L^n)$ are invariant under the symmetry

$$\xi_i^a \longrightarrow \lambda_i\ \xi_i^a, \qquad \eta_i^a \longrightarrow \lambda_i^{-1}\eta_i^a.$$

Therefore we can compute their Poisson brackets on the reduced phase space straightforwardly; restricting ourselves to the manifolds $f_{ii} = \alpha$, the last term in eq.(42) vanishes, and we will systematically drop its contribution in intermediate calculations.

We emphasize that in eq.(42,43) the same r-matrix appears. Moreover it is the term $[\mathcal{C}, F_2^{ab}]$ in eq.(43) which is responsible for the quadratic form of eq.(40), as we shall see in what follows.

Introducing the generating functional $H(z) = tr(\frac{1}{L-z})$ of the Hamiltonians $tr(L^n)$ we compute

$$\begin{aligned}\{\frac{1}{L_1-y}F_1^{ab}, \frac{1}{L_2-z}\} \quad &= \quad -\left[\frac{1}{L_2-z}\; r_{12}^0\; \frac{1}{L_2-z}\;,\; \frac{1}{L_1-y}F_1^{ab}\right] + \left[\frac{1}{L_1-y}\; r_{21}^0\; \frac{1}{L_1-y}F_1^{ab}\;,\; \frac{1}{L_2-z}\right] \\ &\quad + \frac{1}{L_1-y}\frac{1}{L_2-z}\; [\mathcal{C}, F_1^{ab}]\; \frac{1}{L_2-z}.\end{aligned}$$

Taking the trace we obtain

$$\{T^{ab}(y), H(z)\} = tr\left(F^{ab}\left[\frac{1}{L-y}, \frac{1}{(L-z)^2}\right]\right) = 0.$$

This proves the first part of the proposition. To prove the second part we evaluate

$$\begin{aligned}\{\frac{1}{L_1-y}F_1^{ab}, \frac{1}{L_2-z}F_2^{cd}\} \quad &= \quad -\left[\frac{1}{L_2-z}\; r_{12}^0\; \frac{1}{L_2-z}F_2^{cd}\;,\; \frac{1}{L_1-y}F_1^{ab}\right] \\ &\quad + \left[\frac{1}{L_1-y}\; r_{21}^0\; \frac{1}{L_1-y}F_1^{ab}\;,\; \frac{1}{L_2-z}F_2^{cd}\right] \\ &\quad + \frac{1}{L_1-y}\frac{1}{L_2-z}\left(\delta_{ad}F_1^{cb} - \delta_{cb}F_2^{ad}\right)\; \mathcal{C} \\ &\quad + \frac{1}{L_1-y}\frac{1}{L_2-z}\left\{[\mathcal{C}, F_1^{ab}]\; \frac{1}{L_2-z}F_2^{cd} \;-\; [\mathcal{C}, F_2^{cd}]\; \frac{1}{L_1-y}F_1^{ab}\right\}.\end{aligned}$$

Hence taking the trace we get

$$\begin{aligned}\{T^{ab}(y), T^{cd}(z)\} \quad &= \quad tr\left(\frac{1}{L-y}\frac{1}{L-z}(\delta_{ad}F^{cb} - \delta_{cb}F^{ad})\right) \\ &\quad + tr\left(\frac{1}{L-y}\left[\frac{1}{L-z}F^{cd}\frac{1}{L-z}\;,\; F^{ab}\right]\right) - tr\left(\frac{1}{L-z}\left[\frac{1}{L-y}F^{ab}\frac{1}{L-y}\;,\; F^{cd}\right]\right).\end{aligned}$$

Using the cyclicity of the trace and

$$\frac{1}{L-y}\frac{1}{L-z} = \frac{1}{y-z}\left(\frac{1}{L-y} - \frac{1}{L-z}\right)$$

this becomes

$$\begin{aligned}\{T^{ab}(y), T^{cd}(z)\} \quad &= \quad \frac{1}{y-z}\left(\delta_{ad}(T^{cb}(y) - T^{cb}(z)) - \delta_{cb}(T^{ad}(y) - T^{ad}(z))\right) \\ &\quad + \frac{2}{y-z}tr\left(\frac{1}{L-y}F^{cd}\frac{1}{L-z}F^{ab} - \frac{1}{L-y}F^{ab}\frac{1}{L-z}F^{cd}\right).\end{aligned}$$

Remarking that

$$\begin{aligned}tr\left(\frac{1}{L-y}F^{cd}\frac{1}{L-z}F^{ab}\right) \quad &= \quad \sum_{ijkl=1}^{N}\left(\frac{1}{L-y}\right)_{ij}\; \xi_j^d\; \eta_k^c\; \left(\frac{1}{L-z}\right)_{kl}\; \xi_l^b\; \eta_i^a \\ &= \left(\sum_{ij=1}^{N}\left(\frac{1}{L-y}\right)_{ij}\; \xi_j^d\; \eta_i^a\right)\left(\sum_{kl=1}^{N}\left(\frac{1}{L-z}\right)_{kl}\; \xi_l^b\; \eta_k^c\right) \\ &= \left(T^{ad}(y) + \frac{1}{2}\delta_{ad}\right)\left(T^{cb}(z) + \frac{1}{2}\delta_{cb}\right)\end{aligned}$$

we prove the result (40).□

The rational limit is obtained by applying the canonical transformation

$$\begin{array}{rcl} p_i & \longrightarrow & \frac{1}{\epsilon}\, p_i \\ q_i & \longrightarrow & \epsilon\, q_i \end{array}$$

and sending ϵ to zero. In this limit

$$\begin{array}{rcl} L_0 & \longrightarrow & \frac{1}{\epsilon}\, L_{\text{rational}} \\ r^0_{12} & \longrightarrow & \frac{1}{\epsilon}\, r^0_{12\ \text{rational}}. \end{array}$$

The Casimir term drops therefore from eq.(43), leaving us with a linear Poisson algebra

$$\{T(y) \overset{\otimes}{,} T(z)\} = -\frac{1}{2}[R(y,z), T(y)\otimes 1 + 1 \otimes T(z)] \tag{45}$$

which is the result found by Gibbons and Hermsen.

6 Conclusion

The Euler-Calogero-Moser model is becoming more and more interesting. On the one hand the computation of the classical r-matrix is made considerably easier by the existence of the extra variables f_{ij}, the more complicated r-matrix of the Calogero-Moser model following naturally from a Hamiltonian reduction procedure. On the other hand, this model exhibits an exact infinite symmetry which is just a current algebra symmetry in the rational case and becomes an exact Yangian symmetry in the trigonometric case. This structure is very much reminiscent of the one discovered in [13]. Actually the two currents J_0 and J_1 (which generate the full algebra) are identical in the two cases. Indeed, in our case we have

$$J_1^{ab} = \sum_{i=1}^N p_i f_{ii}^{ab} - \sum_{\substack{i,j=1 \\ i\neq j}}^N \frac{\mathrm{e}^{-q_{ij}}}{\sinh(q_{ij})} f_{ij} f_{ji}^{ab}.$$

Setting $(X_i)^{ab} = f_{ii}^{ab}$, $\Theta_{ij} = 2\frac{\mathrm{e}^{2q_j}}{\mathrm{e}^{2q_i}-\mathrm{e}^{2q_j}}$ and using eq.(26) we can rewrite

$$J_1^{ab} = \sum_{i=1}^N p_i X_i^{ab} - \sum_{\substack{i,j=1 \\ i\neq j}}^N \Theta_{ij}(X_i X_j)^{ab}.$$

This is exactly the current found in [13]. In fact the model considered in [14, 15, 13] is a quantum version of our model for a particular choice of orbit.

At this point two interesting problems arise. One is the understanding of the role of the r-matrix in the quantization of these models. The other is the hypothetical extension of these results to the elliptic case, which still remains quite mysterious.

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Appendix

The Weierstrass σ function of periods $2\omega_1, 2\omega_2$ is the entire function defined by

$$\sigma(z) = z \prod_{m,n\neq 0} \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left[\frac{z}{\omega_{mn}} + \frac{1}{2}\left(\frac{z}{\omega_{mn}}\right)^2\right] \tag{46}$$

with $\omega_{mn} = 2m\omega_1 + 2n\omega_2$. The functions ζ and $\wp$ are

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} \ , \qquad \wp(z) = -\zeta'(z), \tag{47}$$

these functions having the symmetries

$$\sigma(-z) = -\sigma(z) \ , \qquad \zeta(-z) = -\zeta(z) \ , \qquad \wp(-z) = \wp(z). \tag{48}$$

Their behaviour at the neighbourhood of zero is

$$\sigma(z) = z + O(z^5) \ , \qquad \zeta(z) = z^{-1} + O(z^3) \ , \qquad \wp(z) = z^{-2} + O(z^2). \tag{49}$$

Setting

$$l(q, \lambda) = -\frac{\sigma(q + \lambda)}{\sigma(q) \ \sigma(\lambda)} \tag{50}$$

it is easy to check that

$$l(-q, \lambda) = -l(q, -\lambda) \ , \qquad l'(q, \lambda) = l(q, \lambda) \ [\zeta(\lambda + q) - \zeta(q)]. \tag{51}$$

We need several non trivial relations:

$$-\frac{\sigma(\lambda - \mu) \ \sigma(\lambda + \mu)}{\sigma^2(\lambda) \ \sigma^2(\mu)} = \wp(\lambda) - \wp(\mu), \tag{52}$$

$$\sigma(x-y)\sigma(x+y)\sigma(z-t)\sigma(z+t) + \sigma(y-z)\sigma(y+z)\sigma(x-t)\sigma(x+t) + \sigma(z-x)\sigma(z+x)\sigma(y-t)\sigma(y+t) = 0, \tag{53}$$

$$\frac{\sigma(2z) \ \sigma(x+y) \ \sigma(x-y)}{\sigma(x+z) \ \sigma(x-z) \ \sigma(y+z) \ \sigma(y-z)} = \zeta(x+z) - \zeta(x-z) + \zeta(y-z) - \zeta(y+z), \tag{54}$$

this last equation becoming, in terms of the $l(q, \lambda)$ function,

$$\frac{l(q, \lambda) \ l(-q, \lambda - \mu)}{l(q, \mu)} = \zeta(\lambda) + \zeta(\mu - \lambda) + \zeta(q) - \zeta(\mu + q). \tag{55}$$

Choosing the periods $\omega_1 = \infty$ and $\omega_2 = i\frac{\pi}{2}$ we obtain the hyperbolic case

$$\sigma(z) = \sinh(z) \exp\left(-\frac{z^2}{6}\right) \ , \qquad \zeta(z) = \coth(z) - \frac{z}{3} \ , \qquad \wp(z) = \frac{1}{\sinh^2(z)} + \frac{1}{3} \tag{56}$$

and

$$l(q, \lambda) = -\frac{\sinh(\lambda + q)}{\sinh(\lambda) \sinh(q)} \exp\left(-\frac{\lambda \ q}{3}\right). \tag{57}$$

All these formulas were collected in [11].

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