Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories

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The correspondence between supergravity (and string theory) on *AdS* space and boundary conformal field theory relates the thermodynamics of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions to the thermodynamics of Schwarzschild black holes in Anti-de Sitter space. In this description, quantum phenomena such as the spontaneous breaking of the center of the gauge group, magnetic confinement, and the mass gap are coded in classical geometry. The correspondence makes it manifest that the entropy of a very large *AdS* Schwarzschild black hole must scale "holographically" with the volume of its horizon. By similar methods, one can also make a speculative proposal for the description of large N gauge theories in four dimensions without supersymmetry.

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1. Introduction

Understanding the large N behavior of gauge theories in four dimensions is a classic and important problem [1]. The structure of the "planar diagrams" that dominate the large N limit gave the first clue that this problem might be solved by interpreting four-dimensional large N gauge theory as a string theory. Attempts in this direction have led to many insights relevant to critical string theory; for an account of the status, see [2]. Recently, motivated by studies of interactions of branes with external probes [3-7], and near-extremal brane geometry [8,9], a concrete proposal in this vein has been made [11], in the context of certain conformally-invariant theories such as $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. The proposal relates supergravity on anti-de Sitter or AdS space (or actually on AdS times a compact manifold) to conformal field theory on the boundary, and thus potentially introduces into the study of conformal field theory the whole vast subject of AdS compactification of supergravity (for a classic review see [12]; see also [13] for an extensive list of references relevant to current developments). Possible relations of a theory on AdS space to a theory on the boundary have been explored for a long time, both in the abstract (for example, see [14]), and in the context of supergravity and brane theory (for example, see [15]). More complete references relevant to current developments can be found in papers already cited and in many of the other important recent papers [16-46] in which many aspects of the CFT/AdS correspondence have been extended and better understood.

In [27,47], a precise recipe was presented for computing CFT observables in terms of AdS space. It will be used in the present paper to study in detail a certain problem in gauge theory dynamics. The problem in question, already discussed in part in section 3.2 of [47], is to understand the high temperature behavior of $\mathcal{N} = 4$ super Yang-Mills theory. As we will see, in this theory, the CFT/AdS correspondence implies, in the infinite volume limit, many expected but subtle quantum properties, including a non-zero expectation value for a temporal Wilson loop [48,49], an area law for spatial Wilson loops, and a mass gap. (The study of Wilson loops is based on a formalism that was introduced recently [37,38].) These expectations are perhaps more familiar for ordinary four-dimensional Yang-Mills theory without supersymmetry – for a review see [50]. But the incorporation of supersymmetry, even $\mathcal{N} = 4$ supersymmetry, is not expected to affect these particular issues, since non-zero temperature breaks supersymmetry explicitly and makes it possible for the spin zero and

spin one-half fields to get mass,[1] very plausibly reducing the high temperature behavior to that of the pure gauge theory. The ability to recover from the CFT/AdS correspondence relatively subtle dynamical properties of high temperature gauge theories, in a situation not governed by supersymmetry or conformal invariance, certainly illustrates the power of this correspondence.

In section 2, we review the relevant questions about gauge theories and the framework in which we will work, and develop a few necessary properties of the Schwarzschild black hole on AdS space. The CFT/AdS correspondence implies readily that in the limit of large mass, a Schwarzschild black hole in AdS space has an entropy proportional to the volume of the horizon, in agreement with the classic result of Bekenstein [51] and Hawking [52]. (The comparison of horizon volume of the AdS Schwarzschild solution to field theory entropy was first made, in the AdS_5 case, in [3], using a somewhat different language. As in some other string-theoretic studies of Schwarzschild black holes [53,54], and some earlier studies of BPS-saturated black holes [55], but unlike some microscopic studies of BPS black holes [56], in our discussion we are not able to determine the constant of proportionality between area and entropy.) This way of understanding black hole entropy is in keeping with the notion of "holography" [57-59]. The result holds for black holes with Schwarzschild radius much greater than the radius of curvature of the AdS space, and so does not immediately imply the corresponding result for Schwarzschild black holes in Minkowski space.

In section 3, we demonstrate, on the basis of the CFT/AdS correspondence, that the $\mathcal{N} = 4$ theory at nonzero temperature has the claimed properties, especially the breaking of the center of the gauge group, magnetic confinement, and the mass gap.

In section 4, we present, using similar ideas, a proposal for studying ordinary large N gauge theory in four dimensions (without supersymmetry or matter fields) via string theory. In this proposal, we can exhibit confinement and the mass gap, precisely by the same arguments used in section 3, along with the expected large N scaling, but we are not able to effectively compute hadron masses or show that the model is asymptotically free.

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[1] The thermal ensemble on a spatial manifold $\mathbf{R}^3$ can be described by path integrals on $\mathbf{R}^3 \times \mathbf{S}^1$, with a radius for the $\mathbf{S}^1$ equal to $\beta = T^{-1}$, with T the temperature. The fermions obey antiperiodic boundary conditions around the $\mathbf{S}^1$ direction, and so get masses of order $1/T$ at tree level. The spin zero bosons get mass at the one-loop level.

2. High Temperatures And *AdS* Black Holes

2.1. $\mathbf{R}^3$ *and* $\mathbf{S}^3$

We will study the $\mathcal{N} = 4$ theory at finite temperature on a spatial manifold $\mathbf{S}^3$ or $\mathbf{R}^3$. $\mathbf{R}^3$ will be obtained by taking an infinite volume limit starting with $\mathbf{S}^3$.

To study the theory at finite temperature on $\mathbf{S}^3$, we must compute the partition function on $\mathbf{S}^3 \times \mathbf{S}^1$ – with supersymmetry-breaking boundary conditions in the $\mathbf{S}^1$ directions. We denote the circumferences of $\mathbf{S}^1$ and $\mathbf{S}^3$ as β and β', respectively. By conformal invariance, only the ratio β/β' matters. To study the finite temperature theory on $\mathbf{R}^3$, we take the large β' limit, reducing to $\mathbf{R}^3 \times \mathbf{S}^1$.

Once we are on $\mathbf{R}^3 \times \mathbf{S}^1$, with circumference β for $\mathbf{S}^1$, the value of β can be scaled out via conformal invariance. Thus, the $\mathcal{N} = 4$ theory on $\mathbf{R}^3$ cannot have a phase transition at any nonzero temperature. Even if one breaks conformal invariance by formulating the theory on $\mathbf{S}^3$ with some circumference β', there can be no phase transition as a function of temperature, since theories with finitely many local fields have in general no phase transitions as a function of temperature.

However, in the large N limit, it is possible to have phase transitions even in finite volume [60]. In section 3.2 of [47], it was shown that the $\mathcal{N} = 4$ theory on $\mathbf{S}^3 \times \mathbf{S}^1$ has in the large N limit a phase transition as a function of β/β'. The large β/β' phase has some properties in common with the usual large β (or small temperature) phase of confining gauge theories, while the small β/β' phase is analogous to a deconfining phase.

When we go to $\mathbf{R}^3 \times \mathbf{S}^1$ by taking $\beta' \to \infty$ for fixed β, we get $\beta/\beta' \to 0$. So the unique nonzero temperature phase of the $\mathcal{N} = 4$ theory on $\mathbf{R}^3$ is on the high temperature side of the phase transition and should be compared to the deconfining phase of gauge theories. Making this comparison will be the primary goal of section 3. We will also make some remarks in section 3 comparing the low temperature phase on $\mathbf{S}^3 \times \mathbf{S}^1$ to the confining phase of ordinary gauge theories. Here one can make some suggestive observations, but the scope is limited because in the particular $\mathcal{N} = 4$ gauge theory under investigation, the low temperature phase on $\mathbf{S}^3$ arises only in finite volume, while most of the deep questions of statistical mechanics and quantum dynamics refer to the infinite volume limit.

2.2. Review Of Gauge Theories

We will now review the relevant expectations concerning finite temperature gauge theories in four dimensions.

Deconfinement at high temperatures can be usefully described, in a certain sense, in terms of spontaneous breaking of the center of the gauge group (or more precisely of the subgroup of the center under which all fields transform trivially). For our purposes, the gauge group will be $G = SU(N)$, and the center is $\Gamma = \mathbf{Z}_N$; it acts trivially on all fields, making possible the following standard construction.

Consider $SU(N)$ gauge theory on $Y \times \mathbf{S}^1$, with Y any spatial manifold. A conventional gauge transformation is specified by the choice of a map $g : Y \times \mathbf{S}^1 \to G$ which we write explicitly as $g(y, z)$, with y and z denoting respectively points in Y and in $\mathbf{S}^1$. (In describing a gauge transformation in this way, we are assuming that the G-bundle has been trivialized at least locally along Y; global properties along Y are irrelevant in the present discussion.) Such a map has $g(y, z + \beta) = g(y, z)$. However, as all fields transform trivially under the center of G, we can more generally consider gauge transformations by gauge functions $g(y, z)$ that obey

$$g(y, z + \beta) = g(y, z)h, \tag{2.1}$$

with h an arbitrary element of the center. Let us call the group of such extended gauge transformations (with arbitrary dependence on z and y and any h) $\overline{G}$ and the group of ordinary gauge transformations (with $h = 1$ but otherwise unrestricted) G'. The quotient $\overline{G}/G'$ is isomorphic to the center Γ of G, and we will denote it simply as Γ. Factoring out G' is natural because it acts trivially on all local observables and physical states (for physical states, G'-invariance is the statement of Gauss's law), while Γ can act nontrivially on such observables.

An order parameter for spontaneous breaking of Γ is the expectation value of a temporal Wilson line. Thus, let C be any oriented closed path of the form $y \times \mathbf{S}^1$ (with again y a fixed point in Y), and consider the operator

$$W(C) = \mathrm{Tr} P \exp \int_C A, \tag{2.2}$$

with A the gauge field and the trace taken in the N-dimensional fundamental representation of $SU(N)$. Consider a generalized gauge transformation of the form (2.1), with h an

N^{th} root of unity representing an element of the center of $SU(N)$. Action by such a gauge transformation multiplies the holonomy of A around C by h, so one has

$$W(C) \to hW(C). \tag{2.3}$$

Hence, the expectation value $\langle W(C)\rangle$ is an order parameter for the spontaneous breaking of the Γ symmetry.

Of course, such spontaneous symmetry breaking will not occur (for finite N) in finite volume. But a nonzero expectation value $\langle W(C)\rangle$ in the infinite volume limit, that is with Y replaced by $\mathbf{R}^3$, is an important order parameter for deconfinement. Including the Wilson line $W(C)$ in the system means including an external static quark (in the fundamental representation of $SU(N)$), so an expectation value for $W(C)$ means intuitively that the cost in free energy of perturbing the system by such an external charge is finite. In a confining phase, this free energy cost is infinite and $\langle W(C)\rangle = 0$. The $\mathcal{N} = 4$ theory on $\mathbf{R}^3$ corresponds to a high temperature or deconfining phase; we will confirm in section 3, using the CFT/AdS correspondence, that it has spontaneous breaking of the center.

Other important questions arise if we take the infinite volume limit, replacing X by $\mathbf{R}^3$. The theory at long distances along $\mathbf{R}^3$ is expected to behave like a pure $SU(N)$ gauge theory in three dimensions. At nonzero temperature, at least for weak coupling, the fermions get a mass at tree level from the thermal boundary conditions in the $\mathbf{S}^1$ direction, and the scalars (those present in four dimensions, as well as an extra scalar that arises from the component of the gauge field in the $\mathbf{S}^1$ direction) get a mass at one loop level; so the long distance dynamics is very plausibly that of three-dimensional gauge fields. The main expected features of three-dimensional pure Yang-Mills theory are confinement and a mass gap. The mass gap means simply that correlation functions $\langle \mathcal{O}(y, z)\mathcal{O}'(y', z)\rangle$ vanish exponentially for $|y - y'| \to \infty$. Confinement is expected to show up in an area law for the expectation value of a spatial Wilson loop. The area law means the following. Let C be now an oriented closed loop encircling an area A in $\mathbf{R}^3$, at a fixed point on $\mathbf{S}^1$. The area law means that if C is scaled up, keeping its shape fixed and increasing A, then the expectation value of $W(C)$ vanishes exponentially with A.

"Confinement" In Finite Volume

Finally, there is one more issue that we will address here. In the large N limit, a criterion for confinement is whether (after subtracting a constant from the ground state energy) the free energy is of order one – reflecting the contributions of color singlet hadrons

– or of order N^2 – reflecting the contributions of gluons. (This criterion has been discussed in [61].) In [47], it was shown that in the $\mathcal{N} = 4$ theory on $\mathbf{S}^3 \times \mathbf{S}^1$, the large N theory has a low temperature phase with a free energy of order 1 – a "confining" phase – and a high temperature phase with a free energy of order N^2 – an "unconfining" phase.

Unconfinement at high temperatures comes as no surprise, of course, in this theory, and since the theory on $\mathbf{R}^3 \times \mathbf{S}^1$, at any temperature, is in the high temperature phase, we recover the expected result that the infinite volume theory is not confining. However, it seems strange that the finite volume theory on $\mathbf{S}^3$, at low temperatures, is "confining" according to this particular criterion.

This, however, is a general property of large N gauge theories on $\mathbf{S}^3$, at least for weak coupling (and presumably also for strong coupling). On a round three-sphere, the classical solution of lowest energy is unique up to gauge transformations (flat directions in the scalar potential are eliminated by the $R\phi^2$ coupling to scalars, R being the Ricci scalar), and is given by setting the gauge field A, fermions ψ, and scalars ϕ all to zero. This configuration is invariant under global $SU(N)$ gauge transformations. The Gauss law constraint in finite volume says that physical states must be invariant under the global $SU(N)$. There are no zero modes for any fields (for scalars this statement depends on the $R\phi^2$ coupling). Low-lying excitations are obtained by acting on the ground state with a finite number of A, ψ, and ϕ creation operators, and then imposing the constraint of global $SU(N)$ gauge invariance. The creation operators all transform in the adjoint representation, and so are represented in color space by matrices $M_1, M_2, \ldots, M_s$. $SU(N)$ invariants are constructed as traces, say $\text{Tr}\, M_1 M_2 \ldots M_s$. The number of such traces is given by the number of ways to order the factors and is independent of N. So the multiplicity of low energy states is independent of N, as is therefore the low temperature free energy.

This result, in particular, is kinematic, and has nothing to do with confinement. To see confinement from the N dependence of the free energy, we must go to infinite volume. On $\mathbf{R}^3$, the Gauss law constraint does not say that the physical states are invariant under global $SU(N)$ transformations, but only that their global charge is related to the electric field at spatial infinity. If the free energy on $\mathbf{R}^3$ is of order 1 (and not proportional to N^2), this actually is an order parameter for confinement.

Now let us go back to finite volume and consider the behavior at high temperatures. At high temperatures, one cannot effectively compute the free energy by counting elementary excitations. It is more efficient to work in the "crossed channel." In $\mathbf{S}^3 \times \mathbf{S}^1$, with circumferences β' and β, if we take $\beta' \to \infty$ with fixed β, the free energy is proportional to

the volume of $\mathbf{S}^3$ times the ground state energy density of the 2+1-dimensional theory that is obtained by compactification on $\mathbf{S}^1$ (with circumference β and supersymmetry-breaking boundary conditions). That free energy is of order N^2 (the supersymmetry breaking spoils the cancellation between bosons and fermions already at the one-loop level, and the one-loop contribution is proportional to N^2). The volume of $\mathbf{S}^3$ is of order $(\beta')^3$. So the free energy on $\mathbf{S}^3 \times \mathbf{S}^1$ scales as $N^2(\beta')^3$ if one takes $\beta' \to \infty$ at fixed β, or in other words as $N^2\beta^{-3}$ if one takes $\beta \to 0$ at fixed β'. Presently we will recover this dependence on β by comparing to black holes.

2.3. AdS Correspondence And Schwarzshild Black Holes

The version of the CFT/AdS correspondence that we will use asserts that conformal field theory on an n-manifold M is to be studied by summing over contributions of Einstein manifolds B of dimension $n+1$ which (in a sense explained in [27,47]) have M at infinity.

We will be mainly interested in the case that $M = \mathbf{S}^{n-1} \times \mathbf{S}^1$, or $\mathbf{R}^{n-1} \times \mathbf{S}^1$. For $\mathbf{S}^{n-1} \times \mathbf{S}^1$, there are two known B's, identified by Hawking and Page [62] in the context of quantum gravity on AdS space. One manifold, X_1, is the quotient of AdS space by a subgroup of $SO(1, n+1)$ that is isomorphic to $\mathbf{Z}$. The metric (with Euclidean signature) can be written

$$ds^2 = \left(\frac{r^2}{b^2} + 1\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2}\right) + 1} + r^2 d\Omega^2, \tag{2.4}$$

with $d\Omega^2$ the metric of a round sphere $\mathbf{S}^{n-1}$ of unit radius. Here t is a periodic variable of arbitrary period. We have normalized (2.4) so that the Einstein equations read

$$R_{ij} = -nb^{-2}g_{ij}; \tag{2.5}$$

here b is the radius of curvature of the anti-de Sitter space. With this choice, n does not appear explicitly in the metric. This manifold can contribute to either the standard thermal ensemble $\mathrm{Tr}\, e^{-\beta H}$ or to $\mathrm{Tr}(-1)^F e^{-\beta H}$, depending on the boundary conditions one uses for fermions in the t direction. The topology of X_1 is $\mathbf{R}^n \times \mathbf{S}^1$, or $\mathbf{B}^n \times \mathbf{S}^1$ ($\mathbf{B}^n$ denoting an n-ball) if we compactify it by including the boundary points at $r = \infty$.

The second solution, X_2, is the Schwarzschild black hole, in AdS space. The metric is

$$ds^2 = \left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}}\right)} + r^2 d\Omega^2. \tag{2.6}$$

Here w_n is the constant

$$w_n = \frac{16\pi G_N}{(n-1)\mathrm{Vol}(\mathbf{S}^{n-1})}. \tag{2.7}$$

Here G_N is the $n+1$-dimensional Newton's constant and $\mathrm{Vol}(\mathbf{S}^{n-1})$ is the volume of a unit $n-1$-sphere; the factor w_n is included so that M is the mass of the black hole (as we will compute later). Also, the spacetime is restricted to the region $r \geq r_+$, with r_+ the largest solution of the equation

$$\frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}} = 0. \tag{2.8}$$

The metric (2.6) is smooth and complete if and only if the period of t is

$$\beta_0 = \frac{4\pi b^2 r_+}{nr_+^2 + (n-2)b^2}. \tag{2.9}$$

For future use, note that in the limit of large M one has

$$\beta_0 \sim \frac{4\pi b^2}{n(w_n b^2)^{1/n} M^{1/n}}. \tag{2.10}$$

As in the $n = 3$ case considered in [62], β_0 has a maximum as a function of r_+, so the Schwarzschild black hole only contributes to the thermodynamics if β is small enough, that is if the temperature is high enough. Moreover, X_2 makes the dominant contribution at sufficiently high temperature, while X_1 dominates at low temperature. The topology of X_2 is $\mathbf{R}^2 \times \mathbf{S}^{n-1}$, or $\mathbf{B}^2 \times \mathbf{S}^{n-1}$ if we compactify it to include boundary points. In particular, X_2 is simply-connected, has a unique spin structure, and contributes to the standard thermal ensemble but not to $\mathrm{Tr}(-1)^F e^{-\beta H}$.

With either (2.4) or (2.6), the geometry of the $\mathbf{S}^{n-1} \times \mathbf{S}^1$ factor at large r can be simply explained: the $\mathbf{S}^1$ has radius approximately $\beta = (r/b)\beta_0$, and the $\mathbf{S}^{n-1}$ has radius $\beta' = r/b$. The ratio is thus $\beta/\beta' = \beta_0$. If we wish to go to $\mathbf{S}^1 \times \mathbf{R}^{n-1}$, we must take $\beta/\beta' \to 0$, that is $\beta_0 \to 0$; this is the limit of large temperatures. (2.9) seems to show that this can be done with either $r_+ \to 0$ or $r_+ \to \infty$, but the $r_+ \to 0$ branch is thermodynamically unfavored [62] (having very large action), so we must take the large r_+ branch, corresponding to large M.

A scaling that reduces (2.9) to a solution with boundary $\mathbf{R}^{n-1} \times \mathbf{S}^1$ may be made as follows. If we set $r = (w_n M/b^{n-2})^{1/n}\rho$, $t = (w_n M/b^{n-2})^{-1/n}\tau$, then for large M we can

reduce $r^2/b^2 + 1 - w_n M/r^{n-2}$ to $(w_n M/b^{n-2})^{2/n}(\rho^2/b^2 - b^{n-2}/\rho^{n-2})$. The period of τ become $\beta_1 = (w_n M/b^{n-2})^{1/n}\beta_0$ or (from (2.10)) for large M

$$\beta_1 = \frac{4\pi b}{n}. \tag{2.11}$$

The metric becomes

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right) d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}} + (w_n M/b^{n-2})^{2/n}\rho^2 \, d\Omega^2. \tag{2.12}$$

The $M^{2/n}$ multiplying the last term means that the radius of $\mathbf{S}^{n-1}$ is of order $M^{1/n}$ and so diverges for $M \to \infty$. Hence, the $\mathbf{S}^{n-1}$ is becoming flat and looks for $M \to \infty$ locally like $\mathbf{R}^{n-1}$. If we introduce near a point $P \in \mathbf{S}^{n-1}$ coordinates y_i such that at P, $d\Omega^2 = \sum_i dy_i^2$, and then set $y_i = (w_n M/b^{n-2})^{1/n}x_i$, then the metric becomes

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right)} + \rho^2 \sum_{i=1}^{n-1} dx_i^2. \tag{2.13}$$

This is the desired solution $\widetilde{X}$ that is asymptotic at infinity to $\mathbf{R}^{n-1} \times \mathbf{S}^1$ instead of $\mathbf{S}^{n-1} \times \mathbf{S}^1$. Its topology, if we include boundary points, is $\mathbf{R}^{n-1} \times \mathbf{B}^2$. The same solution was found recently by scaling of a near-extremal brane solution [43].

2.4. Entropy Of Schwarzschild Black Holes

Following Hawking and Page [62] (who considered the case $n = 3$), we will now describe the thermodynamics of Schwarzschild black holes in AdS_{n+1}. Our normalization of the cosmological constant is stated in (2.5). The bulk Einstein action with this value of the cosmological constant is

$$I = -\frac{1}{16\pi G_N} \int d^{n+1}x \sqrt{g} \left(R + \frac{\frac{1}{2}n(n-1)}{b^2}\right). \tag{2.14}$$

For a solution of the equations of motion, one has $R = -\frac{1}{2}n(n+1)/b^2$, and the action becomes

$$I = \frac{n}{8\pi G_N} \int d^{n+1}x \sqrt{g}, \tag{2.15}$$

that is, the volume of spacetime times $n/8\pi G_N$. The action additionally has a surface term [63,64], but the surface term vanishes for the AdS Schwarzschild black hole, as noted in [62], because the black hole correction to the AdS metric vanishes too rapidly at infinity.

9

Actually, both the AdS spacetime (2.4) and the black hole spacetime (2.6) have infinite volume. As in [62], one subtracts the two volumes to get a finite result. Putting an upper cutoff R on the radial integrations, the regularized volume of the AdS spacetime is

$$V_1(R) = \int_0^{\beta'} dt \int_0^{\infty} dr \int_{\mathbf{S}^{n-1}} d\Omega \, r^{n-1}, \qquad (2.16)$$

and the regularized volume of the black hole spacetime is

$$V_2(R) = \int_0^{\beta_0} dt \int_{r_+}^{\infty} dr \int_{\mathbf{S}^{n-1}} d\Omega \, r^{n-1}. \qquad (2.17)$$

One difference between the two integrals is obvious here: in the black hole spacetime $r \geq r_+$, while in the AdS spacetime $r \geq 0$. A second and slightly more subtle difference is that one must use different periodicities β' and β_0 for the t integrals in the two cases. The black hole spacetime is smooth only if β_0 has the value given in (2.9), but for the AdS spacetime, any value of β' is possible. One must adjust β' so that the geometry of the hypersurface $r = R$ is the same in the two cases; this is done by setting $\beta' \sqrt{(r^2/b^2) + 1} = \beta_0 \sqrt{(r^2/b^2) + 1 - w_n M/r^{n-2}}$. After doing so, one finds that the action difference is

$$I = \frac{n}{8\pi G_N} \lim_{R \to \infty} (V_2(R) - V_1(R)) = \frac{\text{Vol}(\mathbf{S}^{n-1})(b^2 r_+^{n-1} - r_+^{n+1})}{4G_N(nr_+^2 + (n-2)b^2)}. \qquad (2.18)$$

This is positive for small r_+ and negative for large r_+, showing that the phase transition found in [62] occurs for all n.

Then, as in [62], one computes the energy

$$E = \frac{\partial I}{\partial \beta_0} = \frac{(n-1)\text{Vol}(\mathbf{S}^{n-1})(r_+^n b^{-2} + r_+^{n-2})}{16\pi G_N} = M \qquad (2.19)$$

and the entropy

$$S = \beta_0 E - I = \frac{1}{4G_N} r_+^{n-1} \text{Vol}(\mathbf{S}^{n-1}) \qquad (2.20)$$

of the black hole. The entropy can be written

$$S = \frac{A}{4G_N}, \qquad (2.21)$$

with A the volume of the horizon, which is the surface at $r = r_+$.

Comparison To Conformal Field Theory

Now we can compare this result for the entropy to the predictions of conformal field theory.

The black hole entropy should be compared to boundary conformal field theory on $\mathbf{S}^{n-1} \times \mathbf{S}^1$, where the two factors have circumference 1 and β_0/b, respectively. In the limit as $\beta_0 \to 0$, this can be regarded as a high temperature system on $\mathbf{S}^{n-1}$. Conformal invariance implies that the entropy density on $\mathbf{S}^{n-1}$ scales, in the limit of small β_0, as $\beta_0^{-(n-1)}$. According to (2.9), $\beta_0 \to 0$ means $r_+ \to \infty$ with $\beta_0 \sim 1/r_+$. Hence, the boundary conformal field theory predicts that the entropy of this system is of order r_+^{n-1}, and thus asymptotically is a fixed multiple of the horizon volume which appears in (2.21). This is of course the classic result of Bekenstein and Hawking, for which microscopic explanations have begun to appear only recently. Note that this discussion assumes that $\beta_0 << 1$, which means that $r_+ >> b$; so it applies only to black holes whose Schwarzschild radius is much greater than the radius of curvature of AdS space. However, in this limit, one does get a simple explanation of why the black hole entropy is proportional to area. The explanation is entirely "holographic" in spirit [57,59].

To fix the constant of proportionality between entropy and horizon volume (even in the limit of large black holes), one needs some additional general insight, or some knowledge of the quantum field theory on the boundary. For 2+1-dimensional black holes, in the context of an old framework [67] for a relation to boundary conformal field theory which actually is a special case of the general CFT/AdS correspondence, such additional information is provided by modular invariance of the boundary conformal field theory [65,66].

3. High Temperature Behavior Of The $\mathcal{N} = 4$ Theory

In this section we will address three questions about the high temperature behavior of the $\mathcal{N} = 4$ theory that were raised in section 2: the behavior of temporal Wilson lines; the behavior of spatial Wilson lines; and the existence of a mass gap.

In discussing Wilson lines, we use a formalism proposed recently [37,38]. Suppose one is doing physics on a four-manifold M which is the boundary of a five-dimensional Einstein manifold B (of negative curvature). To compute a Wilson line associated with a contour $C \subset M$, we study elementary strings on B with the property that the string worldsheet D has C for its boundary. Such a D has an infinite area, but the divergence is proportional to the circumference of C. One can define therefore a regularized area $\alpha(D)$ by subtracting

from the area of D an infinite multiple of the circumference of C. The expectation value of a Wilson loop $W(C)$ is then roughly

$$\langle W(C) \rangle = \int_D d\mu \, e^{-\alpha(D)}. \tag{3.1}$$

Moreover, according to [37], in the regime in which supergravity is valid (large N and large $g^2 N$), the integral can be evaluated approximately by setting D to the surface of smallest $\alpha(D)$ that obeys the boundary conditions.

The formula (3.1) is oversimplified for various reasons. For one thing, worldsheet fermions must be included in the path integral. Also, the description of the $\mathcal{N} = 4$ theory actually involves not strings on B but strings on the ten-manifold $B \times \mathbf{S}^5$. Accordingly, Maldacena considers in [37] some generalized Wilson loop operators with scalar fields included in the definition; the boundary behavior of D in the $\mathbf{S}^5$ factor depends on which operator one uses. But if all scalars have masses, as they do in the $\mathcal{N} = 4$ theory at positive temperature, the generalized Wilson loop operators are equivalent at long distances to conventional ones. An important conclusion from (3.1) nonetheless stands: Wilson loops on $\mathbf{R}^4$ will obey an area law if, when C is scaled up, the minimum value of $\alpha(D)$ scales like a positive multiple of the area enclosed by C. (3.1) also implies vanishing of $\langle W(C) \rangle$ if suitable D's do not exist, that is, if C is not a boundary in M.

3.1. Temporal Wilson Lines

Our first goal will be to analyze temporal Wilson lines. That is, we take spacetime to be $\mathbf{S}^3 \times \mathbf{S}^1$ or $\mathbf{R}^3 \times \mathbf{S}^1$, and we take $C = P \times \mathbf{S}^1$, with P a point in $\mathbf{S}^3$ or in $\mathbf{R}^3$.

We begin on $\mathbf{S}^3$ in the low temperature phase. We recall that this is governed by a manifold X_1 with the topology of $\mathbf{B}^4 \times \mathbf{S}^1$. In particular, the contour C, which wraps around the $\mathbf{S}^1$, is not homotopic to zero in X_1 and is not the boundary of any D. Thus, the expectation value of a temporal Wilson line vanishes at low temperatures. This is the expected result, corresponding to the fact that the center Γ of the gauge group is unbroken at low temperatures.

Now we move on to the high temperature phase on $\mathbf{S}^3$. This phase is governed by a manifold X_2 that is topologically $\mathbf{S}^3 \times \mathbf{B}^2$. In this phase, $C = P \times \mathbf{S}^1$ is a boundary; in fact it is the boundary of $D = P \times \mathbf{B}^2$. Thus, it appears at first sight that the temporal Wilson line has a vacuum expectation value and that the center of the gauge group is spontaneously broken.

There is a problem here. Though we expect these results in the high temperature phase on $\mathbf{R}^3$, they cannot hold on $\mathbf{S}^3$, because the center (or any other bosonic symmetry) cannot be spontaneously broken in finite volume. The resolution of the puzzle is instructive. The classical solution on X_2 is not unique. We must recall that Type IIB superstring theory has a two-form field B that couples to the elementary string world-sheet D by

$$i \int_D B. \tag{3.2}$$

The gauge-invariant field strength is $H = dB$. We can add to the solution a "world-sheet theta angle," that is a B field of $H = 0$ with an arbitrary value of $\psi = \int_D B$ (here D is any surface obeying the boundary conditions, for instance $D = P \times \mathbf{B}^2$). Since discrete gauge transformations that shift the flux of B by a multiple of 2π are present in the theory, ψ is an angular variable with period 2π.

If this term is included, the path integrand in (3.1) receives an extra factor $e^{i\psi}$. Upon integrating over the space of all classical solutions – that is integrating over the value of ψ – the expectation value of the temporal Wilson line on $\mathbf{S}^3$ vanishes.

Now, let us go to $\mathbf{R}^3 \times \mathbf{S}^1$, which is the boundary of $\mathbf{R}^3 \times \mathbf{B}^2$. In infinite volume, ψ is best understood as a massless scalar field in the low energy theory on $\mathbf{R}^3$. One still integrates over local fluctuations in ψ, but not over the vacuum expectation value of ψ, which is set by the value at spatial infinity. The expectation value of $W(C)$ is nonzero and is proportional to $e^{i\psi}$.

What we have seen is thus spontaneous symmetry breaking: in infinite volume, the expectation value of $\langle W(C) \rangle$ is nonzero, and depends on the choice of vacuum, that is on the value of ψ. The field theory analysis that we reviewed in section 2 indicates that the symmetry that is spontaneously broken by the choice of ψ is the center, Γ, of the gauge group. Since ψ is a continuous angular variable, it seems that the center is $U(1)$. This seems to imply that the gauge group is not $SU(N)$, with center $\mathbf{Z}_N$, but $U(N)$. However, a variety of arguments [47] show that the AdS theory encodes a $SU(N)$ gauge group, not $U(N)$. Perhaps the apparent $U(1)$ center should be understood as a large N limit of $\mathbf{Z}_N$.

't Hooft Loops

We would also like to consider in a similar way 't Hooft loops. These are obtained from Wilson loops by electric-magnetic duality. Electric-magnetic duality of $\mathcal{N} = 4$ arises [68,69] directly from the $\tau \to -1/\tau$ symmetry of Type IIB. That symmetry exchanges

elementary strings with D-strings. So to study the 't Hooft loops we need only, as in [46], replace elementary strings by D-strings in the above discussion.

The $\tau \to -1/\tau$ symmetry exchanges the Neveu-Schwarz two-form B which entered the above discussion with its Ramond-Ramond counterpart B'; the D-brane theta angle $\psi' = \int_D B'$ thus plays the role of ψ in the previous discussion. In the thermal physics on $\mathbf{R}^3 \times \mathbf{S}^1$, the center of the "magnetic gauge group" is spontaneously broken, and the temporal 't Hooft loops have an expectation value, just as we described for Wilson loops. The remarks that we make presently about spatial Wilson loops similarly carry over for spatial 't Hooft loops.

3.2. Spatial Wilson Loops

Now we will investigate the question of whether at nonzero temperature the spatial Wilson loops obey an area law. The main point is to first understand why there is *not* an area law at zero temperature. At zero temperature, one works with the AdS metric

$$ds^2 = \frac{1}{x_0^2} \left(dx_0^2 + \sum_{i=1}^{4} dx_i^2 \right). \tag{3.3}$$

We identify the spacetime M of the $\mathcal{N} = 4$ theory with the boundary at $x_0 = 0$, parametrized by the Euclidean coordinates x_i, $i = 1, \dots, 4$. M has a metric $d\widetilde{s}^2 = \sum_i dx_i^2$ obtained by multiplying ds^2 by x_0^2 and setting $x_0 = 0$. (If we use a function other than x_0^2, the metric on M changes by a conformal transformation.) We take a closed oriented curve $C \subset M$ and regard it as the boundary of an oriented compact surface D in AdS space. The area of D is infinite, but after subtracting an infinite counterterm proportional to the circumference of C, we get a regularized area $\alpha(D)$. In the framework of [37,38], the expectation value of the Wilson line $W(C)$ is proportional to $\exp(-\alpha(D))$, with D chosen to minimize $\alpha(D)$.

Now the question arises: why does not this formalism *always* give an area law? As the area enclosed by C on the boundary is scaled up, why is not the area of D scaled up proportionately? The answer to this is clear from conformal invariance. If we scale up C via $x_i \to tx_i$, with large positive t, then by conformal invariance we can scale up D, with $x_i \to tx_i$, $x_0 \to tx_0$, without changing its area (except for a boundary term involving the regularization). Thus the area of D need not be proportional to the area enclosed by C on the boundary. Since, however, in this process we had to scale $x_0 \to tx_0$ with very large t, the surface D which is bounded by a very large circle C "bends" very far away from the

boundary of AdS space. If such a bending of D were prevented – if D were limited to a region with $x_0 \leq L$ for some cutoff L – then one would get an area law for $W(C)$. This is precisely what will happen at nonzero temperature.

At nonzero temperature, we have in fact the metric (2.13) obtained earlier, with $n = 4$:

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right)} + \rho^2 \sum_{i=1}^{3} dx_i^2. \tag{3.4}$$

The range of ρ is $b \leq \rho \leq \infty$. Spacetime – a copy of $\mathbf{R}^3 \times \mathbf{S}^1$ – is the boundary at $\rho = \infty$. We define a metric on spacetime by dividing by ρ^2 and setting $\rho = \infty$. In this way we obtain the spacetime metric

$$d\tilde{s}^2 = \frac{d\tau^2}{b^2} + \sum_{i=1}^{3} dx_i^2. \tag{3.5}$$

As the period of τ is β_1, the circumference of the $\mathbf{S}^1$ factor in $\mathbf{R}^3 \times \mathbf{S}^1$ is β_1/b and the temperature is

$$T = \frac{b}{\beta_1} = \frac{1}{\pi}. \tag{3.6}$$

Because of conformal invariance, the numerical value of course does not matter.

Now, let C be a Wilson loop in $\mathbf{R}^3$, at a fixed value of τ, enclosing an area A in $\mathbf{R}^3$. A bounding surface D in the spacetime (3.4) is limited to $\rho \geq b$, so the coefficient of $\sum_i dx_i^2$ is always at least b^2. Apart from a surface term that depends on the regularization and the detailed solution of the equation for a minimal surface, the regularized area of D is at least $\alpha(D) = b^2 A$ (and need be no larger than this). The Wilson loops therefore obey an area law, with string tension b^2 times the elementary Type IIB string tension.

We could of course have used a function other than ρ^2 in defining the spacetime metric, giving a conformally equivalent metric on spacetime. For instance, picking a constant s and using $s^2 \rho^2$ instead of ρ^2 would scale the temperature as $T \to T/s$ and would multiply all lengths on $\mathbf{R}^3$ by s. The area enclosed by C would thus become $A' = As^2$. As $\alpha(D)$ is unaffected, the relation betseen $\alpha(D)$ and A' becomes $\alpha(D) = \left(b^2/s^2\right) \cdot A'$. The string tension in the Wilson loop area law thus scales like s^{-2}, that is, like T^2, as expected from conformal invariance.

15

3.3. The Mass Gap

The last issue concerning the $\mathcal{N} = 4$ theory at high temperature that we will discuss here is the question of whether there is a mass gap. We could do this by analyzing correlation functions, using the formulation of [27,47], but it is more direct to use a Hamiltonian approach (discussed at the end of [47]) in which one identifies the quantum states of the supergravity theory with those of the quantum field theory on the boundary.

So we will demonstrate a mass gap by showing that there is a gap, in the three-dimensional sense, for quantum fields propagating on the five-dimensional spacetime

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right)} + \rho^2 \sum_{i=1}^{3} dx_i^2. \tag{3.7}$$

This spacetime is the product of a three-space $\mathbf{R}^3$, parametrized by the x_i, with a two-dimensional "internal space" $\mathbf{W}$, parametrized by ρ and τ. We want to show that a quantum free field propagating on this five-dimensional spacetime gives rise, in the three-dimensional sense, to a discrete spectrum of particle masses, all of which are positive. When such a spectrum is perturbed by interactions, the discreteness of the spectrum is lost (as the very massive particles become unstable), but the mass gap persists.

If $\mathbf{W}$ were compact, then discreteness of the mass spectrum would be clear: particle masses on $\mathbf{R}^3$ would arise from eigenvalues of the Laplacian (and other wave operators) on $\mathbf{W}$. Since $\mathbf{W}$ is not compact, it is at first sight surprising that a discrete mass spectrum will emerge. However, this does occur, by essentially the same mechanism that leads to discreteness of particle energy levels on *AdS* space [70,71] with a certain notion of energy.

For illustrative purposes, we will consider the propagation of a Type IIB dilaton field ϕ on this spacetime. Other cases are similar. The action for ϕ is

$$\begin{aligned} I(\phi) = &\frac{1}{2} \int_b^\infty d\rho \int_0^{\beta_1/b} d\tau \int_{-\infty}^\infty d^3x \ \rho^3 \left(\left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right) \left(\frac{\partial \phi}{\partial \rho} \right)^2 \right. \\ &\left. + \left(\frac{\rho^2}{b^2} - \frac{b^2}{\rho^2} \right)^{-1} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \rho^{-2} \sum_i \left(\frac{\partial \phi}{\partial x_i} \right)^2 \right). \end{aligned} \tag{3.8}$$

Since translation of τ is a symmetry, modes with different momentum in the τ direction are decoupled from one another. The spectrum of such momenta is discrete (as τ is a periodic variable). To simplify things slightly and illustrate the essential point, we will

write the formulas for the modes that are independent of τ; others simply give, by the same argument, additional three-dimensional massive particles with larger masses.

We look for a solution of the form $\phi(\rho, x) = f(\rho)e^{i\vec{k}\cdot x}$, with $\vec{k}$ the momentum in $\mathbf{R}^3$. The effective Lagrangian becomes

$$I(f) = \frac{1}{2} \int_b^\infty d\rho \, \rho^3 \left((\rho^2/b^2 - b^2/\rho^2) \left(\frac{df}{d\rho} \right)^2 + \rho^{-2} k^2 f^2 \right). \tag{3.9}$$

The equation of motion for f is

$$-\rho^{-1} \frac{d}{d\rho} \left(\rho^3 (\rho^2/b^2 - b^2/\rho^2) \frac{df}{d\rho} \right) + k^2 f = 0. \tag{3.10}$$

A mode of momentum k has a mass m, in the three-dimensional sense, that is given by $m^2 = -k^2$. We want to show that the equation (3.10) has acceptable solutions only if m^2 is in a certain discrete set of positive numbers.

Acceptable solutions are those that obey the following boundary conditions:

(1) At the lower endpoint $\rho = b$, we require $df/d\rho = 0$. The reason for this is that ρ behaves near this endpoint as the origin in polar coordinates; hence f is not smooth at this endpoint unless $df/d\rho = 0$ there.

(2) For $\rho \to \infty$, the equation has two linearly independent solutions, which behave as $f \sim$ constant and $f \sim \rho^{-4}$. We want a normalizable solution, so we require that $f \sim \rho^{-4}$.

For given k^2, the equation (3.10) has, up to a constant multiple, a unique solution that obeys the correct boundary condition near the lower endpoint. For generic k^2, this solution will approach a constant for $\rho \to \infty$. As in standard quantum mechanical problems, there is a normalizable solution only if k^2 is such that the solution that behaves correctly at the lower endpoint also vanishes for $\rho \to \infty$. This "eigenvalue" condition determines a discrete set of values of k^2.

The spectrum thus consists entirely of a discrete set of normalizable solutions. There are no such normalizable solutions for $k^2 \geq 0$. This can be proved by noting that, given a normalizable solution f of the equation of motion, a simple integration by parts shows that the action (3.9) vanishes. For $k^2 \geq 0$, vanishing of $I(f)$ implies that $df/d\rho = 0$, whence (given normalizability) $f = 0$. So the discrete set of values of k^2 at which there are normalizable solutions are all negative; the masses $m^2 = -k^2$ are hence strictly positive. This confirms the existence of the mass gap.

To understand the phenomenon better, let us compare to what usually happens in quantum mechanics. In typical quantum mechanical scattering problems, with potentials

that vanish at infinity, the solutions with positive energy (analogous to $m^2 > 0$) are oscillatory at infinity and obey plane wave normalizability. When this is so, both solutions at infinity are physically acceptable (in some situations, for example, they are interpreted as incoming and outgoing waves), and one gets a continuous spectrum that starts at zero energy. The special property of the problem we have just examined is that even for negative k^2, there are no oscillatory solutions at infinity, and instead one of the two solutions must be rejected as being unnormalizable near $\rho = \infty$. This feature leads to the discrete spectrum.

If instead of the spacetime (3.7), we work on AdS spacetime (2.4), there is a continuous spectrum of solutions with plane wave normalizability for all $k^2 < 0$; this happens because for $k^2 < 0$ one gets oscillatory solutions near the lower endpoint, which for the AdS case is at $r = 0$. Like confinement, the mass gap of the thermal $\mathcal{N} = 4$ theory depends on the cutoff at small r.

4. Approach To QCD

One interesting way to study four-dimensional gauge theory is by compactification from a certain exotic six-dimensional theory with $(0, 2)$ supersymmetry. This theory can be realized in Type IIB compactification on K3 [72] or in the dynamics of parallel M-theory fivebranes [73] and can apparently be interpreted [11] in terms of M-theory on $AdS_7 \times \mathbf{S}^4$. This interpretation is effective in the large N limit – as the M-theory radius of curvature is of order $N^{1/3}$. Since compactification from six to four dimensions has been an effective approach to gauge theory dynamics (for instance, in deducing Montonen-Olive duality [72] using a strategy proposed in [74]), it is natural to think of using the solution for the large N limit of the six-dimensional theory as a starting point to understand the four-dimensional theory.

Our basic approach will be as follows. If we compactify the six-dimensional $(0, 2)$ theory on a circle C_1 of radius R_1, with a supersymmetry-preserving spin structure (fermions are periodic in going around the circle), we get a theory that at low energies looks like five-dimensional $SU(N)$ supersymmetric Yang-Mills theory, with maximal supersymmetry and five-dimensional gauge coupling constant $g_5^2 = R_1$. Now compactify on a second circle C_2, orthogonal to the first, with radius R_2. If we take $R_2 >> R_1$, we can determine what the resulting four-dimensional theory is in a two-step process, compactifying to five dimensions on C_1 to get five-dimensional supersymmetric Yang-Mills theory and then compactifying to four-dimensions on C_2.

No matter what spin structure we use on C_2, we will get a four-dimensional $SU(N)$ gauge theory with gauge coupling $g_4^2 = R_1/R_2$. If we take on C_2 (and more precisely, on $C_1 \times C_2$) the supersymmetry-preserving spin structure, then the low energy theory will be the four-dimensional $\mathcal{N} = 4$ theory some of whose properties we have examined in the present paper. We wish instead to break supersymmetry by taking the fermions to be antiperiodic in going around C_2. Then the fermions get masses (of order $1/R_2$) at tree level, and the spin zero bosons very plausibly get masses (of order g_4^2/R_2) at one-loop level. If this is so, the low energy theory will be the pure $SU(N)$ theory without supersymmetry. If $g_4^2 << 1$, the theory will flow at very long distances to strong coupling; at such long distances the spin one-half and spin one fields that receive tree level or one-loop masses will be irrelevant. So this is a possible framework for studying the pure Yang-Mills theory without supersymmetry.

We want to take the large N limit with $g_4 \to 0$ in such a way that $\eta = g_4^2 N$ has a limit. So we need $g_4^2 = \eta/N$, or in other words

$$R_1 = \frac{\eta R_2}{N}. \tag{4.1}$$

We actually want η fixed and small, so that the four-dimensional Yang-Mills theory is weakly coupled at the compactification scale, and flows to strong coupling only at very long distances at which the detailed six-dimensional setup is irrelevant.

To implement this approach, we first look for an Einstein manifold that is asymptotic at infinity to $\mathbf{R}^5 \times C_2$. Though it may seem to reverse the logic of the construction, starting with C_2 first in constructing the solution turns out to be more convenient. The supersymmetry-breaking boundary conditions on C_2 are the right ones for using the spacetime (2.13) that is constructed by scaling of the seven-dimensional AdS Schwarzschild solution:

$$ds^2 = \left(\frac{\rho^2}{b^2} - \frac{b^4}{\rho^4}\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{b^2} - \frac{b^4}{\rho^4}\right)} + \rho^2 \sum_{i=1}^{5} dx_i^2. \tag{4.2}$$

According to [11], we want here

$$b = 2G_N^{1/9}(\pi N)^{1/3}. \tag{4.3}$$

(Here G_N is the eleven-dimensional Newton constant, so $G_N^{1/9}$ has dimensions of length. We henceforth set $G_N = 1$.)

To make the scaling with N clearer, we also set $\rho = 2(\pi N)^{1/3}\lambda$. And – noting from (2.11) that τ has period $4\pi b/n = (4/3)\pi^{4/3}N^{1/3}$ – we set

$$\tau = \theta \cdot \left(\frac{2\pi N}{3}\right)^{1/3}, \tag{4.4}$$

where θ is an ordinary angle, of period 2π. After also a rescaling of the x_i, the metric becomes

$$ds^2 = \frac{4}{9}\pi^{2/3}N^{2/3}\left(\lambda^2 - \frac{1}{\lambda^4}\right)d\theta^2 + 4\pi^{2/3}N^{2/3}\frac{d\lambda^2}{\left(\lambda^2 - \frac{1}{\lambda^4}\right)} + 4\pi^{2/3}N^{2/3}\lambda^2\sum_{i=1}^{5}dx_i^2. \tag{4.5}$$

Now we want to compactify one of the x_i, say x_5, on a second circle whose radius as measured at $\lambda = \infty$ should according to (4.1) should be η/N times the radius of the circle parametrized by θ. To do this, we write $x_5 = (\eta/N)\psi$ with ψ of period 2π. We also now restore the $\mathbf{S}^4$ factor that was present in the original M-theory on $AdS_7 \times \mathbf{S}^4$ and has so far been suppressed. The metric is now

$$\begin{aligned}ds^2 =& \frac{4}{9}\pi^{2/3}N^{2/3}\left(\lambda^2 - \frac{1}{\lambda^4}\right)d\theta^2 + \frac{4}{9}\eta^2\pi^{2/3}N^{-4/3}\lambda^2 d\psi^2 \\ &+ 4N^{2/3}\frac{d\lambda^2}{\left(\lambda^2 - \frac{1}{\lambda^4}\right)} + 4\pi^{2/3}N^{2/3}\rho^2\sum_{i=1}^{4}dx_i^2 + \pi^{2/3}N^{2/3}d\Omega_4^2.\end{aligned} \tag{4.6}$$

At this stage, θ and ψ are both ordinary angular variables of radius 2π, and $d\Omega_4^2$ is the metric on a unit four-sphere.

Now, we want to try to take the limit as $N \to \infty$. The metric becomes large in all directions except that one circle factor – the circle C_1, parametrized by ψ - shrinks. Thus we should try to use the equivalence between M-theory compactified on a small circle and weakly coupled Type IIA superstrings. We see that the radius $R(\lambda)$ of the circle parametrized by C is in fact

$$R(\lambda) = \frac{2}{3}\eta\lambda\pi^{1/3}N^{-2/3}. \tag{4.7}$$

To relate an M-theory compactification on a circle to a Type IIA compactification, we must [75] multiply the metric by R. All factors of N felicitously disappear from the metric, which becomes

$$ds^2 = \frac{8}{27}\eta\lambda\pi\left(\lambda^2 - \frac{1}{\lambda^4}\right)d\theta^2 + \frac{8\pi}{3}\eta\lambda\frac{d\lambda^2}{\left(\lambda^2 - \frac{1}{\lambda^4}\right)} + \frac{8\pi}{3}\eta\lambda^3\sum_{i=1}^{4}dx_i^2 + \frac{2\pi}{3}\eta\lambda d\Omega_4^2. \tag{4.8}$$

The string coupling constant is meanwhile

$$g_{st}^2 = R^{3/2} = \frac{(2/3)^{3/2}\eta^{3/2}\lambda^{3/2}\pi^{1/2}}{N}. \tag{4.9}$$

This result clearly has some of the suspected properties of large N gauge theories. The metric (4.8) is independent of N, so in the weak coupling limit, the spectrum of the string theory will be independent of N. Meanwhile, the string coupling constant (4.9) is of order $1/N$, as expected [1] for the residual interactions between color singlet states in the large N limit. The very ability to get a description such as this one in which $1/N$ only enters as a coupling constant (and not explicitly in the multiplicity of states) is a reflection of confinement. Confinement in the form of an area law for Wilson loops can be demonstrated along the lines of our discussion in section 3: it follows from the fact that the coefficient in the metric of $\sum_{i=1}^{4} dx_i^2$ is bounded strictly above zero. A mass gap likewise can be demonstrated, as in section 3, by using the large λ behavior of the metric.

On the other hand, it is not obvious how one could hope to compute the spectrum or even show asymptotic freedom. Asymptotic freedom should say that as $\eta \to 0$, the particle masses become exponentially small (with an exponent determined by the gauge theory beta function). It is not at all clear how to demonstrate this. A clue comes from the fact that the coupling of the physical hadrons should be independent of η (and of order $1/N$) as $\eta \to 0$. In view of the formula (4.9), this means that we should take $\eta\lambda$ of order one as $\eta \to 0$. If we set $\widetilde{\lambda} = \eta\lambda$, and write the metric in terms of $\widetilde{\lambda}$, then the small η limit becomes somewhat clearer: a singularity develops at small $\widetilde{\lambda}$ for $\eta \to 0$. Apparently, in this approach, the mysteries of four-dimensional quantum gauge theory are encoded in the behavior of string theory near this singularity.

This singularity actually has a very simple and intuitive interpretation which makes it clearer why four-dimensional gauge theory can be described by string theory in the spacetime (4.8). The Euclidean signature Type IIA nonextremal fourbrane solution is described by the metric [76]

$$ds^2 = \left(1 - \left(\frac{r_+}{r}\right)^3\right)\left(1 - \left(\frac{r_-}{r}\right)^3\right)^{-1/2} dt^2 + \left(1 - \left(\frac{r_+}{r}\right)^3\right)^{-1}\left(1 - \left(\frac{r_-}{r}\right)^3\right)^{-5/6} dr^2$$
$$+ \left(1 - \left(\frac{r_-}{r}\right)^3\right)^{1/2} \sum_{i=1}^{4} dx_i^2 + r^2 \left(1 - \left(\frac{r_-}{r}\right)^3\right)^{1/6} d\Omega_4^2, \tag{4.10}$$

with $r_+ > r_- > 0$. The string coupling constant is

$$g_{st}^2 = \left(1 - \left(\frac{r_-}{r}\right)^3\right)^{1/2}. \tag{4.11}$$

The horizon is at $r = r_+$, and the spacetime is bounded by $r \geq r_+$. This spacetime is complete and smooth if t has period

$$T = 12\pi \left(1 - \left(\frac{r_-}{r_+}\right)^3\right)^{-1/6}. \tag{4.12}$$

If one continues (via Lorentzian or complex values of the coordinates) past $r = r_+$, there is a singularity at $r = r_-$. The extremal fourbrane solution is obtained by setting $r_+ = r_-$ and is singular. But this singularity is exactly the singularity that arises in (4.8) upon taking $\eta \to 0$, with $\eta\lambda \sim 1$! In fact, if we set $\lambda^6 = (r^3 - r_-^3)/(r_+^3 - r_-^3)$, identify η with $(1 - (r_-/r_+)^3)^{1/6}$, and take the limit of $r_+ \to r_-$, then (4.10) reduces to (4.8), up to some obvious rescaling. Moreover, according to (4.12), $r_+ \to r_-$ is the limit that T is large, which (as $1/g_4^2 = T/g_5^2$) makes the four-dimensional coupling small.

So in hindsight we could discuss four-dimensional gauge theories in the following way, without passing through the CFT/AdS correspondence. In a spacetime $\mathbf{R}^9 \times \mathbf{S}^1$, consider N Type IIA fourbranes wrapped on $\mathbf{R}^4 \times \mathbf{S}^1$. Pick a spin structure on the $\mathbf{S}^1$ that breaks supersymmetry. This system looks at low energies like four-dimensional $U(N)$ gauge theory, with Yang-Mills coupling $g_4^2 = g_5^2/T = g_{st}/T$. Take $N \to \infty$ with $g_{st}N$ fixed. The D-brane system has both open and closed strings. The dominant string diagrams for large N with fixed $g_5^2 N$ and fixed T are the planar diagrams of 't Hooft [1] – diagrams of genus zero with any number of holes. (This fact was exploited recently [29] in analyzing the beta function of certain field theories.) Summing them up is precisely the long-intractable problem of the $1/N$ expansion.

Now, at least if η is large, supergravity effectively describes the sum of planar diagrams in terms of the metric (4.10) which is produced by the D-branes. This is a smooth metric, with no singularity and no D-branes. So we get a description with closed Type IIA strings only. Thus the old prophecy [1] comes to pass: nonperturbative effects close up the holes in the Feynman diagrams, giving a confining theory with a mass gap, and with $1/N$ as a coupling constant, at least for large η. To understand large N gauge theories, one would want, from this point of view, to show that there is no singularity as a function of η, except at $\eta = 0$, and to exhibit asymptotic freedom and compute the masses for small η. (This

looks like a tall order, given our limited knowledge of worldsheet field theory with Ramond-Ramond fields in the Lagrangian.) The singularity at $\eta = 0$ is simply the singularity of the fourbrane metric at $r_+ = r_-$; it reflects the classical $U(N)$ gauge symmetry of N parallel fourbranes, which disappears quantum mechanically when $\eta \neq 0$ and the singularity is smoothed out.

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