On $SU(2)$ Wess-Zumino-Witten models and stochastic evolutions

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Abstract

It is discussed how stochastic evolutions may be connected to $SU(2)$ Wess-Zumino-Witten models. Transformations of primary fields are generated by the Virasoro group and an affine extension of the Lie group $SU(2)$. The transformations may be treated and linked separately to stochastic evolutions. A combination allows one to associate a set of stochastic evolutions to the affine Sugawara construction. The singular-vector decoupling generating the Knizhnik-Zamolodchikov equations may thus be related to stochastic evolutions. The latter are based on an infinite-dimensional Brownian motion.

Keywords: Conformal field theory, stochastic evolutions, Wess-Zumino-Witten models, Knizhnik-Zamolodchikov equations.

1 Introduction

There is a tradition in the physics community for describing a broad class of two-dimensional critical systems in terms of conformal field theory (CFT). Schramm has introduced the celebrated stochastic Löwner evolutions (SLEs) [1] as a mathematical rigorous way of handling some of these two-dimensional systems at criticality. The method involves the study of stochastic evolutions of conformal maps, and has been developed further in [2, 3]. Recent reviews on SLE may be found in [4, 5]. Applications as well as formal properties and generalizations of SLE are currently being investigated from various points of view.

An intriguing link to CFT has been examined by Bauer and Bernard [6] (see also [7]) in which the SLE differential equation is associated to a particular random walk on the Virasoro group. The relationship can be made more direct by establishing a connection between the representation theory of CFT and entities conserved in mean under the stochastic process. This is based on the existence of level-two singular vectors in highest-weight modules of the Virasoro algebra.

The approach of Bauer and Bernard has been extended from ordinary CFT and SLE to $N = 1$ superconformal field theory and stochastic evolutions in $N = 1$ superspace [8, 9], to logarithmic CFT [10, 11], and to CFT and SLE-type growth processes in smaller regions of the complex plane than ordinary chordal SLE [12]. The present work offers an extension from ordinary CFT to Wess-Zumino-Witten (WZW) models where the conformal symmetry is supplemented by a Lie group symmetry. We are thus led to consider stochastic evolutions of affine Lie group transformations. Most of our results pertain to $SU(2)$ WZW models, and we refer to [13] for a survey on CFT.

The present elevation to WZW models is discussed in the realm of generating-function primary fields, see [14, 15, 16, 17]. They serve as a convenient way of handling the multiplet of Virasoro primary fields comprising the $su(2)$ representation space of a given conformal weight, as discussed below. Transformations of the multiplet of Virasoro primary fields are replaced by transformations of the generating-function primary field. As in ordinary CFT, these transformations may be described in terms of group elements, here elements of the Virasoro group and an affine extension of the Lie group $SU(2)$. Alternatively, they may be described by utilizing that primary fields have tensor-like transformation properties, again as in ordinary CFT. This allows one to link stochastic evolutions of the two kinds of group elements to stochastic evolutions of conformal and affine transformations, respectively. These links may be established separately. By combining them, one may relate the affine Sugawara construction of Virasoro modes (see [18] and references therein) to entities conserved in mean under the combined stochastic process. This process is somewhat formal, though, as it is based on an infinite-dimensional Brownian motion. As the Knizhnik-Zamolodchikov (KZ) equations are generated by one of the conditions appearing in the affine Sugawara construction, they too may be linked to stochastic evolutions.

A brief review of certain aspects of $SU(2)$ WZW models and generating-function primary fields is given in Section 2. The general link between $SU(2)$ WZW models and stochastic evolutions is discussed in Section 3, while Section 4 concerns the special case corresponding to the affine Sugawara construction and the KZ equations. Section 5 contains some concluding remarks.

2 On $SU(2)$ WZW models

We shall discuss WZW models from an algebraic point of view, and are therefore not concerned with their Lagrangian formulation. The conformal symmetry is generated by the Virasoro modes satisfying the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1}$$

Transformations generated by the affine $su(2)_k$ Lie algebra are here referred to as affine transformations. The algebra, including the commutators with the Virasoro modes, reads

$$\begin{aligned} [J_{+,n}, J_{-,m}] &= 2J_{0,n+m} + kn\delta_{n+m,0} \\ [J_{0,n}, J_{\pm,m}] &= \pm J_{\pm,n+m} \\ [J_{0,n}, J_{0,m}] &= \frac{k}{2}n\delta_{n+m,0} \\ [L_n, J_{a,m}] &= -mJ_{a,n+m} \end{aligned} \tag{2}$$

The level of this affine algebra is indicated by k, and we shall assume that it is a non-negative integer. The non-vanishing entries of the Cartan-Killing form of $su(2)$ are given by

$$\kappa_{00} = \frac{1}{2}, \qquad \kappa_{+-} = \kappa_{-+} = 1 \tag{3}$$

and appear as coefficients to the central terms in (2). Its inverse is given by

$$\kappa^{00} = 2, \qquad \kappa^{+-} = \kappa^{-+} = 1 \tag{4}$$

and comes into play when discussing the affine Sugawara construction below.

Virasoro primary fields are defined by their simple transformation properties with respect to the Virasoro algebra:

$$[L_n, \phi(z)] = \left(z^{n+1}\partial_z + \Delta(n+1)z^n\right)\phi(z) \tag{5}$$

Here Δ denotes the conformal weight of ϕ. The Virasoro primary fields of a given conformal weight Δ may be organized in multiplets corresponding to spin-j representations of the $su(2)$ algebra generated by $\{J_{a,0}\}$, where

$$\Delta = \frac{j(j+1)}{k+2} \tag{6}$$

We shall label the members of such a multiplet as in

$$\phi_{j,-j}(z),\ \phi_{j,-j+1}(z),\ \ldots,\ \phi_{j,j-1}(z),\ \phi_{j,j}(z) \tag{7}$$

The field $\phi_{j,m}$ has $J_{0,0}$ eigenvalue m, while a convenient choice of relative normalizations of the fields is indicated by

$$\begin{aligned} [J_{+,0}, \phi_{j,m}(z)] &= (j+m+1)\phi_{j,m+1}(z) \\ [J_{0,0}, \phi_{j,m}(z)] &= m\phi_{j,m}(z) \\ [J_{-,0}, \phi_{j,m}(z)] &= (j-m+1)\phi_{j,m-1}(z) \end{aligned} \tag{8}$$

A generating function for these Virasoro primary fields may be written

$$\phi(z,x) = \sum_{n=0}^{2j} x^n \phi_{j,j-n}(z) \tag{9}$$

Since this field merely is a linear combination of Virasoro primary fields of the same conformal weight, it too transforms as in (5). That is, the transformation generated by the Virasoro group element G simply reads

$$G^{-1}\phi(z,x)G = (\partial_z f(z))^{\Delta}\phi(f(z),x) \tag{10}$$

for some conformal map f.

The action of the affine generators on the generating-function primary field reads

$$[J_{a,n}, \phi(z,x)] = z^n D_a(x)\phi(z,x) \tag{11}$$

where the differential operators D_a are defined by

$$D_+(x) = -x^2\partial_x + 2jx, \qquad D_0(x) = -x\partial_x + j, \qquad D_-(x) = \partial_x \tag{12}$$

The set $\{-D_a\}$ generates the Lie algebra $su(2)$. To derive the transformations generated by affine $SU(2)$ group elements, we first note that

$$e^{-A}Be^{A} = e^{-ad_A}B \tag{13}$$

where $ad_A B = [A,B]$. Using this, one finds that

$$\begin{aligned} e^{-uJ_{+,n}}\phi(z,x)e^{uJ_{+,n}} &= (1-uz^n x)^{2j}\phi(z, \frac{x}{1-uz^n x}) \\ e^{-uJ_{0,n}}\phi(z,x)e^{uJ_{0,n}} &= e^{-ujz^n}\phi(z, e^{uz^n}x) \\ e^{-uJ_{-,n}}\phi(z,x)e^{uJ_{-,n}} &= \phi(z, x-uz^n) \end{aligned} \tag{14}$$

It follows that a general affine $SU(2)$ group element U generates the transformation

$$U^{-1}\phi(z,x)U = (\partial_x y(z,x))^{-j}\phi(z,y(z,x)) \tag{15}$$

where $y(z,x)$ is a Möbius transformation of x with z-dependent coefficients:

$$y(z,x) = \frac{a(z)x + b(z)}{c(z)x + d(z)}, \qquad a(z)d(z) - b(z)c(z) \neq 0 \tag{16}$$

We shall also be interested in combinations of transformations generated by Virasoro and affine $SU(2)$ group elements. In particular, a tranformation generated by one type of group element followed by a transformation generated by a group element of the other type results in

$$G^{-1}U^{-1}\phi(z,x)UG = (\partial_z f(z))^{\Delta}(\partial_x y(z,x))^{-j}\phi(f(z), y(z,x)) \tag{17}$$

or

$$U^{-1}G^{-1}\phi(z,x)GU = (\partial_z f(z))^{\Delta}(\partial_x y(f(z),x))^{-j}\phi(f(z), y(f(z),x)) \tag{18}$$

The affine Sugawara construction of the Virasoro modes in terms of the affine generators is given by

$$L_N = \frac{1}{2(k+2)}\kappa^{ab}\left(\sum_{n\leq -1} J_{a,n}J_{b,N-n} + \sum_{n\geq 0} J_{a,N-n}J_{b,n}\right) \tag{19}$$

Here and in the following we shall use the convention of summing over appropriately repeated group indices, $a = \pm, 0$. Acting on a highest-weight state, the affine Sugawara construction gives rise to singular vectors of the combined algebra. The decoupling of these is trivial for $N > 0$ while for $N = 0$ it merely reproduces the relation (6). The condition corresponding to $N = -1$ leads to the celebrated KZ equations used in discussions of correlation functions.

3 $SU(2)$ WZW models and stochastic evolutions

By considering the Ito differential of both sides of (10) where G and $f(z)$ now are considered as stochastic processes, G_t and $f_t(z)$, one may relate stochastic differentials of Virasoro group elements to stochastic evolutions of conformal maps. We shall allow higher-dimensional Brownian motion satisfying

$$dB_t^{\mu}dB_t^{\nu} = \delta^{\mu\nu}dt, \qquad dB_t^{\mu}dt = dtdt = 0 \tag{20}$$

and $B_0^{\mu} = 0$. We then have

$$G_t^{-1}dG_t = \alpha_t(L)dt + \sum_{\mu}\beta_{\mu,t}(L)dB_t^{\mu}, \qquad G_0 = 1 \tag{21}$$

where α_t and $\beta_{\mu,t}$ are expressions in the Virasoro modes. Similarly, the stochastic evolution of the associated conformal maps may be written

$$df_t(z) = f_{0,t}(z)dt + \sum_{\mu} f_{\mu,t}(z)dB_t^{\mu}, \qquad f_0(z) = z \tag{22}$$

Techniques for computing and comparing the Ito differentials of both sides of (10) are described in [8, 9, 11]. With

$$\begin{aligned}
\beta_{\mu,t}(L) &= \sum_{n\in\mathbb{Z}} l_{\mu,n,t}L_n \\
\alpha_{0,t}(L) &= \alpha_t(L) - \frac{1}{2}\sum_{\mu,\nu}\delta^{\mu\nu}\beta_{\mu,t}(L)\beta_{\nu,t}(L) \;=\; \sum_{n\in\mathbb{Z}} l_{0,n,t}L_n
\end{aligned} \tag{23}$$

one finds

$$\begin{aligned}
f_{\mu,t}(z) &= -\sum_{n\in\mathbb{Z}} l_{\mu,n,t}(f_t(z))^{n+1} \\
f_{0,t}(z) &= -\sum_{n\in\mathbb{Z}} l_{0,n,t}(f_t(z))^{n+1} + \frac{1}{2}\sum_{\mu,\nu}\delta^{\mu\nu}\sum_{n,m\in\mathbb{Z}}(m+1)l_{\mu,n,t}l_{\nu,m,t}(f_t(z))^{n+m+1}
\end{aligned} \tag{24}$$

This provides a general link expressing the stochastic evolution of conformal maps in terms of the stochastic Virasoro differentials.

A similar description of stochastic evolutions of the Möbius transformations (16) in terms of affine $SU(2)$ differentials follows from an evaluation of the Ito differential of both sides of (15). For later convenience, we shall base the analysis on a potentially non-diagonal higher-dimensional Brownian motion with W_t^ρ an invertible linear combination of Brownian motions, satisfying

$$dW_t^\rho dW_t^\sigma = \lambda^{\rho\sigma}dt, \qquad dW_t^\rho dt = dtdt = 0 \tag{25}$$

and $W_0^\rho = 0$. Here λ is a symmetric and invertible matrix. Using the same approach as above, we write

$$U_t^{-1}dU_t = p_t(J)dt + \sum_\rho q_{\rho,t}(J)dW_t^\rho, \qquad U_0 = 1 \tag{26}$$

and

$$dy_t(z,x) = y_{0,t}(z,x)dt + \sum_\rho y_{\rho,t}(z,x)dW_t^\rho, \qquad y_0(z,x) = x \tag{27}$$

The analogue to (23) reads

$$\begin{aligned}
q_{\rho,t}(J) &= \sum_{n\in\mathbb{Z}} j^a_{\rho,n,t}J_{a,n} \\
p_{0,t}(J) &= p_t(J) - \frac{1}{2}\sum_{\rho,\sigma}\lambda^{\rho\sigma}q_{\rho,t}(J)q_{\sigma,t}(J) \;=\; \sum_{n\in\mathbb{Z}} j^a_{0,n,t}J_{a,n}
\end{aligned} \tag{28}$$

and a goal is to express $y_{0,t}(z,x)$ and $y_{\rho,t}(z,x)$ in terms of $j^a_{0,n,t}$, $j^a_{\rho,n,t}$ and $y_t(z,x)$. We thereby find the following general link

$$y_{\rho,t}(z,x) = \sum_{n\in\mathbb{Z}} z^n\left((y_t(z,x))^2 j^+_{\rho,n,t} + y_t(z,x)j^0_{\rho,n,t} - j^-_{\rho,n,t}\right)$$

$$
\begin{aligned}
y_{0,t}(z,x) &= \sum_{n\in\mathbb{Z}} z^n \left((y_t(z,x))^2 j^+_{0,n,t} + y_t(z,x) j^0_{0,n,t} - j^-_{0,n,t}\right) \\
&+ \frac{1}{2}\sum_{\rho,\sigma}\lambda^{\rho\sigma} \sum_{n,m\in\mathbb{Z}} z^{n+m}\left((y_t(z,x))^2 j^+_{\rho,n,t} + y_t(z,x) j^0_{\rho,n,t} - j^-_{\rho,n,t}\right) \\
&\qquad\qquad\qquad \times\left(2y_t(z,x) j^+_{\sigma,m,t} + j^0_{\sigma,m,t}\right)
\end{aligned} \tag{29}
$$

It turns out that the links (24) and (29) are unaltered if one considers the Ito differential of both sides of (17) based on a combination of the two group actions. This is a priori not obvious since one in that case should allow that some of the Brownian motions B_t^μ, appearing in (21) and (22), and W_t^ρ, appearing in (26) and (27), may be related. One would thus have to supplement (20) and (25) by

$$
dB_t^\mu dW_t^\rho = \hat{\lambda}^{\mu\rho} dt \tag{30}
$$

where $\hat{\lambda}$ could be non-vanishing. As already indicated, however, all terms proportional to $\hat{\lambda}$ cancel and one is left with the separate Virasoro and affine $SU(2)$ links. The rationale for making this consistency check is that we shall use the product UG in our discussion of the affine Sugawara construction in the following.

4 Affine Sugawara construction

Our next objective is to relate a set of stochastic evolutions to the affine Sugawara conditions (19). To this end, we consider a general stochastic process F_t with Ito differential

$$
dF_t = u_t dt + \sum_l v_{l,t} dB_t^l \tag{31}
$$

where $B_0^l = 0$. For sufficiently well-behaved functions u_t and $v_{l,t}$, the time evolution of the expectation value of F_t is given by

$$
\partial_t E[F_t] = E[u_t] \tag{32}
$$

and is seen to vanish provided $E[u_t]$ vanishes. A goal is thus to find processes F_t whose associated u_t's (31) correspond to the affine Sugawara conditions. This illustrates a general scenario in which the representation theory or structure of the CFT (here the $SU(2)$ WZW model) allows one to put an entity equal to zero (here represented by (19) and u_t), thereby producing a martingale (here the stochastic process F_t) of the system.

Since the affine Sugawara conditions involve both Virasoro and affine $su(2)$ generators, we ought to look for combinations of Virasoro and affine $SU(2)$ group elements. We therefore consider

$$
(U_tG_t)^{-1}d(U_tG_t) = G_t^{-1}(U_t^{-1}dU_t)G_t + G_t^{-1}dG_t + G_t^{-1}(U_t^{-1}dU_t)G_t(G_t^{-1}dG_t)
$$

$$
\begin{aligned}
&= \left(G_t^{-1}p_t(J)G_t + \alpha_t(L) + \sum_{\rho,\mu}\hat{\lambda}^{\rho\mu}G_t^{-1}q_{\rho,t}(J)G_t\beta_{\mu,t}(L)\right)dt \\
&+ \left(\sum_{\mu}\beta_{\mu,t}(L)\right)dB_t^{\mu} + \left(\sum_{\rho}G_t^{-1}q_{\rho,t}(J)G_t\right)dW_t^{\rho}
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
(G_tU_t)^{-1}d(G_tU_t) &= U_t^{-1}(G_t^{-1}dG_t)U_t + U_t^{-1}dU_t + U_t^{-1}(G_t^{-1}dG_t)U_t(U_t^{-1}dU_t) \\
&= \left(U_t^{-1}\alpha_t(L)U_t + p_t(J) + \sum_{\mu,\rho}\hat{\lambda}^{\mu\rho}U_t^{-1}\beta_{\mu,t}(L)U_tq_{\rho,t}(J)\right)dt \\
&+ \left(\sum_{\mu}U_t^{-1}\beta_{\mu,t}(L)U_t\right)dB_t^{\mu} + \left(\sum_{\rho}q_{\rho,t}(J)\right)dW_t^{\rho}
\end{aligned}
\tag{34}
$$

It is noted that the inter-relating matrix $\hat{\lambda}$ (30) appears in these expressions. Since the affine Sugawara conditions are linear in the Virasoro modes and bilinear in the affine $su(2)$ modes, we shall work with the differential (33) and not (34). The linearity in the Virasoro modes also suggests that we should consider $\beta_{\mu,t}(L) = 0$ and $\alpha_t(L) = \alpha_{0,t}(L) = L_N$, in which case

$$
G_t = e^{tL_N}, \qquad G_t^{-1}dG_t = L_N dt \tag{35}
$$

and

$$
\partial_t E[U_tG_t] = E[U_tG_t(L_N + G_t^{-1}p_t(J)G_t)] \tag{36}
$$

We should thus require that $p_{0,t}(J) = 0$ and

$$
\begin{aligned}
\sum_{\rho,\sigma}\lambda^{\rho\sigma}q_{\rho,t}(J)q_{\sigma,t}(J) &= \frac{\kappa^{ab}}{k+2}\left(\sum_{n\leq -1}J_{a,n}J_{b,N-n} + \sum_{n\geq 0}J_{a,N-n}J_{b,n}\right) \\
&= \delta_{N,0}\frac{\kappa^{ab}}{k+2}\left(J_{a,0}J_{b,0} + 2\sum_{n\geq 1}J_{a,-n}J_{b,n}\right) \\
&+ (1-\delta_{N,0})\frac{\kappa^{ab}}{k+2}\sum_{n,m\in\mathbb{Z}}\delta_{n+m,N}J_{a,n}J_{b,m}
\end{aligned}
\tag{37}
$$

The summation indices ρ and σ are then naturally considered as double indices: $\rho = (a,n)$ where a is a group index taking the values $a = 0, \pm$, and n is an integer. That is,

$$
\sum_{\rho,\sigma}\lambda^{\rho\sigma}q_{\rho,t}(J)q_{\sigma,t}(J) = \sum_{n,m\in\mathbb{Z}}\lambda^{(a,n)(b,m)}q_{(a,n),t}(J)q_{(b,m),t}(J) \tag{38}
$$

and the condition (37) for $N \neq 0$ is satisfied if

$$
\lambda^{(a,n)(b,m)} = \kappa^{ab}\delta_{n+m,N}, \qquad q_{(a,n),t}(J) = \frac{1}{\sqrt{k+2}}J_{a,n} \tag{39}
$$

The $N=0$ condition is not covered by this analysis due to the divergencies appearing in the affine Sugawara construction when the normal ordering of the affine modes (as in (19) and (37)) is omitted. Since the Ito calculus is based on a *symmetric* 'two-form' in (25), $\lambda^{\rho\sigma}=\lambda^{\sigma\rho}$, we are confined to ordinary products and thus seem deprived of the power of normal ordering required in the $N=0$ condition.

The stochastic differential equation of the affine $SU(2)$ group element accompanying (35) for $N\neq 0$ is now seen to be

$$U_t^{-1}dU_t=\frac{\kappa^{ab}}{2(k+2)}\sum_{n\in\mathbb{Z}}J_{a,N-n}J_{b,n}dt+\frac{1}{\sqrt{k+2}}\sum_{n\in\mathbb{Z}}J_{a,n}dW_t^{(a,n)},\qquad U_0=1 \tag{40}$$

This is a somewhat formal expression as it involves an infinite-dimensional Brownian motion. It is emphasized that there is a pair (G_t,U_t) for each of the affine Sugawara conditions (19) with $N\neq 0$. An explicit indication of which condition such a pair refers to has nevertheless been omitted for notational reasons.

The conformal maps $f_t(z)$ associated to the Virasoro group elements G_t given in (35) evolve deterministically as we have

$$l_{\mu,n,t}=0,\qquad l_{0,n,t}=\delta_{n,N} \tag{41}$$

and subsequently from (24)

$$df_t(z)=-(f_t(z))^{N+1}dt,\qquad f_0(z)=z \tag{42}$$

This is solved by

$$f_t(z)=\frac{z}{(1+Ntz^N)^{1/N}},\qquad N\neq 0 \tag{43}$$

or

$$f_t(z)=ze^{-t},\qquad N=0 \tag{44}$$

and can be verified directly using (5), (10) and (13).

To determine the stochastic differentials of the Möbius transformations $y_t(z,x)$ associated to (40), we rely on the link (29). We have

$$j^b_{(a,n),m,t}=\frac{1}{\sqrt{k+2}}\delta_{nm}\delta_a^b,\qquad j^a_{0,n,t}=0 \tag{45}$$

and it follows from (29) that

$$y_{(a,n),t}=\frac{1}{\sqrt{k+2}}z^n\left(y^2\delta_a^+ +y\delta_a^0-\delta_a^-\right),\qquad y_{0,t}=0 \tag{46}$$

and hence

$$dy_t(z,x)=\frac{1}{\sqrt{k+2}}\sum_{n\in\mathbb{Z}}z^n\left(y^2dW_t^{(+,n)}+ydW_t^{(0,n)}-dW_t^{(-,n)}\right),\qquad y_0(z,x)=x \tag{47}$$

Note that the dependence on N is given implicitly via $dW_t^{(a,n)}dW_t^{(b,m)} = \kappa^{ab}\delta_{n+m,N}dt$, cf. (39).

In order to express $dy_t(z,x)$ in terms of ordinary, though infinitely many, 'orthonormal' Brownian motions instead of $\{W^{(a,n)}\}$, we perform some linear transformations. First we introduce the linear combinations

$$
\begin{aligned}
B_t^{(+,n)} &= \frac{1}{\sqrt{2}}\left(B_t^{(1,n)} + iB_t^{(2,n)}\right) \\
B_t^{(0,n)} &= \sqrt{2}B_t^{(3,n)} \\
B_t^{(-,n)} &= \frac{1}{\sqrt{2}}\left(B_t^{(1,n)} - iB_t^{(2,n)}\right)
\end{aligned}
\tag{48}
$$

where we have introduced an infinite set of Brownian motions labeled as $B_t^{(\ell,n)}$. They satisfy the orthonormality conditions

$$
dB_t^{(\ell,n)}dB_t^{(\ell',n')} = \delta_{\ell\ell'}\delta_{nn'}dt, \qquad \ell,\ell' \in \{1,2,3\}, \quad n,n' \in \mathbb{Z}
\tag{49}
$$

and $B_0^{(\ell,n)} = 0$. With our standard convention for the group index, $a = \pm, 0$, we then have

$$
\begin{aligned}
W_t^{(a,n>N/2)} &= \frac{1}{\sqrt{2}}\left(B_t^{(a,n)} + iB^{(a,N-n)}\right) \\
W_t^{(a,n=N/2)} &= B^{(a,N/2)}, \qquad\qquad \text{for } N \text{ even} \\
W_t^{(a,n<N/2)} &= \frac{-i}{\sqrt{2}}\left(B_t^{(a,n)} + iB^{(a,N-n)}\right)
\end{aligned}
\tag{50}
$$

These processes are seen to respect $dW_t^{(a,n)}dW_t^{(b,m)} = \lambda^{(a,n)(b,m)}dt$ with λ given in (39). It is now straightforward to express $dy_t(z,x)$ given in (47) in terms of the orthonormal Brownian differentials $dB_t^{(\ell,n)}$, and we find

$$
\begin{aligned}
dy_t(z,x) &= \frac{1}{2\sqrt{k+2}}\sum_{n>N/2}\left(z^n + z^{N-n}\right)\left((y^2-1)dB_t^{(1,n)} + i(y^2+1)dB_t^{(2,n)} + 2ydB_t^{(3,n)}\right) \\
&+ \frac{i}{2\sqrt{k+2}}\sum_{n<N/2}\left(z^{N-n} - z^n\right)\left((y^2-1)dB_t^{(1,n)} + i(y^2+1)dB_t^{(2,n)} + 2ydB_t^{(3,n)}\right) \\
&+ \frac{(1+(-1)^N)}{2\sqrt{2(k+2)}}z^{N/2}\left((y^2-1)dB_t^{(1,n)} + i(y^2+1)dB_t^{(2,n)} + 2ydB_t^{(3,n)}\right) \\
y_0(z,x) &= x
\end{aligned}
\tag{51}
$$

where the last term proportional to $z^{N/2}$ vanishes for N odd. It is recalled that $N \neq 0$ in this expression. The condition underlying the KZ equations is easily obtained by setting $N = -1$.

5 Conclusion

We have discussed how $SU(2)$ WZW models may be linked to stochastic evolutions. The conformal symmetry is thereby related to stochastic evolutions of conformal maps whereas the affine $SU(2)$ invariance is linked to stochastic evolutions of affine transformations. This extends work done in [6, 12, 8, 9, 10, 11] on connections between CFT and SLE, and generalizations thereof. An objective of the present work was to develop a set of stochastic differential equations corresponding to the affine Sugawara construction of the Virasoro generators in terms of the affine generators. This has been achieved for all Virasoro modes but L_0, in which case our approach seems to be incapable of reproducing the required normal ordering of the affine modes. Since the L_{-1} mode is covered, we have thus obtained a stochastic differential equation describing the condition underlying the KZ equations. The associated stochastic process is somewhat formal, as it is based on an infinite-dimensional Brownian motion. We nevertheless hope that our analysis may prove itself useful.

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