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Affine and Yangian Symmetries in $SU(2)_1$ Conformal Field Theory*

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Abstract

In these lectures, we study and compare two different formulations of $SU(2)$, level $k = 1$, Wess-Zumino-Witten conformal field theory. The first, conventional, formulation employs the affine symmetry of the model; in this approach correlation functions are derived from the so-called Knizhnik-Zamolodchikov equations. The second formulation is based on an entirely different algebraic structure, the so-called Yangian $Y(sl_2)$. In this approach, the Hilbert space of the theory is obtained by repeated application of modes of the so-called spinon field, which has $SU(2)$ spin $j = \frac{1}{2}$ and obeys fractional (semionic) statistics. We show how this new formulation, which can be generalized to many other rational conformal field theories, can be used to compute correlation functions and to obtain new expressions for the Virasoro and affine characters in the theory.

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1. Introduction

Conformal Field Theory (CFT) in two dimensions has found applications in a wide area of topics in theoretical physics. In the context of this Summer School, the first applications that come to mind are in the area of String Theory, where CFT techniques are an essential tool. We would like to stress here, though, that there are a variety of other topics, both within and outside the realm of theoretical High Energy Physics, where CFT has found important applications.

One of the surprising developments of the last few years has been the realization (by many) that there are often a variety of entirely different ways to set up the description of a given CFT. As an illustration, we mention that conformal characters can often be written in different forms. The fact that such different forms give rise to the same q-series has led to non-trivial combinatorical identities generalizing the so-called Rogers-Ramanujan identities. A particularly interesting observation was made in [1], where a new formulation of the $SU(2)$, $k=1$, Wess-Zumino-Witten (WZW) CFT was proposed. This new formulation does not rely on the affine Kac-Moody and Virasoro symmetries, but instead uses invariance under the Yangian $Y(sl_2)$ to give a systematic description of the spectrum in terms of so-called multi-spinon states.

The spinon formulation of the $SU(2)_1$ WZW model has been further worked out in [2, 3], where it was also observed that the new formulation can actually be used to derive alternative expressions for both the affine and the Virasoro characters. These papers thus established a connection with some of the character identities that had been proposed on different grounds and explained their origin.

It is the purpose of these lectures to explain in some detail the new formulation of the $SU(2)_1$ WZW model and to compare this with the conventional approach. We have chosen to focus entirely on this one specific (and well-studied) theory, hoping that this will help to clarify the new issues that we address. Thus we shall only briefly touch on one of the most interesting questions in this field, which is: in what way can our results be extended to more general rational conformal field theories and perturbations thereof? New results in this direction will be published elsewhere [4, 5].

The organization of these lectures will be as follows. In section 2, we review some basic features of CFT in general and of the $SU(2)_1$ WZW model in particular. In section 3 we derive the celebrated Knizhnik-Zamolodchikov (KZ) equations and solve them for the case of the 4-spinon correlation function. In section 4 we focus on the spinon fields and derive generalized commutation relations for their Fourier modes. In section 5 we present the complete spinon formulation $SU(2)_1$ WZW model by specifying the rules for building a complete set of independent multi-spinon states. In section 6 we use spinon formulation to compute multi-spinon correlation functions and to derive new expressions for the Virasoro and affine characters in the theory. Appendix A gives a brief summary of the definition of the Yangian $Y(sl_2)$

The central event of the last week of this Trieste Summer School was the presentation to P. van Nieuwenhuizen of one of the 1994 Dirac Medals, honoring his contribution to the discovery of supergravity in 1976. This year's other recipients, S. Ferrara and D.Z. Freedman, had received their medals on earlier occasions. We would like to take this opportunity to congratulate the laureates with this long-due token of appreciation for their monumental work.

2. The $SU(2)$, level $k = 1$, WZW model

The $SU(2)_1$ WZW model is a particularly simple CFT, which can be used nicely to illustrate a 'quasi-particle approach' to CFT. In this case, the quasi-particles are spinons, which, as we shall show, can be viewed as free particles that obey semionic statistics. Before we come to the spinon formulation, we shall review the conventional formulation of the theory, which uses the affine algebra $A_1^{(1)}$ as its starting point. We shall in particular focus on the structure of the Hilbert space of physical states.

Being a conformal field theory, the $SU(2)_1$ WZW model possesses conformal invariance, with the corresponding currents $T(z)$ satisfying the operator product expansion (OPE)

$$T(z)T(w) = \frac{\frac{1}{2}}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \ldots \tag{2.1}$$

The modes L_n, defined through $T(z) = \sum_n L_n z^{-n-2}$, satisfy the Virasoro algebra

$$[L_m, L_n] = \frac{1}{12}(m^3 - m)\delta_{m+n} + (m-n)L_{m+n} \tag{2.2}$$

with central charge $c = 1$. The full chiral spectrum of the theory can be viewed as a collection of irreducible representations of the Virasoro algebra, each characterized by a conformal dimension h. There is actually an infinite number of such representations L_j, with h taking the values $h_j = j^2$, $j = 0, \frac{1}{2}, 1, \ldots$. In this notation, j indicates the $SU(2)$ spin of the highest weight state of the representation.

The number of states at a given L_0-level in each of the Virasoro representations L_j is summarized in the characters

$$\chi_{j^2}^{\mathrm{Vir}}(q) = \mathrm{tr}_{L_j}(q^{L_0}) = \frac{q^{j^2}(1-q^{2j+1})}{\prod_{m=1}^{\infty}(1-q^m)} \ . \tag{2.3}$$

The structure of these characters is easily understood as follows. The product in the denominator comes from the repeated application of the (bosonic) Virasoro modes L_{-m}, $m = 1, 2, \ldots$, and the subtraction in the numerator corresponds to a null state of dimension $L_0 = (j+1)^2$ in the representation L_j. For example, for $j = 0$ we have the null state $L_{-1}|0\rangle$ with $L_0 = 1$.

A more economical description is obtained by exploiting the invariance of this field theory under the affine Lie algebra $A_1^{(1)}$ with level $k=1$. The corresponding currents $J^a(z)$ satisfy the OPE

$$J^a(z)J^b(w) = \frac{d^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{(z-w)} + \ldots \tag{2.4}$$

The adjoint index takes the values $a = \,\#, 3, =$; the metric is $d^{\#=} = 1$, $d_{=\#} = 1$, $d^{33} = 2$, $d_{33} = \frac{1}{2}$ and the structure constants follow from $f^{\#=}{}_3 = 1$. The modes J^a_m, defined through $J^a(z) = \sum_n J^a_n z^{-n-1}$, satisfy the algebra

$$[J^a_m, J^b_n] = m\, d^{ab}\delta_{m+n} + f^{ab}{}_c J^c_{m+n} \,. \tag{2.5}$$

The full chiral spectrum of the field theory consists of two irreducible representations of the algebra (2.5), the corresponding primary fields being the identity and the field $\phi^\alpha(z)$, of conformal dimension $h = \frac{1}{4}$, which we will call the *spinon field*. The index α is an $SU(2)$ doublet index, taking values $+, -$, and we have the OPE

$$J^a(z)\phi^\alpha(w) = (t^a)^\alpha{}_\beta \frac{\phi^\beta(w)}{(z-w)} + \ldots \,, \tag{2.6}$$

with $(t^{\#})^-{}_+ = (t^=)^+{}_- = 1$, $(t^3)^\pm{}_\pm = \pm 1$.

The characters of the two representations (labeled by $j = 0, \frac{1}{2}$) are given by

$$\begin{aligned} \chi_j^{A_1^{(1)}}(q) &= \frac{q^{-\frac{1}{12}}}{\prod_{m=1}^\infty (1-q^m)^3} \sum_{n\in\mathbb{Z}} (6n+2j+1) q^{\frac{(6n+2j+1)^2}{12}} \\ &= \frac{q^{j^2}}{\prod_{m=1}^\infty (1-q^m)^3} \left[(2j+1) - (5-2j)q^{2-2j} + \ldots \right] . \end{aligned} \tag{2.7}$$

where the dots indicate the subtraction of further null states. These character formulas can be understood as follows: the product in the denominator accounts for the presence of bosonic modes J^a_m and the subtractions correspond to null states in the affine modules. The simplest null states (corresponding to the first correction term in (2.7)) are given by

$$\begin{aligned} j=0: &\qquad J^a_{-1}J^b_{-1}|\,0\,\rangle|_{j=2 \text{ component}} \\ j=\tfrac{1}{2}: &\qquad J^a_{-1}|\,\phi^\alpha\,\rangle|_{j=\frac{3}{2} \text{ component}} \end{aligned} \tag{2.8}$$

where $|\,\phi^\alpha\,\rangle$ denotes the highest weight state of the $j = \frac{1}{2}$ representation, and the subscripts on the states denote projections onto irreducible representations of $SU(2)$.

An alternative form of the affine characters is as follows

$$\chi_j^{A_1^{(1)}}(q) = \frac{\sum_{n\in\mathbb{Z}} q^{(n+j)^2}}{\prod_{m=1}^{\infty}(1-q^m)} \ . \tag{2.9}$$

This form is natural from the point of view of a representation of the $SU(2)_1$ theory in terms of a single bosonic field (compare with section 5.2, where we shall use this bosonic field in the construction of Yangian Highest Weight Vectors).

3. Analyzing the KZ equations

Having discussed the spectrum of physical states we now turn to correlation functions of the fundamental spinon field $\phi^\alpha(z)$. We first observe that for $k = 1$ the existence of the second null state in (2.8) implies that

$$(J^a\phi^\alpha)(z) = 2(t^a)^\alpha{}_\beta \partial\phi^\beta(z) \ . \tag{3.1}$$

Here we use ordinary brackets to denote the normal ordered field product defined by

$$(AB)(z) = \oint_{\mathcal{C}_z} \frac{dx}{2\pi i} \frac{A(x)B(z)}{(x-z)} \ , \tag{3.2}$$

where the contour $\mathcal{C}_z$ encloses the point z counterclockwise. From (3.1) the following identity can be derived

$$2\,(t^a)^{(i)}\,\partial_{z_i} \langle \phi(z_1)\ldots\phi(z_N)\rangle = \sum_{j\neq i} \frac{(t^a)^{(j)}}{z_i - z_j} \langle \phi(z_1)\ldots\phi(z_N)\rangle \ , \tag{3.3}$$

where we suppressed spinor indices. For general level k we have the following equation, which is known as the Knizhnik-Zamolodchikov (KZ) equation [6]

$$0 = \left((k+2)\,\partial_{z_i} - \sum_{j\neq i} \frac{(t_a)^{(i)}(t^a)^{(j)}}{z_i - z_j} \right) \langle \phi(z_1)\ldots\phi(z_N)\rangle \ . \tag{3.4}$$

For $k = 1$ the equation (3.4) follows from (3.3) by taking a contraction.

We shall now derive an explicit expression for the 4-point function

$$G_{\alpha_1\alpha_2}^{\alpha_3\alpha_4}(z_1,z_2,z_3,z_4) = \langle \phi_{\alpha_1}(z_1)\phi_{\alpha_2}(z_2)\phi^{\alpha_3}(z_3)\phi^{\alpha_4}(z_4)\rangle \ , \tag{3.5}$$

by using conformal invariance and the equation (3.4) for $k = 1$. (We raise and lower spinor indices according to $X^\alpha = \epsilon^{\alpha\beta}X_\beta$, $X_\alpha = \epsilon_{\alpha\beta}X^\beta$, where $\epsilon^{+-} = -\epsilon^{-+} = 1$, $\epsilon_{-+} = -\epsilon_{+-} = 1$.) Conformal invariance alone tells us that the 4-point function can be written as follows

$$G_{\alpha_1\alpha_2}^{\alpha_3\alpha_4}(z_1,z_2,z_3,z_4) = (z_{14}z_{23})^{-\frac{1}{2}}\, G_{\alpha_1\alpha_2}^{\alpha_3\alpha_4}(x) \ , \tag{3.6}$$

where we wrote

$$x = \frac{z_{12} z_{34}}{z_{14} z_{32}} \ , \qquad z_{ij} = z_i - z_j \ . \tag{3.7}$$

By exploiting $SU(2)$ invariance we obtain the following general form of the solution

$$G_{\alpha_1 \alpha_2}^{\alpha_3 \alpha_4}(x) = (J_1)_{\alpha_1 \alpha_2}^{\alpha_3 \alpha_4} \, G_1(x) + (J_2)_{\alpha_1 \alpha_2}^{\alpha_3 \alpha_4} \, G_2(x) \ , \tag{3.8}$$

where

$$(J_1)_{\alpha_1 \alpha_2}^{\alpha_3 \alpha_4} = \epsilon_{\alpha_1 \alpha_2} \epsilon^{\alpha_3 \alpha_4} \ , \qquad (J_2)_{\alpha_1 \alpha_2}^{\alpha_3 \alpha_4} = \delta_{\alpha_1}^{\alpha_3} \delta_{\alpha_2}^{\alpha_4} \ . \tag{3.9}$$

Using the identities (suppressing indices)

$$\begin{aligned}
(t_a)^{(1)}(t^a)^{(2)} J_1 &= -\tfrac{3}{2} J_1 \ , & (t_a)^{(1)}(t^a)^{(2)} J_2 &= -J_1 + \tfrac{1}{2} J_2 \ , \\
(t_a)^{(1)}(t^a)^{(3)} J_1 &= \tfrac{1}{2} J_1 - J_2 \ , & (t_a)^{(1)}(t^a)^{(3)} J_2 &= -\tfrac{3}{2} J_2 \ , \\
(t_a)^{(1)}(t^a)^{(4)} J_1 &= -\tfrac{1}{2} J_1 + J_2 \ , & (t_a)^{(1)}(t^a)^{(4)} J_2 &= J_1 - \tfrac{1}{2} J_2 \ ,
\end{aligned} \tag{3.10}$$

one can rewrite the KZ equation (3.4) into the following differential equation

$$-6 \partial_x \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix} = \left[\frac{1}{x} \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} + \frac{1}{x-1} \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} \right] \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix} \ . \tag{3.11}$$

The solution to these equations takes the simple form

$$G_1(x) = (1-x)^{\frac{1}{2}} x^{-\frac{1}{2}} \ , \qquad G_2(x) = (1-x)^{-\frac{1}{2}} x^{\frac{1}{2}} \ . \tag{3.12}$$

Obviously, the final form of the correlation function could have been obtained by using that the $SU(2)_1$ WZW model is a $c = 1$ CFT that is equivalent to a theory of a single real scalar field compactified on a circle of the appropriate radius. However, we presented this derivation to illustrate how the presence of affine symmetry can be exploited to solve for interesting quantities such as correlation functions. In section 6 we shall present yet another derivation of this same result, that time using the systematics of the spinon formulation.

Before closing this section, we would like to digress and show how the result (3.12) for the spinon 4-point function can be used to gain some insight in the structure of the Hilbert space, in particular about 2-spinon states.

To this end we map the complex z-plane into the cylinder, $z_i = \exp(-\frac{2\pi}{l} w_i)$, and write $w_i = \tau_i + i x_i$. Under this conformal transformation, the 4-point function picks up a factor $\prod_i (dz_i/dw_i)^{\frac{1}{4}}$. We impose time ordering on the τ_i, choosing $\tau_{i+1} - \tau_i = 0^+$, and pass to a description using modes ϕ^α_n defined through

$$\phi^\alpha(x,0) = \sum_n \phi^\alpha_n \, e^{i\frac{2\pi}{l} nx} \ . \tag{3.13}$$

With that, the 4-point function can be written as

$$G(x_1,x_2,x_3,x_4) = \sum_{n_1,n_2,n_3,n_4} e^{i\frac{2\pi}{l}(n_1x_1+n_2x_2+n_3x_3+n_4x_4)} \langle\, 0\,|\phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4}\,|\,0\,\rangle \ . \tag{3.14}$$

By decomposing the explicit result (3.12) in a series, we can thus extract exact information on the inner products of pairs of 2-spinon states. Explicitly, we have

$$\begin{aligned} G_1(x_1,x_2,x_3,x_4) &= e^{\frac{2\pi}{l}\frac{1}{4}(x_1-x_2+x_3-x_4)}\,(1+\tfrac{1}{2}e^{\frac{2\pi}{l}x_{23}}+\ldots) \ , \\ G_2(x_1,x_2,x_3,x_4) &= -e^{\frac{2\pi}{l}\frac{1}{4}(x_1+3x_2-3x_3-x_4)}\,(1+\tfrac{1}{2}e^{\frac{2\pi}{l}x_{23}}+\ldots) \ . \end{aligned} \tag{3.15}$$

If we now, for instance, project on the triplet channel for the indices α_3, α_4, implying that we consider G_2 only, we find that the following 2-spinon states are nonvanishing

$$(t^a)_{\alpha\beta}\,\phi^\alpha_{-\frac{3}{4}}\phi^\beta_{-\frac{1}{4}}\,|\,0\,\rangle \ , \qquad (t^a)_{\alpha\beta}\,\phi^\alpha_{-\frac{7}{4}}\phi^\beta_{-\frac{1}{4}}\,|\,0\,\rangle \ , \qquad \text{etc.} \tag{3.16}$$

Similarly, in the singlet channel, where $(2G_1+G_2)$ is the relevant combination, we find the following 2-spinon states

$$\epsilon_{\alpha\beta}\,\phi^\alpha_{\frac{1}{4}}\phi^\beta_{-\frac{1}{4}}\,|\,0\,\rangle \ , \qquad \epsilon_{\alpha\beta}\,\phi^\alpha_{-\frac{7}{4}}\phi^\beta_{-\frac{1}{4}}\,|\,0\,\rangle \ , \qquad \text{etc.} \tag{3.17}$$

We would like to give two comments on these results. First, one notes that the mode-index of the second ϕ factor is shifted from $(-\frac{1}{4} \bmod \mathbb{Z})$ to $(-\frac{3}{4} \bmod \mathbb{Z})$. Second, one observes that the state with mode-indices $(-\frac{3}{4}, -\frac{1}{4})$ vanishes in the singlet channel. The vanishing of specific multi-spinon states can be understood from a 'generalized Pauli principle' for semionic operators, which is to be discussed in sections 4 and 5.

4. Spinon formulation

4.1. Generalized commutation relations

In this section we start our discussion of what we call the 'spinon formulation' of the $SU(2)_1$ WZW model. Since we shall be considering the repeated application of modes of the spinon fields $\phi^\alpha(z)$, we start by deriving generalized commutation relations that are satisfied by these modes.

Our starting point are the following spinon-spinon OPE's

$$\begin{aligned}\phi^\alpha(z)\phi^\beta(w) &= (-1)^q (z-w)^{-\frac{1}{2}} \epsilon^{\alpha\beta} \left(1 + \tfrac{1}{2}(z-w)^2 T(w) + \ldots\right) + \\ &\quad -(-1)^q (z-w)^{\frac{1}{2}} (t_a)^{\alpha\beta} \left(J^a(w) + \tfrac{1}{2}(z-w)\partial J^a(w) + \ldots\right) \ , \end{aligned} \tag{4.1}$$

where $(t_a)^{\alpha\beta} = d_{ab}\epsilon^{\beta\gamma}(t^b)^\alpha{}_\gamma$. In these formulas, q depends on the sector that the OPE's are acting on : $q=0$ on states that are created by an even number of spinons (*i.e.*, the states in the vacuum module of $A_1^{(1)}$) and $q=1$ if the number of spinons is odd (the $j=\frac{1}{2}$ module of $A_1^{(1)}$). The appearance of explicit factors $(-1)^q$ is due to our convention to write upper indices on all spinon fields. The natural convention would be to write lower indices whenever a spinon field acts on a $q=1$ state; raising these indices leads to a relative minus sign between the OPE's in the two sectors $q=0,1$.

The occurence of a factor $(z-w)^{1/2}$ in the spinon-spinon OPE's clearly shows that the braiding properties of the spinons are those of semions or 'half-fermions.' The mode expansions of the spinons are

$$\begin{aligned}\phi^\alpha(z)\chi_q(0) &= \sum_m z^{m+\frac{q}{2}} \phi^\alpha_{-m-\frac{q}{2}-\frac{1}{4}} \chi_q(0) \ , \\ \phi^\alpha_{-m-\frac{q}{2}+\frac{3}{4}} \chi_q(0) &= \oint \frac{dz}{2\pi i} z^{-m-\frac{q}{2}} \phi^\alpha(z)\chi_q(0) \ , \end{aligned} \tag{4.2}$$

where $\chi_q(0)$ is an arbitrary state in the sector indicated by the value of q. On states with $q=0$ we can apply modes $\phi^\pm_{-1/4-n}$ with n integer, and on states with $q=1$ we can apply $\phi^\pm_{-3/4-n}$.

By following a standard procedure (see *e.g.* [7]) we can derive the following relations for the modes of the spinon fields

$$\sum_{l\geq 0} C_l^{(-\frac{1}{2})} \left(\phi^\alpha_{-m-\frac{q+1}{2}-l+\frac{3}{4}} \phi^\beta_{-n-\frac{q}{2}+l+\frac{3}{4}} - \left(\begin{array}{c} \alpha \leftrightarrow \beta \\ m \leftrightarrow n \end{array}\right)\right) = (-1)^q \, \epsilon^{\alpha\beta} \, \delta_{m+n+q-1} \ , \tag{4.3}$$

$$\sum_{l\geq 0} C_l^{(-\frac{3}{2})} \left(\phi^\alpha_{-m-\frac{q+1}{2}-l-\frac{1}{4}} \phi^\beta_{-n-\frac{q}{2}+l+\frac{3}{4}} + \left(\begin{array}{c} \alpha \leftrightarrow \beta \\ m \leftrightarrow n \end{array} \right) \right)$$

$$= (-1)^q \left(-\epsilon^{\alpha\beta}(m+\frac{q}{2})\delta_{m+n+q} - (t_a)^{\alpha\beta} J^a_{-m-n-q} \right) , \tag{4.4}$$

$$\sum_{l\geq 0} C_l^{(-\frac{5}{2})} \left(\phi^\alpha_{-m-\frac{q+1}{2}-l-\frac{5}{4}} \phi^\beta_{-n-\frac{q}{2}+l+\frac{3}{4}} - \left(\begin{array}{c} \alpha \leftrightarrow \beta \\ m \leftrightarrow n \end{array} \right) \right)$$

$$= (-1)^q \Big(\tfrac{1}{2}\,\epsilon^{\alpha\beta}\,(m+\frac{q}{2})(m+1+\frac{q}{2})\delta_{m+n+q+1}$$

$$-\tfrac{1}{2}(t_a)^{\alpha\beta}(n-m) J^a_{-m-n-q-1} + \tfrac{1}{2}\,\epsilon^{\alpha\beta} L_{-m-n-q-1} \Big) . \tag{4.5}$$

In these relations, the coefficients $C_l^{(\alpha)}$ are defined by the expansion

$$(1-x)^\alpha = \sum_{l\geq 0} C_l^{(\alpha)} x^l . \tag{4.6}$$

Algebras of the type (4.3) are known as 'generalized vertex algebras' [8]; other examples include the so-called parafermion algebras [7] or Z-algebras [9].

The relations (4.3) can be interpreted as *generalized canonical commutation relations* of the fundamental spinon fields. The other relations can be used to express the current modes J^a_n and the Virasoro generators L_n as bilinears in spinon modes. We would like to stress that, up to the complication of the infinite series in the mode index l, these relations are very similar to the anticommutation relations for free fermions and to the formulas that express affine and Virasoro currents as fermion bilinears.

With the generalized commutation relations (4.3) in place, we are now ready to consider the Fock space generated by the spinon fields acting on the vacuum. By this we mean that we construct all possible multi-spinon states, taking into account equivalences that follow from (4.3). A surprising result, which we shall further discuss in the next section, is the following

> *The (chiral part of the) Hilbert space of the $SU(2)_1$ WZW model is identical to the spinon Fock space constructed using the generalized commutation relations (4.3).*

To illustrate what this result really means, let us look back at the formulation of the same theory using affine Kac Moody algebras (see section 2). In that formulation, one constructs affine modules by applying affine current modes J^a_m to a highest weight state (giving 'current Fock spaces' as opposed to 'spinon Fock spaces'). An important point was that in those affine modules there were null states, *i.e.*, states whose vanishing does not directly follow from the commutation relations of the J's. (Examples of such null states are the states (2.8). In the spinon formulation of the theory the only relations among the multi-spinon states are those implied by the generalized commutation relations (4.3), which express the semionic nature of the fields. There are no null states, and the theory can be viewed as a theory of *free* spinon fields.

As a comment, we mention that there are many more conformal field theories for which the Hilbert space can be generated by repeatedly acting with modes of one or a few low-dimension primary fields. However, in general one should expect that the Fock space generated from such generalized spinons is larger that the actual Hilbert space of the theory. The systematics of reducing such Fock spaces to the true Hilbert space have not been worked out. What is so special about the $SU(2)_1$ WZW model and about a number of other examples [4, 5], is that the Hilbert space is actually a free Fock space of fractional statistics objects.

4.2. Yangian symmetry

When setting up a description of the $SU(2)_1$ WZW field theory in terms of multi-spinon states, one soon discovers that the affine and Virasoro symmetries are not very convenient tools, the problem being that their action on multi-spinon states is not easily tractable. Interestingly, it has been found that this same field theory admits another highly non-trivial symmetry structure, the so-called Yangian $Y(sl_2)$, which is very natural from the point of the spinon formulation.

In Appendix A we recall the definition of the Yangian $Y(sl_2)$, which is an example of a non-trivial quantum group. Let us now show that this algebra can be represented on the Hilbert space of the $SU(2)_1$ WZW model. Following [1], we make the following definitions

$$Q^a_0 = J^a_0 \, , \quad Q^a_1 = \tfrac{1}{2} f^a{}_{bc} \sum_{m>0} J^b_{-m} J^c_m \; . \tag{4.7}$$

It may be checked that, when acting on integrable highest weight representations of $A^{(1)}_1$ at level $k = 1$, these generators satisfy the defining relations (A.1) of $Y(sl_2)$.

It is easily seen that the Virasoro generator L_0 commutes with the Yangian generators (4.7). It is actually expected [1, 10, 2] that there exists an infinite number of mutually commuting operators H_n, $n = 1, 2, \ldots$, that all commute with the Yangian, the first few examples being $H_1 = L_0$ and

$$H_2 = d_{ab} \sum_{m>0} m \, J^a_{-m} J^b_m \ . \tag{4.8}$$

The existence of Yangian symmetry and of the operators H_n suggests a description of the Hilbert space in terms of irreducible multiplets of the Yangian, which are each characterized by specific eigenvalues of the H_n. As we shall see, such multiplets are naturally described in terms of multi-spinon states.

As an aside we remark that, if the group $SU(2)$ is replaced by $SU(N)$, the expressions (4.7) and (4.8) for Q^a_1 and H_2 have to be modified by additional terms that involve the 3-index d-symbol of $SU(N)$, $N \geq 3$ [11].

5. Structure of the Hilbert space

5.1. Example: 2-spinon states

To illustrate the connection between the spinon formulation and Yangian symmetry, we shall first analyze in some detail the structure of the 2-spinon states in the spectrum.

We introduce the following notations for general 2-spinon states (t, s refer to $SU(2)$ triplet and singlet channels, respectively)

$$\Phi^{t,a}_{n_2,n_1} = (t^a)_{\alpha\beta}\ \phi^{\alpha}_{-3/4-n_2}\phi^{\beta}_{-1/4-n_1}|0\rangle, \quad \Phi^{s}_{n_2,n_1} = \epsilon_{\alpha\beta}\ \phi^{\alpha}_{-3/4-n_2}\phi^{\beta}_{-1/4-n_1}|0\rangle\ . \tag{5.1}$$

The energy eigenvalue of these states is simply

$$L_0 = 1 + n_1 + n_2\ . \tag{5.2}$$

We can now use eq. (3.1) to show that (4.7) leads to

$$\begin{aligned} Q^a_1 \Phi^{t,b}_{n_2,n_1} &= -(n_2+n_1+\tfrac{1}{2}) f^{ab}{}_c \Phi^{t,c}_{n_2,n_1} + (n_2-n_1+1) d^{ab} \Phi^{s}_{n_2,n_1} + d^{ab} \sum_{l>0} \Phi^{s}_{n_2+l,n_1-l} \\ Q^a_1 \Phi^{s}_{n_2,n_1} &= 2(n_2-n_1) \Phi^{t,a}_{n_2,n_1} - 2 \sum_{l>0} \Phi^{t,a}_{n_2+l,n_1-l}\ . \end{aligned} \tag{5.3}$$

The first thing to notice from these formulas is the fact that the Yangian generators Q^a_1 map 2-spinon states into 2-spinon states. This illustrates our claim that the Yangian is entirely natural from the point of view of the spinon formulation. Notice also that the action of Q^a_1 is not diagonal in the indices (n_2, n_1) but rather lower-triangular in the sense that (n_2, n_1) gets mapped into $(n_2 + l, n_1 - l)$ with $l \geq 0$ and $n_1 - l \geq 0$.

From the action of Q^a_1 it is easily seen that the space of all two-spinon states with $n_1 + n_2 = n$ fixed can be decomposed into multiplets of the Yangian. Each multiplet contains a Yangian Highest Weight Vector (YHWV), *i.e.*, a state that is highest weight with respect to $SU(2)$, is an eigenvector for Q^3_1 and is annihilated by Q^{++}_1. These YHWV's are of the form

$$\Phi^{t,++}_{n_2,n_1} + \sum_{l>0} a^{(l)}_{n_2,n_1} \Phi^{t,++}_{n_2+l,n_1-l}\ , \tag{5.4}$$

where the $a^{(l)}_{n_2,n_1}$ are real coefficients. The 2-spinon Yangian multiplets each contain a triplet and a singlet of $SU(2)$, *i.e.*, a total of four states, except if $n_1 = n_2$, when the relation (4.3) can be used to show that the singlet is absent.

We remark that Q^a_1 acts by comultiplication (A.3) on the 2-spinon YHWV, given its action on the 1-spinon states (which are YHWV).

Turning to the operator H_2, using the relation

$$(\partial J^a \phi^\alpha)(z) = \frac{4}{3}(t^a)^\alpha{}_\beta \partial^2 \phi^\beta(z) \ , \tag{5.5}$$

we find that the action on 2-spinon states is as follows

$$\begin{aligned} H_2 \Phi^{t,a}_{n_2,n_1} &= 2\left((n_2+1)(n_2+\frac{1}{2}) + (n_1+\frac{1}{2})n_1\right) \Phi^{t,a}_{n_2,n_1} + \sum_{l>0} l\, \Phi^{t,a}_{n_2+l,n_1-l} \ , \\ H_2 \Phi^{s}_{n_2,n_1} &= 2\left((n_2+1)(n_2+\frac{1}{2}) + (n_1+\frac{1}{2})n_1\right) \Phi^{s}_{n_2,n_1} - 3\sum_{l>0} l\, \Phi^{s}_{n_2+l,n_1-l} \ . \end{aligned} \tag{5.6}$$

This action is lower triangular, and the eigenvalues of the operator H_2 are immediately seen to equal $2\left((n_2+1)(n_2+\frac{1}{2}) + (n_1+\frac{1}{2})n_1\right)$. As H_2 commutes with the Yangian, the H_2 eigenstates group into Yangian multiplets, and the YHWV's are given by the H_2 eigenstates with triplet index $+\!\!+$.

To be completely explicit, we present the example where $n_1 + n_2 = 4$, which are the 2-spinon states with $L_0 = 5$ (from (5.2)). In the following formula we list the labels (n_2, n_1) of the Yangian representation, the H_2-eigenvalues, and the states

$$\begin{array}{lll} (4,0) & H_2 = 45 & \Phi^{t,a}_{4,0} \ , \quad \Phi^{s}_{4,0} \\ (3,1) & H_2 = 31 & \Phi^{t,a}_{3,1} - \frac{1}{14}\Phi^{t,a}_{4,0} \ , \quad \Phi^{s}_{3,1} + \frac{3}{14}\Phi^{s}_{4,0} \\ (2,2) & H_2 = 25 & \Phi^{t,a}_{2,2} - \frac{1}{6}\Phi^{t,a}_{3,1} - \frac{11}{120}\Phi^{t,a}_{4,0} \ . \end{array} \tag{5.7}$$

When acting on the $(2,2)$ YHWV, $Q^{=}_1$ produces a multiple of the $SU(2)$ descendant plus a state proportional to $\Phi^s_{2,2} + \frac{1}{2}\Phi^s_{3,1} + \frac{3}{8}\Phi^s_{4,0}$, which vanishes as a consequence of the commutation relation (4.3).

5.2. N-SPINON STATES

The analysis of the previous subsection can in principle be generalized to the case of general N-spinon states. We shall first focus on the following set of states, which we call *fully polarized N-spinon states*

$$\phi^+_{-\frac{(2N-1)}{4}-n_N} \cdots \phi^+_{-\frac{5}{4}-n_3}\phi^+_{-\frac{3}{4}-n_2}\phi^+_{-\frac{1}{4}-n_1}|0\rangle \,,$$
$$\text{with } \; n_N \geq n_{N-1} \geq \ldots \geq n_2 \geq n_1 \geq 0 \,. \tag{5.8}$$

The eigenvalue of the Virasoro zero mode L_0 on these states is

$$L_0 = \frac{N^2}{4} + \sum_{i=1}^{N} n_i \,. \tag{5.9}$$

One easily derives the following explicit expression for the action of H_2 on a fully polarized N-spinon state. Denoting by $|\chi^{(N-1)}\rangle$ a fully polarized $(N-1)$-spinon state, we have from (4.8), (5.5)

$$[H_2, \phi^+_{-\frac{2N-1}{4}-n_N}]|\chi^{(N-1)}\rangle =$$
$$2\,(n_N + \tfrac{N-1}{2})(n_N + \tfrac{N}{2}))\,\phi^+_{-\frac{2N-1}{4}-n_N}|\chi^{(N-1)}\rangle + \sum_{l>0} l\,\phi^+_{-\frac{2N-1}{4}-n_N-l} J^3_l|\chi^{(N-1)}\rangle \,. \tag{5.10}$$

Since this action is again 'lower triangular,' the eigenvalue of H_2 on the eigenstate labeled by mode-indices $\{n_1, n_2, \ldots, n_N\}$ is found to be

$$H_2 = \sum_{i=1}^{N} 2\,(n_i + \tfrac{1}{2}(i-1))(n_i + \tfrac{1}{2}i) \,. \tag{5.11}$$

The fully polarized N-spinon states that are eigenstates of H_2 are YHWV's. By acting with the Yangian generators Q^a_0 and Q^a_1 we may construct Yangian multiplets. Obviously, all the states in one such multiplet share common eigenvalues for the operators H_n, $n = 1, 2, \ldots$.

The precise structure of the Yangian multiplets has first been explored in the context of the so-called Haldane-Shastry spin chain with inverse square exchange [1].

The $SU(2)_1$ conformal field theory can be viewed as a continuum limit of the Haldane-Shastry chain, and many results, in particular statements about Yangian symmetry and about the structure of the spectrum, carry over to the field theory. Backing all these results is the representation theory of Yangians, which has been worked out in detail in [12].

In the language that we developed, the Yangian multiplets can be characterized as follows. (i): Each Yangian multiplet is characterized by a set of non-decreasing integers $\{n_i\}_{i=1,\ldots,N}$ as in (5.8). (ii): The eigenvalue of L_0 (commuting with the Yangian) on the states in the Yangian multiplet specified by $\{n_i\}_{i=1,\ldots,N}$ is given by (5.9). (iii): When acting on a YHWV (which will be a linear combination of fully polarized states of the form (5.8)), the Yangian generators (4.7) create states of a similar form, which however have some of the $+$ indices replaced by $-$, and which have different coefficients in the linear combination. If we were to act only with Q_0^a we would find a total of $N+1$ such states; the maximal possible number when acting with the full Yangian is 2^N. This maximal number is only realized if the mode-indices n_i are all different. If some of the n_i's are equal, the corresponding product of doublets is projected on the symmetric combination. For example, a 2-spinon Yangian multiplet will have $3+1=4$ states if $n_2 \neq n_1$, but only 3 states if $n_2 = n_1$. The vanishing of some of the singlet channel states is encoded in the generalized commutation relations (4.3).

The union of all Yangian multiplets, whose structure we just described, precisely forms a basis of the Hilbert space of the $SU(2)_1$ WZW model. The 'Yangian rules' for the construction of multi-spinon states can thus be interpreted as a generalized Pauli principle that governs the filling of possible momentum states of the fundamental spinons. As we already mentioned, the full set of rules reflects the semionic nature of the spinons, which can otherwise be viewed as free objects. It is instructive to compare the Pauli priciple for spinons with that for spinful fermions, where, as is well-known, double occupancy of a given momentum state leads to anti-symmetrization of internal indices.

We would now like to give explicit expressions for the fully polarized N-spinon states that are eigenstates of H_2, *i.e.*, the YHWV's. These results, which were first given in [2], have been inspired by the machinery that has recently been developed for the analysis of a variety of exactly solvable quantum mechanical systems with inverse square exchange. The explicit formulas involve so-called Jack polynomials, whose properties have been studied in the mathematical literature (see, *e.g.*, [13]). We will here follow the conventions and normalizations of [2].

The construction in [2] uses an auxiliary free field $\varphi(z) = q - ip\ln z + i\sum_{n\neq 0}\alpha_n \frac{z^{-n}}{n}$, in terms of which the fundamental spinons can be written as

$$\phi^{\pm}(z) = : e^{\pm i \frac{1}{\sqrt{2}}\varphi(z)} : \ . \tag{5.12}$$

The N-spinon YHWV with labels $\{n\} = \{n_N, n_{N-1}, \ldots, 1\}$ can be written as

$$|\{n\}\rangle = (-1)^{|n|}\, P^{(-2)}_{\{n'\}}(p_n)\, e^{i\frac{N}{\sqrt{2}}q}\,|0\rangle \ . \tag{5.13}$$

In this formula, the set $\{n'\}$ is dual to $\{n\}$ in the sense of partitions or, equivalently, Young tableaux and we wrote $|n| = \sum n_i$. The function $P^{(-2)}_{\{n'\}}$ is a Jack polynomial, whose arguments p_n are set equal to the oscillators of the auxiliary boson field $\varphi(z)$,

$$p_n = -\tfrac{1}{2}\sqrt{2}\,\alpha_{-n} \ . \tag{5.14}$$

Let us do an example and show that the explicit 2-spinon YHWV's that we discussed before are indeed reproduced by this general formula. Choosing $n_2 = 2, n_1 = 1$, we have

$$\begin{aligned} |\{2,1\}\rangle &= -P^{(-2)}_{\{2,1\}}(p_n)\, e^{i\sqrt{2}q}\,|0\rangle \\ &= -\tfrac{2}{5}(p_1^3 - \tfrac{1}{2}p_2p_1 - \tfrac{1}{2}p_3)\, e^{i\sqrt{2}q}\,|0\rangle \ . \end{aligned} \tag{5.15}$$

In the language that we used before we would have written this YHWV as

$$\Phi^{t,++}_{2,1} - \tfrac{1}{10}\Phi^{t,++}_{3,0} \tag{5.16}$$

(compare with (5.7)). Explicit algebra gives

$$\begin{aligned} \Phi^{t,++}_{2,1} &= -\left(P^{(-2)}_{\{1,1\}}\,P^{(-2)}_{\{1\}} - \tfrac{1}{2}P^{(-2)}_{\{1,1,1\}}\right)(p_n)\, e^{i\sqrt{2}q}\,|0\rangle \\ \Phi^{t,++}_{3,0} &= -P^{(-2)}_{\{1,1,1\}}(p_n)\, e^{i\sqrt{2}q}\,|0\rangle \end{aligned} \tag{5.17}$$

and the two expressions (5.15), (5.16) for the YHWV are seen to agree by using the recursion relation

$$P^{(-2)}_{\{2,1\}} = P^{(-2)}_{\{1,1\}}P^{(-2)}_{\{1\}} - \tfrac{3}{5}P^{(-2)}_{\{1,1,1\}} \ . \tag{5.18}$$

The YHWV's satisfy the following orthogonality condition

$$\langle\{m\}|\{n\}\rangle = \delta_{\{m\},\{n\}}\, j_{\{n'\}}\ . \tag{5.19}$$

Explicit results for the metric $j_{\{n\}}$ may be found for example in [13]. In the next section, we will use these general results for the explicit computation of $2N$-point correlation functions.

Before closing this section, we present a second way to write a multi-spinon basis for the $SU(2)_1$ WZW model. One considers the states

$$\phi^-_{-\frac{2(N^++N^-)-1}{4}-n^-_{N^-}} \cdots \phi^-_{-\frac{2(N^++1)-1}{4}-n^-_1}\phi^+_{-\frac{2N^+-1}{4}-n^+_{N^+}} \cdots \phi^+_{-\frac{1}{4}-n^+_1}|0\rangle\ ,$$

$$\text{with}\ \ n^+_{N^+} \geq \ldots \geq n^+_2 \geq n^+_1 \geq 0\ , \quad n^-_{N^-} \geq \ldots \geq n^-_2 \geq n^-_1 \geq 0\ . \tag{5.20}$$

Using an induction argument, it can easily be shown that the generalized commutation relations (4.3) allow one to write every mixed index multi-spinon state as a sum of states of the form (5.20). The eigenvalue of L_0 is now given as

$$L_0 = \frac{(N^+ + N^-)^2}{4} + \sum_{i=1}^{N^+} n_i^+ + \sum_{i=1}^{N^-} n_i^-\ . \tag{5.21}$$

6. Applications

6.1. Ward Identities and correlators

In section 3 we discussed the KZ equations for correlation functions in the $SU(2)_1$ CFT and explicitly solved for the 4-spinon correlation functions. In the spinon formulation of the theory, one would like to derive similar results, this time not by using the KZ equations (which have their origin in the affine symmetry of the theory) but rather by using Ward identities that are directly related to the Yangian symmetry and to the existence of the higher conserved quantities H_n. Without derivation (see however [2]), we present the following Ward identities, which are associated with Q_1^a and H_2, respectively

$$
\begin{aligned}
0 &= \left(-2\sum_i (z_i\partial_i)(t^a)^{(i)} + \tfrac{1}{2} f^a{}_{bc} \sum_{i\neq j} \theta_{ij} (t^b)^{(i)} (t^c)^{(j)} \right) \langle \phi(z_1)\dots\phi(z_N)\rangle \ , \\
0 &= \left(2\sum_i ((z_i\partial_i)^2 + \tfrac{1}{2}(z_i\partial_i)) - \sum_{i\neq j} \theta_{ij}\theta_{ji} (t_a)^{(i)} (t^a)^{(j)} \right) \langle \phi(z_1)\dots\phi(z_N)\rangle \ . \quad (6.1)
\end{aligned}
$$

It is easily seen that these two equations, together with the Ward identities coming from translational and scale invariance (*i.e.*, L_{-1} and L_0) and from Q_0^a, are sufficient to completely determine the 4-point correlation functions. One expects that more general correlators can be determined by invoking in addition Ward identities coming from the higher conserved quantities H_n, $n \geq 3$. Rather than pursuing this road, we will now present a direct computation of a general $2N$-spinon correlation function.

Before we come to the correlation functions, let us write an expression for the action of a product of spinon fields $\phi^+(z)$ on the vacuum. We have [2]

$$
\phi^+(z_1)\dots\phi^+(z_N)|0\rangle = \prod_{i<j}(z_i - z_j)^{1/2} \sum_{n_N\geq\dots\geq n_1\geq 0} P^{(-1/2)}_{\{n_N,\dots,n_1\}}(z_1,\dots,z_N)|\{n_N,\dots n_1\}\rangle \ . \quad (6.2)
$$

Notice that the indices of the Jack polynomial in the wave function are $(-\frac{1}{2})$ and $\{n\}$, which are dual to the indices (-2) and $\{n'\}$ on the Jack polynomial used in the construction of the YHWV $|\{n\}\rangle$ in eq. (5.13). The Jack polynomials are now evaluated at argument $p_n = \sum_i z_i^n$.

It will be clear to the reader that the formula (6.2), which decomposes a general spinon field product on an orthogonal basis of YHWV's, is a convenient starting point for the computation of correlation functions.

Let us warm up by computing the two point function. Assuming $|z_1| > |z_2|$, we have

$$\begin{aligned} \langle \phi_+(z_1)\phi^+(z_2)\rangle &= \langle 0|\phi_+(z_1)\phi^+(z_2)|0\rangle \\ &= z_1^{-1/2} \sum_{m,n} \langle m|P^{(-1/2)}_{\{m\}}(\frac{1}{z_1})P^{(-1/2)}_{\{n\}}(z_2)|n\rangle \\ &= \sum_n C_n^{(-\frac{1}{2})}(\frac{z_2}{z_1})^n z_1^{-1/2} = (z_1 - z_2)^{-1/2} \ , \end{aligned} \tag{6.3}$$

where we used that with our choice of normalization $P^{(-1/2)}_{\{n\}}(z) = z^n$ and $j_{\{1^n\}} = C_n^{(-\frac{1}{2})}$.

For a general $2N$-point function, we can avoid using explicit expressions for the metric $j_{\{n\}}$, and instead use the general result

$$\sum_{\{n\}} j_{\{n'\}} P^{(-1/2)}_{\{n\}}(x) P^{(-1/2)}_{\{n\}}(y) = \prod_{i=1}^N \prod_{j=1}^N (1 - x_i y_j)^{-1/2} \ . \tag{6.4}$$

This immediately leads to the following result for the $2N$-point spinon correlator

$$\begin{aligned} &\langle \phi_+(w_1)\ldots\phi_+(w_N)\phi^+(z_1)\ldots\phi^+(z_N)\rangle \\ &= \prod_{i=1}^N w_i^{-1/2} \prod_{i>j}(\frac{1}{w_i} - \frac{1}{w_j})^{1/2} \prod_{i<j}(z_i - z_j)^{1/2} \\ &\quad \times \sum_{\{m\},\{n\}} \langle\{m\}|P^{(-1/2)}_{\{m\}}(\frac{1}{w_1},\ldots,\frac{1}{w_N})P^{(-1/2)}_{\{n\}}(z_1,\ldots,z_N)|\{n\}\rangle \\ &= \prod_{i=1}^N w_i^{-N/2} \prod_{i<j}(w_i - w_j)^{1/2} \prod_{i<j}(z_i - z_j)^{1/2} \\ &\quad \times \sum_{\{n\}} j_{\{n'\}} P^{(-1/2)}_{\{n\}}(\frac{1}{w_1},\ldots,\frac{1}{w_N})P^{(-1/2)}_{\{n\}}(z_1,\ldots,z_N) \\ &= \prod_{i=1}^N w_i^{-N/2} \prod_{i<j}(w_i - w_j)^{1/2} \prod_{i<j}(z_i - z_j)^{1/2} \prod_{i,j}(1 - \frac{z_i}{w_j})^{-1/2} \end{aligned}$$

$$= \left(\frac{\prod_{i<j}(w_i - w_j) \prod_{i<j}(z_i - z_j)}{\prod_{i,j}(w_j - z_i)} \right)^{1/2} . \qquad (6.5)$$

One may check that this result is consistent with the triplet channel 4-point function $G_2(x)$ that we computed in section 3. Once again, there is no novelty in the final expressions, which are Gaussian and can be recovered by using a free boson formalism. Our point here has been to illustrate some techniques which we expect to be valuable in much more general situations.

6.2. Characters

An interesting observation, which we worked out in our paper [3], is that the spinon formulation of the $SU(2)_1$ WZW model directly leads to novel ways to write the characters of the Virasoro and affine modules in this theory. Without repeating the derivation here, we shall here briefly summarize the results of [3].

The Virasoro character for the module characterized the conformal dimension $L_0 = j^2$ (where $j = 0, \frac{1}{2}, 1, \ldots$ can be related to an $SU(2)$ spin) is found to be

$$\begin{aligned} \chi^{\rm Vir}_{j^2}(q) &= q^{-j/2} \sum^{*}_{m_1, m_2, \ldots} q^{\frac{1}{2}(m_1^2 + m_2^2 + \ldots - m_1 m_2 - m_2 m_3 - \ldots)} \\ &\quad \times \frac{1}{(q)_{m_1}} \prod_{a \geq 2} \left[\begin{array}{c} \frac{1}{2}(m_{a-1} + m_{a+1} + \delta_{a,2j+1}) \\ m_a \end{array} \right]_q . \end{aligned} \qquad (6.6)$$

In these formulas the m_i are non-negative integers; the $*$ on the summation symbol indicates that $m_{2j}, m_{2j-2}, \ldots$ are to be odd and the other m_i even. The q-deformed binomial coefficients that feature in this formula are defined as

$$\left[\begin{array}{c} a \\ b \end{array} \right]_q = \frac{(q)_a}{(q)_{a-b}(q)_b}, \quad \text{for} \quad a \geq b , \qquad (6.7)$$

(and zero otherwise), where

$$(q)_a = \prod_{n=1}^{a} (1 - q^n) \qquad (6.8)$$

with $(q)_0 = 1$ and $(q)_{-a} = 0$. In the character formula (6.6), the number m_1 is precisely the number of spinons in a state that contributes to the character. When expanded in a q-series, the character formula (6.6) by construction agrees with the formula (2.3) that we wrote earlier.

The formula (6.6) is a limiting case of a corresponding formula for Virasoro minimal models, which was first conjectured in [14] and later proven in [15].

From the basis of states which we gave at the end of section 5, formula (5.20), one easily derives the following expressions for the (level-1) affine characters in the theory

$$\chi^{A_1^{(1)}}_{j=0}(q) = \sum_{N^+ + N^- \text{ even}} \frac{q^{(N^+ + N^-)^2/4}}{(q)_{N^+}(q)_{N^-}} \ , \qquad \chi^{A_1^{(1)}}_{j=\frac{1}{2}} = \sum_{N^+ + N^- \text{ odd}} \frac{q^{(N^+ + N^-)^2/4}}{(q)_{N^+}(q)_{N^-}} \ . \qquad (6.9)$$

These formulas were first written in [16] and they were related to the spinon picture in [2] (see also [17] for closely related results).

In summary, we have in this section indicated a number of ways in which the spinon formulation and Yangian symmetry of the $SU(2)_1$ CFT can be used. We expect that many more applications are to be found, in particular in the context of the Thermodynamic Bethe Ansatz and of integrable perturbations of CFT. We will report on some such developments elsewhere [5].

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A. The Yangian $Y(sl_2)$

The Yangian $Y(\mathbf{g})$ associated to a Lie algebra $\mathbf{g}$ is a Hopf algebra that is neither commutative nor cocommutative, and as such it can be viewed as a non-trivial example of a quantum group [18]. Its history goes back to the general formalism of the Quantum Inverse Scattering Method (see [19] for an introduction). Indeed, the object $Y(\mathbf{g})$ is directly related to certain rational solutions of the Quantum Yang Baxter Equation, the simplest of which was first obtained by C.N. Yang [20]. For the purpose of these proceedings we shall consider the case $\mathbf{g} = sl_N$.

We write the lowest generators of $Y(sl_N)$ as Q_0^a and Q_1^a; higher generators can be obtained by taking successive commutators with the generators Q_1^a. The defining relations of the algebra $Y(sl_N)$ can be written as follows [18]

$$
\begin{aligned}
&\text{(Y1)} \qquad [Q_0^a, Q_0^b] = f^{ab}{}_c Q_0^c \ , \\
&\text{(Y2)} \qquad [Q_0^a, Q_1^b] = f^{ab}{}_c Q_1^c \ , \\
&\text{(Y3)} \qquad [Q_1^a, [Q_1^b, Q_0^c]] + (\text{cyclic in } a,b,c) = A^{abc}{}_{def}\{Q_0^d, Q_0^e, Q_0^f\} \ , \\
&\text{(Y4)} \qquad [[Q_1^a, Q_1^b], [Q_0^c, Q_1^d]] + [[Q_1^c, Q_1^d], [Q_0^a, Q_1^b]] \\
&\qquad\qquad = \left(A^{abp}{}_{qrs} f^{cd}{}_p + A^{cdp}{}_{qrs} f^{ab}{}_p\right)\{Q_0^q, Q_0^r, Q_1^s\} \ ,
\end{aligned}
\tag{A.1}
$$

where $A^{abp,def} = \frac{1}{4} f^{adp} f^{beq} f^{cfr} f_{pqr}$ and the curly brackets denote a completely symmetrized product. The $SU(N)$ structure constants f^{abc} have been normalized as

$$
f^{abc} f^{d}{}_{bc} = -2N\, d^{ad} \ . \tag{A.2}
$$

The following comultiplications may be used to define the action of the Yangian generators on a tensor product of states

$$
\begin{aligned}
\Delta_\pm(Q_0^a) &= Q_0^a \otimes \mathbf{1} + \mathbf{1} \otimes Q_0^a \ , \\
\Delta_\pm(Q_1^a) &= Q_1^a \otimes \mathbf{1} + \mathbf{1} \otimes Q_1^a \pm \tfrac{1}{2} f^a{}_{bc} Q_0^b \otimes Q_0^c \ .
\end{aligned}
\tag{A.3}
$$

The ‘terrific’ (dixit Drinfel’d [18]) right hand sides of the relations (Y3) and (Y4) can be derived from the homomorphism property of these comultiplications. For $\mathbf{g} = sl_2$, the cubic relation (Y3) is superfluous and for all other algebras (Y4) follows from (Y2) and (Y3).

References

[1] F.D.M. Haldane, Z.N.C. Ha, J.C. Talstra, D. Bernard and V. Pasquier, Phys. Rev. Lett. **69** (1992) 2021.

[2] D. Bernard, V. Pasquier and D. Serban, Nucl. Phys. **B428** (1994) 612, (hep-th/9404050).

[3] P. Bouwknegt, A. Ludwig and K. Schoutens, Phys. Lett. **338B** (1994) 448, (hep-th/9406020).

[4] P. Bouwknegt, A. Ludwig and K. Schoutens, *Spinon basis for higher level $SU(2)$ WZW models*, (hep-th/9412108).

[5] P. Bouwknegt, A. Ludwig and K. Schoutens, in preparation.

[6] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. **B247** (1984) 83.

[7] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. J.E.T.P. **62** (1985) 215; Sov. Phys. J.E.T.P. **63** (1986) 913.

[8] C. Dong and J. Lepowsky, *Generalized vertex algebras and relative vertex operators*, Prog. in Math. **112** (Birkhäuser, Boston, 1993).

[9] J. Lepowsky and R.L. Wilson, Proc. Natl. Acad. Sci. USA **78** (1981) 7254; Adv. Math. **45** (1982) 21; Inv. Math. **77** (1984) 199; Inv. Math. **79** (1985) 417.

[10] J.C. Talstra and F.D.M. Haldane, *Integrals of motion of the Haldane Shastry model*, (cond-mat/9411065).

[11] K. Schoutens, Phys. Lett. **331B** (1994) 335, (hep-th/9401154).

[12] V. Chari and A. Pressley, L'Enseignement Math. **36** (1990) 267; J. reine angew. Math. **417** (1991) 87.

[13] R.P. Stanley, Adv. Math. **77** (1989) 76.

[14] R. Kedem, T. Klassen, B. McCoy and E. Melzer, Phys. Lett. **307B** (1993) 68, (hep-th/9301046).

[15] A. Berkovich, Nucl. Phys. **B431** (1994) 315, (hep-th/9403073); A. Berkovich and B. McCoy, *Continued fractions and fermionic representations for characters of $M(p,p')$ minimal models*, (hep-th/9408113).

[16] R. Kedem, B. McCoy and E. Melzer, *The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in* $1+1$*-dimensional quantum field theory*, (hep-th/9304056); E. Melzer, Lett. Math. Phys. **31** (1994) 233, (hep-th/9312043).

[17] B.L. Feigin and A.V. Stoyanovsky, *Quasi-particles models for the representations of Lie algebras and geometry of flag manifold*, (hep-th/9308079).

[18] V.G. Drinfel'd, Sov. Math. Dokl. **32** (1985) 254; in Proceedings of the International Congress of Mathematicians, Berkeley, California (1986).

[19] V.E. Korepin, G. Izergin and N.M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz*, Cambridge University Press (1992).

[20] C.N. Yang, Phys. Rev. Lett. **19** (1967) 1312.